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COMPARING LIFETIMES OF COHERENT SYSTEMS WITH DEPENDENT COMPONENTS OPERATING IN RANDOM ENVIRONMENTS

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Abstract

We study the impact of a random environment on lifetimes of coherent systems with dependent components. There are two combined sources of this dependence. One results from the dependence of the components of the coherent system operating in a deterministic environment and the other is due to dependence of components of the system sharing the same random environment. We provide different sets of sufficient conditions for the corresponding stochastic comparisons and consider various scenarios, namely, (i) two different (as a specific case, identical) coherent systems operate in the same random environment; (ii) two coherent systems operate in two different random environments; (iii) one of the coherent systems operates in a random environment and the other in a deterministic environment. Some examples are given to illustrate the proposed reasoning.

Keywords: Coherent system; distortion/domination function; dependent component; k-out-of-n system; random environment; stochastic order

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1. Introduction and preliminaries

Most often, the real-world populations of items are heterogeneous and the corresponding homogeneity can be considered as some approximation. There can be different reasons for heterogeneity. For example, the items can be produced by different manufacturers and then mixed by the user. It can happen with one manufacturer as well, as reliability characteristics of manufactured items change with time depending on many factors (e.g. supplied material, human factors, the production condition, etc.). Heterogeneity can be also induced by a random environment in which items are operating. This random environment can be modelled by the corresponding shock process, for example (see e.g. [8] and the references therein). However, the simplest and sometimes the most effective way is to model it by the non-negative environmental random variable Θ that can affect the distribution of a lifetime X in a baseline deterministic environment (denoted by $F_X(x)$). Thus, conditional on a realization $\Theta=\theta$, the corresponding distribution of $X(\theta)$ is $F_X(x\mid\theta)$, where the most popular specific models are the scale model $F_X(\theta x)$, the multiplicative and additive frailty models written as $\theta r_X(x)$ and $r_X(x)+\theta$, where $r_X(x)$ is the corresponding failure rate assuming that it exists.

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The most common example of environment is a stress or load under which technical systems are operating (e.g. an electrical load). Some overall climate or nutrition parameters can also describe environment for organisms. An effect of a random environment on various reliability indices was intensively studied in the literature: see e.g. [11], [16], [22], [30], [31], [34], and the references therein.

While describing the following baseline simplified scenarios to be considered in our paper in much more generality, we will use several basic stochastic orders, to be defined for convenience by Definition 1.1 at the end of this section.

(i) Consider two items (systems) with lifetimes $X(\Theta)$ and $Y(\Theta)$ (later we will use a slightly different notation that is more appropriate for the multicomponent case) operating in the same environment modelled by Θ, and we are interested in stochastic comparisons of these lifetimes. Note that they are dependent via the common environment. Obviously, if we know that its impact is the same on both items (e.g. the multiplicative frailty model), then in order for $X(\Theta) \le_{\rm st} Y(\Theta)$, for example, to hold, it is sufficient for this inequality to be true for the baseline, deterministic environment, i.e. $X \le_{\rm st} Y$, as it will hold in each realization of Θ. For a general case, we must just assume this property in each realization, i.e. $X(\theta) \le_{\rm st} Y(\theta)$, for all θ.

(ii) Now let one item (or two statistically identical items) operate in two environments Θ_1 and Θ_2 with the corresponding lifetimes $X(\Theta_1)$ and $X(\Theta_2)$. There are a number of simple, meaningful results in the literature for the corresponding comparisons. For example, in [12] and [37] it is stated that if $\Theta_1 \le_{\rm hr} \Theta_2$, then $X(\Theta_1) \le_{\rm hr} X(\Theta_2)$, provided that the corresponding failure rate $r(t \mid \theta)$ is ordered in θ for all $t > 0$.

(iii) The last general introductory setting to be considered is when two items with lifetimes $X(\Theta_1)$ and $Y(\Theta_2)$, are operating in different environments modelled by Θ_1 and Θ_2, respectively. A specific case is when, for example, Θ_2 is degenerate, meaning that the second environment is deterministic.

The above scenarios are described with respect to comparisons of lifetimes of two items or systems with a *black box* description. However, our paper deals with these scenarios for the multicomponent systems, namely, *coherent systems* that satisfy two basic requirements: each component is important for operation of a system and the system lifetime should not decrease if we replace any failed component with a ‘new’ one. This class of systems is rather wide and includes, for example, the k-out-of-n systems as a special case (see [5]).

Stochastic comparisons for k-out-of-n systems with independent components are extensively studied in [4], [13], [32], and [33], to name a few. General coherent systems with independent components are considered, for example, in [6], [10], [14], [17], [18], [23], [35], and [36]. Stochastic ordering for coherent systems with dependent components is discussed in [24], [25], [26], [27], and [28]. The impact of a random environment has been studied via multivariate mixture models; references include [2], [3], [7], [19], and [20].

However, to the best of our knowledge, only two papers have been devoted to stochastic comparisons of coherent systems with dependent components operating in different random environments [1, 21]. These publications provide some detailed sufficient conditions for the lifetime of a coherent system operating in one environment (e.g. the more severe) to be smaller than that of this system operating in a milder environment. This setting generalizes scenario (ii) above to the case of systems with dependent components. It should be noted that there are two ‘combined’ sources of this dependence. One results from the dependence of the components in the coherent system operating in a deterministic environment and the other is due to the dependence of components of the system sharing the same random environment.

Inspired by the work of these authors, we present solutions for some *open problems* formulated in [1]. We also generalize their results and present some new comparisons which consider the settings (i) and (iii) applied to coherent systems with dependent components. To be more specific, we provide different sets of sufficient conditions for one system to dominate the other with respect to different stochastic orders, namely, usual stochastic order, hazard rate order, reversed hazard rate order, and likelihood ratio order. It is worth mentioning that the above scenario (ii) considered in [1] for coherent systems can be viewed as a specific case of a more general scenario (iii). Moreover, our methodology for obtaining relevant comparisons also differs from that discussed in [1]. Lastly, although the case when Θ_2 is degenerate (meaning that the second environment is deterministic) is specific for scenario (iii), for technical reasons it is convenient to consider it separately, which is done in Section 5.

After this introductory discussion, we can carry on with some basic facts to be used intensively throughout the paper, starting with the relevant, formal notation that is more convenient for our multivariate setting. For an absolutely continuous random variable W, we denote the probability density function (pdf) by $f_W(\cdot)$, the cumulative distribution function (cdf) by $F_W(\cdot)$, the hazard rate function by $r_W(\cdot)$, the reversed hazard rate function by $\tilde{r}_W(\cdot)$, and the survival/reliability function by $\bar{F}_W(\cdot)$.

Consider a coherent system with lifetime $\tau(\boldsymbol{X})$ formed by n dependent components with the lifetime vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. The dependence among components can be represented by the joint reliability function of $\boldsymbol{X}$,

$$\bar{F}_{\boldsymbol{X}}(x_1, x_2, \ldots, x_n) = \mathbb{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) = C(\bar{F}_{X_1}(x_1), \bar{F}_{X_2}(x_2), \ldots, \bar{F}_{X_n}(x_n)),$$

where $C(\cdot, \cdot, \ldots, \cdot)$ is a survival copula. In the literature, many different types of survival copulas have been studied, for example, the Farlie–Gumbel–Morgenstern (FGM) copula, the Archimedean copula, the Clayton–Oakes (CO) copula, etc. For more discussion on this, we refer the reader to [29]. Based on the above representation, the following fundamental lemma (see [26]) can be formulated.

Lemma 1.1. *Let $\tau(\boldsymbol{X})$ be the lifetime of a coherent system formed by n dependent components with the lifetime vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. Then the system's reliability function can be written as*

$$\bar{F}_{\tau(\boldsymbol{X})}(x) = h(\bar{F}_{X_1}(x), \bar{F}_{X_2}(x), \ldots, \bar{F}_{X_n}(x)),$$

where $h\colon [0, 1]^n \to [0, 1]$, called the domination (or dual distortion) function, depends on the structure function $\phi(\cdot)$ (see [5] for a definition) and on the survival copula C of $X_1, X_2, \ldots, X_n$. Furthermore, $h(\cdot)$ is an increasing continuous function in $[0, 1]^n$ such that $h(0, 0, \ldots, 0) = 0$ and $h(1, 1, \ldots, 1) = 1$. □

The following example (borrowed from [24]) illustrates the meaning of this lemma.

Example 1.1. Let us consider a coherent system with lifetime $\tau(\boldsymbol{X}) = \min\{X_1, \max\{X_2, X_3\}\}$, where $\boldsymbol{X} = (X_1, X_2, X_3)$ is described by the FGM copula (see [29]). Then the minimal path sets (see [5]) of this system are given by $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. Consequently, its reliability function can be obtained as

$$\begin{aligned}\bar{F}_{\tau(\boldsymbol{X})}(x) &= \mathbb{P}(\{X_{P_1} > x\} \cup \{X_{P_2} > x\}) \\ &= \mathbb{P}(X_{P_1} > x) + \mathbb{P}(X_{P_2} > x) - \mathbb{P}(X_{\{1,2,3\}} > x) \\ &= \bar{F}_{\boldsymbol{X}}(x, x, 0) + \bar{F}_{\boldsymbol{X}}(x, 0, x) - \bar{F}_{\boldsymbol{X}}(x, x, x) \\ &= C(\bar{F}_{X_1}(x), \bar{F}_{X_2}(x), 1) + C(\bar{F}_{X_1}(x), 1, \bar{F}_{X_3}(x)) + C(\bar{F}_{X_1}(x), \bar{F}_{X_2}(x), \bar{F}_{X_3}(x)) \\ &= h(\bar{F}_{X_1}(x), \bar{F}_{X_2}(x), \bar{F}_{X_3}(x)),\end{aligned}$$

where

$$\begin{aligned}h(p_1, p_2, p_3) &= C(p_1, p_2, 1) + C(p_1, 1, p_3) - C(p_1, p_2, p_3) \\ &= p_1 p_2 p_3 [1 - x(1-p_1)(1-p_2)(1-p_3)] \quad \text{for } p_i \in (0, 1) \text{ and } x \in [-1, 1].\end{aligned}$$

Let $\tau(\boldsymbol{X}(\Theta))$ be a random variable representing the lifetime of a coherent system $\tau(\boldsymbol{X})$ that is operating in a random environment modelled by a random variable Θ with support $\Omega \subseteq [0, \infty)$. For a given environment $\Theta = \theta$, let $\boldsymbol{X}(\theta) = (X_1(\theta), X_2(\theta), \ldots, X_n(\theta))$ be the vector of lifetimes of components, and let $h(\cdot)$ be the domination function of $\tau(\boldsymbol{X}(\theta))$. Further, let $\bar{F}_{X_i}(\cdot \mid \theta)$, $F_{X_i}(\cdot \mid \theta)$, $f_{X_i}(\cdot \mid \theta)$, $r_{X_i}(\cdot \mid \theta)$, and $\tilde{r}_{X_i}(\cdot \mid \theta)$ be the survival function, the cumulative distribution function, the probability density function, the hazard rate function, and the reversed hazard rate function describing $X_i(\theta)$, respectively, for $i = 1, 2, \ldots, n$. Then the reliability function describing $\tau(\boldsymbol{X}(\Theta))$ can be expressed as the following mixture:

$$\bar{F}_{\tau(\boldsymbol{X}(\Theta))}(x) = \mathbb{P}(\tau(\boldsymbol{X}(\Theta)) > x) = \int_{\Omega} h(\bar{F}_{X_1}(x \mid \theta), \bar{F}_{X_2}(x \mid \theta), \ldots, \bar{F}_{X_n}(x \mid \theta))\, \mathrm{d}F_{\Theta}(\theta), \quad (1.1)$$

where the last equality holds due to Lemma 1.1. Further, its cumulative distribution function is given by

$$F_{\tau(\boldsymbol{X}(\Theta))}(x) = \int_{\Omega} [1 - h(\bar{F}_{X_1}(x \mid \theta), \bar{F}_{X_2}(x \mid \theta), \ldots, \bar{F}_{X_n}(x \mid \theta))]\, \mathrm{d}F_{\Theta}(\theta). \quad (1.2)$$

If all the X_i are identical, then (1.1) and (1.2) reduce to

$$\bar{F}_{\tau(\boldsymbol{X}(\Theta))}(x) = \int_{\Omega} h(\bar{F}_{X_1}(x \mid \theta))\, \mathrm{d}F_{\Theta}(\theta),$$

and

$$F_{\tau(\boldsymbol{X}(\Theta))}(x) = \int_{\Omega} [1 - h(\bar{F}_{X_1}(x \mid \theta))]\, \mathrm{d}F_{\Theta}(\theta).$$

Stochastic orders are frequently used as an effective tool to compare the lifetimes of two systems. Numerous stochastic orders are reported in the literature. Each of them has its own merit. For example, the usual stochastic order compares two reliability functions, the hazard rate order compares two hazard/failure rate functions, whereas the reversed hazard rate order compares two reversed hazard rate functions. For exhaustive details see [37]. For the sake of completeness, we define below the stochastic orders that are used in our paper.

Definition 1.1. Let X and Y be two absolutely continuous random variables with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then, X is said to be smaller than Y in:

(a) the likelihood ratio (lr) order, denoted by $X \leq_{\text{lr}} Y$, if

$$f_Y(t)/f_X(t) \text{ is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y);$$

(b) the hazard rate (hr) order, denoted by $X \leq_{\text{hr}} Y$, if

$$\bar{F}_Y(t)/\bar{F}_X(t) \text{ is increasing in } t \in (-\infty, \max(u_X, u_Y));$$

(c) the reversed hazard rate (rhr) order, denoted by $X \leq_{\text{rhr}} Y$, if

$$F_Y(t)/F_X(t) \text{ is increasing in } t \in (\min(l_X, l_Y), \infty);$$

(d) the usual stochastic (st) order, denoted by $X \leq_{\text{st}} Y$, if

$$\bar{F}_X(t) \leq \bar{F}_Y(t) \quad \text{for all } t \in (-\infty, \infty).$$

Note that the following chain of implications holds among the stochastic orders discussed above:

$$X \leq_{\text{lr}} Y \implies X \leq_{\text{hr [rhr]}} Y \implies X \leq_{\text{st}} Y.$$

The theory of totally positive functions has many applications in different areas of approximation theory and related fields. An encyclopedic information on this topic could be found in [15]. Below we give the definitions of TP_2 and RR_2 functions which will be used in our paper.

Definition 1.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two linearly ordered sets. Then, a real-valued non-negative function $\kappa(\cdot, \cdot)$ defined on $\mathcal{X} \times \mathcal{Y}$ is said to be TP_2 (resp. RR_2) if

$$\kappa(x_1, y_1)\,\kappa(x_2, y_2) \geq (\text{resp.} \leq)\,\kappa(x_1, y_2)\,\kappa(x_2, y_1)$$

for all $x_1 < x_2$ and $y_1 < y_2$. □

Throughout the paper, increasing and decreasing, as usual, mean non-decreasing and non-increasing, respectively. All random variables considered in this paper are assumed to be absolutely continuous and non-negative. We use bold symbols to represent vectors. Further, we use the acronym 'i.i.d.' for 'independent and identically distributed'. We let $\tau_{k:n}$ and $\tau_{l:m}$ denote the lifetimes of k-out-of-n and l-out-of-m systems, respectively.

The rest of the paper is organized as follows. In Section 2 we formulate some useful lemmas which are used in proving the main results. In Section 3 we consider two coherent systems that operate in the same random environment. We provide some sufficient conditions for proving that one coherent system dominates the other with respect to different stochastic orders. In Section 4 we study the same kind of comparisons under the assumption that different coherent systems operate in different random environments. In Section 5 we assume that one of the coherent systems operates in a random environment and the other in a deterministic environment. Concluding remarks are given in Section 6.

To enhance the readability of the paper, all proofs of theorems, wherever given, are deferred to the Appendix.

2. Useful lemmas

In this section we provide some lemmas which will be used intensively for proving the main results. The first lemma describes the TP_2/RR_2 property of the integral of a function. The proof could be completed along the same lines as in [9, Lemma 2.1].

Lemma 2.1. *Let $\phi_i(x, \theta)$, $i = 1, 2$, be a non-negative real-valued function on $\mathbb{R} \times \mathbb{X}$, where $\mathbb{R}$ is the set of real numbers, and $\mathbb{X} \subseteq \mathbb{R}$. Suppose that the following conditions hold.*

(i) *For $\theta \in \mathbb{X}$, $\phi_2(x, \theta)/\phi_1(x, \theta)$ is [increasing, increasing, decreasing, decreasing, resp.] in $x \in \mathbb{R}$.*

(ii) *For $x \in \mathbb{R}$, $\phi_2(x, \theta)/\phi_1(x, \theta)$ is [increasing, decreasing, decreasing, increasing, resp.] in $\theta \in \mathbb{X}$.*

(iii) *Either $\phi_1(x, \theta)$ or $\phi_2(x, \theta)$ is [TP_2, RR_2, TP_2, RR_2, resp.] in $(x, \theta) \in \mathbb{R} \times \mathbb{X}$.*

Then

$$s_i(x) = \int_X \phi_i(x, \theta) w(\theta)\, d\theta \text{ is } [\mathrm{TP}_2, \mathrm{TP}_2, \mathrm{RR}_2, \mathrm{RR}_2, \textit{resp.}] \text{ in } (x, i) \in \mathbb{R} \times \{1, 2\},$$

where $w(\cdot)$ is a continuous function with $\int_{\mathbb{X}} w(\theta)\, d\theta < \infty$. □

In next four lemmas we discuss some properties of the reliability functions of the k-out-of-n and l-out-of-m systems. Lemmas 2.2(i) and 2.4(i) are obtained in [10], whereas Lemmas 2.2(ii) and 2.4(ii) are obtained in [23]. Further, Lemmas 2.2(iii), 2.3, 2.5(i), and 2.6 are obtained in [6]. Furthermore, Lemmas 2.5(ii) and 2.5(iv) are obtained in [14], whereas Lemmas 2.5(iii) and 2.5(v) could be proved along the same lines as Lemmas 5 and 7 of [14].

Lemma 2.2. *Let $h_{k:n}(\cdot)$ be the reliability function of the k-out-of-n system with i.i.d. components. Then the following results hold.*

(i) $\dfrac{p h'_{k:n}(p)}{h_{k:n}(p)}$ *is decreasing in $p \in (0, 1)$.*

(ii) $\dfrac{(1-p) h'_{k:n}(p)}{1 - h_{k:n}(p)}$ *is increasing in $p \in (0, 1)$.*

(iii) *There exists some point $\mu \in (0, 1)$ such that*

   (a) $\dfrac{p h''_{k:n}(p)}{h'_{k:n}(p)}$ *is decreasing and positive for all $p \in (0, \mu)$, and*

   (b) $\dfrac{(1-p) h''_{k:n}(p)}{h'_{k:n}(p)}$ *is decreasing and negative for all $p \in (\mu, 1)$,*

   where $\mu = (k-1)/(n-1)$.

Lemma 2.3. *Let $h_{k:n}(\cdot)$ and $h_{l:m}(\cdot)$ be the reliability functions of the k-out-of-n and the l-out-of-m systems with i.i.d. components, respectively. Then, for $l \le k$ and $n - k \le m - l$,*

(i) *$h_{k:n}(p) \le h_{l:m}(p)$ for all $p \in (0, 1)$,*

(ii) $\dfrac{h_{k:n}(p)}{h_{l:m}(p)}$ *is increasing $p \in (0, 1)$,*

(iii) $\dfrac{1 - h_{k:n}(p)}{1 - h_{l:m}(p)}$ *is increasing $p \in (0, 1)$,*

(iv) $\dfrac{h'_{k:n}(p)}{h'_{l:m}(p)}$ *is increasing $p \in (0, 1)$.*

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Lemma 2.4. *Let $h_{k:n}(\cdot)$ be the reliability function of the k-out-of-n system with independent components. Then*

(i) $\sum_{i=1}^{n} \frac{p_i}{h_{k:n}(\boldsymbol{p})} \frac{\partial h_{k:n}(\boldsymbol{p})}{\partial p_i}$ *is decreasing in each $p_i \in (0, 1)$ for all $i = 1, 2, \ldots, n$,*

(ii) $\sum_{i=1}^{n} \frac{1-p_i}{1-h_{k:n}(\boldsymbol{p})} \frac{\partial h_{k:n}(\boldsymbol{p})}{\partial p_i}$ *is increasing in each $p_i \in (0, 1)$ for all $i = 1, 2, \ldots, n$.*

Lemma 2.5. *Let $h_{k:n}(\cdot)$ and $h_{l:m}(\cdot)$ be the reliability functions of the k-out-of-n and the l-out-of-m systems with independent components, respectively. Then, for $l \le k$ and $n - k \le m - l$,*

(i) $h_{k:n}(\boldsymbol{p}) \le h_{l:m}(\boldsymbol{p})$,

(ii) $\frac{1}{h_{k:n}(\boldsymbol{p})} \frac{\partial h_{k:n}(\boldsymbol{p})}{\partial p_i} \ge \frac{1}{h_{l:m}(\boldsymbol{p})} \frac{\partial h_{l:m}(\boldsymbol{p})}{\partial p_i}$ *for all $i = 1, 2, \ldots, \min\{m, n\}$,*

(iii) $\frac{1}{1-h_{k:n}(\boldsymbol{p})} \frac{\partial h_{k:n}(\boldsymbol{p})}{\partial p_i} \le \frac{1}{1-h_{l:m}(\boldsymbol{p})} \frac{\partial h_{l:m}(\boldsymbol{p})}{\partial p_i}$ *for all $i = 1, 2, \ldots, \min\{m, n\}$,*

(iv) $\sum_{i=1}^{n} \frac{p_i}{h_{k:n}(\boldsymbol{p})} \frac{\partial h_{k:n}(\boldsymbol{p})}{\partial p_i} \ge \sum_{i=1}^{m} \frac{p_i}{h_{l:m}(\boldsymbol{p})} \frac{\partial h_{l:m}(\boldsymbol{p})}{\partial p_i}$,

(v) $\sum_{i=1}^{n} \frac{1-p_i}{1-h_{k:n}(\boldsymbol{p})} \frac{\partial h_{k:n}(\boldsymbol{p})}{\partial p_i} \le \sum_{i=1}^{m} \frac{1-p_i}{1-h_{l:m}(\boldsymbol{p})} \frac{\partial h_{l:m}(\boldsymbol{p})}{\partial p_i}$.

Lemma 2.6. *Let $h_{k:n}(\cdot)$ and $h_{l:m}(\cdot)$ be the reliability functions of the k-out-of-n and the l-out-of-m systems, respectively. Further, let*

$$\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n) \quad \text{and} \quad \boldsymbol{W} = (W_1, W_2, \ldots, W_m)$$

be two sets of independent component lifetimes. Suppose that, for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, $Z_i \le_{\mathrm{lr}} W_j$. Then, for $l \le k$ and $n - k \le m - l$,

$$\frac{\partial h_{l:m}(\boldsymbol{q})}{\partial q_j} \bigg/ \frac{\partial h_{k:n}(\boldsymbol{p})}{\partial p_i} \text{ is increasing in } x,$$

where $p_i = \bar{F}_{Z_i}(x)$ and $q_j = \bar{F}_{W_j}(x)$.

3. Two different coherent systems in the same random environment

In this section we consider two coherent systems with lifetimes $\tau_1(\boldsymbol{X}(\Theta))$ and $\tau_2(\boldsymbol{Y}(\Theta))$ that operate in the same random environment described by a random variable Θ with support Ω. For a given (realization) environment $\Theta = \theta$, we denote the domination functions of $\tau_1(\boldsymbol{X}(\theta))$ and $\tau_2(\boldsymbol{Y}(\theta))$ by $h_1(\cdot)$ and $h_2(\cdot)$, respectively. In what follows, we provide some sufficient conditions for proving that one coherent system dominates the other with respect to different stochastic orders.

3.1. Systems with not necessarily identical components

In this subsection we consider coherent systems that are formed by not necessarily identical components.

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In the following theorem, whose proof is deferred to the Appendix, we compare two coherent systems with respect to the usual stochastic order.

Theorem 3.1. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Suppose that the following conditions hold:*

(i) $h_1(p_1, p_2, \ldots, p_n) \le h_2(p_1, p_2, \ldots, p_m)$,

(ii) $X_i(\theta) \le_{\text{st}} Y_i(\theta)$ *for all* $i = 1, 2, \ldots, \min\{m, n\}$.

Then $\tau_1(\boldsymbol{X}(\Theta)) \le_{\text{st}} \tau_2(\boldsymbol{Y}(\Theta))$. □

In the next theorem (see the Appendix for the proof), we show that, under some sufficient conditions, $\tau_2(\boldsymbol{Y}(\Theta))$ is larger than $\tau_1(\boldsymbol{X}(\Theta))$ with respect to the hazard rate order.

Theorem 3.2. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes, where* $n \ge m$. *Suppose that* {(i), (ii), (iii)} *or* {(i), (ii), (iv)} *holds:*

(i) $\dfrac{1}{h_1(\boldsymbol{p})}\dfrac{\partial h_1(\boldsymbol{p})}{\partial p_i} \ge \dfrac{1}{h_2(\boldsymbol{p})}\dfrac{\partial h_2(\boldsymbol{p})}{\partial p_i}$ *for all* $i = 1, 2, \ldots, m$,

(ii) $\dfrac{p_i}{h_2(\boldsymbol{p})}\dfrac{\partial h_2(\boldsymbol{p})}{\partial p_i}$ *is decreasing in* $\boldsymbol{p} \in (0, 1)^m$ *for all* $i = 1, 2, \ldots, m$,

(iii) $X_i(\theta_1) \le_{\text{hr}} X_i(\theta_2)$, $X_j(\theta) \le_{\text{hr}} Y_j(\theta)$, *and* $Y_j(\theta_2) \le_{\text{hr}} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$,

(iv) $X_i(\theta_1) \ge_{\text{hr}} X_i(\theta_2)$, $X_j(\theta) \le_{\text{hr}} Y_j(\theta)$, *and* $Y_j(\theta_2) \ge_{\text{hr}} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$.

Then $\tau_1(\boldsymbol{X}(\Theta)) \le_{\text{hr}} \tau_2(\boldsymbol{Y}(\Theta))$. □

The following theorem shows that the above result holds for the reversed hazard rate order under some different set of sufficient conditions. The proof could be completed along the same lines as in Theorem 3.2 and is hence omitted.

Theorem 3.3. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes, where* $m \ge n$. *Suppose that* {(i), (ii), (iii)} *or* {(i), (ii), (iv)} *holds:*

(i) $\dfrac{1}{1 - h_1(\boldsymbol{p})}\dfrac{\partial h_1(\boldsymbol{p})}{\partial p_i} \le \dfrac{1}{1 - h_2(\boldsymbol{p})}\dfrac{\partial h_2(\boldsymbol{p})}{\partial p_i}$ *for all* $i = 1, 2, \ldots, n$,

(ii) $\dfrac{1 - p_i}{1 - h_1(\boldsymbol{p})}\dfrac{\partial h_1(\boldsymbol{p})}{\partial p_i}$ *is increasing in* $\boldsymbol{p} \in (0, 1)^n$ *for all* $i = 1, 2, \ldots, n$,

(iii) $X_i(\theta_1) \le_{\text{rhr}} X_i(\theta_2)$, $X_i(\theta) \le_{\text{rhr}} Y_i(\theta)$, *and* $Y_j(\theta_2) \le_{\text{rhr}} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$,

(iv) $X_i(\theta_1) \ge_{\text{rhr}} X_i(\theta_2)$, $X_i(\theta) \le_{\text{rhr}} Y_i(\theta)$, *and* $Y_j(\theta_2) \ge_{\text{rhr}} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$.

Then $\tau_1(\boldsymbol{X}(\Theta)) \le_{\text{rhr}} \tau_2(\boldsymbol{Y}(\Theta))$.

3.2. Systems with identical components

In this subsection we consider coherent systems of identical components. Obviously, this case has its own value when compared with the general case of non-identical components.

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The following theorem is analogous to Theorem 3.1, and the proof also immediately follows from it.

Theorem 3.4. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Assume the* X_i *are identical and the* Y_j *are identical. Suppose that the following conditions hold:*

(i) $h_1(p) \le h_2(p)$ *for all* $p \in (0, 1)$*,*

(ii) $X_1(\theta) \le_{\rm st} Y_1(\theta)$ *for all* $\theta \in \Omega$*.*

Then $\tau_1(\boldsymbol{X}(\Theta)) \le_{\rm st} \tau_2(\boldsymbol{Y}(\Theta))$*.* □

In the next theorem we compare two coherent systems with respect to the hazard rate order. The proof could be completed along the same lines as in Theorem 3.2.

Theorem 3.5. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Assume the* X_i *are identical and the* Y_j *are identical. Suppose that* {(i), (ii), (iii)} *or* {(i), (ii), (iv)} *holds:*

(i) $h_1(p)/h_2(p)$ *is increasing in* $p \in (0, 1)$*,*

(ii) $ph_2'(p)/h_2(p)$ *is decreasing in* $p \in (0, 1)$*,*

(iii) $X_1(\theta_1) \le_{\rm hr} X_1(\theta_2) \le_{\rm hr} Y_1(\theta_2) \le_{\rm hr} Y_1(\theta_1)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$*,*

(iv) $X_1(\theta_2) \le_{\rm hr} X_1(\theta_1) \le_{\rm hr} Y_1(\theta_1) \le_{\rm hr} Y_1(\theta_2)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$*.*

Then $\tau_1(\boldsymbol{X}(\Theta)) \le_{\rm hr} \tau_2(\boldsymbol{Y}(\Theta))$*.* □

In the following theorem we show that the same result as in Theorem 3.5 holds for the reversed hazard rate order. The proof is similar to that of Theorem 3.3.

Theorem 3.6. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Assume the* X_i *are identical and the* Y_j *are identical. Suppose that* {(i), (ii), (iii)} *or* {(i), (ii), (iv)} *holds:*

(i) $(1 - h_1(p))/(1 - h_2(p))$ *is increasing in* $p \in (0, 1)$*,*

(ii) $(1 - p)h_1'(p)/(1 - h_1(p))$ *is increasing in* $p \in (0, 1)$*,*

(iii) $X_1(\theta_1) \le_{\rm rhr} X_1(\theta_2) \le_{\rm rhr} Y_1(\theta_2) \le_{\rm rhr} Y_1(\theta_1)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$*,*

(iv) $X_1(\theta_2) \le_{\rm rhr} X_1(\theta_1) \le_{\rm rhr} Y_1(\theta_1) \le_{\rm rhr} Y_1(\theta_2)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$*.*

Then $\tau_1(\boldsymbol{X}(\Theta)) \le_{\rm rhr} \tau_2(\boldsymbol{Y}(\Theta))$*.*

4. Different coherent systems in different random environments

In this section we consider two coherent systems with lifetimes $\tau_1(\boldsymbol{X}(\Theta_1))$ and $\tau_2(\boldsymbol{Y}(\Theta_2))$, where Θ_1 and Θ_2 are two random variables (with support Ω) that describe two different random environments. For given environments $\Theta_1 = \theta$ and $\Theta_2 = \theta^*$, we denote the domination functions of $\tau_1(\boldsymbol{X}(\theta))$ and $\tau_2(\boldsymbol{Y}(\theta^*))$ by $h_1(\cdot)$ and $h_2(\cdot)$, respectively. We will compare $\tau_1(\boldsymbol{X}(\Theta_1))$ and $\tau_2(\boldsymbol{Y}(\Theta_2))$ with respect to different stochastic orders. It should be noted that the results of this section can be considered as generalizations of the corresponding results given in [1] to the case when there are two different coherent systems; in [1], the case of one system (or of two identical systems) operating in two environments was discussed.

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4.1. Systems with not necessarily identical components

In this subsection we assume that coherent systems are formed by not necessarily identical components.

In the following theorem, we show that under a set of sufficient conditions $\tau_2(\boldsymbol{Y}(\Theta_2))$ dominates $\tau_1(\boldsymbol{X}(\Theta_1))$ with respect to the usual stochastic order. The proof follows from [1, Theorem 3.1] and our Theorem 3.1.

Theorem 4.1. *Let* $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y}=(Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Suppose that* {(i), (ii), (iv)} *or* {(i), (iii), (iv)} *holds:*

(i) $h_1(p_1, p_2, \ldots, p_n) \le h_2(p_1, p_2, \ldots, p_m)$,

(ii) $X_i(\theta_1) \le_{\rm st} X_i(\theta_2)$ *and* $X_j(\theta) \le_{\rm st} Y_j(\theta)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i=1, 2, \ldots, n$ *and* $j=1, 2, \ldots, \min\{m, n\}$,

(iii) $Y_i(\theta_1) \le_{\rm st} Y_i(\theta_2)$ *and* $X_j(\theta) \le_{\rm st} Y_j(\theta)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i=1, 2, \ldots, m$ *and* $j=1, 2, \ldots, \min\{m, n\}$,

(iv) $\Theta_1 \le_{\rm st} \Theta_2$.

Then $\tau_1(\boldsymbol{X}(\Theta_1)) \le_{\rm st} \tau_2(\boldsymbol{Y}(\Theta_2))$. □

Now we compare two coherent systems with respect to the hazard rate order. The proof follows from [1, Theorem 3.2] and our Theorem 3.2.

Theorem 4.2. *Let* $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y}=(Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes, where* $n \ge m$. *Suppose that* {(i), (ii), (iii), (v)} *or* {(i), (ii), (iv), (v)} *holds:*

(i) $\dfrac{1}{h_1(\boldsymbol{p})}\dfrac{\partial h_1(\boldsymbol{p})}{\partial p_i} \ge \dfrac{1}{h_2(\boldsymbol{p})}\dfrac{\partial h_2(\boldsymbol{p})}{\partial p_i}$ *for all* $i=1, 2, \ldots, m$,

(ii) $\dfrac{p_i}{h_2(\boldsymbol{p})}\dfrac{\partial h_2(\boldsymbol{p})}{\partial p_i}$ *is decreasing in* $\boldsymbol{p} \in (0, 1)^m$ *for all* $i=1, 2, \ldots, m$,

(iii) $X_i(\theta_1) \le_{\rm hr} X_i(\theta_2)$, $X_j(\theta) \le_{\rm hr} Y_j(\theta)$, *and* $Y_j(\theta_2) \le_{\rm hr} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i=1, 2, \ldots, n$ *and* $j=1, 2, \ldots, m$,

(iv) $X_i(\theta_1) \ge_{\rm hr} X_i(\theta_2)$, $X_j(\theta) \le_{\rm hr} Y_j(\theta)$, *and* $Y_j(\theta_2) \ge_{\rm hr} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$, *and for all* $i=1, 2, \ldots, n$ *and* $j=1, 2, \ldots, m$,

(v) $\Theta_1 \le_{\rm hr} \Theta_2$.

Then $\tau_1(\boldsymbol{X}(\Theta_1)) \le_{\rm hr} \tau_2(\boldsymbol{Y}(\Theta_2))$. □

Below we give two examples that illustrate conditions (i) and (ii) of Theorem 4.2.

Example 4.1. Consider two coherent systems with lifetimes $\tau_1(\boldsymbol{X})=\min\{X_1, X_2, X_3\}$ and $\tau_2(\boldsymbol{Y})=\min\{X_1, \max\{X_2, X_3\}\}$. Further, let both $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ have the same Gumbel–Barnett copula given by

$$C(p_1, p_2, p_3)=p_1p_2p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}, \quad \alpha > 0, \text{ and } 0 < p_i < 1 \quad \text{for } i=1, 2, 3.$$

Then the domination functions of $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$, respectively, are given by

$$h_1(\boldsymbol{p})=C(p_1, p_2, p_3)=p_1p_2p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}$$

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and

$$h_2(\boldsymbol{p}) = C(p_1, p_2, 1) + C(p_1, 1, p_3) - C(p_1, p_2, p_3) = p_1p_2 + p_1p_3 - p_1p_2p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}.$$

Note that

$$\frac{p_i}{h_1(\boldsymbol{p})} \frac{\partial h_1(\boldsymbol{p})}{\partial p_i} = 1 + \alpha \prod_{\substack{j=1\\ j\neq i}}^{3} \ln p_j \quad \text{for } i = 1, 2, 3, \tag{4.1}$$

and

$$\frac{p_1}{h_2(\boldsymbol{p})} \frac{\partial h_2(\boldsymbol{p})}{\partial p_1} = 1 - \frac{\alpha p_1p_2p_3 \ln p_2 \ln p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}}{p_1p_2 + p_1p_3 - p_1p_2p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}}, \tag{4.2}$$

$$\frac{p_2}{h_2(\boldsymbol{p})} \frac{\partial h_2(\boldsymbol{p})}{\partial p_2} = 1 - \frac{p_1p_3 + \alpha p_1p_2p_3 \ln p_1 \ln p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}}{p_1p_2 + p_1p_3 - p_1p_2p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}}, \tag{4.3}$$

$$\frac{p_3}{h_2(\boldsymbol{p})} \frac{\partial h_2(\boldsymbol{p})}{\partial p_3} = 1 - \frac{p_1p_2 + \alpha p_1p_2p_3 \ln p_2 \ln p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}}{p_1p_2 + p_1p_3 - p_1p_2p_3\, e^{\alpha \ln p_1 \ln p_2 \ln p_3}}. \tag{4.4}$$

Using (4.1)–(4.4), we obtain

$$\frac{1}{h_1(\boldsymbol{p})} \frac{\partial h_1(\boldsymbol{p})}{\partial p_i} \geq \frac{1}{h_2(\boldsymbol{p})} \frac{\partial h_2(\boldsymbol{p})}{\partial p_i} \quad \text{for all } i = 1, 2, 3.$$

Example 4.2. Consider two coherent systems

$$\tau_1(\boldsymbol{X}) = \min\{X_1, X_2, \dots, X_n\}, \qquad \tau_2(\boldsymbol{Y}) = \min\{Y_1, Y_2, \dots, Y_n\},$$

where n is even. Further, let both $\{X_1, X_2, \dots, X_n\}$ and $\{Y_1, Y_2, \dots, Y_n\}$ have the same Gumbel–Barnett copula given by

$$C(p_1, p_2, \dots, p_n) = \prod_{i=1}^{n} p_i\, e^{-\alpha(-1)^n \prod_{j=1}^{n} \ln p_j}, \quad \alpha > 0, \text{ and } 0 < p_i < 1 \text{ for } i = 1, 2, \dots, n.$$

Then $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ have the same domination function:

$$h_i(\boldsymbol{p}) = C(p_1, p_2, \dots, p_n) \quad \text{for } i = 1, 2.$$

Note that, for all $i = 1, 2, \dots, n$,

$$\frac{p_i}{h_2(\boldsymbol{p})} \frac{\partial h_2(\boldsymbol{p})}{\partial p_i} = 1 - \alpha \prod_{\substack{j=1\\ j\neq i}}^{n} \ln p_j,$$

which is decreasing in $\boldsymbol{p} \in (0, 1)^n$. □

In the following theorem we provide a set of sufficient conditions for proving that $\tau_2(\boldsymbol{Y}(\Theta_2))$ is larger than $\tau_1(\boldsymbol{X}(\Theta_1))$ with respect to the reversed hazard rate order. The proof follows from [1, Theorem 3.3] and Theorem 3.3.

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Theorem 4.3. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes, where* $m \geq n$. *Suppose that* {(i), (ii), (iii), (v)} *or* {(i), (ii), (iv), (v)} *holds:*

(i) $\dfrac{1}{1-h_1(\boldsymbol{p})} \dfrac{\partial h_1(\boldsymbol{p})}{\partial p_i} \leq \dfrac{1}{1-h_2(\boldsymbol{p})} \dfrac{\partial h_2(\boldsymbol{p})}{\partial p_i}$ *for all* $i = 1, 2, \ldots, n$,

(ii) $\dfrac{1-p_i}{1-h_1(\boldsymbol{p})} \dfrac{\partial h_1(\boldsymbol{p})}{\partial p_i}$ *is increasing in* $\boldsymbol{p} \in (0, 1)^n$ *for all* $i = 1, 2, \ldots, n$,

(iii) $X_i(\theta_1) \leq_{\text{rhr}} X_i(\theta_2)$, $X_i(\theta) \leq_{\text{rhr}} Y_i(\theta)$, *and* $Y_j(\theta_2) \leq_{\text{rhr}} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \leq \theta_2$, *and for all* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$,

(iv) $X_i(\theta_1) \geq_{\text{rhr}} X_i(\theta_2)$, $X_i(\theta) \leq_{\text{rhr}} Y_i(\theta)$, *and* $Y_j(\theta_2) \geq_{\text{rhr}} Y_j(\theta_1)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \leq \theta_2$, *and for all* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$,

(v) $\Theta_1 \leq_{\text{rhr}} \Theta_2$.

Then $\tau_1(\boldsymbol{X}(\Theta_1)) \leq_{\text{rhr}} \tau_2(\boldsymbol{Y}(\Theta_2))$.

Remark 4.1. It is to be noted that condition (i) of Theorems 4.2 and 4.3 holds for $h_1(\boldsymbol{p}) = h_{k:n}(\boldsymbol{p})$ and $h_2(\boldsymbol{p}) = h_{l:m}(\boldsymbol{p})$, $l \leq k$ and $n - k \leq m - l$, where $h_{k:n}(\cdot)$ and $h_{l:m}(\cdot)$ are the reliability functions of the k-out-of-n and the l-out-of-m systems with independent components, respectively (see Lemma 2.5).

4.2. Systems with identical components

In this subsection we compare two coherent systems that are formed by identical components. Obviously, this is of interest when compared with the general case of non-identical components.

We show that under a set of sufficient conditions $\tau_2(\boldsymbol{Y}(\Theta_2))$ is larger than $\tau_1(\boldsymbol{X}(\Theta_1))$ with respect to the usual stochastic order. The proof follows from [1, Theorem 3.1] and Theorem 3.4.

Theorem 4.4. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Assume the* X_i *are identical and the* Y_j *are identical. Suppose that* {(i), (ii), (iv)} *or* {(i), (iii), (iv)} *holds:*

(i) $h_1(p) \leq h_2(p)$ *for all* $p \in (0, 1)$,

(ii) $X_1(\theta_1) \leq_{\text{st}} X_1(\theta_2)$ *and* $X_1(\theta) \leq_{\text{st}} Y_1(\theta)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \leq \theta_2$,

(iii) $Y_1(\theta_1) \leq_{\text{st}} Y_1(\theta_2)$ *and* $X_1(\theta) \leq_{\text{st}} Y_1(\theta)$ *for all* $\theta, \theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \leq \theta_2$,

(iv) $\Theta_1 \leq_{\text{st}} \Theta_2$.

Then $\tau_1(\boldsymbol{X}(\Theta_1)) \leq_{\text{st}} \tau_2(\boldsymbol{Y}(\Theta_2))$. □

The next theorem discusses the same result as in the above theorem but with respect to the hazard rate order. The proof follows from [1, Corollary 3.1] and Theorem 3.5.

Theorem 4.5. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Assume the* X_i *are identical and the* Y_j *are identical. Suppose that* {(i), (ii), (iii), (v)} *or* {(i), (ii), (iv), (v)} *holds:*

(i) $h_1(p)/h_2(p)$ *is increasing in* $p \in (0, 1)$,

(ii) $ph_2'(p)/h_2(p)$ *is decreasing in* $p \in (0, 1)$,

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(iii) $X_1(\theta_1) \le_{\rm hr} X_1(\theta_2) \le_{\rm hr} Y_1(\theta_2) \le_{\rm hr} Y_1(\theta_1)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$,

(iv) $X_1(\theta_2) \le_{\rm hr} X_1(\theta_1) \le_{\rm hr} Y_1(\theta_1) \le_{\rm hr} Y_1(\theta_2)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$,

(v) $\Theta_1 \le_{\rm hr} \Theta_2$.

Then $\tau_1(\boldsymbol{X}(\Theta_1)) \le_{\rm hr} \tau_2(\boldsymbol{Y}(\Theta_2))$. □

The following example illustrates conditions (i) and (ii) of Theorem 4.5.

Example 4.3. Consider two coherent systems with lifetimes $\tau_1(\boldsymbol{X}) = \min\{X_1, X_2, X_3\}$ and $\tau_2(\boldsymbol{Y}) = \min\{Y_1, \max\{Y_2, Y_3\}\}$, where both $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ are homogeneous and have the same FGM copula given by

$$C(p_1, p_2, p_3) = p_1p_2p_3(1 + x(1 - p_1)(1 - p_2)(1 - p_3)) \quad \text{for } x \in [-1, 1].$$

Then the domination functions of $\tau_1(\boldsymbol{X}) = \min\{X_1, X_2, X_3\}$ and $\tau_2(\boldsymbol{Y}) = \min\{Y_1, \max\{Y_2, Y_3\}\}$ are given by

$$h_1(p) = C(p, p, p) = p^3(1 + x(1 - p)^3),$$

and

$$h_2(p) = 2C(p, p, 1) - C(p, p, p) = 2p^2 - p^3 - xp^3(1 - p)^3,$$

respectively. Further, it could be verified that, for all $x \in [-1, 1]$,

$$\frac{h_1(p)}{h_2(p)} = \frac{p^3(1 + x(1 - p)^3)}{2p^2 - p^3(1 + x(1 - p)^3)} \text{ is increasing in } p \in (0, 1),$$

and

$$p\frac{h_2'(p)}{h_2(p)} = \frac{4 - 3(1 + x)p + 12xp^2 - 15xp^3 + 6xp^4}{2 - (1 + x)p + 3xp^2 - 3xp^3 + xp^4} \text{ is decreasing in } p \in (0, 1),$$

and hence our claim is proved. □

In the next theorem, we prove the same result as in Theorem 4.5, but for the reversed hazard rate order. The proof follows from [1, Corollary 3.2] and Theorem 3.6.

Theorem 4.6. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Assume the* X_i *are identical and the* Y_j *are identical. Suppose that* {(i), (ii), (iii), (v)} *or* {(i), (ii), (iv), (v)} *holds:*

(i) $(1 - h_1(p))/(1 - h_2(p))$ *is increasing in* $p \in (0, 1)$,

(ii) $(1 - p)h_1'(p)/(1 - h_1(p))$ *is increasing in* $p \in (0, 1)$,

(iii) $X_1(\theta_1) \le_{\rm rhr} X_1(\theta_2) \le_{\rm rhr} Y_1(\theta_2) \le_{\rm rhr} Y_1(\theta_1)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$,

(iv) $X_1(\theta_2) \le_{\rm rhr} X_1(\theta_1) \le_{\rm rhr} Y_1(\theta_1) \le_{\rm rhr} Y_1(\theta_2)$ *for all* $\theta_1, \theta_2 \in \Omega$ *such that* $\theta_1 \le \theta_2$,

(v) $\Theta_1 \le_{\rm rhr} \Theta_2$.

Then $\tau_1(\boldsymbol{X}(\Theta_1)) \le_{\rm rhr} \tau_2(\boldsymbol{Y}(\Theta_2))$. □

5. One of the coherent systems in a random environment

In this section as previously, we compare two coherent systems with respect to different stochastic orders. However, we assume that one of them operates in a random environment, whereas the other operates in a deterministic environment. As already mentioned, although this case can be viewed as the special case of the discussion in the previous section, technically it is more convenient to consider it independently. Let $\tau_1(\boldsymbol{X}(\Theta))$ be the lifetime of a coherent system that operates in a random environment modelled by a random variable Θ with support Ω. Further, let $\tau_2(\boldsymbol{Y})$ be the lifetime of the other coherent system that operates in some baseline, deterministic environment. For a given environment $\Theta=\theta$, we denote the domination function of $\tau_1(\boldsymbol{X}(\theta))$ by $h_1(\cdot)$. Further, we denote the domination function of $\tau_2(\boldsymbol{Y})$ by $h_2(\cdot)$.

5.1. Systems with not necessarily identical components

In the following theorem we show that under a set of sufficient conditions $\tau_1(\boldsymbol{X}(\Theta))$ dominates $\tau_2(\boldsymbol{Y})$ with respect to the usual stochastic order. The proof follows along the same lines as in Theorem 3.1 and is hence omitted.

Theorem 5.1. *Let $\boldsymbol{X}=(X_1, X_2, \dots, X_n)$ and $\boldsymbol{Y}=(Y_1, Y_2, \dots, Y_m)$ be two sets of components' lifetimes. Suppose that the following conditions hold:*

(i) $h_1(p_1, p_2, \dots, p_n)\le h_2(p_1, p_2, \dots, p_m)$,

(ii) $X_i(\theta)\le_{\text{st}} Y_i$ *for all* $\theta\in\Omega$, *and for all* $i=1, 2, \dots, \min\{m, n\}$.

Then $\tau_1(\boldsymbol{X}(\Theta))\le_{\text{st}}\tau_2(\boldsymbol{Y})$. □

The following theorem provides some sufficient conditions for proving that $\tau_1(\boldsymbol{X}(\Theta))$ is smaller than $\tau_2(\boldsymbol{Y})$ with respect to the hazard rate order. See the Appendix for the proof.

Theorem 5.2. *Let $\boldsymbol{X}=(X_1, X_2, \dots, X_n)$ and $\boldsymbol{Y}=(Y_1, Y_2, \dots, Y_m)$ be two sets of components' lifetimes. Suppose that* {(i), (ii), (iv)} *or* {(i), (iii), (iv)} *holds:*

(i) $$\sum_{i=1}^{n}\frac{p_i}{h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\ge\sum_{i=1}^{m}\frac{p_i}{h_2(\boldsymbol{p})}\frac{\partial h_2(\boldsymbol{p})}{\partial p_i},$$

(ii) $$\sum_{i=1}^{n}\frac{p_i}{h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\text{ is decreasing in each } p_i,\ i=1, 2, \dots, n,$$

(iii) $$\sum_{i=1}^{m}\frac{p_i}{h_2(\boldsymbol{p})}\frac{\partial h_2(\boldsymbol{p})}{\partial p_i}\text{ is decreasing in each } p_i,\ i=1, 2, \dots, m,$$

(iv) $X_i(\theta)\le_{\text{hr}} Y_j$ *for all* $\theta\in\Omega$, *and for all* $i=1, 2, \dots, n$ *and* $j=1, 2, \dots, m$.

Then $\tau_1(\boldsymbol{X}(\Theta))\le_{\text{hr}}\tau_2(\boldsymbol{Y})$. □

Now we discuss the corresponding result for the reversed hazard rate order. The proof follows along the same lines as in Theorem 5.2 and is hence omitted.

Theorem 5.3. *Let $\boldsymbol{X}=(X_1, X_2, \dots, X_n)$ and $\boldsymbol{Y}=(Y_1, Y_2, \dots, Y_m)$ be two sets of components' lifetimes. Suppose that* {(i), (ii), (iv)} *or* {(i), (iii), (iv)} *holds:*

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(i) $\displaystyle\sum_{i=1}^{n}\frac{1-p_i}{1-h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\le\sum_{i=1}^{m}\frac{1-p_i}{1-h_2(\boldsymbol{p})}\frac{\partial h_2(\boldsymbol{p})}{\partial p_i}$,

(ii) $\displaystyle\sum_{i=1}^{n}\frac{1-p_i}{1-h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}$ *is increasing in each* p_i, $i=1, 2, \ldots, n$,

(iii) $\displaystyle\sum_{i=1}^{m}\frac{1-p_i}{1-h_2(\boldsymbol{p})}\frac{\partial h_2(\boldsymbol{p})}{\partial p_i}$ *is increasing in each* p_i, $i=1, 2, \ldots, m$,

(iv) $X_i(\theta)\le_{\text{rhr}} Y_j$ *for all* $\theta\in\Omega$, *and for all* $i=1, 2, \ldots, n$ *and* $j=1, 2, \ldots, m$.

Then $\tau_1(\boldsymbol{X}(\Theta))\le_{\text{rhr}}\tau_2(\boldsymbol{Y})$. □

Below we discuss the corresponding result for the likelihood ratio order. See the Appendix for the proof.

Theorem 5.4. *Let* $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y}=(Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Suppose that the following conditions hold:*

(i) *for all* $i=1, 2, \ldots, n$ *and* $j=1, 2, \ldots, m$,

$$\frac{\partial h_2(\boldsymbol{q})}{\partial q_j}\bigg/\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\ \text{is increasing in } x \text{ for all } \theta\in\Omega,$$

where $p_i=\bar{F}_{X_i}(x\mid\theta)$, $q_j=\bar{F}_{Y_j}(x)$,

(ii) $X_i(\theta)\le_{\text{lr}} Y_j$ *for all* $\theta\in\Omega$, *and for all* $i=1, 2, \ldots, n$ *and* $j=1, 2, \ldots, m$.

Then $\tau_1(\boldsymbol{X}(\Theta))\le_{\text{lr}}\tau_2(\boldsymbol{Y})$. □

5.2. Systems with identical components

In this subsection we consider coherent systems of identical components. For brevity, we only discuss the corresponding result for the likelihood ratio order. Comparisons with other stochastic orders can easily be derived similar to the previous section. In the following theorems, we compare $\tau_1(\boldsymbol{X}(\Theta))$ and $\tau_2(\boldsymbol{Y})$ with respect to the likelihood ratio order. The proof could be completed along the same lines as in Theorem 5.4 and is hence omitted.

Theorem 5.5. *Let* $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y}=(Y_1, Y_2, \ldots, Y_m)$ *be two sets of components' lifetimes. Assume the* X_i *are identical and the* Y_j *are identical. Suppose that the following conditions hold:*

(i) $h_1'(p)/h_2'(p)$ *is increasing in* $p\in(0, 1)$,

(ii) *for* $k=1$ *or* 2, *there exists some point* $\mu\in(0, 1)$ *such that*

  (a) $ph_k''(p)/h_k'(p)$ *is decreasing and positive for all* $p\in(0, \mu)$, *and*

  (b) $(1-p)h_k''(p)/h_k'(p)$ *is decreasing and negative for all* $p\in(\mu, 1)$,

(iii) $X_1(\theta)\le_{\text{lr}} Y_1$ *for all* $\theta\in\Omega$.

Then $\tau_1(\boldsymbol{X}(\Theta))\le_{\text{lr}}\tau_2(\boldsymbol{Y})$.

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6. Concluding remarks

In this paper we study the impact of a random environment on lifetimes of coherent systems with dependent lifetimes.

Motivated by discussions in [1], we present solutions for some open problems formulated in this paper. We also generalize the results of these authors and present some new comparisons as well. Specifically, we provide different sets of sufficient conditions for one system to dominate the other with respect to different stochastic orders, namely, usual stochastic order, hazard rate order, reversed hazard rate order, and likelihood ratio order.

Even though we have incorporated a large number of new results in this paper, there are still some open problems left over. One of them is to generalize the results discussed in Sections 3 and 4 to the likelihood ratio order.

We conclude our discussion by mentioning the fact that the straightforward corollaries corresponding to Theorems 4.1, 4.4, 4.5, 4.6, 5.1, 5.2, 5.3, and 5.4 could be formulated (by using Lemmas 2.2–2.6) similarly to the corollary given below for Theorem 4.1.

Corollary 6.1. *Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ be two sets of components' lifetimes. Assume the X_i are independent and the Y_j are independent. Suppose that the set of conditions* {(ii), (iv)} *or* {(iii), (iv)} *in Theorem 4.1 holds:*

(i) $\tau_{k:n}(\boldsymbol{X}(\Theta_1)) \leq_{\rm st} \tau_{l:n}(\boldsymbol{Y}(\Theta_2))$ *for* $l \leq k$,

(ii) $\tau_{k:n}(\boldsymbol{X}(\Theta_1)) \leq_{\rm st} \tau_{k:m}(\boldsymbol{Y}(\Theta_2))$ *for* $n \leq m$,

(iii) $\tau_{k:n}(\boldsymbol{X}(\Theta_1)) \leq_{\rm st} \tau_{k-r:n-r}(\boldsymbol{Y}(\Theta_2))$ *for* $r \leq k$,

(iv) $\tau_{k:n}(\boldsymbol{X}(\Theta_1)) \leq_{\rm st} \tau_{l:m}(\boldsymbol{Y}(\Theta_2))$ *for* $l \leq k$ *and* $n-k \leq m-l$.

Appendix A

Proof of Theorem 3.1. Let $k = \min\{m, n\}$. Note that

$$\begin{aligned}
\bar{F}_{\tau_1(\boldsymbol{X}(\Theta))}(x) &= \int_\Omega h_1(\bar{F}_{X_1}(x \mid \theta), \bar{F}_{X_2}(x \mid \theta), \ldots, \bar{F}_{X_n}(x \mid \theta))\, \mathrm{d}F_\Theta(\theta) \\
&\leq \int_\Omega h_1(\bar{F}_{Y_1}(x \mid \theta), \bar{F}_{Y_2}(x \mid \theta), \ldots, \bar{F}_{Y_k}(x \mid \theta))\, \mathrm{d}F_\Theta(\theta) \\
&\leq \int_\Omega h_2(\bar{F}_{Y_1}(x \mid \theta), \bar{F}_{Y_2}(x \mid \theta), \ldots, \bar{F}_{Y_m}(x \mid \theta))\, \mathrm{d}F_\Theta(\theta) \\
&= \bar{F}_{\tau_2(\boldsymbol{Y}(\Theta))}(x),
\end{aligned}$$

where the first inequality follows from condition (ii) and the fact that $h_1(\boldsymbol{p})$ is increasing in each p_i. The second inequality follows from condition (i). Hence the result is proved. □

Proof of Theorem 3.2. We only prove the result under the set of conditions {(i), (ii), (iii)}. The result could be proved along the same lines under the second set of conditions. Note that

$$\sum_{i=1}^m r_{Y_i}(x \mid \theta_2)\left[\frac{p_i}{h_2(\boldsymbol{p})}\frac{\partial h_2(\boldsymbol{p})}{\partial p_i}\right]_{p_i=\bar{F}_{Y_i}(x\mid\theta_2)} \geq \sum_{i=1}^m r_{Y_i}(x \mid \theta_1)\left[\frac{q_i}{h_2(\boldsymbol{q})}\frac{\partial h_2(\boldsymbol{q})}{\partial q_i}\right]_{q_i=\bar{F}_{Y_i}(x\mid\theta_1)},$$

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which follows from [26, Proposition 2.3(ii)] by using conditions (ii) and (iii). The above inequality is equivalent to the fact that

$$\frac{h_2(\bar{F}_{Y_1}(x \mid \theta_2), \bar{F}_{Y_2}(x \mid \theta_2), \ldots, \bar{F}_{Y_m}(x \mid \theta_2))}{h_2(\bar{F}_{Y_1}(x \mid \theta_1), \bar{F}_{Y_2}(x \mid \theta_1), \ldots, \bar{F}_{Y_m}(x \mid \theta_1))} \text{ is decreasing in } x > 0,$$

or equivalently

$$h_2(\bar{F}_{Y_1}(x \mid \theta), \bar{F}_{Y_2}(x \mid \theta), \ldots, \bar{F}_{Y_m}(x \mid \theta)) \text{ is } \mathrm{RR}_2 \text{ in } (x, \theta) \in (0, \infty) \times \Omega. \tag{A.1}$$

Further, we have

$$\begin{aligned}
\sum_{i=1}^{n} r_{X_i}(x \mid \theta)\left[\frac{p_i}{h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\right]_{p_i=\bar{F}_{X_i}(x\mid\theta)} &\geq \sum_{i=1}^{m} r_{X_i}(x \mid \theta)\left[\frac{p_i}{h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\right]_{p_i=\bar{F}_{X_i}(x\mid\theta)} \\
&\geq \sum_{i=1}^{m} r_{Y_i}(x \mid \theta)\left[\frac{p_i}{h_2(\boldsymbol{p})}\frac{\partial h_2(\boldsymbol{p})}{\partial p_i}\right]_{p_i=\bar{F}_{X_i}(x\mid\theta)} \\
&\geq \sum_{i=1}^{m} r_{Y_i}(x \mid \theta)\left[\frac{q_i}{h_2(\boldsymbol{q})}\frac{\partial h_2(\boldsymbol{q})}{\partial q_i}\right]_{q_i=\bar{F}_{Y_i}(x\mid\theta)},
\end{aligned}$$

where the first inequality holds because each term in the summation is non-negative. The second inequality follows from conditions (i) and (iii), whereas the third inequality follows from (ii) and (iii). Thus the above expression can equivalently be written as

$$\frac{h_2(\bar{F}_{Y_1}(x \mid \theta), \bar{F}_{Y_2}(x \mid \theta), \ldots, \bar{F}_{Y_m}(x \mid \theta))}{h_1(\bar{F}_{X_1}(x \mid \theta), \bar{F}_{X_2}(x \mid \theta), \ldots, \bar{F}_{X_n}(x \mid \theta))} \text{ is increasing in } x > 0 \text{ for all } \theta \in \Omega. \tag{A.2}$$

Again, condition (iii) implies that, for all $\theta_1 \leq \theta_2$ and for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$,

$$X_i(\theta_1) \leq_{\text{st}} X_i(\theta_2) \quad \text{and} \quad Y_j(\theta_2) \leq_{\text{st}} Y_j(\theta_1).$$

Using this, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\theta} h_1(\bar{F}_{X_1}(x \mid \theta), \bar{F}_{X_2}(x \mid \theta), \ldots, \bar{F}_{X_n}(x \mid \theta)) = \sum_{i=1}^{n} \left[\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}\theta}\right]_{p_i=\bar{F}_{X_i}(x\mid\theta)} \geq 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta} h_2(\bar{F}_{Y_1}(x \mid \theta), \bar{F}_{Y_2}(x \mid \theta), \ldots, \bar{F}_{Y_m}(x \mid \theta)) = \sum_{j=1}^{m} \left[\frac{\partial h_2(\boldsymbol{p})}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}\theta}\right]_{p_i=\bar{F}_{Y_i}(x\mid\theta)} \leq 0,$$

or equivalently

$$\frac{1}{h_1(\bar{F}_{X_1}(x \mid \theta), \bar{F}_{X_2}(x \mid \theta), \ldots, \bar{F}_{X_n}(x \mid \theta))} \text{ is decreasing in } \theta \in \Omega$$

and

$$h_2(\bar{F}_{Y_1}(x \mid \theta), \bar{F}_{Y_2}(x \mid \theta), \ldots, \bar{F}_{Y_m}(x \mid \theta)) \text{ is decreasing in } \theta \in \Omega.$$

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Combining these two, we obtain

$$\frac{h_2(\bar{F}_{Y_1}(x\,|\,\theta), \bar{F}_{Y_2}(x\,|\,\theta), \ldots, \bar{F}_{Y_m}(x\,|\,\theta))}{h_1(\bar{F}_{X_1}(x\,|\,\theta), \bar{F}_{X_2}(x\,|\,\theta), \ldots, \bar{F}_{X_n}(x\,|\,\theta))} \text{ is decreasing in } \theta \in \Omega \text{ for all } x > 0. \tag{A.3}$$

Using (A.1), (A.2), and (A.3) in Lemma 2.1, we obtain, for $x_1 \le x_2$,

$$\frac{\int_\Omega h_1(\bar{F}_{X_1}(x_2|\theta), \bar{F}_{X_2}(x_2|\theta), \ldots, \bar{F}_{X_n}(x_2|\theta))\, \mathrm{d}F_\Theta(\theta)}{\int_\Omega h_2(\bar{F}_{Y_1}(x_2|\theta), \bar{F}_{Y_2}(x_2|\theta), \ldots, \bar{F}_{Y_m}(x_2|\theta))\, \mathrm{d}F_\Theta(\theta)}$$
$$\le \frac{\int_\Omega h_1(\bar{F}_{X_1}(x_1|\theta), \bar{F}_{X_2}(x_1|\theta), \ldots, \bar{F}_{X_n}(x_2|\theta))\, \mathrm{d}F_\Theta(\theta)}{\int_\Omega h_2(\bar{F}_{Y_1}(x_1|\theta), \bar{F}_{Y_2}(x_1|\theta), \ldots, \bar{F}_{Y_m}(x_1|\theta))\, \mathrm{d}F_\Theta(\theta)},$$

or equivalently

$$\frac{\bar{F}_{\tau_1(\boldsymbol{X}(\Theta))}(x)}{\bar{F}_{\tau_2(\boldsymbol{Y}(\Theta))}(x)} = \frac{\int_\Omega h_1(\bar{F}_{X_1}(x\,|\,\theta), \bar{F}_{X_2}(x\,|\,\theta), \ldots, \bar{F}_{X_n}(x\,|\,\theta))\, \mathrm{d}F_\Theta(\theta)}{\int_\Omega h_2(\bar{F}_{Y_1}(x\,|\,\theta), \bar{F}_{Y_2}(x\,|\,\theta), \ldots, \bar{F}_{Y_m}(x\,|\,\theta))\, \mathrm{d}F_\Theta(\theta)} \text{ is decreasing in } x > 0,$$

and hence $\tau_1(\boldsymbol{X}(\Theta)) \le_{\mathrm{hr}} \tau_2(\boldsymbol{Y}(\Theta))$. □

Proof of Theorem 5.2. We only prove the result under the condition {(i), (iii), (iv)}. The result follows similarly for the other case. Now, from condition (iv), we have

$$\min_{1\le i\le n} r_{X_i}(x\,|\,\theta) \ge \max_{1\le i\le m} r_{Y_i}(x) \quad \text{for all } \theta \in \Omega. \tag{A.4}$$

Then

$$\begin{aligned}
&\sum_{i=1}^{n} r_{X_i}(x\,|\,\theta)\left[\frac{p_i}{h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\right]_{p_i=\bar{F}_{X_i}(x|\theta)} \\
&\quad\ge \min_{1\le i\le n} r_{X_i}(x\,|\,\theta)\sum_{i=1}^{n}\left[\frac{p_i}{h_1(\boldsymbol{p})}\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\right]_{p_i=\bar{F}_{X_i}(x|\theta)} \\
&\quad\ge \max_{1\le i\le m} r_{Y_i}(x)\sum_{i=1}^{m}\left[\frac{p_i}{h_2(\boldsymbol{p})}\frac{\partial h_2(\boldsymbol{p})}{\partial p_i}\right]_{\substack{p_i=\bar{F}_{X_i}(x|\theta),\ i=1,2,\ldots,\min\{m,n\} \\ \{p_i=\bar{F}_{Y_i}(x),\ i=n+1,\ldots,m\}I_{[m>n]}}} \\
&\quad\ge \max_{1\le i\le m} r_{Y_i}(x)\sum_{i=1}^{m}\left[\frac{q_i}{h_2(\boldsymbol{q})}\frac{\partial h_2(\boldsymbol{q})}{\partial q_i}\right]_{q_i=\bar{F}_{Y_i}(x)} \\
&\quad\ge \sum_{i=1}^{m} r_{Y_i}(x)\left[\frac{q_i}{h_2(\boldsymbol{q})}\frac{\partial h_2(\boldsymbol{q})}{\partial q_i}\right]_{q_i=\bar{F}_{Y_i}(x)},
\end{aligned}$$

where the first and the fourth inequalities are obvious. The second inequality follows from (A.4) and condition (i), whereas the third inequality follows from conditions (iii) and (iv). Now, the above inequality implies that

$$h_2(\bar{F}_{Y_1}(x), \bar{F}_{Y_2}(x), \ldots, \bar{F}_{Y_m}(x))\int_\Omega\left(\sum_{i=1}^{n} r_{X_i}(x\,|\,\theta)\left[p_i\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\right]_{p_i=\bar{F}_{X_i}(x|\theta)}\right)\mathrm{d}F_\Theta(\theta)$$
$$\ge \left(\sum_{i=1}^{m} r_{Y_i}(x)\left[q_i\frac{\partial h_2(\boldsymbol{q})}{\partial q_i}\right]_{q_i=\bar{F}_{Y_i}(x)}\right)\int_\Omega h_1(\bar{F}_{X_1}(x\,|\,\theta), \bar{F}_{X_2}(x\,|\,\theta), \ldots, \bar{F}_{X_n}(x\,|\,\theta))\,\mathrm{d}F_\Theta(\theta),$$

which is equivalent to the fact that

$$\frac{\bar{F}_{\tau_1(\boldsymbol{X}(\Theta))}(x)}{\bar{F}_{\tau_2(\boldsymbol{Y})}(x)} = \frac{\int_\Omega h_1(\bar{F}_{X_1}(x\mid\theta), \bar{F}_{X_2}(x\mid\theta), \ldots, \bar{F}_{X_n}(x\mid\theta))\, \mathrm{d}F_\Theta(\theta)}{h_2(\bar{F}_{Y_1}(x), \bar{F}_{Y_2}(x), \ldots, \bar{F}_{Y_m}(x))} \text{ is decreasing in } x > 0,$$

and hence $\tau_1(\boldsymbol{X}(\Theta)) \leq_{\text{hr}} \tau_2(\boldsymbol{Y})$. □

Proof of Theorem 5.4. Note that $\tau_1(\boldsymbol{X}(\Theta)) \leq_{\text{lr}} \tau_2(\boldsymbol{Y})$ holds if

$$\frac{f_{\tau_1(\boldsymbol{X}(\Theta))}(x)}{f_{\tau_2(\boldsymbol{Y})}(x)} = \left(\int_\Omega \left[\sum_{i=1}^{n}\left(f_{X_i}(x\mid\theta)\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\right)\right]\mathrm{d}F_\Theta(\theta)\right)\Bigg/\left(\sum_{i=1}^{m}\left(f_{Y_i}(x)\frac{\partial h_2(\boldsymbol{q})}{\partial q_i}\right)\right)$$
$$\text{is decreasing in } x > 0,$$

or equivalently

$$\left(\sum_{i=1}^{n}\left[\int_\Omega\left(f_{X_i}(x\mid\theta)\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\right)\mathrm{d}F_\Theta(\theta)\right]\right)\Bigg/\left(\sum_{i=1}^{m}\left(f_{Y_i}(x)\frac{\partial h_2(\boldsymbol{q})}{\partial q_i}\right)\right) \text{ is decreasing in } x > 0.$$

This holds if, for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$,

$$\int_\Omega\left(\frac{f_{X_i}(x\mid\theta)}{f_{Y_j}(x)}\right)\left(\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\Bigg/\frac{\partial h_2(\boldsymbol{q})}{\partial q_j}\right)\mathrm{d}F_\Theta(\theta) \text{ is decreasing in } x > 0,$$

which holds if, for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$,

$$\frac{f_{X_i}(x\mid\theta)}{f_{Y_j}(x)} \text{ is decreasing in } x > 0 \text{ for all } \theta \in \Omega$$

and

$$\frac{\partial h_1(\boldsymbol{p})}{\partial p_i}\Bigg/\frac{\partial h_2(\boldsymbol{q})}{\partial q_j} \text{ is decreasing in } x > 0,$$

which are true because of conditions (i) and (ii). □

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References

[1] Amini-Seresht, E., Zhang, Y. and Balakrishnan, N. (2018). Stochastic comparisons of coherent systems under different random environments. *J. Appl. Prob.* **55**, 459–472.

[2] Badía, F. G., Sangüesa, C. and Cha, J. H. (2014). Stochastic comparison of multivariate conditionally dependent mixtures. *J. Multivar. Anal.* **129**, 82–94.

[3] Balakrishnan, N., Barmalzan, G. and Haidari, A. (2016). Multivariate stochastic comparisons of multivariate mixture models and their applications. *J. Multivar. Anal.* **145**, 37-43.

https://doi.org/10.1017/jpr.2019.53 Published online by Cambridge University Press

[4] BALAKRISHNAN, N. AND ZHAO, P. (2013). Ordering properties of order statistics from heterogeneous populations: a review with an emphasis on some recent developments. *Probab. Engrg. Inform. Sci.* **27**, 403–443.
[5] BARLOW, R. E. AND PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart and Winston, New York.
[6] BELZUNCE, F., FRANCO, M., RUIZ, J. M. AND RUIZ, M. C. (2001). On partial orderings between coherent systems with different structures. *Probab. Engrg. Inform. Sci.* **15**, 273–293.
[7] BELZUNCE, F., MERCADER, J. A., RUIZ, J. M. AND SPIZZICHINO, F. (2009). Stochastic comparisons of multivariate mixture models. *J. Multivar. Anal.* **100**, 1657–1669.
[8] CHA, J. H. AND FINKELSTEIN, M. (2018). *Point Processes for Reliability Analysis. Shocks and Repairable Systems*. Springer, London.
[9] DEWAN, I. AND KHALEDI, B. E. (2014). On stochastic comparisons of residual lifetime at random time. *Statist. Probab. Lett.* **88**, 73–79.
[10] ESARY, J. D. AND PROSCHAN, F. (1963). Reliability between system failure rate and component failure rates. *Technometrics* **5**, 183–189.
[11] FINKELSTEIN, M. (1999). Wearing-out of components in a variable environment. *Reliab. Engrg. System Safety* **66**, 235–242.
[12] FINKELSTEIN, M. (2008). *Failure Rate Modeling for Reliability and Risk*. Springer, London.
[13] HAZRA, N. K., KUITI, M. R., FINKELSTEIN, M. AND NANDA, A. K. (2017). On stochastic comparisons of maximum order statistics from the location-scale family of distributions. *J. Multivar. Anal.* **160**, 31–41.
[14] HAZRA, N. K. AND NANDA, A. K. (2016). Stochastic comparisons between used systems and systems made by used components. *IEEE Trans. Rel.* **65**, 751–762.
[15] KARLIN, S. (1968). *Total Positivity*. Stanford University Press, CA.
[16] KENZIN, M. AND FROSTIG, E. (2009). M-out-of-n inspected systems subject to shocks in random environment. *Reliab. Engrg. System Safety* **94**, 1322–1330.
[17] KOCHAR, S., MUKERJEE, H. AND SAMANIEGO, F. J. (1999). The 'signature' of a coherent system and its application to comparisons among systems. *Naval Res. Logist.* **46**, 507–523.
[18] LINDQVIST, B. H., SAMANIEGO, F. J. AND HUSEBY, A. B. (2016). On the equivalence of systems of different sizes, with applications to system comparisons. *Adv. Appl. Prob.* **48**, 332–348.
[19] MARSHALL, A. W. AND OLKIN, I. (2007). *Life Distributions.* Springer, New York.
[20] MISRA, A. K. AND MISRA N. (2012). Stochastic properties of conditionally independent mixture models. *J. Statist. Plann. Inference* **142**, 1599–1607.
[21] MULERO, J., PELLERY, F. AND RODRÍGUEZ-GRIÑOLO, R. (2010). Negative aging and stochastic comparisons of residual lifetimes in multivariate frailty models. *J. Statist. Plann. Inference* **140**, 1594–1600.
[22] NAKAGAWA, T. (1979). Further results of replacement problem of a parallel system in random environment. *J. Appl. Prob.* **16**, 923–926.
[23] NANDA, A. K., JAIN, K. AND SINGH, H. (1998). Preservation of some partial orderings under the formation of coherent systems. *Statist. Probab. Lett.* **39**, 123–131.
[24] NAVARRO, J., ÁGUILA, Y. D., SORDO, M. A. AND SUÁREZ-LIORENS, A. (2013). Stochastic ordering properties for systems with dependent identical distributed components. *Appl. Stoch. Models Bus. Ind.* **29**, 264–278.
[25] NAVARRO, J., ÁGUILA, Y. D., SORDO, M. A. AND SUÁREZ-LIORENS, A. (2014). Preservation of reliability classes under the formation of coherent systems. *Appl. Stoch. Models Bus. Ind.* **30**, 444–454.
[26] NAVARRO, J., ÁGUILA, Y. D., SORDO, M. A. AND SUÁREZ-LIORENS, A. (2016). Preservation of stochastic orders under the formation of generalized distorted distributions: applications to coherent systems. *Methodol. Comput. Appl. Probab.* **18**, 529–545.
[27] NAVARRO, J., PELLEREY, F. AND DI CRESCENZO, A. (2015). Orderings of coherent systems with randomized dependent components. *Eur. J. Operat. Res.* **240**, 127–139.
[28] NAVARRO, J. AND RUBIO, R. (2010). Comparisons of coherent systems using stochastic precedence. *TEST* **19**, 469–486.
[29] NELSEN, R. B. (1999). *An Introduction to Copulas*. Springer, New York.
[30] PERSONA, A., SGARBOSSA, F. AND PHAM, H. (2016). Systemability: a new reliability function for different environments. In *Quality and Reliability Management and Its Applications*, pp. 145–193. Springer, London.
[31] PETAKOS, K. AND TSAPELAS, T. (1997). Reliability analysis for systems in a random environment. *J. Appl. Prob.* **34**, 1021–1031.
[32] PLEDGER, P. AND PROSCHAN, F. (1971). Comparisons of order statistics and of spacings from heterogeneous distributions. In *Optimizing Methods in Statistics*, ed. J. S. Rustagi, pp. 89–113. Academic Press, New York.
[33] PROSCHAN, F. AND SETHURAMAN, J. (1976). Stochastic comparisons of order statistics from heterogeneous populations, with applications in reliability. *J. Multivar. Anal.* **6**, 608–616.

[34] RÅDE, L. (1976). Reliability systems in random environment. *J. Appl. Prob.* **13**, 407–410.
[35] SAMANIEGO, F. J. (2007). *System Signature and Their Applications in Engineering Reliability*. Springer, New York.
[36] SAMANIEGO, F. J. AND NAVARRO, J. (2016). On comparing coherent systems with heterogeneous components. *Adv. Appl. Prob.* **48**, 88–111.
[37] SHAKED, M. AND SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. Springer, New York.