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INEQUALITIES FOR DRAGOMIR'S MAPPINGS VIA STIELTJES INTEGRALS

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To the memory of my student Tomasz Kołodziej

Abstract

We present some inequalities for the mappings defined by Dragomir ['Two mappings in connection to Hadamard's inequalities', *J. Math. Anal. Appl.* **167** (1992), 49–56]. We analyse known inequalities connected with these mappings using a recently developed method connected with stochastic orderings and Stieltjes integrals. We show that some of these results are optimal and others may be substantially improved.

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1. Introduction

We study some inequalities for mappings defined by Dragomir [1, 2]. Detailed proofs of the inequalities obtained by Dragomir and some other authors may be found in the master's thesis of Kołodziej [5] who passed away before the defence of the thesis and the start of his planned PhD studies.

We begin by outlining the methods we will exploit. First we cite Ohlin's lemma which was recently used to obtain numerous inequalities for convex functions (see, for example, [9, 11–15]).

LEMMA 1.1 (Ohlin [8]). *Let X_1, X_2 be two random variables such that $\mathbb{E}X_1 = \mathbb{E}X_2$ and let F_1, F_2 be their distribution functions. If F_1, F_2 satisfy the inequalities*

$$F_1(x) \le F_2(x) \text{ if } x < x_0 \quad \text{and} \quad F_1(x) \ge F_2(x) \text{ if } x > x_0 \tag{1.1}$$

for some x_0, then

$$\mathbb{E}f(X_1) \le Ef(X_2)$$

for all continuous and convex functions $f : \mathbb{R} \to \mathbb{R}$.

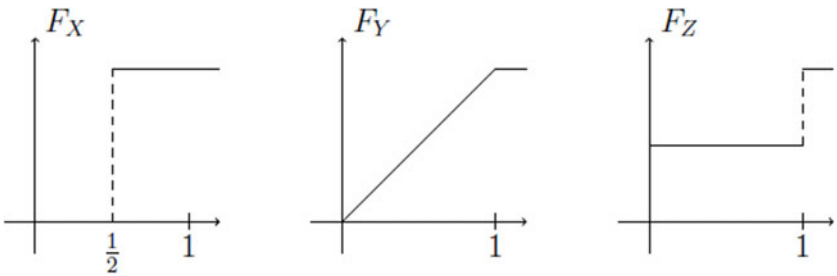

FIGURE 1. The graphs of functions used to prove the Hermite–Hadamard inequality.

It was observed by Rajba in [11] that the classical Hermite–Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

(satisfied by every convex function $f : [a, b] \to \mathbb{R}$) follows immediately from Ohlin's lemma. Indeed, it is enough to take $\mu_X = \delta_{(a+b)/2}$, μ_Y as a measure that is uniformly distributed in the interval $[a, b]$ and $\mu_Z = (\delta_a + \delta_b)/2$. The cumulative distribution functions connected with these measures are presented in Figure 1 (with $a = 0$ and $b = 1$) and it can be seen that both pairs F_X, F_Y and F_Y, F_Z satisfy (1.1) with $x_0 = \frac{1}{2}$.

Ohlin's lemma provides only a sufficient condition for an inequality to hold. For example, if we consider the probability measure equally distributed on the interval $[a, b]$ and the measure given by the formula

$$\lambda(\delta_x + \delta_y) + (1-\lambda)\delta_{(x+y)/2}$$

for some $\lambda \in (0, 1)$, then the corresponding cumulative distribution functions will have more than one crossing point and, consequently, we will not be able to decide whether the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \lambda\frac{f(a)+f(b)}{2} + (1-\lambda)f\Big(\frac{a+b}{2}\Big) \tag{1.2}$$

holds for all convex functions using Ohlin's lemma. Therefore, as in [9] and [14], we use a more general result from [6] (see also [7, Theorem 4.2.7]).

THEOREM 1.2 (Levin and Stečkin [6]). *Let $F_1, F_2 : [a, b] \to \mathbb{R}$ be two functions with bounded variation such that $F_1(a) = F_2(a)$. Then, in order that*

$$\int_a^b f(x)\,dF_1(x) \le \int_a^b f(x)\,dF_2(x)$$

holds for all continuous and convex functions $f : [a, b] \to \mathbb{R}$ it is necessary and

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sufficient that F_1 and F_2 satisfy the three conditions:

$$\text{(i)}\quad F_1(b) = F_2(b),$$
$$\text{(ii)}\quad \int_a^x F_1(t)\,dt \le \int_a^x F_2(t)\,dt, \qquad x \in (a,b),$$
$$\text{(iii)}\quad \int_a^b F_1(t)\,dt = \int_a^b F_2(t)\,dt.$$

The following corollary is a particular case of Theorem 1.2 (see [9]).

COROLLARY 1.3. *Let $F_1, F_2 : [a,b] \to \mathbb{R}$ be two functions with bounded variation such that $F_1(a) = F_2(a)$, $F_1(b) = F_2(b)$ and*

$$\int_a^b F_1(t)\,dt = \int_a^b F_2(t)\,dt.$$

If F_1 and F_2 have three crossing points $x_0, x_1, x_2 \in (a,b)$ and $F_1(x) \le F_2(x)$ for $x \in [a, x_0]$, then the inequality

$$\int_a^b f(x)\,dF_1(x) \le \int_a^b f(x)\,dF_2(x)$$

is satisfied for all continuous and convex functions $f : [a,b] \to \mathbb{R}$ if and only if

$$\int_a^{x_0} (F_2(x) - F_1(x))\,dx \ge \int_{x_0}^{x_1} (F_1(x) - F_2(x))\,dx. \tag{1.3}$$

Using this corollary, it is easily seen that the inequality (1.2) is satisfied by all convex functions if and only if $\lambda \ge \frac{1}{2}$. This, clearly, is not a new result and the inequality obtained for $a = \frac{1}{2}$ is known as Bullen's inequality. Further, since (1.3) is not satisfied for $\lambda > \frac{1}{2}$, the optimal constant in (1.2) is $\lambda = \frac{1}{2}$. Clearly, it is possible to show that (1.2) is satisfied for some $\lambda < \frac{1}{2}$, but such a result will not be optimal and will follow from the optimal one. In the next part of the paper we examine in a similar way some other known results. In some cases, it is possible to provide stronger (and optimal) versions of these results.

2. Auxiliary results

The following mappings were considered among others in [1, 2] (see also [4] for a systematic collection of definitions and results around the Hermite–Hadamard inequality).

DEFINITION 2.1. Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Define the mappings

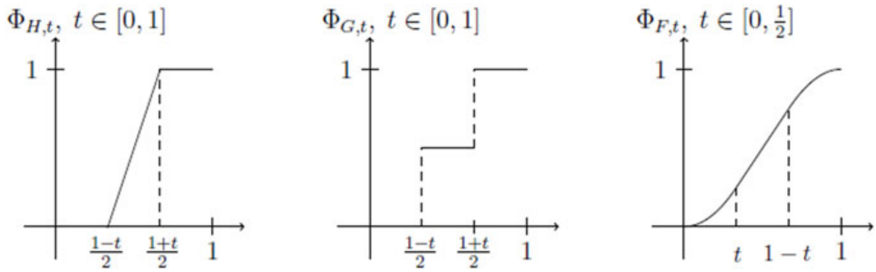

FIGURE 2. The graphs of the functions $\Phi_{H,t}$, $\Phi_{G,t}$ and $\Phi_{F,t}$.

$F, G, H : [0, 1] \to \mathbb{R}$ by

$$F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\,dx\,dy, \tag{2.1}$$

$$G(t) := \frac{1}{2}\Big(f\Big(ta + (1-t)\frac{a+b}{2}\Big) + f\Big(tb + (1-t)\frac{a+b}{2}\Big)\Big), \tag{2.2}$$

$$H(t) := \frac{1}{(b-a)} \int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)\,dx. \tag{2.3}$$

Later in the paper we shall have a closer look at inequalities involving these mappings. However, before doing so, we need to make some simple observations. In the following remark, $f^{(-n)}$ stands for any function satisfying $(f^{(-n)})^{(n)} = f$.

REMARK 2.2. If $f : [a, b] \to \mathbb{R}$ is integrable, then F and H are given by the formulas

$$H(t) = \frac{1}{t}\,\frac{f^{(-1)}(\frac{1}{2}a(1-t) + \frac{1}{2}b(1+t)) - f^{(-1)}(\frac{1}{2}a(1+t) + \frac{1}{2}b(1-t))}{b-a}, \tag{2.4}$$

$$F(t) = \frac{1}{t(1-t)}\,\frac{f^{(-2)}(a) - f^{(-2)}(tb + (1-t)a) - f^{(-2)}(ta + (1-t)b) + f^{(-2)}(b)}{(b-a)^2}. \tag{2.5}$$

We use this remark in the following lemma to show that F, G, H may be represented in the form of Stieltjes integrals. For simplicity we work on the interval $[0, 1]$ instead of $[a, b]$; it is obvious that these functions may easily be transferred to any other interval. The graphs of the functions occurring in Lemma 2.3 are shown in Figure 2.

LEMMA 2.3. *Let $f : [0, 1] \to \mathbb{R}$ be an integrable function and define the mappings $F, G, H : [0, 1] \to \mathbb{R}$ by (2.1), (2.2) and (2.3), respectively. Then, for all $t \in [0, \frac{1}{2}]$,*

$$F(t) = \int_0^1 f(s)\,d\Phi_{F,t}(s), \quad G(t) = \int_0^1 f(s)\,d\Phi_{G,t}(s), \quad H(t) = \int_0^1 f(s)\,d\Phi_{H,t}(s), \tag{2.6}$$

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where

$$\Phi_{F,t}(s) = \begin{cases} \frac{1}{2t(1-t)}s^2 & s \in [0,t], \\ \frac{1}{1-t}s - \frac{t}{2(1-t)} & s \in (t, 1-t], \\ -\frac{1}{2t(1-t)}s^2 + \frac{1}{t(1-t)}s + \frac{-2t^2+2t-1}{2t(1-t)} & t \in (1-t, 1], \end{cases}$$

$$\Phi_{G,t}(s) = \begin{cases} 0 & s \in \left[0, \frac{1-t}{2}\right], \\ \frac{1}{2} & s \in \left(\frac{1-t}{2}, \frac{1+t}{2}\right], \\ 1 & t \in \left(\frac{1+t}{2}, 1\right] \end{cases}$$

and

$$\Phi_{H,t}(s) = \begin{cases} 0 & s \in \left[0, \frac{1-t}{2}\right], \\ \frac{s}{t} + \frac{t-1}{2t} & s \in \left[\frac{1-t}{2}, \frac{1+t}{2}\right], \\ 1 & t \in \left(\frac{1+t}{2}, 1\right]. \end{cases}$$

PROOF. The equality for G is obvious; thus we start with the assertion for F. Since $\Phi_{F,t}$ is continuous, we may write

$$\begin{aligned} \int_0^1 f(s)\,d\Phi_{F,t}(s) &= \frac{1}{t(1-t)}\int_0^t sf(s)\,ds + \frac{1}{1-t}\int_t^{1-t} f(s)\,ds \\ &\quad - \frac{1}{t(1-t)}\int_{1-t}^1 sf(s)\,ds + \frac{1}{t(1-t)}\int_{1-t}^1 f(s)\,ds \\ &= \frac{f^{(-1)}(t)}{1-t} - \frac{1}{t(1-t)}\int_0^t f^{(-1)}(s)\,ds + \frac{f^{(-1)}(1-t) - f^{(-1)}(t)}{1-t} \\ &\quad - \frac{f^{(-1)}(1) - (1-t)f^{(-1)}(1-t)}{t(1-t)} + \frac{1}{t(1-t)}\int_{1-t}^1 f^{(-1)}(s)\,ds \\ &\quad + \frac{f^{(-1)}(1) - f^{(-1)}(1-t)}{t(1-t)} \\ &= \frac{f^{(-2)}(0) - f^{(-2)}(t) - f^{(-2)}(1-t) + f^{(-2)}(1)}{t(1-t)}, \end{aligned}$$

which, together with (2.5), shows that the equation for F in (2.6) is satisfied.

To see that the equation for H is true it is enough to write

$$\int_0^1 f(s)\,d\Phi_{H,t}(s) = \int_{(1-t)/2}^{(1+t)/2} \frac{f(s)}{t}\,ds = \frac{1}{t}\left(f^{(-1)}\left(\frac{1+t}{2}\right) - f^{(-1)}\left(\frac{1-t}{2}\right)\right),$$

which, in view of (2.4) finishes the proof. □

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Remark 2.4. In [1] it was proved that H is increasing. Now it is easy to see this from Ohlin's lemma. Indeed, for $t_1, t_2 \in [0,1]$ with $t_1 < t_2$,

$$\Phi_{H,t_1}(s) \le \Phi_{H,t_2}(s) \text{ for } s \in [0, \tfrac{1}{2}) \quad \text{and} \quad \Phi_{H,t_1}(s) \le \Phi_{H,t_2}(s) \text{ for } s \in (\tfrac{1}{2}, 1].$$

3. Applications

In this section we analyse known inequalities involving the mappings F, G and H using the representations obtained in Lemma 2.3. To visualise the methods which will be used in the remainder of the paper, we begin with an inequality which is optimal and cannot be strengthened. Thus we provide here only a new (very short) proof of the existing result.

Remark 3.1. Let $f : [a,b] \to \mathbb{R}$ be a convex function and define the function H by (2.3). It is known (see [4]) that

$$H(t) \le t \frac{1}{b-a} \int_a^b f(s)\, ds + (1-t) f\Big(\frac{a+b}{2}\Big). \tag{3.1}$$

To see that this inequality is true it is enough to consider the function

$$\Phi_{K,t}(s) = \begin{cases} ts & s \in [0, \frac{1}{2}], \\ ts + (1-t) & s \in (\frac{1}{2}, 1]. \end{cases}$$

Then

$$\int_0^1 f\, d\Phi_{K,t} = t \int_0^1 f(x)\, dx + (1-t) f\Big(\frac{1}{2}\Big).$$

The functions $\Phi_{K,t}$ and $\Phi_{H,t}$ have three crossing points and the second crossing point is $\frac{1}{2}$. Since

$$\int_0^{1/2} \Phi_{K,t}(s)\, ds = \int_0^{1/2} \Phi_{H,t}(s)\, ds = \frac{t}{8},$$

we can use Corollary 1.3 to see that (3.1) holds and that the constant t occurring on the right-hand side of this inequality is optimal.

We now proceed to some examples where we can prove stronger versions of known inequalities. Dragomir in [1] proved that

$$F(t) \ge H(t) \quad \text{for all } t \in [0,1]. \tag{3.2}$$

In the next theorem we show that it is possible to prove a stronger inequality.

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and define the mappings F, H by (2.1) and (2.3), respectively. Then $F(t) \ge H(\psi(t))$ for $t \in [0, \frac{1}{2}]$, where the function $\psi : [0,1] \to [0,1]$ is given by*

$$\psi(t) := \begin{cases} \dfrac{4t^2 - 6t + 3}{3(1-t)} & t \in \Big[0, \dfrac{1}{2}\Big), \\[2ex] \dfrac{4t^2 - 2t + 1}{3t} & t \in \Big[\dfrac{1}{2}, 1\Big]. \end{cases}$$

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PROOF. Assume first that $t \le \frac{1}{2}$. It was observed in Remark 3.1 that

$$\int_0^{1/2} \Phi_{H,t}(s)\,ds = \frac{t}{8}.$$

On the other hand,

$$\int_0^{1/2} \Phi_{F,t}(s)\,ds = \int_0^t \frac{1}{2t(1-t)} s^2\,ds + \int_t^{1/2} \frac{1}{1-t} s - \frac{t}{2(1-t)}\,ds$$
$$= \frac{t^2}{6(1-t)} + \frac{1}{8(1-t)} - \frac{t^2}{2(1-t)} - \frac{t(\frac{1}{2}-t)}{2(1-t)} = \frac{4t^2-6t+3}{24(1-t)}. \tag{3.3}$$

Take $r = (4t^2 - 6t + 3)/3(1-t)$. Then we may write

$$\int_0^{1/2} \Phi_{H,r}(s)\,ds = \frac{r}{8} = \frac{4t^2-6t+3}{24(1-t)} = \int_0^{1/2} \Phi_{F,t}(s)\,ds. \tag{3.4}$$

Furthermore, it can be seen (see Figure 2) that the functions $\Phi_{H,r}$ and $\Phi_{F,t}$ have three crossing points: $x_1 = \frac{1}{2}$ and two other points x_0, x_2 symmetrical with respect to $\frac{1}{2}$. In view of (3.4),

$$\int_0^{x_0} (\Phi_{F,t}(s) - \Phi_{H,r}(s))\,ds = \int_{x_0}^{1/2} (\Phi_{H,r}(s) - \Phi_{F,t}(s))\,ds.$$

Therefore we can use Corollary 1.3 to finish the proof in this case.

Now, consider the case $t > \frac{1}{2}$. Since $F(t) = F(1-t)$ from the definition of F and $1 - t < \frac{1}{2}$, we may use the inequality already proved to arrive at

$$F(t) = F(1-t) \ge H\Big(\frac{4(1-t)^2 - 6(1-t) + 3}{3(1-(1-t))}\Big) = H\Big(\frac{4t^2+2t+1}{3t}\Big). \qquad \square$$

REMARK 3.3. It is easy to see that $\psi(t) > t$ for all $t \in (0,1)$. The difference between the graph of the function ψ and the diagonal is visualised in Figure 3. On the other hand, in Remark 2.4 it was observed that H is increasing. This means that Theorem 3.2 is indeed stronger than the original inequality $F(t) \ge H(t)$ from [1].

In [10] the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \le \min\{t, 1-t\}\Big(\frac{f(x)+f(y)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\Big)$$

was obtained (see also [4]). For $t \le \frac{1}{2}$, the inequality can be reformulated as

$$\frac{1}{b-a}\int_a^b f(s)\,ds \le \frac{t}{1+t}\frac{f(a)+f(b)}{2} + \frac{1}{1+t}F(t),$$

whereas for $t \ge \frac{1}{2}$ it can be written as

$$\frac{1}{b-a}\int_a^b f(s)\,ds \le \frac{1-t}{2-t}\frac{f(a)+f(b)}{2} + \frac{1}{2-t}F(t).$$

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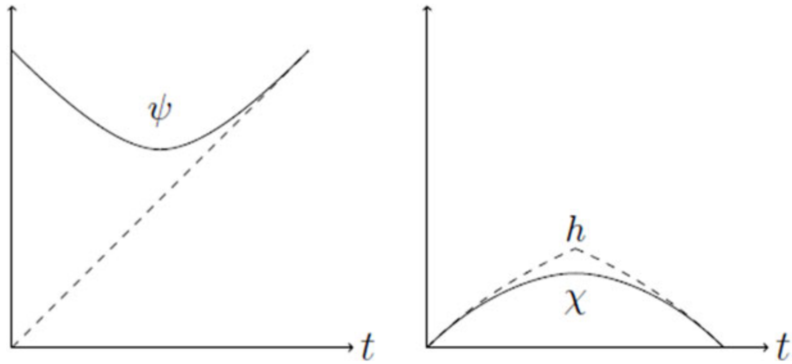

FIGURE 3. Comparison of ψ and χ with the identity function and h, respectively.

Summarising, these inequalities yield

$$\frac{1}{b-a}\int_a^b f(s)\,ds \le h(t)\frac{f(a)+f(b)}{2} + (1-h(t))F(t),$$

where

$$h(t) := \begin{cases} \dfrac{t}{1+t} & t \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{1-t}{2-t} & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{3.5}$$

In the following theorem we give an analogous inequality with much better (and optimal) functions on the right-hand side.

THEOREM 3.4. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and define $F : [0, 1] \to \mathbb{R}$ by (2.1). Then*

$$\frac{1}{b-a}\int_a^b f(s)\,ds \le \chi(t)\frac{f(a)+f(b)}{2} + (1-\chi(t))F(t), \tag{3.6}$$

where

$$\chi(t) = \begin{cases} \dfrac{-4t^2+3t}{-4t^2+3} & t \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{-4t^2+5t-1}{-4t^2+8t-1} & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

PROOF. First we consider the case $t \le \frac{1}{2}$, and as always we work on the interval $[0, 1]$. Let $c \in [0, 1]$ be given and define a function $\Psi_c : [0, 1] \to \mathbb{R}$ by

$$\Psi_c(s) = \begin{cases} 0 & s = 0, \\ c/2 + (1-c)\Phi_{F,t}(s) & s \in (0, 1), \\ 1 & s = 1. \end{cases}$$

It is easy to verify that

$$\int_0^1 f(s)\,d\Psi_c(s) = c\frac{f(0)+f(1)}{2} + (1-c)F(t)$$

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and of course $\int_0^1 f(s)\,ds$ is a Stieltjes integral with respect to the identity function. Therefore, in view of Corollary 1.3, we need to find a c such that

$$\int_0^{1/2} \Psi_c(s)\,ds = \int_0^{1/2} s\,ds. \tag{3.7}$$

From (3.3),

$$\int_0^{1/2} \Phi_{F,t}(s)\,ds = \frac{4t^2 - 6t + 3}{24(1-t)},$$

so (3.7) becomes

$$\frac{c}{4} + (1-c)\frac{4t^2 - 6t + 3}{24(1-t)} = \frac{1}{8},$$

that is,

$$c\Big(2 - \frac{4t^2 - 6t + 3}{3(1-t)}\Big) = 1 - \frac{4t^2 - 6t + 3}{3(1-t)}$$

and, finally,

$$c = \frac{1 - \frac{4t^2-6t+3}{3(1-t)}}{2 - \frac{4t^2-6t+3}{3(1-t)}} = \frac{-4t^2 + 3t}{-4t^2 + 3}.$$

Now, assume that $t > \frac{1}{2}$. Then $1 - t < \frac{1}{2}$ and, from the first part of the proof,

$$\frac{1}{b-a}\int_a^b f(s)\,ds \le \chi(1-t)\frac{f(a)+f(b)}{2} + (1 - \chi(1-t))F(1-t).$$

However, from the definition of F it is clear that $F(t) = F(1-t)$ and so, for $t > \frac{1}{2}$,

$$\chi(t) = \frac{-4(1-t)^2 + 3(1-t)}{-4(1-t)^2 + 3} = \frac{-4t^2 + 5t - 1}{-4t^2 + 8t - 1},$$

as claimed. □

REMARK 3.5. It is easy to see that, for all $t \in (0,1)$, we have $\chi(t) < h(t)$ (as we see in the second part of Figure 3). It is clear that

$$\frac{f(a)+f(b)}{2} \ge F(t), \qquad t \in (0,1).$$

This means that (3.6) is indeed stronger than (3.5). Moreover, the value $\chi(t)$ for each t is the smallest possible (if we want inequality (3.6) to be satisfied by every convex function f).

We end the paper with some results of Bullen type. The classical Bullen inequality

$$\frac{1}{b-a}\int_a^b f(s)\,ds \le \frac{f(a) + 2f((a+b)/2) + f(b)}{4}$$

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(mentioned at the beginning of the paper) is a refinement of the second Hermite–Hadamard inequality. In [3] it was proved that

$$H(t) \le G(t). \tag{3.8}$$

The graphs of functions $\Phi_{H,t}$ and $\Phi_{G,t}$ have exactly one crossing point, and (3.8) follows from Ohlin's lemma. Moreover, using Corollary 1.3, we immediately see that inequality (3.8) cannot be improved as we did for (3.2). But we can obtain a refinement of (3.8) and an inequality in the opposite direction.

Remark 3.6. Let $f : [a, b] \to \mathbb{R}$ be a convex function and $H, G : [0, 1] \to \mathbb{R}$ be given by (2.3) and (2.2), respectively. Then, for all $t \in (0, 1)$,

$$G\Big(\frac{t}{2}\Big) \le H(t) \le \frac{G(t) + f((a + b)/2)}{2}. \tag{3.9}$$

Looking at the graphs of $\Phi_{H,t}$, $\Phi_{G,t}$ and $\Phi_{G,t/2}$ (Figure 2), we can see that inequalities (3.9) follow directly from Corollary 1.3. Moreover, the second inequality is stronger than (3.8), since $f((a + b)/2) \le G(t)$.

References

[1] S. S. Dragomir, 'Two mappings in connection to Hadamard's inequalities', *J. Math. Anal. Appl.* **167** (1992), 49–56.

[2] S. S. Dragomir, 'Further properties of some mappings associated with Hermite–Hadamard inequalities', *Tamkang J. Math.* **34**(1) (2003), 49–56.

[3] S. S. Dragomir, D. S. Milošević and J. Sándor, 'On some refinements of Hadamard's inequalities and applications', *Univ. Belgrad, Publ. Elek. Fak. Sci. Math.* **4** (1993), 21–24.

[4] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, RGMIA Monographs (Victoria University, Melbourne, 2000).

[5] T. Kołodziej, *Nierówności typu Hermite'a–Hadamarda dla odwzorowań Dragomira*, Master's Thesis, University of Silesia, Katowice, Poland (in Polish).

[6] V. I. Levin and S. B. Stečkin, 'Inequalities', *Amer. Math. Soc. Transl. (2)* **14** (1960), 1–29.

[7] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23 (Springer, New York, 2006).

[8] J. Ohlin, 'On a class of measures of dispersion with application to optimal reinsurance', *ASTIN Bulletin* **5** (1969), 249–266.

[9] A. Olbryś and T. Szostok, 'Inequalities of the Hermite–Hadamard type involving numerical differentiation formulas', *Results Math.* **67** (2015), 403–416.

[10] J. E. Pečarić, 'Remarks on two interpolations of Hadamard's inequalities', *Contributions Macedonian Acad. of Sci. and Arts, Sect. of Math. and Technical Sciences (Scopje)* **13** (1992), 9–12.

[11] T. Rajba, 'On the Ohlin lemma for Hermite–Hadamard–Fejer type inequalities', *Math. Inequal. Appl.* **17**(2) (2014), 557–571.

[12] T. Rajba, 'On a generalization of a theorem of Levin and Stečkin and inequalities of the Hermite–Hadamard type', *Math. Inequal. Appl.* **20**(2) (2017), 363–375.

[13] T. Szostok, 'Ohlin's lemma and some inequalities of the Hermite–Hadamard type', *Aequationes Math.* **89** (2015), 915–926.

[14] T. Szostok, 'Levin Steckin theorem and inequalities of the Hermite–Hadamard type', Preprint, arXiv:1411.7708.

[15] T. Szostok, 'Functional inequalities involving numerical differentiation formulas of order two', *Bull. Malays. Math. Sci. Soc.* **41**(4) (2018), 2053–2066.

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