J. Inst. Math. Jussieu (2020) **19**(1), 137–173
doi:10.1017/S1474748017000469

GENERALISED DIVISOR SUMS OF BINARY FORMS OVER NUMBER FIELDS

CHRISTOPHER FREI[1] AND EFTHYMIOS SOFOS[2]

[1] *University of Manchester, School of Mathematics, Oxford Road, Manchester, M13 9PL, UK* (christopher.frei@manchester.ac.uk)
[2] *Max Planck Institute for Mathematics, Vivatsgasse 7, Bonn, 53111, Germany* (sofos@mpim-bonn.mpg.de)

(Received 17 November 2016; revised 17 October 2017; accepted 21 October 2017; first published online 16 November 2017)

Abstract Estimating averages of Dirichlet convolutions $1 * \chi$, for some real Dirichlet character χ of fixed modulus, over the sparse set of values of binary forms defined over $\mathbb{Z}$ has been the focus of extensive investigations in recent years, with spectacular applications to Manin's conjecture for Châtelet surfaces. We introduce a far-reaching generalisation of this problem, in particular replacing χ by Jacobi symbols with both arguments having varying size, possibly tending to infinity. The main results of this paper provide asymptotic estimates and lower bounds of the expected order of magnitude for the corresponding averages. All of this is performed over arbitrary number fields by adapting a technique of Daniel specific to $1 * 1$. This is the first time that divisor sums over values of binary forms are asymptotically evaluated over any number field other than $\mathbb{Q}$. Our work is a key step in the proof, given in subsequent work, of the lower bound predicted by Manin's conjecture for all del Pezzo surfaces over all number fields, under mild assumptions on the Picard number.

Keywords: number theory; divisor sums; binary forms; sums of arithmetic functions

2010 *Mathematics subject classification*: Primary 11N37
Secondary 11N56; 11N64

Contents

1. Introduction

Our aim in this paper is to study averages of arithmetic functions that generalise the divisor function over values of binary forms, defined over arbitrary number fields.

1.1. Divisor sums

Estimating averages of arithmetic functions is among the primary objects of analytic number theory and its applications to surrounding areas. Owing to their connection with L-functions, two of the most studied examples are the divisor and the representation function of sums of two integer squares, respectively given by

$$\tau(n) := \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} 1 \quad \text{and} \quad r(n) := 4 \sum_{\substack{d\in\mathbb{N}\\ d \text{ odd}\\ d\mid n}} \left(\frac{-1}{d}\right),$$

where $(\frac{-1}{\cdot})$ denotes the Jacobi symbol, see for example [30, Chapter XII]. It is possible to obtain level of distribution results, a problem first studied by Hooley, Linnik and Selberg (all in unpublished manuscripts, see the results and the references in [31] and [22] for recent developments). Research on this problem is currently active due to advances in estimating sums of trace functions over finite fields, see for example [13], where the ternary divisor function is studied.

Asymptotically estimating the average of these functions over the sparse set of values of general integer polynomials in a single variable is naturally harder. It is only the case of degree 1 and 2 polynomials that has been settled, see the work of Hooley [20] and of Duke *et al.* [5]. The closely related problem regarding integer binary forms was studied later. Let us introduce some notation to help us describe previous work on this area. For a positive integer n and each $1 \leqslant i \leqslant n$, let $F_i \in \mathbb{Z}[s,t]$ be forms, coprime in pairs, and for any constants $c_i \in \{1,-1\}$ set $\mathfrak{C} = \{(F_i, c_i), i = 1, \dots, n\}$ and

$$D(\mathfrak{C}; X) := \sum_{\substack{(s,t)\in(\mathbb{Z}\cap[-X,X])^2\\ F_i(s,t)\neq 0}} \prod_{i=1}^{n} \Bigg(\sum_{\substack{d_i\in\mathbb{N}\\ d_i \text{ odd}\\ d_i\mid F_i(s,t)}} \left(\frac{c_i}{d_i}\right)\Bigg), \tag{1.1}$$

where the restriction to odd d_i is present only when $c_i = -1$. The case of degree 3 was first studied by Greaves [18], who obtained an asymptotic for $D(\mathfrak{C}; X)$ when $\mathfrak{C} = \{(F, 1)\}$ and F is any irreducible form with $\deg(F) = 3$ via the use of exponential sums.

Extending this result to higher degrees was considered intractable for a long time until the highly influential work of Daniel [3], who employed geometry of numbers to treat the case $\mathfrak{C} = \{(F, 1)\}$ for any irreducible form F with $\deg(F) = 4$. Developing this approach to allow negative c_i, Heath-Brown [19] later tackled the case where $n = 4$, each c_i is -1 and all forms F_i are linear.

It was subsequently realised that proving asymptotics whenever $\sum_{i=1}^{n} \deg(F_i) = 4$ would constitute a key step towards the resolution of Manin's conjecture for Châtelet surfaces over $\mathbb{Q}$. This is a conjecture in arithmetic geometry and regards counting rational points of bounded height on Fano varieties defined over arbitrary number fields; it was introduced by Manin and his collaborators [14] in 1989 and has subsequently given rise to a long-standing research programme that still continues. Thus, Browning and de la Bretèche reworked later the case $\mathfrak{C} = \{(L_i, -1) : 1 \leqslant i \leqslant 4\}$, where each

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

form L_i is linear in [7], the case $\mathfrak{C} = \{(C, -1), (L, -1)\}$, where $\deg(C) = 3$, $\deg(L) = 1$ in [10], and recently Destagnol settled the case $\mathfrak{C} = \{(Q, -1), (L_1, -1), (L_2, -1)\}$ with $\deg(Q) = 2$, $\deg(L_i) = 1$ in [4]. In addition, Browning and de la Bretèche treated the case $\mathfrak{C} = \{(Q, 1), (L_1, 1), (L_2, 1)\}$ with $\deg(Q) = 2$, $\deg(L_i) = 1$ in [8]; this investigation formed a significant part in their proof of Manin's conjecture for a smooth quartic del Pezzo surface for the first time [9]. The remaining cases in the divisor sum problem with $\sum_{i=1}^{n} \deg(F_i) = 4$ require a further development of Daniel's approach, one that necessitates the use of a generalisation of Hooley's delta function [21]. This was achieved independently by Brüdern [1] and de la Bretèche with Tenenbaum [11], enabling the settling of the cases $\mathfrak{C} = \{(F_1, -1)\}$ and $\mathfrak{C} = \{(F_2, -1), (F_3, -1)\}$, where the forms satisfy $\deg(F_1) = 4$ and $\deg(F_2) = \deg(F_3) = 2$ in [12]. In these works, whenever $c_i = -1$ then F_i was irreducible over $\mathbb{Q}(\sqrt{-1})$, since otherwise the corresponding term would be of the form $\tau(F(s, t))$, this would decrease the difficulty of obtaining an asymptotic.

It should be remarked that each work following Daniel came into fruition only for integer forms F_i fulfilling a list of necessary assumptions regarding the small prime divisors and the sign of the integers $F_i(s, t)$ as (s, t) ranges through certain regions in $\mathbb{R}^2$, see for example the normalisation hypotheses (iii) and (iv) in [7, p. 1375], where $c_i = -1$. These conditions are related to quadratic reciprocity and analogues of these will appear in our work, see § 1.3.3, where $\mathfrak{W}$ is to be thought of as the product of small primes.

It will be crucial for our work that Daniel's approach is able of providing a polynomial saving in the error term if $\sum_{i=1}^{n} \deg(F_i) = 3$ but not when $\sum_{i=1}^{n} \deg(F_i) = 4$, while it has never been extended to any case with $\sum_{i=1}^{n} \deg(F_i) > 4$. Lastly, the spectacular work of Matthiesen [24–26], using tools from additive combinatorics, tackled all cases where $\sum_{i=1}^{n} \deg(F_i)$ can be arbitrarily large under the restriction that each F_i is linear. Naturally, this approach does not yield an explicit error term.

1.2. Generalised divisor sums

In our forthcoming joint work [16] with Loughran, we study Manin's conjecture in dimension 2. As a special corollary we obtain the lower bound predicted by Manin for all del Pezzo surfaces over all number fields, only under mild assumptions regarding the Picard number. For del Pezzo surfaces of degree 1 in particular, tight lower bounds were not known before, not even in special cases. The underlying strategy is to use algebro-geometric arguments to translate the problem into one of estimating averages that are a vast generalisation of the ones appearing in (1.1). The success of this strategy therefore relies heavily on a very general conjecture concerning the growth order of our divisor sums; its precise statement will be introduced in Conjecture 1. In this paper we prove it in all cases that we need for our applications to Manin's conjecture, see Theorem 1.1. In the very special case that the base field is $\mathbb{Q}$, dealing with a del Pezzo surface of degree $1 \leqslant d \leqslant 5$ gives birth to averages of the rough shape

$$\sum_{\substack{(s,t)\in(\mathbb{Z}\cap[-X,X])^2\\ F_i(s,t)\neq 0\\ (s,t)\equiv(\sigma,\tau) \bmod q}} \prod_{i=1}^{n} h(F_i(s,t)) \left(\sum_{\substack{d_i\in\mathbb{N}\\ d_i \text{ odd}\\ d_i \mid F_i(s,t)}} \left(\frac{G_i(s,t)}{d_i} \right) \right), \tag{1.2}$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

where σ, τ, q are positive integers, h is a 'small' arithmetic function, each F_i, G_i is an integer binary form with $\deg(G_i)$ divisible by 2, all forms F_i irreducible and satisfying

$$\sum_{i=1}^{n} \deg(F_i) = 8 - d,$$

which is an integer in the range $\{3, \dots, 7\}$. Our assumption on h is that it can be written as $h = 1 * f$, where $*$ denotes the Dirichlet convolution and f is a multiplicative function on $\mathbb{N}$ that satisfies $f(m) = O(\frac{1}{m})$ for $m \in \mathbb{N}$. We shall call a sum as in (1.2) a *generalised divisor sum*. This is because G_i are not constants and hence the terms are no more a product of multiplicative functions on $\mathbb{N}$ restricted at values of binary forms. A further new trait lies in the fact that a *level of distribution* result is required with respect to the modulus q, such a result has not appeared previously for divisor sums over values of polynomials or forms. In particular, we shall be able to handle the case $h(n) = 1$ for all $n \in \mathbb{N}$, thus our results are a true generalisation of previous work and not a different problem.

A supplementary aspect of our work is that we estimate asymptotically, for the first time, divisor sums over values of binary forms in arbitrary number fields, see Theorem 1.2. Thus, one of the central innovations in our work lies in revealing how to extend Daniel's approach to this setting. We shall rely on a lattice point counting theorem of Barroero and Widmer [2], based on the framework of o-minimal structures. It is important to note here that the essence of Daniel's approach lies in taking advantage of the, possibly large on average, size of the first successive minima to produce a sufficiently small error term. Directly adapting this approach to number fields yields an error term whose order supersedes the main term; this would preclude the proof of both Theorems 1.1 and 1.2. We shall introduce an artifice that overcomes this difficulty, namely we shall modify Daniel's method by taking into account not only the first, but also higher successive minima of the lattice.

Let us finally state that it is not clear what is the expected growth order for generalised divisor sums. We shall see that one rôle of Conjecture 1 is to provide an answer in terms of various number fields generated by roots of $F_i(s, 1)$. It is important to note that our conjecture will turn out to be in agreement with the growth order predicted by Manin's conjecture for surfaces; this will be revealed in [16].

1.3. Statement of our set-up

Throughout this paper, K will be a number field of degree $m = [K : \mathbb{Q}]$, whose ring of integers is denoted by $\mathscr{O}_K$. By $\mathfrak{p}$ and $\mathfrak{p}_i$ we always denote non-zero prime ideals of $\mathscr{O}_K$ and $v_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic exponential evaluation.

1.3.1. Systems of binary forms. We consider finite sets of pairs of binary forms

$$\mathfrak{F} = \{(F_i, G_i), i = 1, \dots, n\},$$

where each $F_i, G_i \in \mathscr{O}_K[s, t]$ is such that F_i is irreducible and does not divide G_i in $K[s, t]$. Moreover, we assume that all F_i are coprime over K in pairs and that each $\deg(G_i)$ is even.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

We next define the *rank* of $\mathfrak{F}$, which will be an invariant of $\mathfrak{F}$ that will characterise the growth order in Conjecture 1. If F_i is proportional to t, we denote $\boldsymbol{\theta}_i := (1, 0)$. Otherwise, letting $\overline{K}$ be a fixed algebraic closure of K, we set $\theta_i \in \overline{K}$ to be a fixed root of $F_i(x, 1)$, and $\boldsymbol{\theta}_i := (\theta_i, 1)$. Let $K(\boldsymbol{\theta}_i)$ be the subfield of $\overline{K}$ generated by K and the coordinates of $\boldsymbol{\theta}_i$. We define the **rank** of $\mathfrak{F}$ to be the cardinality

$$\rho(\mathfrak{F}) := \sharp\{1 \leqslant i \leqslant n : G_i(\boldsymbol{\theta}_i) \in K(\boldsymbol{\theta}_i)^{\times 2}\},$$

where, for any field k, we denote the set of its non-zero squares by $k^{\times 2}$.

1.3.2. The group $\mathscr{U}_K$. The terms involving the function h in (1.2) have the rôle of insignificant modifications. We proceed to introduce them precisely. Letting $\mathscr{I}_K$ denote the monoid of non-zero integral ideals of $\mathscr{O}_K$, $\mathfrak{Na}$ be the absolute norm of $\mathfrak{a} \in \mathscr{I}_K$ and μ_K the Möbius function on $\mathscr{I}_K$ allows us to introduce the set of functions

$$\mathscr{Z}_K := \left\{ f : \mathscr{I}_K \to (-1, \infty) : \begin{array}{l} f \text{ multiplicative,} \\ f(\mathfrak{p}) \ll_f \frac{1}{\mathfrak{Np}} \text{ for all } \mathfrak{p}, \\ f(\mathfrak{a}) = 0 \text{ if } \mu_K(\mathfrak{a}) = 0 \end{array} \right\}.$$

For each $f \in \mathscr{Z}_K$, we subsequently define another function $\mathbf{1}_f : \mathscr{I}_K \to (0, \infty)$ given by

$$\mathbf{1}_f(\mathfrak{a}) := \prod_{\mathfrak{p}|\mathfrak{a}} (1 + f(\mathfrak{p})) = (1 * f)(\mathfrak{a}).$$

This then allows us to form the following set of positive multiplicative functions on $\mathscr{I}_K$,

$$\mathscr{U}_K := \{\mathbf{1}_f : f \in \mathscr{Z}_K\}. \tag{1.3}$$

The growth condition placed on f indicates that $\mathbf{1}_f$ behaves on average like a constant function. Note that for all $f \in \mathscr{Z}_K$ and $\varepsilon > 0$ we have

$$\mathbf{1}_f(\mathfrak{a}) \ll_{f,\varepsilon} \mathfrak{Na}^{\varepsilon}, \tag{1.4}$$

and moreover, that the set $\mathscr{U}_K$ forms a group under pointwise multiplication. This will be used often with the aim of simplifying the exposition, for example via replacing terms like $\mathbf{1}_{f_1}\mathbf{1}_{f_2}$ or $1/\mathbf{1}_{f_3}$, where $f_i \in \mathscr{Z}_K$, by $\mathbf{1}_f$ for some $f \in \mathscr{Z}_K$.

1.3.3. $\mathfrak{F}$-admissibility. As usual, we shall identify all completions K_v at archimedean places v with $\mathbb{R}$ or $\mathbb{C}$. We shall thus let $K_\infty := K \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty} K_v$, which we identify with $\mathbb{R}^m$ via $\mathbb{C} \cong \mathbb{R}^2$. In addition, we shall denote by $\mathscr{D}$ a set of the form $\mathscr{D} = \prod_{v|\infty} \mathscr{D}_v$, where $\mathscr{D}_v \subseteq K_v^2$ is a compact ball of positive radius. Fixing an integral ideal $\mathfrak{r} \in \mathscr{I}_K$, we shall consider $\mathfrak{r}$-*primitive* points $(s, t) \in \mathscr{O}_K^2$, by which we mean that $s\mathscr{O}_K + t\mathscr{O}_K = \mathfrak{r}$. For a nonzero ideal $\mathfrak{W}$ of $\mathscr{O}_K$ divisible by $2\mathfrak{r}$, and $\mathfrak{a} \in \mathscr{I}_K$, we define the ideal

$$\mathfrak{a}^\flat := \prod_{\mathfrak{p} \nmid \mathfrak{W}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}, \tag{1.5}$$

and for $a \in \mathscr{O}_K \setminus \{0\}$, we let $a^\flat := (a\mathscr{O}_K)^\flat$. Keep in mind that this notion depends on $\mathfrak{W}$. Let $\sigma, \tau \in \mathscr{O}_K$ be such that $\sigma\mathscr{O}_K + \tau\mathscr{O}_K = \mathfrak{r}$. The symbol $\mathscr{P}$ will refer exclusively throughout this paper to triplets of the form

$$\mathscr{P} = (\mathscr{D}, (\sigma, \tau), \mathfrak{W}),$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

where $\mathscr{D}, (\sigma, \tau), \mathfrak{W}$ are as above. Given any system of forms $\mathfrak{F}$ as in § 1.3.1, a triplet $\mathscr{P}$ and a parameter $X \geqslant 1$, we let

$$M^*(\mathscr{P}, X) := \{(s,t) \in \mathfrak{r}^2 \cap X^{1/m}\mathscr{D} : (s,t) \equiv (\sigma,\tau) \bmod \mathfrak{W},\ s\mathscr{O}_K + t\mathscr{O}_K = \mathfrak{r}\}$$

and

$$M^*(\mathscr{P}, \infty) := \bigcup_{X \geqslant 1} M^*(\mathscr{P}, X).$$

We shall say that $\mathscr{P}$ is $\mathfrak{F}$-*admissible* if each of the following conditions (1.6)–(1.8) holds:

$$F_i(\sigma,\tau) \neq 0 \quad \text{for all } 1 \leqslant i \leqslant n, \tag{1.6}$$

and whenever $(s,t) \in M^*(\mathscr{P}, \infty)$ we have

$$F_i(s,t) \neq 0 \quad \text{for all } 1 \leqslant i \leqslant n, \tag{1.7}$$

as well as

$$\left(\frac{G_i(s,t)}{F_i(s,t)^\flat}\right) = 1 \quad \text{for all } 1 \leqslant i \leqslant n. \tag{1.8}$$

In the last condition, we used the *Jacobi symbol* for K, which is defined as follows: for $a \in \mathscr{O}_K$ and a non-zero ideal $\mathfrak{b} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_l^{e_l}$, with distinct prime ideals $\mathfrak{p}_i$, none of which lies above 2, we let

$$\left(\frac{a}{\mathfrak{b}}\right) := \prod_{i=1}^{l} \left(\frac{a}{\mathfrak{p}_i}\right)^{e_i},$$

where $(\frac{a}{\mathfrak{p}})$ is the Legendre quadratic residue symbol for K.

1.4. Lower bound conjecture for generalised divisor sums

For any $\mathfrak{F}$ as in § 1.3.1, any function $f \in \mathscr{Z}_K$ and any $\mathfrak{F}$-admissible triplet $\mathscr{P}$, we define the function $r : M^*(\mathscr{P}, \infty) \to [0, \infty)$ by

$$r(s,t) = r(\mathfrak{F}, f, \mathscr{P}; s, t) := \prod_{i=1}^{n} \mathbf{1}_f(F_i(s,t)^\flat) \left(\sum_{\mathfrak{d}_i \mid F_i(s,t)^\flat} \left(\frac{G_i(s,t)}{\mathfrak{d}_i}\right)\right).$$

We are now in the position to introduce *generalised divisor sums* as averages of the form

$$D(\mathfrak{F}, f, \mathscr{P}; X) := \sum_{(s,t) \in M^*(\mathscr{P}, X)} r(\mathfrak{F}, f, \mathscr{P}; s, t).$$

The special case of the following claim corresponding to each G_i being constant and $K = \mathbb{Q}$ ought to be familiar, at least among experts, but has not yet appeared in text.

Conjecture 1 (Lower bound conjecture for divisor sums). *Let K be a number field, fix $\mathfrak{r} \in \mathscr{I}_K$, let $f \in \mathscr{Z}_K$, and let $\mathfrak{F}$ be a system of forms as in § 1.3.1. Then there exists a finite set $S_{bad} = S_{bad}(\mathfrak{F}, f, \mathfrak{r})$ of prime ideals in $\mathscr{O}_K$, such that for all $\mathfrak{F}$-admissible triplets $\mathscr{P}$ with $\mathfrak{W}$ being divisible by each $\mathfrak{p} \in S_{bad}$, we have*

$$D(\mathfrak{F}, f, \mathscr{P}; X) \gg X^2 (\log X)^{\rho(\mathfrak{F})}, \quad \text{as } X \to \infty.$$

The implicit constant may depend on every parameter except X.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

It should be stated that the appearance of G_i, f and $\mathscr{P}$ in Conjecture 1, as well as the consideration of arbitrary number fields, are absolutely necessary for our applications to Manin's conjecture in [16]. The presence of the set of bad primes S_{bad} can be avoided; it is only included here to minimise the technical details in the present work.

We next supply heuristical evidence supporting that Conjecture 1 does in fact provide the true order of magnitude of $D(\mathfrak{F}, f, \mathscr{P}; X)$. Firstly, there are about X^2 summands and each term $\mathbf{1}_f(F_i(s,t)^\flat)$ behaves as a constant on average, since our conditions on $\mathfrak{F}$ suggest that the integral ideals $F_i(s,t)^\flat$ behave randomly. Secondly, as we shall see in Lemma 3.2, if the index i contributes towards the rank $\rho(\mathfrak{F})$ then the Jacobi symbols $\left(\frac{G_i(s,t)}{\mathfrak{d}_i}\right)$ assume the value 1, while in the opposite case they take both values 1 and -1 with equal probability. Consequently, in the former case the sum over $\mathfrak{d}_i | F_i(s,t)^\flat$ will resemble the divisor function in $\mathscr{I}_K$, thus contributing a logarithm, while in the latter case it will be approximated by a constant on average owing to the cancellation of the Jacobi symbols. A subtle point here is that if one does not impose condition (1.8) then the implied constant in the lower bound could vanish, so the restriction to admissible triplets is necessary. Furthermore, each work referenced in § 1.1 is in agreement with Conjecture 1 when $K = \mathbb{Q}$ and $G_i = \pm 1$. Lastly, the work of de la Bretèche and Browning [6] can be used to provide a matching upper bound over $\mathbb{Q}$ whenever each G_i is constant.

The main purpose of this paper is to prove Conjecture 1 under a condition regarding only the *complexity* of $\mathfrak{F}$, which we define by

$$c(\mathfrak{F}) := \sum_{\substack{1 \leqslant i \leqslant n \\ G_i(\boldsymbol{\theta}_i) \notin K(\boldsymbol{\theta}_i)^{\times 2}}} \deg F_i,$$

but without a restriction on the value of $\sum_{i=1}^n \deg(F_i)$ or the factorisation type of $\prod_{i=1}^n F_i$.

Theorem 1.1. *Conjecture 1 holds for all K, $\mathfrak{r}$, f and systems of forms $\mathfrak{F}$ with $c(\mathfrak{F}) \leqslant 3$.*

Theorem 1.1 will be reduced to Theorem 1.2, whose statement is given in § 1.5.

Remark 1.1. As an immediate consequence of [16, Theorem 1.6], we see that Conjecture 1 implies Zariski density of rational points on conic bundle surfaces over number fields, under the necessary assumption that there is a rational point on a smooth fibre. This well-known problem is currently open in most cases, see the recent work of Kollár and Mella [23].

1.5. Skeleton of the paper and further results

The preliminary parts, §§ 2.1 and 2.2, respectively, provide general counting results, that are not limited to our applications, for points of certain lattices and averaging results concerning coefficients of Artin L-functions.

The reduction of Theorem 1.1 to Theorem 1.2 below will take place in § 3, while the proof of the latter theorem will be given in § 4. It provides asymptotics in cases where $\sum_{i=1}^n \deg F_i \leqslant 3$ and $G_i(\boldsymbol{\theta}_i) \notin K(\boldsymbol{\theta}_i)^{\times 2}$ for all i, under some further assumptions.

It is worth following the strategy laid out in our proof of Theorem 1.2 to show that, for any positive integers σ, τ, d and fixed irreducible binary forms F_i with $\sum_{i=1}^n \deg(F_i) \leqslant 3$,

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

an asymptotic estimate with a power saving in terms of X and a polynomial dependence on d in the error term holds for the analogue of the classical divisor sums

$$\sum_{\substack{(s,t)\in(\mathbb{Z}\cap[-X,X])^2\\ F_i(s,t)\neq 0\\ (s,t)\equiv(\sigma,\tau) \bmod d}} \prod_{i=1}^{n} \Bigg(\sum_{\substack{d_i\in\mathbb{N}\\ d_i\mid F_i(s,t)}} 1 \Bigg)$$

over any number field. We refrain from this task in the present work to shorten the exposition.

We proceed by providing the statement of our second theorem. We say that an $\mathfrak{F}$-admissible triplet $\mathscr{P}=(\mathscr{D},(\sigma,\tau),\mathfrak{W})$ is *strongly $\mathfrak{F}$-admissible*, if, in addition, for all $1\leqslant i\leqslant n$ and $(s,t)\in M^*(\mathscr{P},\infty)$ one has

$$F_i(\sigma,\tau)\not\equiv 0 \bmod \mathfrak{W} \quad \text{and} \quad v_{\mathfrak{p}}(F_i(s,t))=v_{\mathfrak{p}}(F_i(\sigma,\tau)) \quad \text{for all } \mathfrak{p}\mid\mathfrak{W}. \tag{1.9}$$

Theorem 1.2. *Let K be a number field, $\mathfrak{r}\in\mathscr{I}_K$ and $f\in\mathscr{Z}_K$. Let $\mathfrak{F}$ be a system of forms with $\rho(\mathfrak{F})=0$ and $c(\mathfrak{F})\leqslant 3$. Then there is a non-zero ideal $\mathfrak{W}_0$ of $\mathscr{O}_K$ and constants $\beta_1,\beta_2>0$, such that the following statement holds.*

For every strongly $\mathfrak{F}$-admissible triplet $\mathscr{P}=(\mathscr{D},(\sigma,\tau),\mathfrak{W})$ fulfilling $\mathfrak{W}_0\mid\mathfrak{W}$, there are $\beta_0>0$ and a function $f_0\in\mathscr{Z}_K$, depending only on $\mathfrak{r}, f, \mathfrak{F}, \mathscr{D}, \mathfrak{W}$, such that for each $\mathfrak{d}\in\mathscr{I}_K$ for which the triplet $\mathscr{P}_{\mathfrak{d}}:=(\mathscr{D},(\sigma,\tau),\mathfrak{d}\mathfrak{W})$ satisfies

$$\prod_{i=1}^{n} F_i(s,t)\mathfrak{W}+\mathfrak{d}=\mathscr{O}_K \qquad \text{for all } (s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},\infty), \tag{1.10}$$

the asymptotic

$$\sum_{(s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X)} r(\mathfrak{F},f,\mathscr{P};s,t)=\beta_0\frac{\mathbf{1}_{f_0}(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}^2}X^2+O(X^{2-\beta_1}\mathfrak{N}\mathfrak{d}^{\beta_2})$$

holds with an implied constant independent of $\mathfrak{d},\sigma,\tau$ and X.

This is the first time that any divisor sum over values of binary forms is asymptotically evaluated over any number field other than $\mathbb{Q}$. Even over $\mathbb{Q}$, both Theorems 1.1 and 1.2 are novel due to the appearance of the forms G_i. Furthermore, the extra condition that (s,t) lies in a progression, whose modulus is explicitly recorded in the error term, gives rise to a new level of distribution result, since an asymptotic holds when $\mathfrak{N}\mathfrak{d}\leqslant X^{\beta}$ for all $0<\beta<\beta_1/\beta_2$.

The power saving in the error term of Theorem 1.2 is crucial for deducing Theorem 1.1 from it, and therefore for the application to Manin's conjecture. Even in the simple case $K=\mathbb{Q}$, such a strong error term can presently only be obtained under the assumption $\sum_{i=1}^{n}\deg(F_i)\leqslant 3$, which is the reason for the restriction placed on the complexity $c(\mathfrak{F})$.

As a first step for the proof of Theorem 1.2, we use Dirichlet's hyperbola trick and partition the variables in the summation into a small number of lattices; this is exposed in § 4.1. The next part, residing in § 4.2, consists of counting points on these lattices; it is here that the main step towards the power saving in the error term in Theorem 1.2

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

takes place. Finally, in §§ 4.3–4.6 we prove that the average of the contribution of each lattice alluded to above gives the main term as stated in Theorem 1.2, this part contains the treatment of volumes of slightly awkward regions introduced by the consideration of arbitrary number fields.

Notation. The set of places of the number field K will be denoted Ω_K and for each $v \in \Omega_K$ we shall let $m_v := [K_v : \mathbb{Q}_w]$, where w is the place of $\mathbb{Q}$ below v. For $a \in \mathscr{O}_K$, we write $\mathfrak{N}(a) := \mathfrak{N}(a\mathscr{O}_K) = \prod_{v\in\Omega_\infty} |a|_v^{m_v}$ for the absolute value of its norm. For $s \in K_\infty = \prod_{v\in\Omega_\infty} K_v$ and $v \in \Omega_\infty$, we write $s_v \in K_v$ for the projection of s to K_v. Furthermore, for any prime ideal $\mathfrak{p}$ the $\mathfrak{p}$-adic exponential valuation on ideals (and elements) of $\mathscr{O}_K$ will be denoted by $v_\mathfrak{p}$. As usual, the resultant of two binary forms $F, G \in \mathscr{O}_K[s,t]$ will be represented by $\mathrm{Res}(F,G) \in \mathscr{O}_K$, while Euler's totient function and the divisor function for non-zero ideals of $\mathscr{O}_K$ will be denoted by ϕ_K and τ_K. Lastly, we shall choose a system of integral representatives $\mathscr{C} = \{\mathfrak{r}_1, \dots, \mathfrak{r}_h\}$ for the ideal class group of $\mathscr{O}_K$ and fix it once and for all. Unless the contrary is explicitly stated, the implicit constants in Landau's O-notation and Vinogradov's $\ll$-notation are allowed to depend on $K, \mathscr{C}, \mathfrak{r}, f, \mathfrak{F}$ and $\mathscr{P}$ but no other parameters. The exact value of a small positive constant ε will be allowed to vary from expression to expression throughout our work.

2. Preliminaries

2.1. Lattice point counting

For any lattice $\Lambda \subset K_\infty^2 = \mathbb{R}^{2m}$, we denote its ith successive minimum (with respect to the unit ball) by $\lambda^{(i)}(\Lambda)$. We write $\|\cdot\|$ for the Euclidean norm on $\mathbb{R}^{2m}$. For $\mathfrak{a}, \mathfrak{d} \in \mathscr{I}_K$ and $\gamma \in \mathscr{O}_K$, we define the lattice

$$\Lambda(\mathfrak{a}, \mathfrak{d}, \gamma) := \{(s,t) \in \mathfrak{a}^2 : s \equiv \gamma t \bmod \mathfrak{d}\}.$$

It has determinant proportional to $\mathfrak{N}(\mathfrak{a}^2\mathfrak{d}(\mathfrak{a}+\mathfrak{d})^{-1})$, and we write $\lambda^{(i)}(\mathfrak{a}, \mathfrak{d}, \gamma) := \lambda^{(i)}(\Lambda(\mathfrak{a}, \mathfrak{d}, \gamma))$ for its ith successive minimum. Recall that $\mathscr{C} = \{\mathfrak{r}_1, \dots, \mathfrak{r}_h\}$ is a fixed system of integral representatives of the class group of K. Let us prove some facts about the minima $\lambda^{(i)}(\mathfrak{a}, \mathfrak{d}, \gamma)$.

Lemma 2.1. *Let $\mathfrak{a}, \mathfrak{d} \in \mathscr{I}_K$, $\gamma \in \mathscr{O}_K$ and $1 \leqslant i \leqslant 2m$.*

(1) *Whenever $[\mathfrak{a}] = [\mathfrak{r}_q]$ for $1 \leqslant q \leqslant h$, we have*

$$\mathfrak{N}\mathfrak{a}^{1/m}\lambda^{(i)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{d}(\mathfrak{a}+\mathfrak{d})^{-1}, \gamma) \ll \lambda^{(i)}(\mathfrak{a}, \mathfrak{d}, \gamma) \ll \mathfrak{N}\mathfrak{a}^{1/m}\lambda^{(i)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{d}(\mathfrak{a}+\mathfrak{d})^{-1}, \gamma).$$

(2) *For any non-zero ideal $\mathfrak{b}$ of $\mathscr{O}_K$, the following estimate holds,*

$$\lambda^{(i)}(\mathfrak{a}, \mathfrak{d}, \gamma) \ll \lambda^{(i)}(\mathfrak{a}, \mathfrak{b}\mathfrak{d}, \gamma) \ll \mathfrak{N}(\mathfrak{b})^{1/m}\lambda^{(i)}(\mathfrak{a}, \mathfrak{d}, \gamma).$$

(3) *We have $\lambda^{(i)}(\mathfrak{a}, \mathfrak{d}, \gamma) \ll \mathfrak{N}(\mathfrak{a}^2\mathfrak{d}(\mathfrak{a}+\mathfrak{d})^{-1})^{1/(2m-i+1)}$.*

Proof. Let $a \in K \setminus \{0\}$ such that $\mathfrak{a} = a\mathfrak{r}_q$. Then the elements $(s,t) \in \mathfrak{a}^2$ with $s \equiv \gamma t \bmod \mathfrak{d}$ are exactly those of the form $(s,t) = a(s_1, t_1)$, with $(s_1, t_1) \in \Lambda(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{d}(\mathfrak{a}+\mathfrak{d})^{-1}, \gamma) =: \Lambda'$.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

By Dirichlet's unit theorem, we can choose our generator a to satisfy $|a|_v \ll \mathfrak{N}\mathfrak{a}^{1/m} \ll |a|_v$ for all $v \in \Omega_\infty$. Then, for any $(s_1, t_1) \in \Lambda'$ we have

$$\mathfrak{N}\mathfrak{a}^{1/m}\|(s_1, t_1)\| \ll \|a(s_1, t_1)\| \ll \mathfrak{N}\mathfrak{a}^{1/m}\|(s_1, t_1)\|,$$

which shows claim (1). The first inequality of (2) is clear. For the remaining one, let $b \in \mathfrak{b}$ such that $|b|_v \ll \mathfrak{N}\mathfrak{b}^{1/m} \ll |b|_v$ for all $v \in \Omega_\infty$ and let $(s, t) \in \Lambda(\mathfrak{a}, \mathfrak{d}, \gamma)$. This implies that $(bs, bt) \in \Lambda(\mathfrak{a}, \mathfrak{b}\mathfrak{d}, \gamma)$ and $\|(bs, bt)\| \ll \mathfrak{N}\mathfrak{b}^{1/m}\|(s, t)\|$. Assertion (3) flows directly from Minkowski's second theorem combined with the obvious fact that $\lambda^{(1)}(\mathfrak{a}, \mathfrak{d}, \gamma) \gg 1$. □

We use the framework of [2], built on o-minimality, to count points of $\Lambda(\mathfrak{a}, \mathfrak{d}, \gamma)$ in fairly general domains. For an introduction to o-minimality, we refer to [32] and the introductory section of [2]. Assume we are given an o-minimal structure that extends the semialgebraic structure. Let $\mathscr{R} \subset \mathbb{R}^{k+2m}$ be a definable family, such that for each $T \in \mathbb{R}^k$ the fibre

$$\mathscr{R}_T := \{(s, t) \in \mathbb{R}^{2m} \mid (T, s, t) \in \mathscr{R}\}$$

is contained in a ball, not necessarily zero-centred, of radius $\ll X_T^{1/m}$ for some $X_T \geqslant 1$. The first part of Lemma 2.1 makes the following lemma an immediate consequence of [2, Theorem 1.3].

Lemma 2.2. *Whenever* $[\mathfrak{a}] = [\mathfrak{r}_q]$ *and* $T \in \mathbb{R}^k$*, the quantity* $\sharp(\Lambda(\mathfrak{a}, \mathfrak{d}, \gamma) \cap \mathscr{R}_T)$ *equals*

$$\frac{c_K \operatorname{vol} \mathscr{R}_T}{\mathfrak{N}(\mathfrak{a}^2\mathfrak{d}(\mathfrak{a}+\mathfrak{d})^{-1})} + O\left(\sum_{j=0}^{2m-1} \frac{X_T^{j/m}}{\mathfrak{N}\mathfrak{a}^{j/m} \prod_{i=1}^{j} \lambda^{(i)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{d}(\mathfrak{a}+\mathfrak{d})^{-1}, \gamma)}\right),$$

with an explicit positive constant c_K *depending only on* K*. The implicit constant in the error term may depend on* $K, \mathscr{R}$*, but not on* $T, \mathfrak{a}, \mathfrak{d}, \gamma$*.*

Still keeping the notation from above, we now fix an ideal $\mathfrak{r} \in \mathscr{I}_K$ and assume that $\mathfrak{r} \mid \mathfrak{a}$ and that $\mathfrak{a} + \mathfrak{d} = \mathscr{O}_K$. Let $\sigma, \tau \in \mathfrak{r}$ such that $\sigma\mathscr{O}_K + \tau\mathscr{O}_K + \mathfrak{a} = \mathfrak{r}$ and define a discrete subset of $K_\infty^2 = \mathbb{R}^{2m}$ by

$$\Lambda^*(\mathfrak{a}, (\sigma, \tau), \mathfrak{d}, \gamma) := \left\{ \begin{array}{c} (s, t) \equiv (\sigma, \tau) \bmod \mathfrak{a}, \\ (s, t) \in \mathfrak{r}^2 : s\mathscr{O}_K + t\mathscr{O}_K = \mathfrak{r}, \\ s \equiv \gamma t \bmod \mathfrak{d} \end{array} \right\}. \tag{2.1}$$

Moreover, we require now that each $\mathscr{R}_T$ is contained in a *zero-centred* ball of radius $\ll X_T^{1/m}$.

Lemma 2.3. *We have*

$$\sharp(\Lambda^*(\mathfrak{a}, (\sigma, \tau), \mathfrak{d}, \gamma) \cap \mathscr{R}_T) - \frac{c_K \operatorname{vol} \mathscr{R}_T}{\zeta_K(2)\mathfrak{N}(\mathfrak{d}\mathfrak{a}^2)} \prod_{\mathfrak{p} \mid \mathfrak{a}\mathfrak{r}^{-1}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right)^{-1} \prod_{\mathfrak{p} \mid \mathfrak{d}} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1}$$
$$\ll \sum_{j=0}^{m-1} \frac{X_T^{1+j/m} (\log X_T) \tau_K(\mathfrak{d})}{\min_{1 \leqslant q \leqslant h} \{\lambda^{(1)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{d}, \gamma)^m \lambda^{(m+1)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{d}, \gamma)^j\}}.$$

Here, ζ_K *is the Dedekind zeta function of* K *and* τ_K *is the divisor function on* $\mathscr{I}_K$*. The implicit constant in the error term depends on* $K, \mathfrak{r}, \mathscr{R}$*, but not on* $T, \mathfrak{a}, \sigma, \tau, \mathfrak{d}$ *or* γ*.*

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Proof. After Möbius inversion the quantity under consideration becomes equal to

$$\sum_{\mathfrak{e}\mid\mathfrak{d}} \sum_{\substack{\mathfrak{b}\in\mathscr{I}_K\\ \mathfrak{b}+\mathfrak{d}=\mathfrak{e}\\ \mathfrak{b}+\mathfrak{a}\mathfrak{r}^{-1}=\mathscr{O}_K}} \mu_K(\mathfrak{b})\sharp\{(s,t)\in(\mathfrak{r}\mathfrak{b})^2\cap(\mathscr{R}_T\smallsetminus\{0\}) : (s,t)\equiv(\sigma,\tau) \bmod \mathfrak{a}, s\equiv\gamma t \bmod \mathfrak{d}\}.$$

Writing $\mathfrak{b}=\mathfrak{b}'\mathfrak{e}$, we see that $\mathfrak{b}'+\mathfrak{d}=\mathscr{O}_K$ whenever $\mu(\mathfrak{b})\neq 0$, thus the sum becomes

$$\sum_{\mathfrak{e}\mid\mathfrak{d}}\mu_K(\mathfrak{e}) \sum_{\substack{\mathfrak{b}'\in\mathscr{I}_K\\ \mathfrak{b}'+\mathfrak{a}\mathfrak{r}^{-1}\mathfrak{d}=\mathscr{O}_K}} \mu_K(\mathfrak{b}')\ \sharp\ \{(s,t)\in(\mathfrak{r}\mathfrak{b}'\mathfrak{e})^2\cap(\mathscr{R}_T\smallsetminus\{0\}) : (s,t)\equiv(\sigma,\tau) \bmod \mathfrak{a}, s\equiv\gamma t \bmod \mathfrak{d}\}.$$

Since the set counted in the inner summand is contained in $\Lambda(\mathfrak{r}\mathfrak{b}'\mathfrak{e},\mathfrak{d},\gamma)\cap(\mathscr{R}_T\smallsetminus\{0\})$, the summand is zero unless $\lambda^{(1)}(\mathfrak{r}\mathfrak{b}'\mathfrak{e},\mathfrak{d},\gamma)\ll X_T^{1/m}$. Using Lemma 2.1, this condition implies that

$$\mathfrak{N}\mathfrak{b}'\ll\frac{X_T}{\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)\}^m\mathfrak{N}\mathfrak{r}}. \tag{2.2}$$

Let $\tilde\sigma,\tilde\tau\in\mathfrak{r}\mathfrak{b}'\mathfrak{e}$ such that $(\tilde\sigma,\tilde\tau)\equiv(\sigma,\tau) \bmod \mathfrak{a}$. We have $(\sigma,\tau)\equiv(0,0) \bmod (\mathfrak{r}\mathfrak{b}'\mathfrak{e}+\mathfrak{a})=\mathfrak{r}$, hence, such $(\tilde\sigma,\tilde\tau)$ exist. The Chinese remainder theorem allows us to transform our sum to

$$\sum_{\mathfrak{e}\mid\mathfrak{d}}\mu_K(\mathfrak{e}) \sum_{\substack{(2.2)\\ \mathfrak{b}'\in\mathscr{I}_K\\ \mathfrak{b}'+\mathfrak{a}\mathfrak{r}^{-1}\mathfrak{d}=\mathscr{O}_K}} \mu_K(\mathfrak{b}')\sharp\{(s,t)\in((\tilde\sigma,\tilde\tau)+(\mathfrak{a}\mathfrak{b}'\mathfrak{e})^2)\cap(\mathscr{R}_T\smallsetminus\{0\}) : s\equiv\gamma t \bmod \mathfrak{d}\}.$$

Next, we replace (s,t) by $(s_1,t_1):=(s-\tilde\sigma,t-\tilde\tau)$, so that the inner cardinality becomes

$$\sharp\{(s_1,t_1)\in(\mathfrak{a}\mathfrak{b}'\mathfrak{e})^2\cap((\mathscr{R}_T\smallsetminus\{0\})-(\tilde\sigma,\tilde\tau)) : s_1+\tilde\sigma-\gamma\tilde\tau\equiv\gamma t_1 \bmod \mathfrak{d}\}.$$

Since $\tilde\sigma-\gamma\tilde\tau\equiv 0 \bmod \mathfrak{e}=\mathfrak{a}\mathfrak{b}'\mathfrak{e}+\mathfrak{d}$, we can find $\delta\in\mathfrak{a}\mathfrak{b}'\mathfrak{e}$ with $\delta\equiv\tilde\sigma-\gamma\tilde\tau \bmod \mathfrak{d}$. The replacement of s_1 by $s_2:=s_1+\delta$ transforms the count to

$$\begin{aligned}&\sharp\{(s_2,t_1)\in(\mathfrak{a}\mathfrak{b}'\mathfrak{e})^2\cap((\mathscr{R}_T\smallsetminus\{0\})-(\tilde\sigma,\tilde\tau)+(\delta,0)) : s_2\equiv\gamma t_1 \bmod \mathfrak{d}\}\\ &\quad=\sharp(\Lambda(\mathfrak{a}\mathfrak{b}'\mathfrak{e},\mathfrak{d},\gamma)\cap((\mathscr{R}_T\smallsetminus\{0\})-(\tilde\sigma,\tilde\tau)+(\delta,0))).\end{aligned} \tag{2.3}$$

Clearly, we can extend our family $\mathscr{R}$ to a definable family $\widetilde{\mathscr{R}}\subseteq\mathbb{R}^{(k+2m)+2m}$, whose fibre $\widetilde{\mathscr{R}}_{(T,\sigma,\tau)}$, for $(T,\sigma,\tau)\in\mathbb{R}^{k+2m}$, is the translate $\mathscr{R}_T+(\sigma,\tau)$. Lemma 2.2 thus allows us to approximate the quantity in (2.3) by

$$\frac{c_K\operatorname{vol}\mathscr{R}_T}{\mathfrak{N}(\mathfrak{a}^2\mathfrak{b}'^2\mathfrak{e}\mathfrak{d})}+O\left(\sum_{j=0}^{2m-1}\frac{X_T^{j/m}}{\mathfrak{N}(\mathfrak{a}\mathfrak{b}'\mathfrak{e})^{j/m}\min_{1\leqslant q\leqslant h}\{\prod_{i=1}^j\lambda^{(i)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}\mathfrak{e}^{-1},\gamma)\}}\right). \tag{2.4}$$

Summing the main term over $\mathfrak{e}$ and $\mathfrak{b}'$ gives

$$\frac{c_K\operatorname{vol}\mathscr{R}_T}{\mathfrak{N}(\mathfrak{a}^2\mathfrak{d})}\sum_{\mathfrak{e}\mid\mathfrak{d}}\frac{\mu_K(\mathfrak{e})}{\mathfrak{N}\mathfrak{e}}\sum_{\substack{\mathfrak{b}'\in\mathscr{I}_K\\ \mathfrak{b}'+\mathfrak{a}\mathfrak{d}\mathfrak{r}^{-1}=\mathscr{O}_K\\ (2.2)}}\frac{\mu_K(\mathfrak{b}')}{\mathfrak{N}\mathfrak{b}'^2}.$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

The desired main term is obtained by removing condition (2.2), present in the inner sum. This introduces an error of size

$$\ll \frac{\operatorname{vol}\mathscr{R}_T}{X_T\mathfrak{N}\mathfrak{d}}\sum_{\mathfrak{e}\mid\mathfrak{d}}\mathfrak{N}\mathfrak{r}\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)\}^m \ll \frac{\tau_K(\mathfrak{d})\operatorname{vol}\mathscr{R}_T\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)\}^m\mathfrak{N}\mathfrak{r}}{X_T\mathfrak{N}\mathfrak{d}}$$
$$\ll \frac{X_T\tau_K(\mathfrak{d})\mathfrak{N}\mathfrak{r}}{\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)\}^m},$$

where the third part of Lemma 2.1 has been used to obtain the last inequality. Summing the summand for j in the error term of (2.4) over $\mathfrak{e}$ and $\mathfrak{b}'$ gives a total error

$$\ll X_T^{j/m}\sum_{\mathfrak{e}\mid\mathfrak{d}}\frac{1}{\mathfrak{N}\mathfrak{e}^{j/m}\min_{1\leqslant q\leqslant h}\{\prod_{i=1}^{j}\lambda^{(i)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}\mathfrak{e}^{-1},\gamma)\}}\sum_{\substack{\mathfrak{b}'\in\mathscr{I}_K\\ (2.2)}}\frac{1}{\mathfrak{N}\mathfrak{b}'^{j/m}} \tag{2.5}$$

and

$$\sum_{\substack{\mathfrak{b}'\in\mathscr{I}_K\\ (2.2)}}\frac{1}{\mathfrak{N}\mathfrak{b}'^{j/m}} \ll \left(\frac{X_T}{\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)\}^m}\right)^{\max\{0,1-j/m\}}(\log X_T).$$

Observe, moreover, that $\mathfrak{N}\mathfrak{e}^{1/m}\lambda^{(i)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}\mathfrak{e}^{-1},\gamma)\gg\lambda^{(i)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)$, by Lemma 2.1. Thus, for $j\geqslant m$ the expression in (2.5) is

$$\ll \frac{X_T^{j/m}(\log X_T)\tau(\mathfrak{d})}{\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)^m\lambda^{(m+1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)^{j-m}\}},$$

which, upon replacing j by $j-m$, is covered by the lemma's error term. For $j<m$, the expression in (2.5) is at most $\ll X_T(\log X_T)\tau(\mathfrak{d})(\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d},\gamma)^m\})^{-1}$. □

2.2. Averages of certain arithmetic functions related to Artin L-functions

We shall provide asymptotic estimates for averages of functions that will later appear in the treatment of the main term in Theorem 1.2.

Lemma 2.4. *Let $a:\mathbb{N}\to\mathbb{C}$ be an arithmetic function with associated Dirichlet series $A(s)=\sum_{n\in\mathbb{N}}a(n)n^{-s}$. Let $\delta,C>0$, $\lambda>2$ and assume that*

$$|a(n)|\leqslant Cn^{\delta}, \tag{2.6}$$
$$A(\cdot)\ \textit{has an analytic continuation to}\ \Re(s)>1/2, \tag{2.7}$$
$$|A(s)|\leqslant C(1+|\Im(s)|)^{1/2},\quad \textit{for}\ \Re(s)\geqslant 1-1/\lambda. \tag{2.8}$$

Then

$$\sum_{n\leqslant X}a(n)\ll CX^{1-1/(2\lambda)+2\delta},$$

for $X\geqslant 1$, where the implicit constant may depend at most on λ and δ.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Proof. The Dirichlet series defining $A(s)$ converges absolutely for $\Re(s) > 1+\delta$, thanks to (2.6). Let $\sigma_0 := 1+2\delta$ and $T := X^{1/\lambda}$. We shall make use of Perron's formula (see for example [29, Corollary 5.3]) to obtain

$$\sum_{n\leqslant X} a(n) - \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} A(s)\frac{X^s}{s}\,ds \ll \sum_{X/2<n<2X} |a(n)| \min\left\{1, \frac{X}{T|X-n|}\right\} + \frac{4^{\sigma_0}+X^{\sigma_0}}{T}\sum_{n\in\mathbb{N}}\frac{|a(n)|}{n^{\sigma_0}}.$$

Replacing the minimum by its second term unless $|X-n|<1$, the first error term becomes

$$\ll CX^{\delta}\left(1+\frac{X}{T}\sum_{1\leqslant m\leqslant 2X}\frac{1}{m}\right) \ll_\delta CX^{1-1/\lambda+2\delta},$$

while the second error term is $\ll CX^{1-1/\lambda+2\delta}\sum_{n\in\mathbb{N}} n^{-1-\delta} \ll_\delta CX^{1-1/\lambda+2\delta}$. Shifting the line of integration to the left, we see that the main term equals

$$\left(-\int_{1-1/\lambda-iT}^{\sigma_0-iT} + \int_{1-1/\lambda-iT}^{1-1/\lambda+iT} + \int_{1-1/\lambda+iT}^{\sigma_0+iT}\right) A(s)\frac{X^s}{s}\,ds.$$

The first and third integral are bounded by

$$\ll CT^{-1/2}\int_{u=1-1/\lambda}^{\sigma_0} X^u\,du \ll CT^{-1/2}X^{\sigma_0} = CX^{1-1/(2\lambda)+2\delta}$$

and the second integral attains a value

$$\ll CX^{1-1/\lambda}\int_{t=-T}^{T}\frac{(1+|t|)^{1/2}}{|1-1/\lambda+it|}\,dT \ll CX^{1-1/\lambda}\left(1+\int_{t=1}^{T} t^{-1/2}\,dt\right)$$
$$\ll CX^{1-1/\lambda}T^{1/2} \ll CX^{1-1/(2\lambda)}.$$

□

Lemma 2.5. *Let $\rho : \mathscr{I}_K \to \mathbb{C}$ be a multiplicative function whose associated Dirichlet series is $D_\rho(s) = \sum_{\mathfrak{a}\in\mathscr{I}_K}\rho(\mathfrak{a})\mathfrak{N}\mathfrak{a}^{-s}$. Let $\mathfrak{W}\in\mathscr{I}_K$, $\lambda>2$, and $f\in\mathscr{Z}_K$. Assume that the following conditions hold:*

$$\rho(\mathfrak{a}) = 0 \text{ unless } \mathfrak{a}+\mathfrak{W} = \mathscr{O}_K, \tag{2.9}$$
$$\rho(\mathfrak{p}^k) \ll_\rho 1 \text{ for all prime ideals } \mathfrak{p}\nmid\mathfrak{W} \text{ and all } k\geqslant 0, \tag{2.10}$$
$$D_\rho(\cdot) \text{ has an analytic continuation to } \Re(s) > 1/2, \tag{2.11}$$
$$D_\rho(s) \ll_\rho (1+|\Im(s)|)^{1/2} \text{ for } \Re(s)\geqslant 1-1/\lambda, \tag{2.12}$$
$$\left|\sum_{k=1}^{\infty}\frac{\rho(\mathfrak{p}^k)}{\mathfrak{N}\mathfrak{p}^{ks}}\right| < \frac{1}{2} \text{ for all prime ideals } \mathfrak{p}\nmid\mathfrak{W} \text{ and } \Re(s)>1/2, \tag{2.13}$$
$$\left|\mathbf{1}_f(\mathfrak{p})\sum_{k=1}^{\infty}\frac{\rho(\mathfrak{p}^k)}{\mathfrak{N}\mathfrak{p}^{ks}}\right| < \frac{1}{2} \text{ for all prime ideals } \mathfrak{p}\nmid\mathfrak{W} \text{ and } \Re(s)>1/2. \tag{2.14}$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Then there is $\beta > 0$ and $\gamma \in \mathscr{Z}_K$, such that, for any $\mathfrak{c} \in \mathscr{I}_K$ with $\mathfrak{c} + \mathfrak{W} = \mathscr{O}_K$, we have

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leqslant X \\ \mathfrak{a}+\mathfrak{c}\mathfrak{W}=\mathscr{O}_K}} \frac{\mathbf{1}_f(\mathfrak{a})\rho(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} = D_\rho(1)\beta \mathbf{1}_\gamma(\mathfrak{c}) + O(\mathfrak{N}\mathfrak{c}^{\varepsilon} X^{-1/(2\lambda)+\varepsilon}),$$

for all $\varepsilon > 0$. The implicit constant is allowed to depend on $\varepsilon, \rho, \mathfrak{W}, f, \lambda$, but not on $\mathfrak{c}, X$.

Proof. For $\mathfrak{p} \nmid \mathfrak{W}$ let $\Phi_\mathfrak{p}(s) := \sum_{k=1}^{\infty} \rho(\mathfrak{p}^k)\mathfrak{N}\mathfrak{p}^{-ks}$, which is bounded in absolute value by $1/2$ whenever $\Re(s) > 1/2$, due to (2.13). Moreover, condition (2.10) implies that

$$\Phi_\mathfrak{p}(s) \ll_\rho \mathfrak{N}\mathfrak{p}^{-s} \quad \text{for } \Re(s) > 1/2. \tag{2.15}$$

Define formally the Dirichlet series

$$\begin{aligned} D_\mathfrak{c}(s) &:= \sum_{\substack{\mathfrak{a} \in \mathscr{I}_K \\ \mathfrak{a}+\mathfrak{c}\mathfrak{W}=\mathscr{O}_K}} \frac{\mathbf{1}_f(\mathfrak{a})\rho(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^s} = \prod_{\mathfrak{p} \nmid \mathfrak{c}\mathfrak{W}} (1 + \mathbf{1}_f(\mathfrak{p})\Phi_\mathfrak{p}(s)), \\ \Psi_\mathfrak{c}(s) &:= \prod_{\mathfrak{p} \mid \mathfrak{c}} (1 + \mathbf{1}_f(\mathfrak{p})\Phi_\mathfrak{p}(s))^{-1} \quad \text{and} \\ \Phi(s) &:= \prod_{\mathfrak{p} \nmid \mathfrak{W}} \frac{1 + \mathbf{1}_f(\mathfrak{p})\Phi_\mathfrak{p}(s)}{1 + \Phi_\mathfrak{p}(s)} = \prod_{\mathfrak{p} \nmid \mathfrak{W}} \left(1 + \frac{f(\mathfrak{p})\Phi_\mathfrak{p}(s)}{1 + \Phi_\mathfrak{p}(s)}\right), \end{aligned}$$

to obtain a factorisation

$$D_\mathfrak{c}(s) = D_\rho(s)\Phi(s)\Psi_\mathfrak{c}(s). \tag{2.16}$$

By (2.15), the Euler products for $D_\mathfrak{c}(s)$ and $D_\rho(s)$ converge absolutely and define holomorphic functions for $\Re(s) > 1$, while (2.15) and (2.13) guarantee that $\Phi(s)$ converges absolutely and defines a holomorphic function on $\Re(s) > 1/2$. Moreover, (2.14) ensures that all factors of the finite product $\Psi_\mathfrak{c}(s)$ are defined and holomorphic for $\Re(s) > 1/2$. Consequently, the factorisation (2.16) holds for $\Re(s) > 1$ and, using (2.11), provides an analytic continuation of $D_\mathfrak{c}(s)$ to $\Re(s) > 1/2$. For $\Re(s) \geqslant 1 - 1/\lambda$, we obtain by (2.12) and (2.13) that

$$|D_\mathfrak{c}(s)| \ll_\rho (1 + |\Im(s)|)^{1/2} \left(\prod_{\mathfrak{p} \mid \mathfrak{c}} 2\right) |\Phi(s)| \ll_{\varepsilon, f, \rho, \lambda} \mathfrak{N}\mathfrak{c}^{\varepsilon}(1 + |\Im(s)|)^{1/2}.$$

Since moreover $\sum_{\mathfrak{N}\mathfrak{a}=k} \mathbf{1}_f(\mathfrak{a})\rho(\mathfrak{a}) \ll_{\varepsilon, f, \rho} k^{\varepsilon}$, we may apply Lemma 2.4 to obtain for any $\varepsilon > 0$,

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leqslant X \\ \mathfrak{a}+\mathfrak{c}\mathfrak{W}=\mathscr{O}_K}} \mathbf{1}_f(\mathfrak{a})\rho(\mathfrak{a}) \ll_{\varepsilon, f, \rho, \lambda} \mathfrak{N}\mathfrak{c}^{\varepsilon} X^{1-1/(2\lambda)+\varepsilon}.$$

Partial summation reveals that the series defining $D_\mathfrak{c}(s)$ converges for $s = 1$ and

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leqslant X \\ \mathfrak{a}+\mathfrak{c}\mathfrak{W}=\mathscr{O}_K}} \frac{\mathbf{1}_f(\mathfrak{a})\rho(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} = D_\rho(1)\Phi(1)\Psi_\mathfrak{c}(1) + O(\mathfrak{N}\mathfrak{c}^{\varepsilon} X^{-1/(2\lambda)+\varepsilon}).$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Conditions (2.13) and (2.14) show that $\beta := \Phi(1) > 0$. We finish our proof with the observation $\Psi_{\mathfrak{c}}(1) = \mathbf{1}_\gamma(\mathfrak{c})$, where

$$\gamma(\mathfrak{p}) := (1 + \mathbf{1}_f(\mathfrak{p})\Phi_{\mathfrak{p}}(1))^{-1} - 1 = \sum_{k=1}^{\infty} (\mathbf{1}_f(\mathfrak{p})\Phi_{\mathfrak{p}}(1))^k.$$

In particular, $|\gamma(\mathfrak{p})| < 1$ and $\gamma(\mathfrak{p}) \ll \mathfrak{N}\mathfrak{p}^{-1}$, so $\gamma \in \mathscr{Z}_K$. □

In our proof of Theorem 1.2, we shall apply the above result for Dirichlet series $D_\rho(s)$ of the following form. Let (F, G) be a pair of binary forms in $\mathscr{O}_K[s, t]$, such that F is irreducible in $K[s, t]$, not proportional to t, and does not divide G in $K[s, t]$. We assume furthermore that G is of even degree, and that $G(\theta, 1) \notin K(\theta)^{\times 2}$, where $\theta \in \overline{K}$ is a root of $F(s, 1)$.

Fix $\mathfrak{W} \in \mathscr{I}_K$ with $2 \mid \mathfrak{W}$. We define, for $\mathfrak{a} \in \mathscr{I}_K$, the multiplicative function $\rho_{(F,G)}(\mathfrak{a})$ by

$$\rho_{(F,G)}(\mathfrak{a}) := \sum_{\substack{\lambda \bmod \mathfrak{a} \\ F(\lambda,1) \equiv 0 \bmod \mathfrak{a}}} \left(\frac{G(\lambda, 1)}{\mathfrak{a}}\right) \quad \text{if } \mathfrak{a} + \mathfrak{W} = \mathscr{O}_K,$$

and $\rho_{(F,G)}(\mathfrak{a}) = 0$ otherwise. We assume that $\mathfrak{W}$ is divisible by enough small prime ideals to ensure that $2 \cdot |\rho_{(F,G)}(\mathfrak{p})| < \mathfrak{N}\mathfrak{p}^{1/2}$ for all prime ideals $\mathfrak{p}$.

Lemma 2.6. *The Dirichlet series of $\rho_{(F,G)}$, given by*

$$D_{(F,G)}(s) := \sum_{\mathfrak{a} \in \mathscr{I}_K} \frac{\rho_{(F,G)}(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^s},$$

defines a holomorphic function in $\Re(s) > \frac{1}{2}$ that does not vanish at $s = 1$. We furthermore have $|D_{(F,G)}(s)| \ll (1 + |\Im(s)|)^{1/2}$ in the region $\Re(s) > 1 - 1/\lambda$, where $\lambda = 1 + 2m \deg F$.

Proof. Let $a := F(1, 0) \in \mathscr{O}_K \setminus \{0\}$. Then $F(s, at) = a\widehat{F}(s, t)$, where $\widehat{F}(s, 1) \in \mathscr{O}_K[s]$ is monic and irreducible. Note that the constant $\widehat{\theta} := a\theta$ is a root of $\widehat{F}(s, 1)$. Define the number field $H := K(\theta, \sqrt{G(\widehat{\theta}, a)}) = K(\theta, \sqrt{G(\theta, 1)})$, which clearly fulfils $[H : K(\theta)] = 2$.

The non-trivial representation of $\mathrm{Gal}(H/K(\theta))$ gives rise to the Artin L-function

$$L(s, \chi) = \prod_{\mathfrak{P}} (1 - \chi(\mathfrak{P})\mathfrak{N}_{K(\theta)/\mathbb{Q}}\mathfrak{P}^{-s})^{-1},$$

with the product running over the non-zero prime ideals $\mathfrak{P}$ of $K(\theta)$. The character $\chi(\mathfrak{P})$ is 0 if $\mathfrak{P}$ is ramified in $H/K(\theta)$ and 1 or -1 according to whether $\mathfrak{P}$ is split or inert in $H/K(\theta)$. This L-function is entire and does not vanish at $s = 1$. The usual argument about split primes shows that

$$\prod_{\mathfrak{P} \mid \mathfrak{p}} (1 + \chi(\mathfrak{P})\mathfrak{N}_{K(\theta)/\mathbb{Q}}\mathfrak{P}^{-s}) = 1 + \left(\sum_{\substack{\mathfrak{P} \mid \mathfrak{p} \\ f(\mathfrak{P}/\mathfrak{p}) = 1}} \chi(\mathfrak{P}) \right) \mathfrak{N}\mathfrak{p}^{-s} + O(\mathfrak{N}\mathfrak{p}^{-2s}),$$

for every prime ideal $\mathfrak{p}$ of $\mathscr{O}_K$, where $f(\mathfrak{P}/\mathfrak{p})$ is the inertia degree.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

In the following considerations, we assume that $\mathfrak{p}$ is relatively prime to a and to the conductors of the orders $\mathscr{O}_K[\widehat{\theta}]$ in $K(\theta)$ and $\mathscr{O}_{K(\theta)}[\sqrt{G(\widehat{\theta},a)}]$ in H. Then the primes $\mathfrak{P}$ in $K(\theta)$ above $\mathfrak{p}$ with $f(\mathfrak{P}/\mathfrak{p})=1$ are parameterised by the roots λ of $\widehat{F}(s,1)$ modulo $\mathfrak{p}$. If $\mathfrak{P}$ corresponds to the root λ, then we have an isomorphism $\mathscr{O}_{K(\theta)}/\mathfrak{P}\to\mathscr{O}_K/\mathfrak{p}$ given by $\widehat{\theta}\mapsto\lambda$. Consequently,

$$\chi(\mathfrak{P})=\left(\frac{G(\widehat{\theta},a)}{\mathfrak{P}}\right)=\left(\frac{G(\lambda,a)}{\mathfrak{p}}\right)$$

and in particular,

$$\sum_{\substack{\mathfrak{P}\mid\mathfrak{p}\\ f(\mathfrak{P}/\mathfrak{p})=1}}\chi(\mathfrak{P})=\sum_{\substack{\lambda \bmod \mathfrak{p}\\ \widehat{F}(\lambda,1)\equiv 0 \bmod \mathfrak{p}}}\left(\frac{G(\lambda,a)}{\mathfrak{p}}\right)=\sum_{\substack{\lambda \bmod \mathfrak{p}\\ F(\lambda,1)\equiv 0 \bmod \mathfrak{p}}}\left(\frac{G(a\lambda,a)}{\mathfrak{p}}\right)=\rho_{(F,G)}(\mathfrak{p}),$$

where we again relied on the fact that G is of even degree. Let $\mathfrak{W}_1$ be the product of all the prime ideals excluded above. We have shown that

$$L(s,\chi)=g_0(s)\prod_{\mathfrak{p}\nmid\mathfrak{W}_1\mathfrak{W}}\left(1+\frac{\rho_{(F,G)}(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^s}\right)=g_1(s)\prod_{\mathfrak{p}\nmid\mathfrak{W}}\left(1+\frac{\rho_{(F,G)}(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^s}\right)=g_2(s)D_{(F,G)}(s),$$

where g_0,g_1,g_2 are holomorphic functions and have absolutely convergent Euler products on $\Re(s)>1/2$ that do not vanish there. Hence, for $\Re(s)>1/2+\varepsilon$. we have $1\ll_\varepsilon g_2(s)\ll_\varepsilon 1$.

Convexity bounds, for example [27, Theorem III.14 A] with $\eta=1/(2m\deg F)$, show that

$$L(s,\chi)\ll(1+|\Im(s)|)^{1/2}\quad\text{in } 1-\eta\leqslant\Re(s)\leqslant 1+\eta,$$

which extends to the region $1-\eta\leqslant\Re(s)$ by absolute convergence of $L(s,\chi)$ in $\Re(s)>1$. □

We shall need to handle averages of volumes of certain regions (see (4.11)). The next version of Abel's sum formula is optimally tailored for this task.

Lemma 2.7. *Let $g,\omega:\mathbb{N}\to\mathbb{C}$ be functions, and write $G(u):=\sum_{n\leqslant u}g(n)$. Let $X\geqslant 1$, $A,B\geqslant 0$ with $A+B<1$, and assume that:*

(1) *$\omega(n)=0$ for $n\geqslant X$;*
(2) *there is $Q\geqslant 0$ such that $|\omega(n)-\omega(n+1)|\leqslant Qn^{-B}$ holds for all $n\in\mathbb{N}$;*
(3) *there are $\lambda_0\in\mathbb{C}$, $M\geqslant 0$, such that $|G(n)-\lambda_0|\leqslant Mn^{-A}$ holds for all $n\in\mathbb{N}$.*

Then

$$\left|\sum_{n\leqslant X}g(n)\omega(n)-\lambda_0\omega(1)\right|\leqslant MQ\left(1+\frac{X^{1-A-B}}{1-A-B}\right).$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Proof. Telescoping and using assumption (1), we see that

$$\sum_{n\leqslant X} g(n)\omega(n) = \sum_{n\leqslant X} G(n)(\omega(n)-\omega(n+1))$$
$$= \lambda_0 \sum_{n\leqslant X} (\omega(n)-\omega(n+1)) + \sum_{n\leqslant X} (G(n)-\lambda_0)(\omega(n)-\omega(n+1)).$$

The first summand is equal to $\lambda_0\omega(1)$, and, using assumptions (2) and (3), the last sum has absolute value at most

$$MQ\sum_{n\leqslant X} n^{-A-B} \leqslant MQ\left(1+\int_1^X \frac{du}{u^{A+B}}\right) \leqslant MQ\left(1+\frac{X^{1-A-B}}{1-A-B}\right). \qquad \square$$

3. Proof of Theorem 1.1

In this section we assume the validity of Theorem 1.2 and we prove Theorem 1.1 from it. The finite set S_{bad} will contain all prime ideals that we want to exclude at various steps of our argument. It will grow during the proof, but it will never depend on anything but K, $\mathfrak{r}$, $\mathfrak{F}$ and f. In Theorems 1.1 and 1.2, we always assume that none of the forms $F_i(s,t)$ is proportional to t. This can be achieved by a unimodular transformation $\phi_a : K^2 \to K^2$, $(s,t)\mapsto(s,as+t)$, for suitable $a\in\mathscr{O}_K$. This map ϕ_a extends to $K_\infty^2 \to K_\infty^2$ in an obvious way, transforming $\mathscr{D}$ to $\phi_a(\mathscr{D})$. Clearly, all our hypotheses are still satisfied.

3.1. Simple reductions

Lemma 3.1. *Let $\mathscr{P}=(\mathscr{D},(\sigma,\tau),\mathfrak{W})$ be an $\mathfrak{F}$-admissible triplet, and $k\in\mathbb{N}$. Then*

$$\mathscr{P}^k := (\mathscr{D},(\sigma,\tau),\mathfrak{W}^k)$$

is also an $\mathfrak{F}$-admissible triplet and $D(\mathfrak{F},f,\mathscr{P};X) \gg_k D(\mathfrak{F},f,\mathscr{P}^k;X)$.

Proof. Since $\mathfrak{W}$ and $\mathfrak{W}^k$ have the same prime factors, the ideals $\mathfrak{a}^\flat$, for $\mathfrak{a}\in\mathscr{I}_K$, are the same for $\mathfrak{W}$ and $\mathfrak{W}^k$. Moreover, $M^*(\mathscr{P}^k,X)\subseteq M^*(\mathscr{P},X)$. This shows that, $\mathscr{P}^k$ is admissible, and moreover $r(\mathfrak{F},f,\mathscr{P};s,t)=r(\mathfrak{F},f,\mathscr{P}^k;s,t)$. The lemma follows immediately, since $r(\mathfrak{F},f,\mathscr{P};s,t)\geqslant 0$. $\square$

It is enough to prove Conjecture 1 for all strongly $\mathfrak{F}$-admissible triplets. Indeed, given any $\mathfrak{F}$-admissible triplet $\mathscr{P}=(\mathscr{D},(\sigma,\tau),\mathfrak{W})$, we may assume it to be strongly $\mathfrak{F}$-admissible. To this end, we may replace $\mathfrak{W}$ by any positive power of itself, thanks to Lemma 3.1. By (1.6), we can find $k\in\mathbb{N}$, such that $\mathscr{P}^k$ satisfies (1.9).

By including in S_{bad} enough small prime ideals and replacing $\mathfrak{W}$ by a high enough power, we can moreover assume that

$$2\mathfrak{r}\prod_i F_i(1,0)\prod_{i\neq j}\mathrm{Res}(F_i,F_j) \mid \mathfrak{W}. \tag{3.1}$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

3.2. Eclipsing the trivial G_i

Lemma 3.2. *Whenever $i \in \{1, \dots, n\}$ is such that $G_i(\boldsymbol{\theta}_i) \in K(\boldsymbol{\theta}_i)^{\times 2}$, then for all $s, t \in \mathscr{O}_K$ with $s\mathscr{O}_K + t\mathscr{O}_K = \mathfrak{r}$ we have*

$$\sum_{\mathfrak{d}_i \mid F_i(s,t)^\flat} \left(\frac{G_i(s,t)}{\mathfrak{d}_i}\right) = \tau_K(F_i(s,t)^\flat).$$

Proof. The isomorphism $K[S]/F_i(S,1) \to K(\boldsymbol{\theta}_i)$, $S \mapsto \theta_i$, sends $G_i(S,1)$ to $G_i(\boldsymbol{\theta}_i)$. Hence,

$$G_i(S,1) = h(S)^2 + c(S)F_i(S,1),$$

with polynomials $h(S), c(S) \in K[S]$, such that $F_i(S,1) \nmid h(S)$. Let d be the maximum of the degrees of $G_i(S,1)$, $h(S)^2$, $c(S)F_i(S,1)$. Rehomogenising, we obtain

$$G_i(S,T)T^{d-\deg G_i} = H(S,T)^2 T^{d-2\deg H} + C(S,T)T^{d-\deg C-\deg F_i} F_i(S,T),$$

with forms $H, C \in K[S,T]$. Letting $b \in \mathscr{O}_K$ such that $bH(S,T) \in \mathscr{O}_K[S,T]$, we find that $\operatorname{Res}(bH(S,T), F(S,T)) \in \mathscr{O}_K \smallsetminus \{0\}$. After adding to S_{bad} all prime ideals that divide $b\operatorname{Res}(bH(S,T), F(S,T))$, and all modulo which the form C cannot be reduced, we obtain, for all $s, t \in \mathscr{O}_K$ and all $\mathfrak{p} \mid F_i(s,t)^\flat$,

$$\left(\frac{G_i(s,t)t^{d-\deg G}}{\mathfrak{p}}\right) = \left(\frac{H(s,t)^2 t^{d-2\deg H}}{\mathfrak{p}}\right).$$

Using $s\mathscr{O}_K + t\mathscr{O}_K = \mathfrak{r}$ and $\mathfrak{p} \nmid F_i(1,0)$, we see that if $\mathfrak{p} \mid t$ then $\mathfrak{p} \mid s$, which shows that $\mathfrak{p} \mid \mathfrak{r} \mid \mathfrak{W}$, a contradiction. Hence, t is invertible modulo $\mathfrak{p}$ and using that $\deg G$ is even, we derive

$$\left(\frac{G_i(s,t)}{\mathfrak{p}}\right) = \left(\frac{H(s,t)}{\mathfrak{p}}\right)^2 \left(\frac{t}{\mathfrak{p}}\right)^{\deg G - 2\deg H} = \left(\frac{H(s,t)}{\mathfrak{p}}\right)^2 = 1.$$

In the last equality, we were allowed to exclude the case $H(s,t) \equiv 0 \bmod \mathfrak{p}$ due to the condition $\mathfrak{p} \nmid \operatorname{Res}(bH(S,T), F_i(S,T))$. □

By possibly reordering the $(F_i, G_i) \in \mathfrak{F}$, we may assume that

$$G_i(\boldsymbol{\theta}_i) \begin{cases} \in K(\boldsymbol{\theta}_i)^{\times 2} & \text{for } 1 \leqslant i \leqslant \rho(\mathfrak{F}), \\ \notin K(\boldsymbol{\theta}_i)^{\times 2} & \text{for } \rho(\mathfrak{F}) + 1 \leqslant i \leqslant n. \end{cases}$$

We define $f'(\mathfrak{p}) := 0$ if $\mathfrak{p} \in S_{\mathrm{bad}}$ and $f'(\mathfrak{p}) := 2f(\mathfrak{p})$ otherwise. Note that choosing S_{bad} large enough ensures that $f' \in \mathscr{Z}_K$. All n factors in the definition of $r(s,t)$ are non-negative and for $1 \leqslant i \leqslant \rho(\mathfrak{F})$ we see by Lemma 3.2 that

$$\begin{aligned} \mathbf{1}_f(F_i(s,t)^\flat) \sum_{\mathfrak{d}_i \mid F_i(s,t)^\flat} \left(\frac{G_i(s,t)}{\mathfrak{d}_i}\right) &= \prod_{\mathfrak{p} \mid F_i(s,t)^\flat} (1 + f(\mathfrak{p}))(v_{\mathfrak{p}}(F_i(s,t)) + 1) \\ &\geqslant \prod_{\mathfrak{p} \mid F_i(s,t)^\flat} (1 + (1 + 2f(\mathfrak{p}))) = \sum_{\substack{\mathfrak{d}_i \mid F_i(s,t) \\ \mathfrak{d}_i + \mathfrak{W} = \mathscr{O}_K}} \mu_K^2(\mathfrak{d}_i)\mathbf{1}_{f'}(\mathfrak{d}_i). \end{aligned}$$

If $\rho(\mathfrak{F}) < n$, we let $\mathfrak{F}' := \{(F_{\rho(\mathfrak{F})+1}, G_{\rho(\mathfrak{F})+1}), \dots, (F_n, G_n)\}$ comprise those pairs in $\mathfrak{F}$ with $G_i(\boldsymbol{\theta}_i) \notin K(\boldsymbol{\theta}_i)^{\times 2}$. Then $\rho(\mathfrak{F}') = 0$ and $c(\mathfrak{F}') = c(\mathfrak{F}) \leqslant 3$. Clearly, the strongly $\mathfrak{F}$-admissible triplet $\mathscr{P}$ is also strongly $\mathfrak{F}'$-admissible.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Lemma 3.3. *Let $\rho(\mathfrak{F}) < n$. Then, for any $\varepsilon \in (0,1)$, the sum $D(\mathfrak{F}, f, \mathscr{P}; X)$ is $\ll$*

$$\sum_{\substack{\mathfrak{d}_1,\dots,\mathfrak{d}_{\rho(\mathfrak{F})} \in \mathscr{I}_K \\ \mathfrak{N}\mathfrak{d}_i \leqslant X^{\varepsilon} \ \forall i \\ \mathfrak{d}_i + \mathfrak{W} = \mathscr{O}_K \ \forall i \\ \mathfrak{d}_i + \mathfrak{d}_j = \mathscr{O}_K \ \forall i \neq j}} \left(\prod_{i=1}^{\rho(\mathfrak{F})} \mu_K^2(\mathfrak{d}_i) \mathbf{1}_{f'}(\mathfrak{d}_i) \right) \sum_{\substack{(\sigma_i,\tau_i) \bmod \mathfrak{d}_i \ \forall i \\ \sigma_i \mathscr{O}_K + \tau_i \mathscr{O}_K + \mathfrak{d}_i = \mathscr{O}_K \\ F_i(\sigma_i,\tau_i) \equiv 0 \bmod \mathfrak{d}_i \ \forall i}} \sum_{\substack{(s,t) \in M^*(\mathscr{P},X) \\ (s,t) \equiv (\sigma_i,\tau_i) \bmod \mathfrak{d}_i \ \forall i}} r(\mathfrak{F}', f, \mathscr{P}; s, t). \tag{3.2}$$

In these sums, the quantifiers $\forall i$ run over all $i \in \{1, \dots, \rho(\mathfrak{F})\}$.

Proof. This stems upon reordering the sum with respect to the factors $\mathfrak{d}_i \mid F_i(s,t)$ and splitting into congruence classes $\bmod \mathfrak{d}_i$. Since $r(s,t) \geqslant 0$, we are allowed to impose additional restrictions on the $\mathfrak{d}_i$, such as $\mathfrak{N}\mathfrak{d}_i \leqslant X^{\varepsilon}$. □

Lemma 3.4. *Let $\mathfrak{r}, \mathfrak{a} \in \mathscr{I}_K$, $\mathfrak{r} \mid \mathfrak{a}$, and let $(\tilde{\sigma}, \tilde{\tau}) \in \mathfrak{r}^2$ such that $\tilde{\sigma}\mathscr{O}_K + \tilde{\tau}\mathscr{O}_K + \mathfrak{a} = \mathfrak{r}$. Then there is $(\sigma, \tau) \in \mathfrak{r}^2$ satisfying $(\sigma, \tau) \equiv (\tilde{\sigma}, \tilde{\tau}) \bmod \mathfrak{a}$ and $\sigma\mathscr{O}_K + \tau\mathscr{O}_K = \mathfrak{r}$.*

Proof. Let $\mathfrak{b} \in \mathscr{I}_K$ such that $\mathfrak{b}\mathfrak{a} = w\mathscr{O}_K$ is a principal ideal, and such that any prime ideal $\mathfrak{p}$ dividing $\tilde{\sigma}$ divides $\mathfrak{b}$ if and only if it does not divide $\tilde{\tau}\mathfrak{r}^{-1}$. We may then choose $\sigma := \tilde{\sigma}$ and $\tau := \tilde{\tau} + w$. □

We next deploy Theorem 1.2 to estimate the innermost sum in Lemma 3.3.

Lemma 3.5. *Let $\rho(\mathfrak{F}) < n$. There is a function $f_0 \in \mathscr{Z}_K$ and $\beta_0, \beta_1, \beta_2 > 0$, such that the following holds: for any $\mathfrak{d}_1, \dots, \mathfrak{d}_{\rho(\mathfrak{F})} \in \mathscr{I}_K$ and $(\sigma_i, \tau_i) \bmod \mathfrak{d}_i$, satisfying the conditions under the first two sums in (3.2), we have, with $\mathfrak{d} := \mathfrak{d}_1 \cdots \mathfrak{d}_{\rho(\mathfrak{F})}$, the asymptotic*

$$\sum_{\substack{(s,t) \in M^*(\mathscr{P},X) \\ (s,t) \equiv (\sigma_i,\tau_i) \bmod \mathfrak{d}_i \ \forall i}} r(\mathfrak{F}', f, \mathscr{P}; s, t) = \beta_0 X^2 \frac{\mathbf{1}_{f_0}(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}^2} + O(X^{2-\beta_1} \mathfrak{N}\mathfrak{d}^{\beta_2}). \tag{3.3}$$

The implicit constant in the error term is independent of all $\mathfrak{d}_i$, (σ_i, τ_i).

Proof. The Chinese remainder theorem and the coprimality conditions on $\mathfrak{d}_1, \dots, \mathfrak{d}_{\rho(\mathfrak{F})}$, $\mathfrak{W}$ allow us to express the congruences $(s,t) \equiv (\sigma, \tau) \bmod \mathfrak{W}$ and $(s,t) \equiv (\sigma_i, \tau_i) \bmod \mathfrak{d}_i$ for all i as one congruence $(s,t) \equiv (\tilde{\sigma}, \tilde{\tau}) \bmod \mathfrak{d}\mathfrak{W}$. The pair $(\tilde{\sigma}, \tilde{\tau}) \in \mathscr{O}_K^2$ then necessarily satisfies $\tilde{\sigma}\mathscr{O}_K + \tilde{\tau}\mathscr{O}_K + \mathfrak{d}\mathfrak{W} = \mathfrak{r}$. Using Lemma 3.4, we may thus assume that $\tilde{\sigma}\mathscr{O}_K + \tilde{\tau}\mathscr{O}_K = \mathfrak{r}$.

The triplet $\mathscr{P}' := (\mathscr{D}, (\tilde{\sigma}, \tilde{\tau}), \mathfrak{W})$ is strongly $\mathfrak{F}'$-admissible. Moreover $\mathfrak{d}$ satisfies the condition (1.10) in Theorem 1.2, since $\mathfrak{r} \prod_{i,j} \mathrm{Res}(F_i, F_j) | \mathfrak{W}$, and since $\mathfrak{d}_i + \mathfrak{W} = \mathscr{O}_K$ for all i.

The sum in the lemma equals

$$\sum_{(s,t) \in M^*(\mathscr{P}'_{\mathfrak{d}}, X)} r(\mathfrak{F}', f, \mathscr{P}'; s, t),$$

so the lemma stems from Theorem 1.2, once we enlarge S_{bad} and replace $\mathfrak{W}$ by a sufficiently high power to ensure that $\mathfrak{W}_0 \mid \mathfrak{W}$. □

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Using the bound $|\mathbf{1}_{f'}(\mathfrak{d}_i)| \ll \mathfrak{N}\mathfrak{d}_i$, we see that the error terms arising from substituting (3.3) into (3.2) are $\ll X^{2-\beta_1+\varepsilon\rho(\mathfrak{F})(\beta_2+3)}$. Finally, choosing ε small enough makes the exponent smaller than 2.

Let us consider the main term. For a form $F \in \mathscr{O}_K[s,t]$, irreducible over K and not divisible by t and for $\mathfrak{d} \in \mathscr{I}_K$ we define

$$\tau_F(\mathfrak{d}) := \sharp\{\mu \in \mathscr{O}_K/\mathfrak{d} : F(\mu,1) \equiv 0 \bmod \mathfrak{d}\}. \tag{3.4}$$

Using (3.1), we obtain for all $\mathfrak{d} \in \mathscr{I}_K$ with $\mathfrak{d}+\mathfrak{W} = \mathscr{O}_K$,

$$\sum_{\substack{(\sigma,\tau) \bmod \mathfrak{d} \\ F(\sigma,\tau)\equiv 0 \bmod \mathfrak{d} \\ \sigma\mathscr{O}_K+\tau\mathscr{O}_K+\mathfrak{d}=\mathscr{O}_K}} 1 = \tau_F(\mathfrak{d})\phi_K(\mathfrak{d}).$$

Let us now introduce the function

$$L(\mathfrak{d}) := \mathbf{1}_{f'}(\mathfrak{d})\mathbf{1}_{f_0}(\mathfrak{d})\mathfrak{N}(\mathfrak{d})^{-1}\phi_K(\mathfrak{d}) \sum_{\mathfrak{d}_1\cdots\mathfrak{d}_{\rho(\mathfrak{F})}=\mathfrak{d}} \prod_{i=1}^{\rho(\mathfrak{F})} \tau_{F_i}(\mathfrak{d}_i).$$

To finish the proof of Theorem 1.1 in the case $\rho(\mathfrak{F}) < n$, it remains to show that

$$\sum_{\substack{\mathfrak{N}\mathfrak{d}\leqslant X^{\varepsilon} \\ \mathfrak{d}+\mathfrak{W}=\mathscr{O}_K}} \mu_K^2(\mathfrak{d})\frac{L(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} \gg (\log X)^{\rho(\mathfrak{F})}.$$

This bound can be proved in a straightforward manner by alluding to the generalisation of Wirsing's theorem to all number fields as supplied in [17, Lemma 2.2]. The required estimate

$$\sum_{\mathfrak{N}\mathfrak{p}\leqslant X} \frac{\tau_{F_i}(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}} \log \mathfrak{N}\mathfrak{p} = \log X + O(1)$$

follows from the prime ideal theorem for the number field $K(\boldsymbol{\theta}_i)$.

Finally, if $\rho(\mathfrak{F}) = n$, we proceed as in Lemma 3.3 to obtain a lower bound for $D(\mathfrak{F}, f, \mathscr{P}; s, t)$ as in (3.2), but with $r(\mathfrak{F}', f, \mathscr{P}; s, t)$ replaced by 1. Arguing as in Lemma 3.5 and using Möbius inversion as in the proof of Lemma 2.3, the innermost sum then becomes

$$\sum_{(s,t)\in M^*(\mathscr{P}'_{\mathfrak{d}},X)} 1 = \sum_{\substack{\mathfrak{a}\in\mathscr{I}_K \\ \mathfrak{a}+\mathfrak{d}\mathfrak{W}\mathfrak{r}^{-1}=\mathscr{O}_K \\ \mathfrak{N}\mathfrak{a}\ll X}} \sharp(((\sigma^*,\tau^*)+(\mathfrak{a}\mathfrak{r}\mathfrak{d}\mathfrak{W})^2)\cap X^{1/m}\mathscr{D}),$$

for some $(\sigma^*, \tau^*) \in \mathscr{O}_K^2$. By lattice point counting, the summand for $\mathfrak{a}$ is

$$\sharp((\mathfrak{a}\mathfrak{r}\mathfrak{d}\mathfrak{W})^2 \cap (-(\sigma^*,\tau^*)+X^{1/m}\mathscr{D})) = \frac{c_K X^2 \operatorname{vol}\mathscr{D}}{\mathfrak{N}(\mathfrak{a}\mathfrak{r}\mathfrak{d}\mathfrak{W})^2} + O\left(\left(\frac{X}{\mathfrak{N}\mathfrak{a}}\right)^{2-1/m}+1\right).$$

Summing this over all $\mathfrak{a}$ yields a positive constant $\beta_0 = \beta_0(\mathfrak{r}, \mathscr{D}, \mathfrak{W})$, such that

$$\sum_{(s,t)\in M^*(\mathscr{P}'_{\mathfrak{d}},X)} 1 = \beta_0\frac{X^2}{\mathfrak{N}\mathfrak{d}^2} + O(X^{2-1/m}\log X).$$

We may use this asymptotic instead of Lemma 3.5 to proceed as in the case $\rho(\mathfrak{F}) < n$. This completes our proof of Theorem 1.1.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

4. Proof of Theorem 1.2: asymptotics for divisor sums

Recall that we have shown that it is sufficient to consider the case when none of the forms F_i is proportional to t. The ideal $\mathfrak{W}_0$ will be modified throughout the proof, but it will only depend on $K, \mathfrak{r}, \mathfrak{F}, f$. We start by assuming that $\mathfrak{W}_0$ satisfies (3.1). Let $\mathfrak{F}$ be a system of forms as in the theorem, and $\mathfrak{W}$ be a strongly $\mathfrak{F}$-admissible triplet with $\mathfrak{W}_0 \mid \mathfrak{W}$. Moreover, let $\mathfrak{d} \in \mathscr{I}_K$ satisfy (1.10).

4.1. The Dirichlet hyperbola trick

Let us recall that the expression

$$\sum_{(s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X)} r(\mathfrak{F}, f, \mathscr{P}; s, t)$$

can be recast as

$$\sum_{(s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X)} \prod_{i=1}^{n} \mathbf{1}_f(F_i(s,t)^\flat) \left(\sum_{\mathfrak{c}_i \mid F_i(s,t)^\flat} \left(\frac{G_i(s,t)}{\mathfrak{c}_i} \right) \right). \tag{4.1}$$

Defining $\mathfrak{W}_i := \prod_{\mathfrak{p}\mid\mathfrak{W}} \mathfrak{p}^{v_{\mathfrak{p}}(F_i(\sigma,\tau))}$ makes apparent, once (1.9) has been taken into account, that $F_i(s,t)^\flat = F_i(s,t)\mathfrak{W}_i^{-1}$. Furthermore, for each $(s,t) \in M^*(\mathscr{P}_{\mathfrak{d}}, X)$ we have the following inequalities,

$$\mathfrak{N}F_i(s,t)^\flat = \mathfrak{N}\mathfrak{W}_i^{-1} \prod_{v\in\Omega_\infty} |F_i(s,t)|_v^{m_v} \ll \prod_{v\in\Omega_\infty} \max\{|s|_v, |t|_v\}^{m_v \deg F_i} \ll X^{\deg F_i},$$

thus for each index i there exists $c_i > 1$, independent of X, such that whenever $X > 1$ and $(s,t) \in M^*(\mathscr{P}_{\mathfrak{d}}, X)$ then $\mathfrak{N}F_i(s,t)^\flat < c_i X^{\deg F_i}$. We let $Y_i := c_i X^{\deg F_i}$. Suppressing the dependence on $\mathfrak{W}$ in the notation, we define the arithmetic functions

$$r_i^-(s,t) := \sum_{\substack{\mathfrak{c}_i \mid F_i(s,t)^\flat \\ \mathfrak{N}\mathfrak{c}_i < \sqrt{Y_i}}} \left(\frac{G_i(s,t)}{\mathfrak{c}_i} \right) \quad \text{and} \quad r_i^+(s,t) := \sum_{\substack{\mathfrak{c}_i^* \mid F_i(s,t)^\flat \\ \mathfrak{N}\mathfrak{c}_i^* < \sqrt{Y_i} \\ \mathfrak{N}\mathfrak{c}_i^* \sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i \leqslant \mathfrak{N}(F_i(s,t))}} \left(\frac{G_i(s,t)}{\mathfrak{c}_i^*} \right),$$

an action which, upon writing $F_i(s,t)^\flat = \mathfrak{c}_i\mathfrak{c}_i^*$ and using assumption (1.8), allows us to obtain the validity of

$$\sum_{\mathfrak{c}_i \mid F_i(s,t)^\flat} \left(\frac{G_i(s,t)}{\mathfrak{c}_i} \right) = r_i^-(s,t) + r_i^+(s,t).$$

Let us introduce for every $\mathbf{v} \in [0,\infty)^n$ and $\boldsymbol{\psi} = (\psi_1, \dots, \psi_n) \in \{0,1\}^n$ the region

$$\mathscr{D}_{\boldsymbol{\psi}}(X; \mathbf{v}) := \bigcap_{i=1}^{n} \{(s,t) \in X^{1/m}\mathscr{D} : \mathfrak{N}(F_i(s,t)) \geqslant \psi_i v_i \sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i\} \subseteq K_\infty^2. \tag{4.2}$$

Here X is considered as fixed and the dependence on $\mathbf{v}$ is what we are interested in. Define $\omega_{\boldsymbol{\psi}}(X; \mathbf{v}) : \mathbb{R}^n \to \mathbb{R}$ through

$$\mathbf{v} \mapsto \mathrm{vol}(\mathscr{D}_{\boldsymbol{\psi}}(X; \mathbf{v})). \tag{4.3}$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

For $\underline{\mathfrak{c}} = (\mathfrak{c}_1, \dots, \mathfrak{c}_n) \in \mathscr{I}_K^n$ we use the abbreviation $\mathfrak{N}\underline{\mathfrak{c}} := (\mathfrak{N}\mathfrak{c}_1, \dots, \mathfrak{N}\mathfrak{c}_n) \in (0,\infty)^n$ and arrive at the equality of the quantity in (4.1) with

$$\sum_{(s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X)} \prod_{i=1}^{n} \mathbf{1}_f(F_i(s,t)^\flat)(r_i(s,t)^- + r_i(s,t)^+),$$

which can be reshaped into

$$\sum_{\psi\in\{0,1\}^n} \sum_{\substack{\underline{\mathfrak{c}}\in\mathscr{I}_K^n \\ \mathfrak{N}\mathfrak{c}_i<\sqrt{Y_i}\ \forall i \\ \prod_{i=1}^n \mathfrak{c}_i+\mathfrak{d}\mathfrak{W}=\mathscr{O}_K \\ \mathfrak{c}_i+\mathfrak{c}_j=\mathscr{O}_K\ \forall i\neq j}} \sum_{\substack{(s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X) \\ (s,t)\in\mathscr{D}_\psi(X;\mathfrak{N}\underline{\mathfrak{c}}) \\ \mathfrak{c}_i\mid F_i(s,t)\ \forall i}} \prod_{i=1}^{n} \left(\frac{G_i(s,t)}{\mathfrak{c}_i}\right) \mathbf{1}_f(F_i(s,t)^\flat).$$

Here we added the coprimality condition $\prod_{i=1}^n \mathfrak{c}_i + \mathfrak{d}\mathfrak{W} = \mathscr{O}_K$ due to (1.10) and the assumptions $\mathfrak{c}_i + \mathfrak{c}_j = \mathscr{O}_K$ for $i \neq j$ due to (3.1). The identity

$$\mathbf{1}_f(F_i(s,t)^\flat) = \sum_{\substack{\mathfrak{b}_i\mid F_i(s,t) \\ \mathfrak{b}_i+\mathfrak{W}=\mathscr{O}_K}} f(\mathfrak{b}_i)$$

reveals that, with

$$S_\psi := \sum_{\substack{\underline{\mathfrak{b}},\underline{\mathfrak{c}}\in\mathscr{I}_K^n \\ \mathfrak{N}\mathfrak{b}_i<Y_i,\mathfrak{N}\mathfrak{c}_i<\sqrt{Y_i}\ \forall i \\ \prod_{i=1}^n \mathfrak{b}_i\mathfrak{c}_i+\mathfrak{d}\mathfrak{W}=\mathscr{O}_K \\ \mathfrak{c}_i+\mathfrak{c}_j=\mathfrak{b}_i+\mathfrak{b}_j=\mathfrak{b}_i+\mathfrak{c}_j=\mathscr{O}_K\ \forall i\neq j}} \prod_{i=1}^{n} f(\mathfrak{b}_i) \sum_{\substack{(s,t)\in\mathscr{D}_\psi(X;\mathfrak{N}\underline{\mathfrak{c}}) \\ (s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X) \\ (\mathfrak{b}_i\cap\mathfrak{c}_i)\mid F_i(s,t)\ \forall i}} \prod_{i=1}^{n} \left(\frac{G_i(s,t)}{\mathfrak{c}_i}\right),$$

one has

$$\sum_{(s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X)} r(\mathfrak{F},f,\mathscr{P};s,t) = \sum_{\psi\in\{0,1\}^n} S_\psi. \tag{4.4}$$

For any $\mathfrak{a} \in \mathscr{I}_K$ we let $\langle\mathfrak{a}\rangle \subset \mathscr{I}_K$ denote the monoid generated by the prime ideals dividing $\mathfrak{a}$. We collect here some conditions on n-tuples $\underline{\mathfrak{a}}, \underline{\mathfrak{b}}''', \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''' \in \mathscr{I}_K^n$ for later reference:

$$\forall i : \mathfrak{a}_i + \mathfrak{d}\mathfrak{W} = \mathscr{O}_K \quad \text{and} \quad \mathfrak{a}_i + \prod_{j<i} \mathfrak{a}_j = \mathscr{O}_K, \tag{4.5}$$

$$\forall i : \mathfrak{N}\mathfrak{a}_i\mathfrak{b}_i''' < Y_i, \quad \mathfrak{b}_i''' + \mathfrak{d}\mathfrak{W}\prod_{j=1}^{n} \mathfrak{a}_j\mathfrak{c}_j''' = \mathscr{O}_K \quad \text{and} \quad \mathfrak{b}_i''' + \prod_{j<i} \mathfrak{b}_j''' = \mathscr{O}_K, \tag{4.6}$$

$$\forall i : \mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''' < \sqrt{Y_i}, \quad \mathfrak{c}_i'' \in \langle\mathfrak{a}_i\rangle, \quad \mathfrak{c}_i''' + \mathfrak{d}\mathfrak{W}\prod_{j=1}^{n} \mathfrak{a}_j = \mathscr{O}_K \quad \text{and} \quad \mathfrak{c}_i''' + \prod_{j<i} \mathfrak{c}_j''' = \mathscr{O}_K. \tag{4.7}$$

Recall the definition of $\Lambda^*(\mathfrak{a},(\sigma,\tau),\mathfrak{d},\gamma)$ in (2.1).

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Lemma 4.1. *We have*

$$S_\Psi = \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}}''',\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''\in\mathscr{I}_K^n\\(4.5),(4.6),(4.7)}}\left(\prod_{i=1}^n f(\mathfrak{a}_i\mathfrak{b}_i''')\right)\sum_{\substack{\lambda_i \bmod \mathfrak{d}_i\ \forall i\\ \mathfrak{d}_i\mid F_i(\lambda_i,1)}}\left(\prod_{i=1}^n\left(\frac{G_i(\lambda_i,1)}{\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''}\right)\right)$$
$$\times\,|\Lambda^*(\mathfrak{d}\mathfrak{W},(\sigma,\tau),\mathfrak{d}',\lambda)\cap\mathscr{D}_\Psi(X;(\mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''')_{i=1}^n)|,$$

where the quantities $\mathfrak{d}_i,\mathfrak{d}'$ *and* λ *are defined through* $\underline{\mathfrak{a}},\underline{\mathfrak{b}}''',\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''$ *as follows,*

$$\mathfrak{d}_i := \mathfrak{a}_i\mathfrak{b}_i'''\mathfrak{c}_i''\mathfrak{c}_i''',\ \mathfrak{d}' := \prod_{i=1}^n\mathfrak{d}_i$$

and λ *is the, unique modulo* $\mathfrak{d}'$*, solution of the system* $\lambda\equiv\lambda_i \bmod \mathfrak{d}_i$ *for all* i*.*

Proof. For each pair of ideals $\mathfrak{b}_i,\mathfrak{c}_i$ in the definition of S_Ψ we let $\mathfrak{a}_i := \mathfrak{b}_i+\mathfrak{c}_i$. Therefore $\mathfrak{b}_i=\mathfrak{a}_i\mathfrak{b}_i'$ and $\mathfrak{c}_i=\mathfrak{a}_i\mathfrak{c}_i'$ for some coprime ideals $\mathfrak{b}_i',\mathfrak{c}_i'$ which satisfy $\mathfrak{b}_i\cap\mathfrak{c}_i=\mathfrak{a}_i\mathfrak{b}_i'\mathfrak{c}_i'$. We may further decompose $\mathfrak{b}_i'$ and $\mathfrak{c}_i'$ uniquely as $\mathfrak{b}_i'=\mathfrak{b}_i''\mathfrak{b}_i'''$ and $\mathfrak{c}_i'=\mathfrak{c}_i''\mathfrak{c}_i'''$, where $\mathfrak{b}_i'',\mathfrak{b}_i''',\mathfrak{c}_i'',\mathfrak{c}_i'''\in\mathscr{I}_K$ and for all non-zero prime ideals $\mathfrak{p}$ we have

$$\mathfrak{p}\mid\mathfrak{b}_i''\mathfrak{c}_i''\Rightarrow\mathfrak{p}\mid\mathfrak{a}_i\quad\text{and}\quad\mathfrak{p}\mid\mathfrak{b}_i'''\mathfrak{c}_i'''\Rightarrow\mathfrak{p}\nmid\mathfrak{a}_i.$$

Since the function f is supported on square-free ideals, the only relevant value for $\mathfrak{b}_i''$ in S_Ψ is $\mathfrak{b}_i''=\mathscr{O}_K$. Taking into account the conditions (4.5)–(4.7) we have thus obtained the following factorisation for the $\mathfrak{b}_i,\mathfrak{c}_i$ in the sum S_Ψ,

$$\mathfrak{b}_i=\mathfrak{a}_i\mathfrak{b}_i'''\quad\text{and}\quad\mathfrak{c}_i=\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''.$$

We are therefore led to the equality of S_Ψ with

$$\sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}}''',\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''\in\mathscr{I}_K^n\\(4.5),(4.6),(4.7)}}\left(\prod_{i=1}^n f(\mathfrak{a}_i\mathfrak{b}_i''')\right)\sum_{\substack{(s,t)\in M^*(\mathscr{P}_\mathfrak{d},X)\\(s,t)\in\mathscr{D}_\Psi(X;(\mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''')_{i=1}^n)\\ \mathfrak{a}_i\mathfrak{b}_i'''\mathfrak{c}_i''\mathfrak{c}_i'''\mid F_i(s,t)\ \forall i}}\prod_{i=1}^n\left(\frac{G_i(s,t)}{\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''}\right).$$

For any pair (s,t) in the inner sum we have $t\mathscr{O}_K+\mathfrak{d}_i=\mathscr{O}_K$, since if $\mathfrak{p}\mid t\mathscr{O}_K+\mathfrak{d}_i$ then $\mathfrak{p}\nmid\mathfrak{W}$ and hence $\mathfrak{p}\nmid F_i(1,0)$. This implies that $\mathfrak{p}\mid s$ and thus $\mathfrak{p}\mid s\mathscr{O}_K+t\mathscr{O}_K=\mathfrak{r}\mid\mathfrak{W}$, a contradiction. Hence, letting $\lambda_i := st^{-1} \bmod \mathfrak{d}_i$ we obtain the congruence $s\equiv\lambda_i t \bmod \mathfrak{d}_i$. Note that each G_i has even degree and therefore

$$\left(\frac{G_i(s,t)}{\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''}\right)=\left(\frac{G_i(\lambda_i,1)}{\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''}\right),$$

an equality which can be exploited to transform S_Ψ into

$$\sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}}''',\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''\in\mathscr{I}_K^n\\(4.5),(4.6),(4.7)}}\left(\prod_{i=1}^n f(\mathfrak{a}_i\mathfrak{b}_i''')\right)\sum_{\substack{\lambda_i \bmod \mathfrak{d}_i\ \forall i\\ \mathfrak{d}_i\mid F_i(\lambda_i,1)}}\prod_{i=1}^n\left(\frac{G_i(\lambda_i,1)}{\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''}\right)\sum_{\substack{(s,t)\in M^*(\mathscr{P}_\mathfrak{d},X)\\(s,t)\in\mathscr{D}_\Psi(X;(\mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''')_{i=1}^n)\\ s\equiv\lambda_i t \bmod \mathfrak{d}_i\ \forall i}}1.$$

Since the $\mathfrak{d}_i$ are relatively prime in pairs, we may combine the congruences under the innermost sum to a single congruence of the form $s\equiv\lambda t \bmod \mathfrak{d}'$ and our lemma is furnished upon tautologically reformulating the innermost sum. □

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

4.2. Application of lattice point counting

Let us define the multiplicative function on $\mathscr{I}_K$,

$$\eta(\mathfrak{a}) := \frac{\mu_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \prod_{\mathfrak{p}\mid\mathfrak{a}} \left(1+\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1},$$

which is supported on square-free ideals and satisfies $|\eta(\mathfrak{p})| < 1/\mathfrak{N}\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. We use the symbols $\mathfrak{d}_i, \mathfrak{d}', \lambda$ with the same meaning as in Lemma 4.1. For any $\boldsymbol{\psi} \in \{0,1\}^n$, let

$$M_{\psi} := \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}}''',\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''\in\mathscr{I}_K^n \\ (4.5),(4.6),(4.7)}} \omega_{\boldsymbol{\psi}}(X;(\mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''')_{i=1}^n) \prod_{i=1}^n \left(\frac{f(\mathfrak{a}_i\mathfrak{b}_i''')\mathbf{1}_{\eta}(\mathfrak{a}_i\mathfrak{b}_i'''\mathfrak{c}_i''\mathfrak{c}_i''')}{\mathfrak{N}\mathfrak{a}_i\mathfrak{b}_i'''\mathfrak{c}_i''\mathfrak{c}_i'''} \sum_{\substack{\lambda_i \bmod \mathfrak{d}_i \\ \mathfrak{d}_i\mid F_i(\lambda_i,1)}} \left(\frac{G_i(\lambda_i,1)}{\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''}\right)\right).$$

Lemma 4.2. *For all $\varepsilon > 0$ we have*

$$\sum_{(s,t)\in M^*(\mathscr{P}_{\mathfrak{d}},X)} r(\mathfrak{F},f,\mathscr{P};s,t) = \frac{c_K'}{\mathfrak{N}(\mathfrak{d}\mathfrak{W})^2} \prod_{\mathfrak{p}\mid\mathfrak{d}\mathfrak{W}\mathfrak{r}^{-1}} \left(1-\frac{1}{\mathfrak{N}\mathfrak{p}^2}\right)^{-1} \sum_{\boldsymbol{\psi}\in\{0,1\}^n} M_{\boldsymbol{\psi}} + O_{\varepsilon}(X^{2-1/(4m)+\varepsilon}).$$

Here, c_K' is a positive constant depending only on K and the implied constant in the error term depends only on $K, \mathfrak{r}, \mathscr{D}, \mathfrak{W}, \mathfrak{F}, f, \varepsilon$.

Proof. Recall that $\mathscr{C} = \{\mathfrak{r}_1,\dots,\mathfrak{r}_h\}$ is a fixed system of integral representatives of the class group of K. By possibly modifying $\mathfrak{W}_0$, we may assume that $\mathfrak{r}_1\cdots\mathfrak{r}_h \mid \mathfrak{W}$.

Since $\mathscr{D} \subseteq K_{\infty}^2 = \mathbb{R}^{2m}$ is a cartesian product of balls in $K_v^2 = \mathbb{R}^{2m_v}$, it is clear that the sets $\mathscr{D}_{\boldsymbol{\psi}}(X;\mathbf{v}) \subseteq \mathbb{R}^{2m}$, for $X > 0$ and $\mathbf{v} \in \mathbb{R}^n$ are fibres of a definable family with parameters $(X,\mathbf{v},\boldsymbol{\psi}) \in \mathbb{R}^{1+2n}$ in the o-minimal structure $\mathbb{R}_{\mathrm{alg}}$ of semialgebraic sets. Moreover, $\mathscr{D}_{\boldsymbol{\psi}}(X;\mathbf{v}) \subseteq X^{1/m}\mathscr{D}$, which is contained in a zero-centred ball of radius $\ll X^{1/m}$.

Injecting the estimate of Lemma 2.3 into Lemma 4.1 yields the desired main term. The sum over the error terms in Lemma 2.3 can be bounded by $\ll E_0 + \cdots + E_{m-1}$, where, for $0 \leqslant j \leqslant m-1$,

$$E_j := \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}}''',\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''\in\mathscr{I}_K^n \\ \mathfrak{N}\mathfrak{a}_i\mathfrak{b}_i'''\leqslant Y_i \\ \mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''\leqslant\sqrt{Y_i} \\ \mathfrak{a}_i\mathfrak{b}_i'''\mathfrak{c}_i''\mathfrak{c}_i'''+\mathfrak{W}=\mathscr{O}_K \\ \mathfrak{b}_i'''+\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''=\mathscr{O}_K}} \prod_{i=1}^n \frac{1}{\mathfrak{N}\mathfrak{a}_i\mathfrak{b}_i'''} \sum_{\substack{\lambda_i \bmod \mathfrak{d}_i \\ \mathfrak{d}_i\mid F_i(\lambda_i,1)}} \forall i \frac{X^{1+j/m+\varepsilon}}{\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}',\lambda)^m\lambda^{(m+1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}',\lambda)^j\}}.$$

Let us bound E_j. The Chinese remainder theorem allows us to separate the sum over $\lambda_i \bmod \mathfrak{d}_i$ into a sum over $\lambda_i \bmod \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''$ and a sum over $\lambda_i \bmod \mathfrak{b}_i'''$. Write $\mathfrak{d}'' := \prod_{i=1}^n \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''$ and let $\lambda' \equiv \lambda_i \bmod \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''$ for all i. Since $\Lambda(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}',\lambda) \subset \Lambda(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')$, we obtain

$$\lambda^{(i)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}',\lambda) \geqslant \lambda^{(i)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

for all $1 \leqslant i \leqslant 2m$. This allows us to sum over $\underline{\mathfrak{b}}'''$, obtaining the estimate

$$E_j \ll \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''\in\mathscr{I}_K^n \\ \mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''\leqslant\sqrt{Y_i} \\ \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''+\mathfrak{W}=\mathscr{O}_K}} \prod_{i=1}^{n} \frac{1}{\mathfrak{N}\mathfrak{a}_i} \sum_{\substack{\lambda_i \bmod \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''' \ \forall i \\ \mathfrak{a}_i\mathfrak{c}_i\mathfrak{c}_i'''\mid F_i(\lambda_i,1)}} \frac{X^{1+j/m+\varepsilon}}{\min_{1\leqslant q\leqslant h}\{\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')^m\lambda^{(m+1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')^j\}}. \tag{4.8}$$

Let

$$Y := \prod_{i=1}^{n} Y_i.$$

Each first successive minimum $\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')$ is attained by a point $\mathbf{v}=(v_1,v_2)$ in the lattice $\Lambda(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda') \subseteq \mathscr{O}_K^2 \subset K_\infty^2$, of euclidean norm bounded by

$$\|\mathbf{v}\| \ll \mathfrak{N}\mathfrak{d}''^{1/(2m)} \ll Y^{1/(4m)},$$

due to Lemma 2.1. Let

$$E_j(\mathbf{v}) := \sum_{q=1}^{h} \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{c}}'',\underline{\mathfrak{c}}'''\in\mathscr{I}_K^n \\ \mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''\leqslant\sqrt{Y_i} \\ \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''+\mathfrak{W}=\mathscr{O}_K}} \prod_{i=1}^{n} \frac{1}{\mathfrak{N}\mathfrak{a}_i} \sum_{\substack{\lambda_i \bmod \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''' \ \forall i \\ \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''\mid F_i(\lambda_i,1) \\ \mathbf{v}\in\Lambda(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda') \\ \|\mathbf{v}\|=\lambda^{(1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')}} \frac{1}{\|\mathbf{v}\|^m\lambda^{(m+1)}(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')^j}.$$

Sorting the expression in (4.8) by the first successive minimum, we see that

$$E_j \ll \sum_{\substack{\mathbf{v}\in\mathscr{O}_K^2\smallsetminus\{0\} \\ \|\mathbf{v}\|\ll Y^{1/(4m)}}} X^{1+j/m+\varepsilon}E_j(\mathbf{v}).$$

For $\mathbf{v}\in\mathscr{O}_K^2$ to be an element of the lattice $\Lambda(\mathfrak{r}_q,\mathfrak{r}_q\mathfrak{d}'',\lambda')$, it is necessary that $v_1 \equiv \lambda' v_2 \bmod \mathfrak{d}''$, so in particular $v_1 \equiv \lambda_i v_2 \bmod \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''$ and hence $\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''' \mid F_i(\mathbf{v})$. This allows us to conclude that

$$E_j(\mathbf{v}) \ll \frac{\prod_{i=1}^{n}\mathfrak{N}(F_i(\mathbf{v}))^\varepsilon}{\|\mathbf{v}\|^{m+j}} \ll \frac{X^\varepsilon}{\|\mathbf{v}\|^{m+j}},$$

whenever $F_i(\mathbf{v})\neq 0$ holds for all $1\leqslant i\leqslant n$. The sum of $E_j(\mathbf{v})$ over all such $\mathbf{v}$ is

$$\ll X^{1+j/m+\varepsilon} \sum_{\substack{\mathbf{v}\in\mathscr{O}_K^2\smallsetminus\{0\} \\ \|\mathbf{v}\|\ll Y^{1/(4m)}}} \frac{1}{\|\mathbf{v}\|^{m+j}} \ll X^{1+j/m+\varepsilon}Y^{1/2(1-(m+j)/(2m))} \ll X^{1+j/m+\varepsilon}Y^{1/4-j/(4m)}.$$

Recalling our assumption that $c(\mathfrak{F})\leqslant 3$ and the fact that $Y\ll X^{c(\mathfrak{F})}$, we see that this error term does not exceed

$$X^{2-1/4+j/(4m)+\varepsilon} \leqslant X^{2-1/(4m)+\varepsilon}.$$

It remains to bound the sum over those $\mathbf{v}$ for which $F_k(\mathbf{v})=0$ for some $1\leqslant k\leqslant n$. Since $F_k(s,t)$ is irreducible, this necessarily implies that $F_k(s,t)$ is linear and since the forms

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

$F_i(s,t)$ are pairwise coprime, we conclude that $F_i(\mathbf{v}) \neq 0$ for all $i \neq k$. This allows us to bound the number of $\mathfrak{a}_i, \mathfrak{c}_i'', \mathfrak{c}_i''', \lambda_i$, for $i \neq k$, as before by $\prod_{i\neq k} \mathfrak{N}(F_i(\mathbf{v}))^{\varepsilon} \ll X^{\varepsilon}$. Writing temporarily

$$F_k(s,t) = as - bt,$$

with $a \neq 0$ and $a \mid \mathfrak{W}_0 \mid \mathfrak{W}$, we see that the equality $F_k(\lambda_k, 1) \equiv 0 \bmod \mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k'''$ is equivalent to $\lambda_k = a^{-1}b \bmod \mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k'''$. Moreover, $\Lambda(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{d}'', \lambda') \subseteq \Lambda(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k)$. We may thus bound

$$E_j(\mathbf{v}) \ll \sum_{q=1}^{h} \sum_{\substack{\mathfrak{a}_k, \mathfrak{c}_k'', \mathfrak{c}_k''' \in \mathscr{I}_K \\ \mathfrak{N}\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''' \ll \sqrt{X} \\ \mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''' + \mathfrak{W} = \mathscr{O}_K}} \frac{X^{\varepsilon}}{\|\mathbf{v}\|^m \lambda^{(m+1)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k)^j}.$$

Let $\alpha_1, \dots, \alpha_m$ be $\mathbb{Z}$-linearly independent elements of $\mathfrak{r}_q$ with $\|\alpha_i\| \asymp \lambda^{(i)}(\mathfrak{r}_q) \asymp 1$ and let $\beta_1, \dots, \beta_m$ be $\mathbb{Z}$-linearly independent in $\mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k'''$ with $\|\beta_i\| \asymp \lambda^{(i)}(\mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''') \asymp \mathfrak{N}(\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''')^{1/m}$. To estimate the successive minima, we used Minkowski's second theorem and the fact that $\lambda^{(1)}(\mathfrak{a}) \gg \mathfrak{N}\mathfrak{a}^{1/m}$ holds for any $\mathfrak{a} \in \mathscr{I}_K$ (see, e.g. [28, Lemma 5] or [15, Lemma 5.1]). This provides us with the linearly independent lattice points

$$\begin{pmatrix} b\alpha_1 \\ a\alpha_1 \end{pmatrix}, \dots, \begin{pmatrix} b\alpha_m \\ a\alpha_m \end{pmatrix}, \quad \begin{pmatrix} \beta_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \beta_m \\ 1 \end{pmatrix} \in \Lambda(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k).$$

The first m of these have norm $\asymp 1$, whereas the latter m ones have norm $\asymp \mathfrak{N}(\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''')^{1/m}$, so the product of their norms is $\asymp \mathfrak{N}(\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''') \asymp \det\Lambda(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k)$. Using again Minkowski's second theorem, this shows that the successive minima of $\Lambda(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k)$ satisfy

$$\begin{aligned} \lambda^{(1)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k), \dots, \lambda^{(m)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k) &\asymp 1, \\ \lambda^{(m+1)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k), \dots, \lambda^{(2m)}(\mathfrak{r}_q, \mathfrak{r}_q\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''', \lambda_k) &\asymp \mathfrak{N}(\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''')^{1/m}. \end{aligned}$$

As a result, we obtain the bound

$$E_j(\mathbf{v}) \ll \sum_{\substack{\mathfrak{a}_k, \mathfrak{c}_k'', \mathfrak{c}_k''' \in \mathscr{I}_K \\ \mathfrak{N}\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''' \ll \sqrt{X}}} \frac{X^{\varepsilon}}{\|\mathbf{v}\|^m \mathfrak{N}(\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''')^{j/m}}.$$

In addition, we observe that any $\mathbf{v} = (v_1, v_2) \in \mathscr{O}_K^2$ with $F_k(\mathbf{v}) = 0$ is uniquely determined by v_2. Consequently,

$$\begin{aligned} \sum_{\substack{\mathbf{v}\in\mathscr{O}_K^2 \smallsetminus \{0\} \\ \|\mathbf{v}\| \ll Y^{1/(4m)} \\ F_k(\mathbf{v})=0}} X^{1+j/m+\varepsilon} E_j(\mathbf{v}) &\ll X^{1+j/m+\varepsilon} \sum_{\substack{v_2 \in \mathscr{O}_K \smallsetminus \{0\} \\ \|v_2\| \ll Y^{1/(4m)}}} \frac{1}{\|v_2\|^m} \sum_{\substack{\mathfrak{a}_k, \mathfrak{c}_k'', \mathfrak{c}_k''' \in \mathscr{I}_K \\ \mathfrak{N}\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''' \ll \sqrt{X}}} \frac{1}{\mathfrak{N}(\mathfrak{a}_k\mathfrak{c}_k''\mathfrak{c}_k''')^{j/m}} \\ &\ll X^{1+j/m+\varepsilon} (\log Y)^m X^{1/2(1-j/m)+\varepsilon} \ll X^{3/2+j/(2m)+\varepsilon} \ll X^{2-1/(2m)+\varepsilon}. \end{aligned}$$

□

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

4.3. Controlling the main term

Let $\rho_i(\mathfrak{a}) := \rho_{(F_i,G_i)}(\mathfrak{a})$, as defined prior to Lemma 2.6 and moreover recall (3.4).

Lemma 4.3. *The arithmetic factor in the definition of M_Ψ decomposes as follows:*

$$\sum_{\substack{\lambda_i \bmod \mathfrak{d}_i\\ \mathfrak{d}_i \mid F_i(\lambda_i,1)}} \left(\frac{G_i(\lambda_i,1)}{\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i'''}\right) = \rho_i(\mathfrak{a}_i\mathfrak{c}_i'')\tau_{F_i}(\mathfrak{b}_i''')\rho_i(\mathfrak{c}_i''').$$

Proof. Recall that we set $\mathfrak{d}_i = \mathfrak{a}_i\mathfrak{b}_i\mathfrak{c}_i''\mathfrak{c}_i'''$, and that the ideals $\mathfrak{a}_i\mathfrak{c}_i''$, $\mathfrak{b}_i'''$, $\mathfrak{c}_i'''$ are coprime in pairs due to (4.5)–(4.7). The Chinese remainder theorem, jointly with multiplicativity properties of the Jacobi symbol, yields

$$\sum_{\substack{\lambda_i \bmod \mathfrak{d}_i\\ \mathfrak{d}_i \mid F_i(\lambda_i,1)}} \left(\frac{G_i(\lambda_i,1)}{\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i'''}\right) = \sum_{\substack{\lambda_i' \bmod \mathfrak{a}_i\mathfrak{c}_i''\\ \mathfrak{a}_i\mathfrak{c}_i'' \mid F_i(\lambda_i',1)}} \left(\frac{G_i(\lambda_i',1)}{\mathfrak{a}_i \mathfrak{c}_i''}\right) \sum_{\substack{\lambda_i'' \bmod \mathfrak{b}_i'''\\ \mathfrak{b}_i''' \mid F_i(\lambda_i'',1)}} 1 \sum_{\substack{\lambda_i''' \bmod \mathfrak{c}_i'''\\ \mathfrak{c}_i''' \mid F_i(\lambda_i''',1)}} \left(\frac{G_i(\lambda_i''',1)}{\mathfrak{c}_i'''}\right).$$

□

Letting $\mathfrak{B} := \mathfrak{d}\mathfrak{W}\prod_{j=1}^n \mathfrak{a}_j\mathfrak{c}_j'''$, we define $M(\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''')$ as

$$\sum_{\substack{\mathfrak{b}_1''' \in \mathscr{I}_K\\ \mathfrak{N}\mathfrak{b}_1''' < Y_1/\mathfrak{N}\mathfrak{a}_1\\ \mathfrak{b}_1''' + \mathfrak{B} = \mathscr{O}_K}} \frac{\mathbf{1}_\eta(\mathfrak{b}_1''')f(\mathfrak{b}_1''')\tau_{F_1}(\mathfrak{b}_1''')}{\mathfrak{N}\mathfrak{b}_1'''} \sum_{\substack{\mathfrak{b}_2''' \in \mathscr{I}_K\\ \mathfrak{N}\mathfrak{b}_2''' < Y_2/\mathfrak{N}\mathfrak{a}_2\\ \mathfrak{b}_2''' + \mathfrak{B}\mathfrak{b}_1''' = \mathscr{O}_K}} \frac{\mathbf{1}_\eta(\mathfrak{b}_2''')f(\mathfrak{b}_2''')\tau_{F_2}(\mathfrak{b}_2''')}{\mathfrak{N}\mathfrak{b}_2'''} \cdots \sum_{\substack{\mathfrak{b}_n''' \in \mathscr{I}_K\\ \mathfrak{N}\mathfrak{b}_n''' < Y_n/\mathfrak{N}\mathfrak{a}_n\\ \mathfrak{b}_n''' + \mathfrak{B}\prod_{j<n}\mathfrak{b}_j''' = \mathscr{O}_K}} \frac{\mathbf{1}_\eta(\mathfrak{b}_n''')f(\mathfrak{b}_n''')\tau_{F_n}(\mathfrak{b}_n''')}{\mathfrak{N}\mathfrak{b}_n'''},$$

a definition that makes the succeeding equality valid,

$$M_\Psi = \sum_{\substack{\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''' \in \mathscr{I}_K^n\\ (4.5),(4.7)}} \omega_\Psi(X; (\mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i''')_{i=1}^n))M(\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''') \prod_{i=1}^n \frac{f(\mathfrak{a}_i)\mathbf{1}_\eta(\mathfrak{a}_i\mathfrak{c}_i''')\rho_i(\mathfrak{a}_i\mathfrak{c}_i'')\rho_i(\mathfrak{c}_i''')}{\mathfrak{N}\mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''}. \tag{4.9}$$

Let us bring into play the multiplicative function γ, supported on square-free ideals, by letting $\gamma(\mathfrak{p}) := 0$ for $\mathfrak{p} \mid \mathfrak{W}$ and in the remaining case, $\mathfrak{p} \nmid \mathfrak{W}$, we define

$$\gamma(\mathfrak{p}) := -1 + \left(1 + \frac{(1+\eta(\mathfrak{p}))f(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}} \sum_{i=1}^n \tau_{F_i}(\mathfrak{p})\right)^{-1}.$$

Including enough small prime ideals in the factorisation of $\mathfrak{W}_0$, we can ensure that $\mathbf{1}_\gamma \in \mathscr{U}_K$.

Lemma 4.4. *Let $\gamma_0 := \prod_{\mathfrak{p}\nmid\mathfrak{W}}(1+\gamma(\mathfrak{p}))^{-1}$ and suppose that $\mathfrak{N}\mathfrak{a}_i \leqslant Y_i$ for all $1 \leqslant i \leqslant n$. Then*

$$M(\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''') = \gamma_0\mathbf{1}_\gamma(\mathfrak{d})\prod_{i=1}^n \mathbf{1}_\gamma(\mathfrak{a}_i)\mathbf{1}_\gamma(\mathfrak{c}_i''') + O_\varepsilon\left(X^\varepsilon \max_{i=1,\dots,n}\left\{\frac{\mathfrak{N}\mathfrak{a}_i}{Y_i}\right\}\right).$$

The implied constant is independent of $\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}'''$, $\mathfrak{d}$, and X.

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Proof. The bound bestowed upon f by (1.3) shows that each sum over $\mathfrak{b}_i'''$ in $M(\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''')$ forms an absolutely convergent series. We may complete the summation step by step for $i = n, n-1, \dots, 1$. The bounds

$$\mathbf{1}_\eta(\mathfrak{b}_i'''),\ |\mathfrak{N}\mathfrak{b}_i''' f(\mathfrak{b}_i''')|,\ \tau_{F_i}(\mathfrak{b}_i''') \ll_\varepsilon \mathfrak{N}\mathfrak{b}_i'''^{\varepsilon} \quad \text{and} \quad \sum_{\mathfrak{N}\mathfrak{b}_i > Y_i/\mathfrak{N}\mathfrak{a}_i} \frac{\mathfrak{N}\mathfrak{b}_i^{\varepsilon}}{\mathfrak{N}\mathfrak{b}_i^2} \ll X^{\varepsilon} \frac{\mathfrak{N}\mathfrak{a}_i}{Y_i}$$

reveal that the error introduced by this process is $\ll_\varepsilon X^{\varepsilon} \max\{\mathfrak{N}\mathfrak{a}_i/Y_i : i = 1, \dots, n\}$, thus acquiring the main term

$$\sum_{\substack{\mathfrak{b}_1''' \in \mathscr{I}_K \\ \mathfrak{b}_1''' + \mathfrak{B} = \mathscr{O}_K}} \frac{\mathbf{1}_\eta(\mathfrak{b}_1''') f(\mathfrak{b}_1''') \tau_{F_1}(\mathfrak{b}_1''')}{\mathfrak{N}\mathfrak{b}_1'''} \cdots \sum_{\substack{\mathfrak{b}_n''' \in \mathscr{I}_K \\ \mathfrak{b}_n''' + \mathfrak{B} \prod_{j<n} \mathfrak{b}_j''' = \mathscr{O}_K}} \frac{\mathbf{1}_\eta(\mathfrak{b}_n''') f(\mathfrak{b}_n''') \tau_{F_n}(\mathfrak{b}_n''')}{\mathfrak{N}\mathfrak{b}_n'''}.$$

Grouping all n-tuples $\underline{\mathfrak{b}}'''$ according to the value of $\mathfrak{b} := \prod_{i=1}^n \mathfrak{b}_i'''$ and letting

$$g(\mathfrak{b}) := \mathbf{1}_\eta(\mathfrak{b}) \sum_{\substack{\underline{\mathfrak{b}}''' \in \mathscr{I}_K^n \\ \mathfrak{b} = \mathfrak{b}_1''' \dots \mathfrak{b}_n''' \\ \mathfrak{b}_i''' + \mathfrak{b}_j''' = \mathscr{O}_K \ \forall i \neq j}} \prod_{i=1}^n f(\mathfrak{b}_i''') \tau_{F_i}(\mathfrak{b}_i'''),$$

the main term becomes

$$\sum_{\substack{\mathfrak{b} \in \mathscr{I}_K \\ \mathfrak{b} + \mathfrak{B} = \mathscr{O}_K}} \frac{g(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}} = \prod_{\mathfrak{p} \nmid \mathfrak{B}} \left(1 + \frac{g(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) = \prod_{\mathfrak{p} \nmid \mathfrak{B}} (1 + \gamma(\mathfrak{p}))^{-1}.$$

Here, we used the observation that $1 + \gamma(\mathfrak{p}) = \left(1 + \frac{g(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right)^{-1}$ holds for all $\mathfrak{p} \nmid \mathfrak{W}$. □

We may now plant Lemma 4.4 into (4.9) to show that M_Ψ equals

$$\gamma_0 \mathbf{1}_\gamma(\mathfrak{d}) \sum_{\substack{\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''' \in \mathscr{I}_K^n \\ (4.5), (4.7)}} \omega_\Psi(X; (\mathfrak{N}\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i''')_{i=1}^n)) \prod_{i=1}^n \frac{f(\mathfrak{a}_i) \mathbf{1}_\eta(\mathfrak{a}_i \mathfrak{c}_i''') \rho_i(\mathfrak{a}_i \mathfrak{c}_i'') \rho_i(\mathfrak{c}_i''') \mathbf{1}_\gamma(\mathfrak{a}_i) \mathbf{1}_\gamma(\mathfrak{c}_i''')}{\mathfrak{N}\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i'''}$$

up to an error of size

$$\ll_\varepsilon X^{\varepsilon} \sum_{\substack{\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''' \in \mathscr{I}_K^n \\ (4.5), (4.7)}} \omega_\Psi(X; (\mathfrak{N}\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i''')_{i=1}^n)) \left(\prod_{i=1}^n \frac{|f(\mathfrak{a}_i)| \mathbf{1}_\eta(\mathfrak{a}_i \mathfrak{c}_i''') \rho_i(\mathfrak{a}_i \mathfrak{c}_i'') \rho_i(\mathfrak{c}_i''')}{\mathfrak{N}\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i'''} \right) \max_{1 \leqslant i \leqslant n} \left\{ \frac{\mathfrak{N}\mathfrak{a}_i}{Y_i} \right\}.$$

Using the inequalities $Y_i \gg X$, $\max\{\mathbf{1}_\eta(\mathfrak{a}), \rho_i(\mathfrak{a}), f(\mathfrak{a})\mathfrak{N}\mathfrak{a}\} \ll_\varepsilon \mathfrak{N}\mathfrak{a}^{\varepsilon}$,

$$\max_{1 \leqslant i \leqslant n} \{\mathfrak{N}\mathfrak{a}_i\} \leqslant \prod_{i=1}^n \mathfrak{N}\mathfrak{a}_i, \quad \text{and} \quad \omega_\Psi(X; (\mathfrak{N}\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i''')_{i=1}^n)) \leqslant \mathrm{vol}(X^{1/m} \mathscr{D}) \ll X^2,$$

we find that the sum in the error term is

$$\ll_\varepsilon X^{1+\varepsilon} \sum_{\substack{\underline{\mathfrak{a}}, \underline{\mathfrak{c}}'', \underline{\mathfrak{c}}''' \in \mathscr{I}_K^n \\ \mathfrak{N}\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i''' \leqslant \sqrt{Y_i}}} \prod_{i=1}^n \frac{1}{\mathfrak{N}\mathfrak{a}_i \mathfrak{c}_i'' \mathfrak{c}_i'''} \ll_\varepsilon X^{1+\varepsilon}.$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

To analyse the main term further, we define on $\mathscr{I}_K$ the multiplicative functions

$$g_i(\mathfrak{c}_i) := \sum_{\substack{\mathfrak{a}_i,\mathfrak{c}_i'',\mathfrak{c}_i'''\in\mathscr{I}_K\\ \mathfrak{a}_i\mathfrak{c}_i''\mathfrak{c}_i'''=\mathfrak{c}_i\\ \mathfrak{c}_i''\in\langle\mathfrak{a}_i\rangle\\ \mathfrak{a}_i+\mathfrak{c}_i'''=\mathscr{O}_K}} f(\mathfrak{a}_i)\mathbf{1}_\eta(\mathfrak{a}_i\mathfrak{c}_i''')\rho_i(\mathfrak{a}_i\mathfrak{c}_i'')\rho_i(\mathfrak{c}_i''')\mathbf{1}_\gamma(\mathfrak{a}_i\mathfrak{c}_i'''),$$

which satisfy, for prime ideals $\mathfrak{p}$ and positive integers k,

$$g_i(\mathfrak{p}^k) = \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{Z}_{\geqslant 0}\\ \alpha+\beta+\gamma=k\\ \beta>0\Rightarrow\alpha>0\\ \alpha\gamma=0}} f(\mathfrak{p}^\alpha)\mathbf{1}_\eta(\mathfrak{p}^{\alpha+\gamma})\rho_i(\mathfrak{p}^{\alpha+\beta})\rho_i(\mathfrak{p}^\gamma)\mathbf{1}_\gamma(\mathfrak{p}^{\alpha+\gamma}).$$

Since f is supported on square-free ideals the only candidate values for (α,β,γ) are $(0,0,k)$ and $(1,k-1,0)$. Let us mention that the group structure of $\mathscr{U}_K$ provides us with a function δ fulfilling $\mathbf{1}_f\cdot\mathbf{1}_\eta\cdot\mathbf{1}_\gamma=\mathbf{1}_\delta$. We are therefore afforded with the equality $g_i(\mathfrak{p}^k)=\rho_i(\mathfrak{p}^k)\mathbf{1}_\delta(\mathfrak{p}^k)$, which, upon introducing

$$g(\underline{\mathfrak{c}}) := \prod_{i=1}^n \frac{g_i(\mathfrak{c}_i)}{\mathfrak{N}\mathfrak{c}_i}\cdot\begin{cases}1 & \text{if } \mathfrak{c}_i+\mathfrak{c}_j=\mathscr{O}_K\ \forall i\neq j,\\ 0 & \text{otherwise,}\end{cases} \tag{4.10}$$

makes the ensuing estimate available,

$$M_\Psi = \gamma_0\mathbf{1}_\gamma(\mathfrak{d}) \sum_{\substack{\underline{\mathfrak{c}}\in\mathscr{I}_K^n\\ \mathfrak{N}\mathfrak{c}_i<\sqrt{Y_i}\\ \mathfrak{c}_i+\mathfrak{d}\mathfrak{W}=\mathscr{O}_K}} \omega_\Psi(X;\mathfrak{N}\underline{\mathfrak{c}})g(\underline{\mathfrak{c}}) + O_\varepsilon(X^{1+\varepsilon}).$$

4.4. Excluding small conjugates

For $X, Z\geqslant 1$, $w\in\Omega_\infty$ and a separable form $F\in K_w[s,t]$, let

$$\mathscr{B}_{F,w}(X;Z) := \{(s,t)\in K_w^2 : |s|_w, |t|_w\leqslant X^{1/m} \text{ and } |F(s,t)|_w\leqslant Z^{1/m}\}.$$

Lemma 4.5. *We have*

$$\operatorname{vol}\mathscr{B}_{F,w}(X;Z) \ll_F \begin{cases}(XZ)^{m_w/m} & \text{if } 1\leqslant\deg F<3,\\ Z^{2m_w/(m\deg(F))} & \text{if } \deg F\geqslant 3.\end{cases}$$

Proof. First, let $\deg F=1$. The bound claimed in the lemma is obvious if F is proportional to t. If F is not proportional to t, then the linear transformation $L: K_w^2\to K_w^2$ given by $L(s,t)=(F(s,t),t)$ is an isomorphism and thus

$$\operatorname{vol}\mathscr{B}_{F,w}(X;Z)\ll_F \operatorname{vol}\{(s,t)\in K_w^2 : |s|_w\leqslant Z^{1/m}, |t|_w\ll X^{1/m}\}\ll (XZ)^{m_w/m}.$$

Next, let us consider the case where F is a quadratic form equivalent to s^2-t^2 over K_w. Then we can find an invertible linear transformation $L: K_w^2\to K_w^2$ with $F(L(s,t))=st$,

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

and hence

$$\begin{aligned}\operatorname{vol}\mathscr{B}_{F,w}(X;Z) &\ll_F \operatorname{vol}\{(s,t)\in K_w^2 : |s|_w, |t|_w \ll_F X^{1/m}, |st|_w \leqslant Z^{1/m}\}\\ &\ll_F X^{m_w/m} + Z^{m_w/m}\log(X) \ll (XZ)^{m_w/m}.\end{aligned}$$

If F is a quadratic form equivalent to s^2+t^2 over $K_w=\mathbb{R}$, then we get

$$\operatorname{vol}\mathscr{B}_{F,w}(X;Z) \ll_F \operatorname{vol}\{(s,t)\in\mathbb{R}^2 : s^2+t^2\leqslant Z^{1/m}\} \ll Z^{1/m} \ll (XZ)^{m_w/m}.$$

It remains to consider the case where $\deg F\geqslant 3$. In this case, F is the product of at least three non-proportional linear factors in $\mathbb{C}$ and therefore

$$V_{w,F} := \operatorname{vol}\{(s,t)\in K_w^2 : |F(s,t)|_w \leqslant 1\} < \infty.$$

We procure the validity of

$$\operatorname{vol}\mathscr{B}_{F,w}(X;Z) \ll \operatorname{vol}(Z^{1/(m\deg(F))}V_{w,F}) \ll_F Z^{2m_w/(m\deg(F))}. \qquad \square$$

For any non-constant separable form $F\in K_w[s,t]$, let

$$\mathscr{D}_{F,w}^{<}(X) := \{(s,t)\in X^{1/m}\mathscr{D} : |F(s_w,t_w)|_w < 1\}.$$

Using Lemma 4.5 validates the next estimate

$$\operatorname{vol}\mathscr{D}_{F,w}^{<}(X) \ll_{\mathscr{D}} X^{2-2m_w/m}\cdot\operatorname{vol}\mathscr{B}_{F,w}(X,1) \ll_F X^{2-2m_w/m}\cdot X^{m_w/m},$$

thus providing the proof of the next lemma.

Lemma 4.6. *For $X\geqslant 1$ we have $\operatorname{vol}\mathscr{D}_{F,w}^{<}(X) \ll_{\mathscr{D},F} X^{2-m_w/m}$.*

For every $w\in\Omega_\infty$ we choose a finite set $\mathscr{H}_w$ of forms in $K_w[s,t]$, whose absolute values we want to prevent from becoming too small. For all $w\in\Omega_\infty$, the set $\mathscr{H}_w$ contains s, t, and the forms F_i for $1\leqslant i\leqslant n$. Additionally, for each form F_i that is of degree 2 and reducible over K_w, we choose a factorisation $F_i = G_{i,w}H_{i,w}$ and also include $G_{i,w}$, $H_{i,w}$ in $\mathscr{H}_w$.

Recall the definition of $\mathscr{D}_{\boldsymbol{\psi}}(X;\mathbf{v})$ in (4.2). For $\boldsymbol{\psi}\in\{0,1\}^n$ and $\mathbf{v}\in\mathbb{R}^n$, let

$$\mathscr{D}_{\boldsymbol{\psi}}^{*}(X;\mathbf{v}) := \{(s,t)\in\mathscr{D}_{\boldsymbol{\psi}}(X;\mathbf{v}) : |H_w(s_w,t_w)|_w \geqslant 1\ \forall w\in\Omega_\infty,\ \forall H_w\in\mathscr{H}_w\}$$

and

$$\omega_{\boldsymbol{\psi}}^{*}(X;\mathbf{v}) := \operatorname{vol}\mathscr{D}_{\boldsymbol{\psi}}^{*}(X;\mathbf{v}). \tag{4.11}$$

We obtain that

$$|\omega_{\boldsymbol{\psi}}(X;\mathbf{v}) - \omega_{\boldsymbol{\psi}}^{*}(X;\mathbf{v})| \leqslant \sum_{w\in\Omega_\infty}\sum_{H_w\in\mathscr{H}_w}\operatorname{vol}\mathscr{D}_{H_w,w}^{<}(X)$$

and thus

$$\omega_{\boldsymbol{\psi}}(X;\mathbf{v}) = \omega_{\boldsymbol{\psi}}^{*}(X;\mathbf{v}) + O(X^{2-1/m}).$$

We can now bring into play the entity

$$\mathscr{M}_{\boldsymbol{\psi}} := \sum_{\substack{\underline{\mathfrak{c}}\in\mathscr{I}_K^n\\ \mathfrak{N}\mathfrak{c}_i<\sqrt{Y_i}\\ \mathfrak{c}_i+\mathfrak{d}\mathfrak{W}=\mathscr{O}_K}} \omega_{\boldsymbol{\psi}}^{*}(X;\mathfrak{N}\underline{\mathfrak{c}})g(\underline{\mathfrak{c}}), \tag{4.12}$$

something which instantly permits us to infer the asymptotic relationship

$$M_{\boldsymbol{\psi}} = \gamma_0\mathbf{1}_\gamma(\mathfrak{d})\mathscr{M}_{\boldsymbol{\psi}} + O_\varepsilon(X^{2-1/m+\varepsilon}). \tag{4.13}$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

4.5. Volume computations

In this section we provide estimates of the correct order of magnitude regarding the volumes $\omega_{\Psi}^{*}(X;\mathbf{v})$ appearing in $\mathscr{M}_{\Psi}$. The assumption $c(\mathfrak{F}) \leqslant 3$ will not be used. Let us write $d_i := \deg F_i$ for $1 \leqslant i \leqslant n$ and consider, for $q \in \mathbb{N}$ and $T > 0$, the real integral

$$I_q(T) := \int_{\substack{x_1,\dots,x_q \geqslant 1 \\ x_1 \cdots x_q < T}} 1 \, dx_1 \cdots dx_q.$$

One can show that in the range $T \geqslant 1$ the equality

$$I_q(T) = (-1)^q + \sum_{j=1}^{q} \frac{(-1)^{q-j}}{(j-1)!} T (\log T)^{j-1}$$

holds via induction coupled with $I_{q+1}(T) = \int_1^T I_q(T/x)\, dx$, thus furnishing the succeeding result.

Lemma 4.7. *There is a polynomial $P_q(T) \in \mathbb{Q}[T]$ of degree $q-1$ and with leading coefficient $1/(q-1)!$ such that for $T \geqslant 1$ one has $I_q(T) = T P_q(\log T) + (-1)^q$.*

For $Z \geqslant 1$ and $1 \leqslant i \leqslant n$ with $\deg F_i(s,t) \geqslant 3$ we let

$$\mathscr{D}_i^{*}(Z) := \{(s,t) \in K_\infty^2 : |F_i(s_w,t_w)|_w \geqslant 1 \text{ for all } w \in \Omega_\infty \text{ and } \mathfrak{N}(F_i(s,t)) < Z\}$$

and

$$\mathscr{D}_s^{*}(Z) := \{s \in K_\infty : |s_w|_w \geqslant 1 \text{ for all } w \in \Omega_\infty \text{ and } \mathfrak{N}(s) < Z\}.$$

Letting $\Omega' \subseteq \Omega_\infty$ be a set of real places, we write $\Omega'' := \Omega_\infty \smallsetminus \Omega'$ and subsequently define $\mathscr{D}_{\Omega',\Omega''}^{*}(Z)$ through

$$\left\{ ((s_w,t_w)_{w\in\Omega'}, (s_w)_{w\in\Omega''}) \in \prod_{w\in\Omega'} K_w^2 \times \prod_{w\in\Omega''} K_w : \begin{array}{l} \left|s_w^2 + t_w^2\right|_w \geqslant 1 \text{ for all } w \in \Omega', \\ |s_w|_w \geqslant 1 \text{ for all } w \in \Omega'', \\ \displaystyle\prod_{w\in\Omega'} \left|s_w^2 + t_w^2\right|_w^{m_w} \cdot \prod_{w\in\Omega''} |s_w|^{m_w} < Z \end{array} \right\}.$$

Lemma 4.8. *Let $q := |\Omega_\infty|$. There are positive constants $c_i, c_s, c_{\Omega',\Omega''}$, such that*

$$\begin{aligned} \operatorname{vol} \mathscr{D}_s^{*}(Z) &= c_s I_q(Z), \\ \operatorname{vol} \mathscr{D}_i^{*}(Z) &= c_i I_q(Z^{2/d_i}), \\ \operatorname{vol} \mathscr{D}_{\Omega',\Omega''}^{*}(Z) &= c_{\Omega',\Omega''} I_q(Z). \end{aligned}$$

Proof. Let $C = \prod_{v\in\Omega_\infty}(a_v,b_v] \subseteq [0,\infty)^{\Omega_\infty}$, $V_{w,i} := \operatorname{vol}\{(s,t) \in K_w^2 : |F_i(s,t)|_w \leqslant 1\} < \infty$ and consider the measurable functions

$$\begin{aligned} \Phi_i &: K_\infty^2 \to [0,\infty)^{\Omega_\infty}, \quad (s,t) \mapsto (|F_i(s_w,t_w)|_w^{2m_w/d_i})_{w\in\Omega_\infty}, \\ \Phi_s &: K_\infty \to [0,\infty)^{\Omega_\infty}, \quad s \mapsto (|s_w|_w^{m_w})_{w\in\Omega_\infty}, \end{aligned}$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

$$\Phi_{\Omega',\Omega''} : \prod_{w\in\Omega'} K_w^2 \cdot \prod_{w\in\Omega''} K_w \to [0,\infty)^{\Omega_\infty}, ((s_w,t_w)_w,(s_w)_w)$$
$$\mapsto ((|s^2+t^2|_w^{m_w})_{w\in\Omega'}, (|s|_w^{m_w})_{w\in\Omega''}).$$

By homogeneity we see that $\operatorname{vol}\Phi_i^{-1}(C)$ equals

$$\prod_{w\in\Omega_\infty} \operatorname{vol}\{(s_w,t_w)\in K_w^2 : a_w < |F_i(s_w,t_w)|_v^{2m_w/d_i} \leqslant b_w\}$$
$$= \prod_{w\in\Omega_\infty} V_{w,i}(b_w-a_w) = \left(\prod_{w\in\Omega_\infty} V_{w,i}\right)\cdot \operatorname{vol} C.$$

In like manner, letting $V_{w,s} := \operatorname{vol}\{s\in K_w : |s|_w \leqslant 1\} < \infty$ and

$$V_{w,s^2+t^2} := \operatorname{vol}\{(s,t)\in K_w^2 : |s^2+t^2|_w \leqslant 1\},$$

we observe that V_{w,s^2+t^2} is finite if w is a real place and

$$\operatorname{vol}\Phi_s^{-1}(C) = \left(\prod_{w\in\Omega_\infty} V_{w,s}\right)\cdot \operatorname{vol} C,$$
$$\operatorname{vol}\Phi_{\Omega',\Omega''}^{-1}(C) = \left(\prod_{w\in\Omega'} V_{w,s^2+t^2}\cdot \prod_{w\in\Omega''} V_{w,s}\right)\cdot \operatorname{vol} C.$$

This shows that the pushforward measures $\Phi_{i,*}(\mathrm{vol})$, $\Phi_{s,*}(\mathrm{vol})$, $\Phi_{\Omega',\Omega'',*}(\mathrm{vol})$ are constant multiples of the Lebesgue measure on $[0,\infty)^{\Omega_\infty}$. Let $\mathscr{H}(T)$ be given by

$$\left\{(x_w)_{w\in\Omega_\infty} : x_w \geqslant 1 \text{ for all } w \text{ and } \prod_{w\in\Omega_\infty} x_w < T\right\}.$$

Then $\operatorname{vol}\mathscr{H}(T) = I_q(T)$, $\mathscr{D}_i^*(Z) = \Phi_i^{-1}(\mathscr{H}(Z^{2/d_i}))$, $\mathscr{D}_s^*(Z) = \Phi_s^{-1}(\mathscr{H}(Z))$, as well as $\mathscr{D}_{\Omega',\Omega''}^*(Z) = \Phi_{\Omega',\Omega''}^{-1}(\mathscr{H}(Z))$, from which the lemma flows immediately. □

For $1\leqslant i\leqslant n$, $1\leqslant Z_1\leqslant Z_2$ and $X\geqslant 1$ let

$$\mathscr{R}_i(X;Z_1,Z_2) := \left\{(s,t)\in X^{1/m}\mathscr{D} : \begin{array}{l} |H_w(s_w,t_w)|_w \geqslant 1 \ \forall w\in\Omega_\infty \ \forall H_w\in\mathscr{H}_w \\ Z_1\leqslant \Re(F_i(s,t)) < Z_2\end{array}\right\}.$$

Lemma 4.9. *Denoting $|\Omega_\infty|$ by q we have*

$$\operatorname{vol}\mathscr{R}_i(X;Z_1,Z_2) \ll \begin{cases} X(I_q(Z_2)-I_q(Z_1)) & \text{if } d_i=1,\\ I_q(Z_2^{2/d_i})-I_q(Z_1^{2/d_i}) & \text{if } d_i\geqslant 3.\end{cases}$$

If $d_i=2$, let Ω' be the set of real $w\in\Omega_\infty$ for which F_i is irreducible over K_w and define $\Omega'' := \Omega_\infty\smallsetminus\Omega'$. Then $\operatorname{vol}\mathscr{R}_i(X;Z_1,Z_2)$ is bounded by

$$\ll \int_{\substack{t_w\in K_w \ \forall w\in\Omega'' \\ |t_w|_w\geqslant 1 \ \forall w\in\Omega''}} \left(I_q\left(Z_2\prod_{w\in\Omega''}|t_w|_w^{-m_w}\right) - I_q\left(Z_1\prod_{w\in\Omega''}|t_w|_w^{-m_w}\right)\right)\prod_{w\in\Omega''} dt_w.$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

Proof. We deploy Lemma 4.8 throughout the proof. Assume first that $d_i \geqslant 3$. Then

$$\operatorname{vol}\mathscr{R}_i(X; Z_1, Z_2) \ll \operatorname{vol}(\mathscr{D}_i^*(Z_2) \smallsetminus \mathscr{D}_i^*(Z_1)) = c_i(I_q(Z_2^{2/d_i}) - I_q(Z_1^{2/d_i})).$$

Next, assume that $d_i = 1$. Since F_i is not proportional to t, the linear transformation $L : K^2 \to K^2$ given by $L(s,t) = (F_i(s,t), t)$ is invertible and provides us with the estimate

$$\begin{aligned}\operatorname{vol}\mathscr{R}_i(X; Z_1, Z_2) &\ll \operatorname{vol}\{(s,t) \in K_\infty^2 : |s_w|_w \geqslant 1,\ |t_w|_w \ll X^{1/m}\ \forall w \text{ and } Z_1 < \mathfrak{N}(s) \leqslant Z_2\}\\ &\ll X \operatorname{vol}(\mathscr{D}_s^*(Z_2) \smallsetminus \mathscr{D}_s^*(Z_1)) \ll X(I_q(Z_2) - I_q(Z_1)).\end{aligned}$$

We are left with the case $d_i = 2$. For each $w \in \Omega'$, there is a linear transformation $L_w : K_w^2 \to K_w^2$ such that $F_i(L_w(s,t)) = s^2 + t^2$. For $w \in \Omega''$, we have $F_i = G_{i,w}H_{i,w}$ for linear forms $G_{i,w}, H_{i,w} \in \mathscr{H}_w$. The linear map $K_w^2 \to K_w^2$, $(s,t) \mapsto (G_{i,w}(s,t), H_{i,w}(s,t))$ has an inverse L_w because F_i is separable. We combine all these linear maps to an invertible $\mathbb{R}$-linear map $L = (L_w)_{w\in\Omega_\infty} : K_\infty^2 \to K_\infty^2$, which we apply to obtain

$$\begin{aligned}\operatorname{vol}\mathscr{R}_i(X; Z_1, Z_2) &\ll \operatorname{vol}\left\{(s,t) \in K_\infty^2 : \begin{array}{rl} \left|s_w^2 + t_w^2\right|_w \geqslant 1 & \text{for all } w \in \Omega' \\ |s_w|_w\,, |t_w|_w \geqslant 1 & \text{for all } w \in \Omega'' \\ Z_1 < \displaystyle\prod_{w\in\Omega'} \left|s_w^2 + t_w^2\right|_w^{m_w} \prod_{w\in\Omega''} |s_w t_w|_w^{m_w} \leqslant Z_2 & \end{array}\right\}\\ &= \int_{\substack{t_w\in K_w\ \forall w\in\Omega'' \\ |t_w|_w\geqslant 1\ \forall w\in\Omega''}} \operatorname{vol}\left(\mathscr{D}^*_{\Omega',\Omega''}\left(Z_2 \prod_{w\in\Omega''} |t_w|_w^{-m_w}\right) \smallsetminus \mathscr{D}^*_{\Omega',\Omega''}\left(Z_1 \prod_{w\in\Omega''} |t_w|_w^{-m_w}\right)\right) \prod_{w\in\Omega''} dt_w.\end{aligned}$$

□

Lemma 4.10. *For each $\boldsymbol{\psi} \in \{0,1\}^n$ we have*

$$\omega_{\boldsymbol{\psi}}^*(X; (1,\dots,1)) = X^2 \operatorname{vol}(\mathscr{D}) + O_\varepsilon(X^{2-1/m} + X^{3/2+\varepsilon}).$$

Proof. Let us begin by observing that

$$\begin{aligned}|X^2 \operatorname{vol}(\mathscr{D}) - \omega_{\boldsymbol{\psi}}^*(X; (1,\dots,1))| &\ll \sum_{w\in\Omega_\infty} \sum_{H_w\in\mathscr{H}_w} \operatorname{vol}\mathscr{D}^<_{H_w,w}(X) + \sum_{i=1}^n \operatorname{vol}\mathscr{R}_i(X; 1, \sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i)\\ &\ll X^{2-1/m} + \sum_{i=1}^n \operatorname{vol}\mathscr{R}_i(X; 1, \sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i).\end{aligned}$$

We now use Lemmas 4.9 and 4.7 to estimate the $\operatorname{vol}\mathscr{R}_i(X; 1, \sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i)$. If $d_i = 1$, then $\operatorname{vol}\mathscr{R}_i(X; 1, \sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i) \ll X\sqrt{Y_i}^{1+\varepsilon} \ll X^{3/2+\varepsilon}$, while, if $d_i \geqslant 3$, we acquire

$$\operatorname{vol}\mathscr{R}_i(X; 1, \sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i) \ll Y_i^{1/d_i+\varepsilon} \ll X^{1+\varepsilon}.$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

In the remaining case, $d_i = 2$, we get

$$\begin{aligned}\operatorname{vol}\mathscr{R}_i(X;1,\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i) &\ll \int_{\substack{t_w\in K_w\ \forall w\in\Omega''\\ 1\leqslant|t_w|_w\ll\sqrt{Y_i}\ \forall w\in\Omega''}} \frac{\sqrt{Y_i}^{1+\varepsilon}}{\prod_{w\in\Omega''}|t_w|_w^{m_w}}\prod_{w\in\Omega''}dt_w \ll \sqrt{Y_i}^{1+\varepsilon}\\ &\ll X^{1+\varepsilon}.\end{aligned}$$

□

For a function $\omega:\mathbb{R}^n\to\mathbb{R}$ and $1\leqslant i\leqslant n$, we write $\Delta_i\omega(\mathbf{v}) := \omega(\mathbf{v}+\mathbf{e}_i)-\omega(\mathbf{v})$, where $\mathbf{e}_i$ is the ith vector in the standard basis of $\mathbb{R}^n$.

Lemma 4.11. *Let $\boldsymbol{\psi}\in\{0,1\}^n$, $1\leqslant i\leqslant n$ and $\mathbf{v}\in\mathbb{R}^n$ be given such that $v_j\in[0,\infty)$ for all $j\neq i$. Then $\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v})$, considered as a function of v_i, is non-increasing and satisfies*

$$\Delta_i\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v}) \ll X^{1+\varepsilon}\begin{cases} X^{1/2} & \text{if } d_i=1,\\ v_i^{2/d_i-1} & \text{otherwise,}\end{cases} \tag{4.14}$$

in the interval $1\leqslant v_i\leqslant\sqrt{Y_i}$, with the implied constant independent of $\mathbf{v}$ and X.

Proof. Monotonicity is obvious. Let us prove the estimate (4.14). If $\psi_i=0$, then $\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v})$ is constant in v_i. Let $\psi_i=1$, then

$$|\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v}+\mathbf{e}_i)-\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v})| \ll \mathscr{R}_i(X;\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i v_i,\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i(v_i+1)).$$

Using Lemma 4.9 and the mean value theorem to bound the latter quantity, we obtain in the case $d_i=1$ that, for some $\tilde{v}_i\in[v_i,v_i+1]$,

$$\begin{aligned}\Delta_i\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v}) &\ll \frac{\partial}{\partial V}(XI_q(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i V))|_{V=\tilde{v}_i} \ll X\sqrt{Y_i}\frac{\partial}{\partial V}(VP_q(\log(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i V)))|_{V=\tilde{v}_i}\\ &\ll X^{3/2+\varepsilon}.\end{aligned}$$

When $d_i\geqslant 3$, we get

$$\begin{aligned}\Delta_i\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v}) &\ll \frac{\partial}{\partial V}I_q((\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i V)^{2/d_i})|_{V=\tilde{v}_i}\\ &\ll Y_i^{1/d_i}\frac{\partial}{\partial V}V^{2/d_i}P_q(2/d_i\log(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i V))|_{V=\tilde{v}_i}\\ &\ll X_i^{1+\varepsilon}\tilde{v}_i^{2/d_i-1}.\end{aligned}$$

When $d_i=2$, the quantity $\Delta_i\omega^*_{\boldsymbol{\psi}}(X;\mathbf{v})$ is

$$\begin{aligned}\ll \int_{\substack{t_w\in K_w\ \forall w\in\Omega''\\ |t_w|_w\geqslant 1\ \forall w\in\Omega''}} &I_q\left(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i(v_i+1)\prod_{w\in\Omega''}|t_w|_w^{-m_w}\right)\\ &-I_q\left(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i v_i\prod_{w\in\Omega''}|t_w|_w^{-m_w}\right)\prod_{w\in\Omega''}dt_w.\end{aligned}$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

The integrand is zero, unless $\prod_{w\in\Omega''}|t_w|_w^{m_w}\leqslant\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_i(v_i+1)$. In that case, the mean value theorem allows us to find for any $(t_w)_w$ a number $\tilde{v}_i\in(v_i,v_i+1)$, such that the integrand is

$$\begin{aligned}&\frac{\partial}{\partial V}\left(I_q\left(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_iV\prod_{w\in\Omega''}|t_w|_w^{-m_w}\right)\right)\Bigg|_{V=\tilde{v}_i}\\&=\frac{\partial}{\partial V}\left(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_iV\prod_{w\in\Omega''}|t_w|_w^{-m_w}\,P_q\left(\log\left(\sqrt{Y_i}\mathfrak{N}\mathfrak{W}_iV\prod_{w\in\Omega''}|t_w|_w^{-m_w}\right)\right)\right)\Bigg|_{V=\tilde{v}_i}\\&\ll\sqrt{Y_i}X^{\varepsilon}\prod_{w\in\Omega''}|t_w|_w^{-m_w}\ll X^{1+\varepsilon}\prod_{w\in\Omega''}|t_w|_w^{-m_w}.\end{aligned}$$

This shows that $\Delta_i\omega_{\psi}^*(X;\mathbf{v})\ll X^{1+\varepsilon}$, which concludes our proof. □

4.6. The ending moves towards Theorem 1.2

We are now ready to estimate the sum $\mathscr{M}_{\psi}$ that was introduced in (4.12).

Lemma 4.12. *Let $\delta:=\max_{1\leqslant i\leqslant n}\{4+8m\deg F_i\}$. For any $0\leqslant i\leqslant n$, there are functions $\gamma^{(i)},\delta_1^{(i)},\dots,\delta_i^{(i)}\in\mathscr{Z}_K$, and a positive constant $\mu^{(i)}$, such that*

$$\begin{aligned}\mathscr{M}_{\psi}=\mu^{(i)}\mathbf{1}_{\gamma^{(i)}}(\mathfrak{d})\sum_{\substack{\mathfrak{N}\mathfrak{c}_1\leqslant\sqrt{Y_1}\\ \mathfrak{c}_1+\mathfrak{d}\mathfrak{W}=\mathscr{O}_K}}\frac{\rho_1(\mathfrak{c}_1)\mathbf{1}_{\delta_1^{(i)}}(\mathfrak{c}_1)}{\mathfrak{N}\mathfrak{c}_1}\cdots\sum_{\substack{\mathfrak{N}\mathfrak{c}_i\leqslant\sqrt{Y_i}\\ \mathfrak{c}_i+\mathfrak{c}_1\cdots\mathfrak{c}_{i-1}\mathfrak{d}\mathfrak{W}=\mathscr{O}_K}}\frac{\rho_i(\mathfrak{c}_i)\mathbf{1}_{\delta_i^{(i)}}(\mathfrak{c}_i)}{\mathfrak{N}\mathfrak{c}_i}\\\times\omega_{\psi}^*(X;(\mathfrak{N}\mathfrak{c}_1,\dots,\mathfrak{N}\mathfrak{c}_i,1,\dots,1))+O_{\varepsilon}(\mathfrak{N}\mathfrak{d}^{\varepsilon}X^{2-1/\delta+\varepsilon}).\end{aligned}\tag{4.15}$$

Proof. For $i=n$ our lemma holds with vanishing error term by the definition of g in (4.10). We proceed by backward induction from i to $i-1$. Lemma 2.5 provides the existence of $\beta^{(i)}>0$ and $\gamma'^{(i)}\in\mathscr{Z}_K$ such that, for all $U\geqslant1$,

$$\sum_{\substack{\mathfrak{N}\mathfrak{c}_i\leqslant U\\ \mathfrak{c}_i+\mathfrak{c}_1\cdots\mathfrak{c}_{i-1}\mathfrak{d}\mathfrak{W}=\mathscr{O}_K}}\frac{\rho_i(\mathfrak{c}_i)\mathbf{1}_{\delta_i^{(i)}}(\mathfrak{c}_i)}{\mathfrak{N}\mathfrak{c}_i}=\beta^{(i)}\mathbf{1}_{\gamma'^{(i)}}(\mathfrak{c}_1\cdots\mathfrak{c}_{i-1}\mathfrak{d})+O_{\varepsilon}(\mathfrak{N}(\mathfrak{c}_1\cdots\mathfrak{c}_{i-1}\mathfrak{d})^{\varepsilon}U^{-1/(2\lambda)+\varepsilon}),\tag{4.16}$$

where $\lambda=1+2m\deg F_i$. Indeed, the hypotheses of Lemma 2.5 are satisfied by Lemma 2.6 and Hensel's lemma, once we ensure that $\mathfrak{W}_0$, and hence $\mathfrak{W}$, is divisible by enough small prime ideals.

We write $\omega(\theta):=\omega_{\psi}^*(X;(\mathfrak{N}\mathfrak{c}_1,\dots,\mathfrak{N}\mathfrak{c}_{i-1},\theta,1,\dots,1))$. Assume first that $\deg F_i=1$. In this case, the bounds (4.14) and (4.16) allow us to apply Lemma 2.7 with $A=1/(2\lambda)$, $B=0$,

$$M\ll_{\varepsilon}\mathfrak{N}(\mathfrak{c}_1\cdots\mathfrak{c}_{i-1}\mathfrak{d})^{\varepsilon}X^{\varepsilon}\quad\text{and}\quad Q\ll_{\varepsilon}X^{3/2+\varepsilon},$$

https://doi.org/10.1017/S1474748017000469 Published online by Cambridge University Press

thus leading to

$$\sum_{\substack{\mathfrak{N}\mathfrak{c}_i \leqslant \sqrt{Y_i} \\ \mathfrak{c}_i + \mathfrak{c}_1 \cdots \mathfrak{c}_{i-1}\mathfrak{d}\mathfrak{W} = \mathscr{O}_K}} \frac{\rho_i(\mathfrak{c}_i)\mathbf{1}_{\delta_i^{(i)}}(\mathfrak{c}_i)}{\mathfrak{N}\mathfrak{c}_i}\omega(\mathfrak{N}\mathfrak{c}_i) = \beta^{(i)}\mathbf{1}_{\gamma^{(i)\prime}}(\mathfrak{c}_1 \cdots \mathfrak{c}_{i-1}\mathfrak{d})\omega(1) + O_\varepsilon(\mathfrak{N}(\mathfrak{c}_1 \cdots \mathfrak{c}_{i-1}\mathfrak{d})^\varepsilon X^{2-1/(4\lambda)+\varepsilon}). \quad (4.17)$$

If $\deg F_i \geqslant 2$, we use Lemma 2.7 with the same bounds for M, A and

$$Q \ll_\varepsilon X^{1+\varepsilon}, \quad B = 1 - 2/(\deg F_i)$$

to obtain an estimate identical to (4.17). Injecting this in (4.15) proves our claim for $i-1$. □

The case $i = 0$ of the last lemma shows that $\mathscr{M}_\psi = \mu^{(0)}\mathbf{1}_{\gamma^{(0)}}(\mathfrak{d})\operatorname{vol}\mathscr{D}X^2 + O(\mathfrak{N}\mathfrak{d}^\varepsilon X^{2-1/\delta+\varepsilon})$. Conjuring up (4.13) and Lemma 4.2 completes the undertaking of validating Theorem 1.2.

Acknowledgements. We are grateful to Tim Browning and Roger Heath-Brown for helpful suggestions relating to the proof of Theorem 1.2. The authors would furthermore like to thank Daniel Loughran for useful discussions concerning the presentation of our results. A part of this work was completed while the second author was supported by London's Mathematical Society's *150th Anniversary Postdoctoral Mobility Grant* to visit Göttingen University, the hospitality of which is gratefully acknowledged.

References

1. J. Brüdern, Daniel's twists of Hooley's delta function, in *Contributions in Analytic and Algebraic Number Theory*, Springer Proc. Math., Volume 9, pp. 31–82 (Springer, New York, 2012).
2. F. Barroero and M. Widmer, Counting lattice points and o-minimal structures, *Int. Math. Res. Not. IMRN* **2014**(18) (2014), 4932–4957.
3. S. Daniel, On the divisor-sum problem for binary forms, *J. Reine Angew. Math.* **507** (1999), 107–129.
4. K. Destagnol, La conjecture de Manin pour certaines surfaces de Châtelet, *Acta Arith.* **174**(1) (2016), 31–97.
5. W. Duke, J. B. Friedlander and H. Iwaniec, A quadratic divisor problem, *Invent. Math.* **115**(2) (1994), 209–217.
6. R. de la Bretèche and T. D. Browning, Sums of arithmetic functions over values of binary forms, *Acta Arith.* **125**(3) (2006), 291–304.
7. R. de la Bretèche and T. D. Browning, Binary linear forms as sums of two squares, *Compos. Math.* **144**(6) (2008), 1375–1402.
8. R. de la Bretèche and T. D. Browning, Le problème des diviseurs pour des formes binaires de degré 4, *J. Reine Angew. Math.* **646** (2010), 1–44.
9. R. de la Bretèche and T. D. Browning, Manin's conjecture for quartic del Pezzo surfaces with a conic fibration, *Duke Math. J.* **160**(1) (2011), 1–69.
10. R. de la Bretèche and T. D. Browning, Binary forms as sums of two squares and Châtelet surfaces, *Israel J. Math.* **191**(2) (2012), 973–1012.
11. R. de la Bretèche and G. Tenenbaum, Oscillations localisées sur les diviseurs, *J. Lond. Math. Soc. (2)* **85**(3) (2012), 669–693.

12. R. de la Bretèche and G. Tenenbaum, Sur la conjecture de Manin pour certaines surfaces de Châtelet, *J. Inst. Math. Jussieu* **12**(4) (2013), 759–819.
13. É. Fouvry, E. Kowalski and P. Michel, On the exponent of distribution of the ternary divisor function, *Mathematika* **61**(1) (2015), 121–144.
14. J. Franke, Y. I. Manin and Y. Tschinkel, Rational points of bounded height on Fano varieties, *Invent. Math.* **95**(2) (1989), 421–435.
15. C. Frei, Counting rational points over number fields on a singular cubic surface, *Algebra Number Theory* **7**(6) (2013), 1451–1479.
16. C. Frei, D. Loughran and E. Sofos, Rational points of bounded height on general conic bundle surfaces, preprint, 2016, arXiv:1609.04330.
17. C. Frei and E. Sofos, Counting rational points on smooth cubic surfaces, *Math. Res. Lett.* **23** (2016), 127–143.
18. G. Greaves, On the divisor-sum problem for binary cubic forms, *Acta Arith.* **17** (1970), 1–28.
19. D. R. Heath-Brown, Linear relations amongst sums of two squares, in *Number Theory and Algebraic Geometry*, London Mathematical Society Lecture Note Series, Volume 303, pp. 133–176 (Cambridge University Press, Cambridge, 2003).
20. C. Hooley, On the number of divisors of a quadratic polynomial, *Acta Math.* **110** (1963), 97–114.
21. C. Hooley, On a new technique and its applications to the theory of numbers, *Proc. Lond. Math. Soc. (3)* **38**(1) (1979), 115–151.
22. A. J. Irving, The divisor function in arithmetic progressions to smooth moduli, *Int. Math. Res. Not. IMRN* (15) (2015), 6675–6698.
23. J Kollár and M. Mella, Quadratic families of elliptic curves and unirationality of degree 1 conic bundles, *Amer. J. Math.* **139**(4) (2017), 915–936.
24. L. Matthiesen, Correlations of the divisor function, *Proc. Lond. Math. Soc. (3)* **104**(4) (2012), 827–858.
25. L. Matthiesen, Linear correlations amongst numbers represented by positive definite binary quadratic forms, *Acta Arith.* **154**(3) (2012), 235–306.
26. L. Matthiesen, Correlations of representation functions of binary quadratic forms, *Acta Arith.* **158**(3) (2013), 245–252.
27. C. J. Moreno, Advanced analytic number theory: *L*-functions, in *Mathematical Surveys and Monographs*, Volume 115 (American Mathematical Society, Providence, RI, 2005).
28. D. Masser and J. D. Vaaler, Counting algebraic numbers with large height II, *Trans. Amer. Math. Soc.* **359**(1) (2007), 427–445 (electronic).
29. H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, vol. 97 (Cambridge University Press, Cambridge, 2007).
30. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, second ed (The Clarendon Press, Oxford University Press, New York, 1986). Edited and with a preface by D. R. Heath-Brown.
31. D. I. Tolev, On the remainder term in the circle problem in an arithmetic progression, *Tr. Mat. Inst. Steklova* **276** (2012), 266–279.
32. A. J. Wilkie, o-minimal structures, *Astérisque* **326** (2009), Exp. No. 985, vii, 131–142 (2010), Séminaire Bourbaki. Vol. 2007/2008.