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A uniqueness result in an inverse hyperbolic problem with analyticity

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We prove the uniqueness for the inverse problem of determining a coefficient $q(x)$ in $\partial_t^2 u(x,t) = \Delta u(x,t) - q(x)u(x,t)$ for $x \in R^n$ and $t > 0$, from observations of $u|_{\Gamma\times(0,T)}$ and the normal derivative $\frac{\partial u}{\partial \nu}|_{\Gamma\times(0,T)}$ where Γ is an arbitrary C^∞-hypersurface. Our main result asserts the uniqueness of q over R^n provided that $T > 0$ is sufficiently large and q is analytic near Γ and outside a ball. The proof depends on Fritz John's global Holmgren theorem and the uniqueness by a Carleman estimate.

1 Introduction

We consider a Cauchy problem for a hyperbolic equation:

$$\partial_t^2 u(x,t) = \Delta u(x,t) - q(x)u(x,t), \quad x = (x_1,\ldots,x_n) \in R^n, 0 < t < T, \tag{1.1}$$

$$u(x, T/2) = a(x), \quad \partial_t u(x, T/2) = 0, \qquad x \in R^n. \tag{1.2}$$

Here and henceforth we set

$$\partial_t u = \frac{\partial u}{\partial t}, \quad \partial_t^2 u = \frac{\partial^2 u}{\partial t^2},$$

$$\partial_i u = \frac{\partial u}{\partial x_i}, \quad \partial_i^2 u = \frac{\partial^2 u}{\partial x_i^2}, \quad 1 \leqslant i \leqslant n, \quad \Delta = \sum_{i=1}^n \partial_i^2.$$

Throughout this paper, we fix $a \in C^\infty(R^n)$ and we denote the solution u to (1.1) and (1.2) by $u(q) = u(q)(x,t)$, because it is unique if exists. For example, if $q \in C^\infty(R^n)$, then we know that there exists a unique solution $u(q) \in C^\infty(R^n \times [0,T))$ (e.g. Bers *et al.* [4]).

Our problem is an inverse problem of determining a coefficient $q = q(x)$ from measurements of $u(x,t)$. The coefficient $q(x)$ describes a physical property of the medium (e.g. the elastic modulus in Hooke's law) in (1.1), and our inverse problem is the determination of such a property. We will formulate our inverse problem.

Let $\Gamma \subset R^n$ be a hypersurface of class C^∞ and $T > 0$ be fixed. We consider *uniqueness in an inverse hyperbolic problem.* Is the correspondence

$$\left\{ u|_{\Gamma\times(0,T)}, \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma\times(0,T)} \right\} \longleftrightarrow q$$

injective?

Here $\nu = \nu(x)$ denotes the outward unit normal vector to Γ at x and $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ on Γ, and we observe u and $\frac{\partial u}{\partial \nu}$ on $\Gamma \times (0, T)$. From the theoretical point of view, the uniqueness is the first step in our inverse problem.

For a similar kind of inverse problems, we refer to Bukhgeim [5], Bukhgeim & Klibanov [7], Imanuvilov & Yamamoto [8, 9], Isakov [10, 11], Khaĭdarov [12], Klibanov [13] and Yamamoto [18], which are based on Carleman estimates. As for the determination of a damping coefficient p in $\partial_t^2 u = \Delta u + p(x)\partial_t u$ for $t > 0$, we refer to Bukhgeim *et al.* [6], which relies on the Carleman estimate too. By their results, we can prove the uniqueness in a convex domain D_Γ whose boundary contains Γ, provided that $|a| > 0$ on $\overline{D_\Gamma}$. In particular, if Γ is flat and the principal term of the hyperbolic equation is constant, then the existing results do not yield any uniqueness outside Γ. For general $q \in C^2(R^n)$, no uniqueness is known over R^n. In the flat case, under the condition that unknown coefficients are constant outside a fixed bounded domain, Rakesh [14] and Romanov [16] show the uniqueness in inverse problems with the formulations by Dirichlet to Neumann map [14], where boundary measurements are repeated infinitely.

In this paper, we establish the uniqueness over R^n under the assumption that q is analytic outside $B_R \equiv \{x; |x| < R\}$ and $\Gamma \subset R^n \setminus \overline{B_R}$. Our assumption is satisfied if q is a constant function outside B_R, and such homogeneity of the medium far from 0 is physically acceptable. This assumption generalizes the settings in Rakesh [14] and Romanov [16]. We notice that we do not assume any analyticity inside B_R.

As for the uniqueness within analyticity or piecewise analyticity, we refer to Anikonov [1], Anikonov *et al.* [2] and Berezanskĭi [3]. In particular, in Anikonov *et al.* [2], we consider a hyperbolic equation

$$\begin{aligned}&\partial_t^2 u(x_1, x_2, x_3, t) = \Delta u(x_1, x_2, x_3, t)\\&-q(x_1, x_2, x_3)u(x_1, x_2, x_3, t) + G(x_1, x_2, x_3, t), \quad (x_1, x_2) \in R^2,\ x_3 > 0,\ t > 0\end{aligned}$$

with the zero initial condition and the boundary condition

$$(\partial_3 u)(x_1, x_2, 0, t) = 0, \qquad (x_1, x_2) \in R^2,\ t > 0,$$

and discuss an inverse problem of determining $q = q(x_1, x_2, x_3)$ from boundary observation $u(x_1, x_2, 0, t)$ where (x_1, x_2) varies over a suitable domain and $T > 0$ is sufficiently large, provided that q belongs to a set of piecewise analytic functions: $\{q \in C^2$; there exist $0 \equiv \eta_0 < \eta_1 < \cdots < \eta_\ell < \eta_{\ell+1} \equiv \infty$ such that q is analytic in$\{(x_1, x_2, x_3); (x_1, x_2) \in R^2, \eta_{j-1} < x_3 < \eta_j, 1 \leqslant j \leqslant \ell + 1\}$. In Anikonov *et al.* [2], the uniqueness in determining q is proved to be true in a neighbourhood of $\{x_3 = 0\}$ where the external force G is positive. In Berezanskĭi [3], the uniqueness is proved in determining a potential in a Schrödinger equation within a similar set of piecewise analytic functions from all the eigenvalues and boundary values of the eigenfunctions. In many models, the mediums are assumed to be stratified, that is, coefficients under consideration are piecewise constant, or piecewise analytic in a more general case. Therefore, as a set of unknown coefficients for inverse problems, the class of piecewise analytic functions is reasonable.

The case of flat Γ appears when we consider observations on a flat ground for inverse problems in geophysical prospecting and such a case is important.

This paper is composed of four sections; the main result in § 2, the proof in § 3, and concluding remarks in § 4.

2 Main result

First we note that there exists a unique solution $u(q) \in C^2(R^n \times [0, T))$ to (1.1) and (1.2) for $a \in C^\infty(R^n)$ and $q \in C^\infty(R^n)$ (e.g. Bers *et al.* [4]).

Let us set

$$B_r = \{x \in R^n; |x| < r\}$$

for $r > 0$. We fix $R > 0$ and set

$$\mathcal{Q} = \{q \in C(R^n); q \text{ is analytic in } R^n \setminus \overline{B_R}\}. \tag{2.1}$$

Let $\Gamma \subset R^n$ be a hypersurface of class C^∞ such that

$$\Gamma \subset R^n \setminus \overline{B_R}. \tag{2.2}$$

We set

$$d_c(x, g) = \text{the shortest length among all the continuous curves in } R^n \setminus \overline{B_R} \text{ connecting } x \text{ and } g \tag{2.3}$$

for $x, g \in R^n \setminus B_R$. We note that $d_c(x, g) = |x - g|$ if the segment connecting x and g is in $R^n \setminus \overline{B_R}$.

We are ready to state our main result.

Theorem 2.1 *We assume that $a \in C^\infty(R^n)$ satisfies*

$$|a| > 0 \qquad \text{on} \quad \overline{B_R} \cup \Gamma \tag{2.4}$$

and

$$T > 2\Big(R + \sup_{x \in \partial B_R} \inf_{g \in \Gamma} d_c(x, g)\Big). \tag{2.5}$$

If $u(q_1), u(q_2) \in C^3(R^n \times [0, T))$ satisfy

$$u(q_1) = u(q_2), \quad \frac{\partial u(q_1)}{\partial \nu} = \frac{\partial u(q_2)}{\partial \nu} \qquad \text{on} \quad \Gamma \times (0, T) \tag{2.6}$$

for $q_1, q_2 \in \mathcal{Q}$, then

$$q_1 = q_2 \qquad \text{in} \quad R^n. \tag{2.7}$$

Remark In the case $\Gamma = \{x; |x| = R_1\}$ with $R_1 > R > 0$, the condition (2.5) is reduced to $T > 2R_1$ which can be interpreted from the viewpoint of travelling time and coincides with the condition in earier work [6, 8, 9, 11, 13, 18].

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3 Proof

First step We prove the uniqueness of q_1 and q_2 outside B_R. By the argument in Bers *et al.* [4], we can take a small open set $\mathscr{O}$ with $\mathscr{O} \supset \Gamma$ such that

$$\overline{\mathscr{O}} \subset R^n \setminus \overline{B_R} \tag{3.1}$$

and

$$u(q_1), u(q_2) \in C^\infty(\overline{\mathscr{O}} \times [0, T)). \tag{3.2}$$

Then we will show

Lemma 1 *We consider*

$$\begin{cases} \partial_t^2 u(x,t) = \Delta u(x,t) - q(x)u(x,t), & x \in \mathscr{O},\ 0 < t < T, \\ u(x, T/2) = a(x), \quad \partial_t u(x, T/2) = 0, & x \in \mathscr{O}, \\ u(x,t) = \varphi(x,t), \quad \frac{\partial u}{\partial \nu}(x,t) = \psi(x,t), & x \in \Gamma,\ 0 < t < T. \end{cases} \tag{3.3}$$

Here u, q, a, φ and ψ are of class C^∞ in $(x,t) \in \overline{\mathscr{O}} \times [0,T]$. We assume that $|a| > 0$ on $\overline{\mathscr{O}}$. Then all the derivatives of q in x on Γ can be determined uniquely by a, φ and ψ.

Proof of Lemma 1 We set $u_2(x) = (\partial_t^2 u)(x, T/2)$. Then, by (3.3), we have

$$q(x) = \frac{-u_2(x) + \Delta a(x)}{a(x)}, \qquad x \in \mathscr{O} \tag{3.4}$$

and

$$\Delta u(x,t) = \partial_t^2 u(x,t) + \frac{-u_2(x) + \Delta a(x)}{a(x)} u(x,t), \qquad x \in \mathscr{O},\ 0 < t < T. \tag{3.5}$$

By $u \in C^\infty(\overline{\mathscr{O}} \times [0,T])$, we obtain

$$u_2(x) = \left(\partial_t^2 \varphi\right)(x, T/2), \qquad x \in \Gamma$$

and we see from (3.5) that $\partial_i \partial_j u_2(x, T/2)$, $x \in \Gamma$, $1 \leqslant i,j \leqslant n$, are uniquely calculated by means of φ, ψ and a. Therefore

$$q(x) = \frac{-\partial_t^2 \varphi(x, T/2) + \Delta a(x)}{a(x)}, \qquad x \in \Gamma$$

and $\partial_i \partial_j u_2(x, T/2)$, $x \in \Gamma$, $1 \leqslant i,j \leqslant n$, are uniquely determined. We can continue this argument in view of $u \in C^\infty(\overline{\mathscr{O}} \times [0,T])$. Thus the proof of the lemma is complete.

By (3.2), we apply Lemma 1 to $u(q_1)$ and $u(q_2)$, so that all the derivatives of q_1 and q_2 are equal each other on $\Gamma \subset R^n \setminus \overline{B_R}$. Noting that $q_1, q_2 \in \mathscr{Q}$, we obtain

$$q_1(x) = q_2(x), \qquad x \in R^n \setminus B_R. \tag{3.6}$$

Second step In this step, we prove that

$$u(q_1)(x,t) = u(q_2)(x,t),\ \ \frac{\partial u(q_1)}{\partial \nu}(x,t) = \frac{\partial u(q_2)}{\partial \nu}(x,t), \qquad x \in \partial B_R,\ \ t_0 \leqslant t \leqslant T - t_0, \tag{3.7}$$

where t_0 satisfies

$$2R < T - 2t_0. \tag{3.8}$$

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This will be proved by a global version of the classical Holmgren theorem. Such generalization is not trivial and we will give the proof for completeness. As for generalization of the Holmgren theorem, we refer to Tataru [17]. In any case, for the proof of (3.7) with (3.8), some work is required.

Here $\frac{\partial}{\partial \nu}$ denotes the normal derivative on ∂B_R. Setting $v = u(q_1) - u(q_2)$ in $(R^n \setminus \overline{B_R}) \times (0, T)$, in view of (3.6), we have

$$\partial_t^2 v(x,t) = \Delta v(x,t) - q_1(x) v(x,t), \qquad x \in R^n \setminus \overline{B_R},\ 0 < t < T, \tag{3.9}$$

and

$$v(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0, \qquad x \in \Gamma,\ 0 < t < T. \tag{3.10}$$

We show

Lemma 2 *Let* $x_0, \xi \in R^n \setminus \overline{B_R}$ *satisfy*

$$\bigcup_{0 \leqslant \eta \leqslant 1} \{x; |x - (\eta\xi + (1-\eta)x_0)| < \rho\} \subset R^n \setminus \overline{B_R}. \tag{3.11}$$

Let $T > 0,\ \rho > 0,\ \mu \geqslant 0$ *satisfy*

$$T > 2(\rho + \mu) + 2|x_0 - \xi|. \tag{3.12}$$

We assume

$$v(x,t) = 0, \qquad |x - \xi| < \rho, \quad \mu < t < T - \mu. \tag{3.13}$$

Then

$$\begin{aligned} v(x,t) = 0 \qquad & \text{if } |x - x_0| < \rho \text{ and} \\ & \rho + \mu + |x_0 - \xi| \leqslant t \leqslant T - \rho - \mu - |x_0 - \xi|. \end{aligned} \tag{3.14}$$

Proof of Lemma 2 Without loss of generality, we may assume that $\xi = 0$. We can take a domain D and a hyperplane π such that $\xi = 0 \in D$ and D is on π,

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } D \times (\mu, T - \mu) \tag{3.15}$$

and

$$\overrightarrow{0x_0} \perp \pi. \tag{3.16}$$

Henceforth we set

$$x = (x_1, x_2, \ldots, x_n) = (x_1, x'), \quad x_0 = \left(x_0^1, x_0^2, \ldots, x_0^n\right) = \left(x_0^1, x_0'\right).$$

Since the Laplacian Δ is invariant with respect to a rotation and parallel displacement in x, we may assume that $\pi \subset \{(0, x'); x' \in R^{n-1}\}$ and that

$$x_0^1 > 0, \qquad x_0 = \left(x_0^1, 0, \ldots, 0\right), \quad D = \{(0, x'); |x'| < \rho\}. \tag{3.17}$$

We take an open subset $U \subset R^n \times (\mu, T - \mu)$:

$$U = \left\{(x_1, x', t); |x'| < \rho,\ 0 \leqslant x_1 \leqslant \frac{T - 2\mu}{2} - \left|t - \frac{T}{2}\right|\right\}. \tag{3.18}$$

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In other words, U is a triangular prism with the base

$$\left\{(x_1, t); 0 \leqslant x_1 \leqslant \frac{T-2\mu}{2} - \left|t - \frac{T}{2}\right|\right\},$$

by regarding the x'-direction as the height.

For any $\rho_1 \in (0, \rho)$, we choose a function $\chi = \chi(x')$ such that

$$\begin{cases} \chi \in C_0^\infty(R^{n-1}), \quad \operatorname{supp}\chi \subset \{x'; |x'| \leqslant \rho\}, \\ \chi(x') = 1, \quad |x'| \leqslant \rho_1, \\ \chi(x') = 0, \quad |x'| = \rho, \qquad 0 \leqslant \chi \leqslant 1. \end{cases} \tag{3.19}$$

Moreover, we can take a family of C^∞-functions $x_1 = F(\lambda, t)$ with parameters $\lambda \in [0, 1)$ such that

$$\begin{cases} F(0,t) = 0, \quad F(1,t) = \frac{T-2\mu}{2} - \left|t - \frac{T}{2}\right|, \\ 0 < F(\lambda, t) < F(1,t), \quad \mu < t < T-\mu, \; 0 < \lambda < 1, \\ \lim_{\lambda \uparrow 1} F(\lambda, t) = F(1,t) \quad \text{for fixed } t \in (\mu, T-\mu), \\ F(\lambda, \mu) = F(\lambda, T-\mu) = 0, \qquad 0 < \lambda < 1, \\ |(\partial_t F)(\lambda, t)| < 1, \qquad \mu < t < T-\mu, \; 0 < \lambda < 1. \end{cases} \tag{3.20}$$

We set

$$G(\lambda, x_1, x', t) = x_1 - \chi(x')F(\lambda, t) \tag{3.21}$$

and

$$\Sigma_\lambda = \{(x_1, x', t); |x'| < \rho, \; \mu < t < T-\mu, \; G(\lambda, x_1, x', t) = 0\}. \tag{3.22}$$

Then $\Sigma_0 = \{(0, x', t); |x'| < \rho, \; \mu < t < T-\mu\} = D \times (\mu, T-\mu)$ and for $0 < \lambda < 1$, the hypersurface Σ_λ is noncharacteristic with respect to $\partial_t^2 - \Delta$. In fact,

$$\begin{aligned} |\partial_t G|^2 - \sum_{j=1}^n |\partial_j G|^2 &= |\chi|^2 |\partial_t F|^2 - 1 - \left(\sum_{j=2}^n |\partial_j \chi|^2\right) F^2 \\ &\leqslant |\partial_t F|^2 - 1 - \left(\sum_{j=2}^n |\partial_j \chi|^2\right) F^2 < -\left(\sum_{j=2}^n |\partial_j \chi|^2\right) F^2 \leqslant 0 \end{aligned}$$

by the last conditions in (3.19) and (3.20). Hence the noncharacteristic hypersurfaces Σ_λ sweep out a domain starting Σ_0. In view of (3.15), Fritz John's global Holmgren theorem [4, 15] yields

$$v(x,t) = 0, \qquad (x,t) \in \overline{\bigcup_{0 \leqslant \lambda \leqslant 1} \Sigma_\lambda}. \tag{3.23}$$

Hence, since $\rho_1 > 0$ in (3.19) is arbitrary, we obtain

$$v(x,t) = 0 \quad \text{if } |x'| \leqslant \rho \text{ and } 0 \leqslant x_1 \leqslant \tfrac{T-2\mu}{2} - \left|t - \tfrac{T}{2}\right|. \tag{3.24}$$

On the other hand, if

$$\rho + \mu + |x_0| \leqslant t \leqslant T - \rho - \mu - |x_0|, \tag{3.25}$$

then

$$|x_0| + \rho \leqslant \min\{T - \mu - t, t - \mu\}.$$

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Therefore, if $|x - x_0| < \rho$ and (3.25) holds, then by $x_0 = (x_0^1, 0, \dots, 0)$, we see that $|x'| < \rho$ and that

$$0 \leqslant x_1 \leqslant |x| \leqslant |x - x_0| + |x_0| \leqslant |x_0| + \rho \leqslant \min\{T - \mu - t, t - \mu\}$$
$$= \frac{T - 2\mu}{2} - \left| t - \frac{T}{2} \right|.$$

Hence the conclusion (3.14) follows from (3.24). Thus the proof of the lemma is complete.

Now we will proceed to the proof of (3.7) with (3.8). We take $\delta > 0$, $\delta_0 > 0$, $\epsilon_0 > 0$ and $\epsilon > 0$ sufficiently small, so that

$$0 < \epsilon_0 < \frac{\delta_0}{2} \tag{3.26}$$

and

$$T > 2R + 2 \sup_{x \in \partial B_{R+\delta_0}} \inf_{g \in \Gamma} d_c(x, g) + \delta + \delta_0 + 4(\epsilon + \epsilon_0) + 4\epsilon_0. \tag{3.27}$$

By (2.5) such $\epsilon_0 > 0$, $\epsilon > 0$, $\delta_0 > 0$ and $\delta > 0$ exist.

Let $y \in \partial B_{R+\delta_0}$ be arbitrary. Henceforth $|l|$ denotes the length of the curve l. By the definition (2.3) of d_c, for $\epsilon > 0$, we can take $\widetilde{\xi} \in \Gamma$ and a continuous curve $\widetilde{l} \setminus \{y\} \subset R^n \setminus \overline{B_{R+\delta_0}}$ connecting $\widetilde{\xi}$ and y such that $|\widetilde{l}| \leqslant \inf_{g \in \Gamma} d_c(y, g) + \epsilon$.

By (3.9) and (3.10), we can apply Fritz John's global Holmgren theorem (e.g. [15]), so that for $\epsilon_0 > 0$, we have $\epsilon_1 > 0$ and $\xi \in R^n \setminus \overline{B_{R+\delta_0}}$ such that $|\xi - \widetilde{\xi}| < \epsilon_1$ and

$$v(x, t) = 0, \quad |x - \xi| < \epsilon_1, \, \epsilon_0 < t < T - \epsilon_0. \tag{3.28}$$

If $\epsilon_1 \geqslant \epsilon_0$, then we reset $\epsilon_1 = \epsilon_0$ in (3.28), so that we may assume that $\epsilon_1 \leqslant \epsilon_0$.

We denote the continuous curve $\widetilde{l} \cup \{x = (1 - \eta)\xi + \eta\widetilde{\xi}; 0 \leqslant \eta \leqslant 1\}$ by l. Then

$$|l| \leqslant \inf_{g \in \Gamma} d_c(y, g) + \epsilon + \epsilon_0. \tag{3.29}$$

By the definition of the length of a curve, there exist a natural number $N \geqslant 2$ and $x^1, \dots, x^{N-1} \in l \subset R^n \setminus \overline{B_{R+\delta_0}}$ such that

$$\sum_{j=1}^{N} |x^j - x^{j-1}| \leqslant |l| + \epsilon,$$

where we set $x^0 = \xi$ and $x^N = y$. Hence

$$\sum_{j=1}^{N} |x^j - x^{j-1}| \leqslant \inf_{g \in \Gamma} d_c(y, g) + 2(\epsilon + \epsilon_0),$$

and (3.27) implies

$$\sum_{j=1}^{N} |x^j - x^{j-1}| < \frac{T}{2} - R - \frac{\delta + \delta_0}{2} - 2\epsilon_0. \tag{3.30}$$

Moreover by (3.26), we have

$$\bigcup_{0 \leqslant \eta \leqslant 1} \{x; |x - (\eta x^{k+1} + (1 - \eta)x^k)| < \epsilon_0\} \subset R^n \setminus \overline{B_R}, \quad 0 \leqslant k \leqslant N - 1. \tag{3.31}$$

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First we apply Lemma 2 with $x_0 = x^1$, $\mu = \epsilon_0$ and $\rho = \frac{\epsilon_1}{2}$, so that

$$v(x,t) = 0 \qquad \text{if } |x - x^1| < \tfrac{\epsilon_1}{2} \text{ and}$$
$$\left(\epsilon_0 + \frac{1}{2}\epsilon_1\right) + |x^1 - x^0| \leqslant t \leqslant T - \left(\epsilon_0 + \frac{1}{2}\epsilon_1\right) - |x^1 - x^0|.$$

By $\epsilon_1 \leqslant \epsilon_0$, we see that

$$v(x,t) = 0 \qquad \text{if } |x - x^1| < \tfrac{\epsilon_1}{2} \text{ and}$$
$$\left(1 + \frac{1}{2}\right)\epsilon_0 + |x^1 - x^0| \leqslant t \leqslant T - \left(1 + \frac{1}{2}\right)\epsilon_0 - |x^1 - x^0|. \tag{3.32}$$

By (3.26) and (3.30),

$$T > 2\left\{\left(1 + \frac{1}{2}\right)\epsilon_0 + \left(\frac{1}{2}\right)^2 \epsilon_1\right\} + 2(|x^1 - x^0| + |x^2 - x^1|).$$

Consequently, setting $\xi = x^1$, $x_0 = x^2$, $\mu = \left(1 + \frac{1}{2}\right)\epsilon_0 + |x^1 - x^0|$ and $\rho = \left(\frac{1}{2}\right)^2 \epsilon_1$, in view of (3.31), we can apply Lemma 2, so that by (3.26), we obtain

$$v(x,t) = 0 \qquad \text{if } |x - x^2| \leqslant \left(\tfrac{1}{2}\right)^2 \epsilon_1 \text{ and}$$
$$\epsilon_0\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2\right) + |x^1 - x^0| + |x^2 - x^1| \leqslant t$$
$$\leqslant T - \left\{\epsilon_0\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2\right) + |x^1 - x^0| + |x^2 - x^1|\right\}. \tag{3.33}$$

Changing $\rho = \left(\frac{1}{2}\right)^k \epsilon_1$, $k = 3, 4, \ldots, N$ and taking $x_0 = x^k$, in view of (3.26), (3.30) and (3.31), we apply Lemma 2 for $k = 1, \ldots, N$ to obtain

$$v(x,t) = 0 \qquad \text{if } |x - x^k| < \left(\tfrac{1}{2}\right)^k \epsilon_1 \text{ and}$$
$$\epsilon_0\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^k\right) + \sum_{j=1}^{k} |x^j - x^{j-1}|$$
$$\leqslant t \leqslant T - \left\{\epsilon_0\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^k\right) + \sum_{j=1}^{k} |x^j - x^{j-1}|\right\}. \tag{3.34}$$

In particular, we have

$$v(x,t) = 0 \qquad \text{if } |x - x^N| < \left(\tfrac{1}{2}\right)^N \epsilon_1 \text{ and}$$
$$\epsilon_0\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^N\right) + \sum_{j=1}^{N} |x^j - x^{j-1}|$$
$$\leqslant t \leqslant T - \left\{\epsilon_0\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^N\right) + \sum_{j=1}^{N} |x^j - x^{j-1}|\right\}.$$

Consequently we have

$$v(x,t) = 0 \qquad \text{if } |x-y| < \left(\tfrac{1}{2}\right)^N \epsilon_1 \text{ and}$$
$$\kappa < t < T - \kappa. \tag{3.35}$$

Here by (3.30), we have

$$\kappa \equiv \sum_{j=1}^{N} |x^j - x^{j-1}| + 2\epsilon_0 < \frac{T}{2} - R - \frac{\delta + \delta_0}{2}. \tag{3.36}$$

Setting

$$t_0 = \frac{T}{2} - R - \frac{\delta}{2}, \tag{3.37}$$

we see that $t_0 > 0$ satisfies (3.8). Therefore, in view of (3.36), we have

$$v(y,t) = |\nabla v(y,t)| = 0, \qquad y \in \partial B_{R+\delta_0},\ t_0 - \frac{\delta_0}{2} < t < T - \left(t_0 - \frac{\delta_0}{2}\right). \tag{3.38}$$

Since $\delta_0 > 0$ can be taken arbitrarily small, we obtain

$$v(x,t) = |\nabla v(x,t)| = 0, \qquad x \in \partial B_R,\ t_0 \leqslant t \leqslant T - t_0.$$

That is,

$$v(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0, \qquad x \in \partial B_R,\ t_0 \leqslant t \leqslant T - t_0.$$

Thus the proof of (3.7) with (3.8) is complete.

Third step In this step, we complete the proof of the theorem. In addition to (3.7), for $j = 1, 2$, we have

$$\partial_t^2 u(q_j)(x,t) = \Delta u(q_j)(x,t) - q_j(x)u(q_j)(x,t), \quad x \in B_R,\ t_0 < t < T - t_0, \tag{3.39}$$

$$u(q_j)(x, T/2) = a(x), \quad \partial_t u(q_j)(x, T/2) = 0, \qquad x \in B_R. \tag{3.40}$$

By (3.8), we have

$$\frac{T - 2t_0}{2} > R.$$

Therefore by the uniqueness in the inverse problem for (3.7), (3.39) and (3.40) [8, 9, 10, 11, 13, 18], we see that $q_1(x) = q_2(x)$, $x \in B_R$. In view of (3.6), the proof of the theorem is complete.

4 Concluding remarks

In this paper, we have proved the uniqueness in determining a coefficient $q(x)$ in a hyperbolic equation (1.1) by observation on any small hypersurface Γ along a sufficiently long time interval $(0, T)$, provided that q is analytic outside a ball containing Γ. Moreover, for the uniqueness, our proof requires the strict positivity (2.4) of the initial value $u(\cdot, 0)$.

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Furthermore there is a stability problem in determining q from our observations of u and $\frac{\partial u}{\partial \nu}$ on $\Gamma \times (0, T)$, and in principle, we can apply the method here with suitable modifications but we will omit the details.

Finally we will list important open questions:

- Is the same uniqueness true without analyticity outside the ball containing Γ?
- Can we relax the strict positivity of $u(\cdot, 0)$? For example, is the uniqueness true if $u(\cdot, 0)$ does not vanish identically? This question is very difficult even though we assume the same condition on the analyticity of q.

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