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A class of resistive axisymmetric magnetohydrodynamic equilibria in a periodic cylinder

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Abstract. We consider a model of viscoresistive incompressible magnetohydrodynamics in a periodic cylinder, with boundary conditions meant to idealize in a tractable way those of a laboratory plasma. The resistivity is described by a tensor presenting a field-dependent anisotropic part suggested by kinetic theory, controlled by a certain anisotropy parameter. An explicit analytical description of the corresponding axisymmetric zero-flow equilibria is given, and it is shown how tokamak-like or paramagnetic-pinch-like field profiles are obtained as the anisotropy parameter is changed. The study of the stability properties of such equilibria is deferred to a later paper.

1. Introduction

A familiar approach in fusion plasma theory has been to seek solutions of the governing dynamical equations and then to study their stability properties. One of the main activities of fusion plasma theory has been to seek stationary states that represent as well as possible the experimental conditions that will prevail under thermonuclear conditions. Even treatments of turbulence, disruptions, crashes, and so on have often started from steady states, or 'equilibria', which satisfy to some approximation dynamical systems such as the equations of magnetohydrodynamics (MHD).

Most often, ideal or non-dissipative MHD steady states have been chosen as starting points (e.g. Freidberg 1987; Goldston and Rutherford 1995). Here, we seek one of a considerably more complicated class of MHD steady states, and work at the level of viscoresistive MHD, neglecting electron inertia so that the only velocity field that enters the description is the fluid velocity field $\mathbf{v}$. We utilize, in particular, an anisotropic tensor resistivity with a variable degree of anisotropy. The transport coefficients, strictly speaking, should be calculated self-consistently, depending as they do on such spatially variable fields as the density and temperature (e.g. Braginskii 1965; Balescu 1988). This procedure introduces an energy or temperature equation as well as an equation of motion, and has been pursued by Goodman (1992, 1993, 1998), but leads to calculations of great complexity. Here, we remain at a level where the transport coefficients (in particular the tensor resistivity) are treated as prescribed functions of space that reflect what are believed to be qualitatively correct spatial temperature profiles. A similar program has been described by, for example, Montgomery et al. (1999) for toroidal resistive equilibria with scalar resistivity (see also Kamp et al. 1998; Bates and Montgomery 1998). This choice is motivated by the need for tractability and the urge to produce specific solutions.

In the present paper, we are concerned with the problem of finding a family of axisymmetric zero-flow ($\mathbf{v} = 0$) equilibria in a straight periodic cylinder with boundary conditions chosen to approximate as well as possible, though imperfectly, those of a real confining chamber. We intend in a later paper to discuss the stability of the equilibria described here. One of our main present conclusions is that the resulting equilibrium profiles are strongly dependent upon the degree of anisotropy chosen for the resistivity. The equilibria are obtained in analytical form in terms of transcendental functions. By varying the available free parameters, we can test the physical consistency of all the relevant quantities except the density (which will be assumed to be spatially uniform). The magnetic field, current density, pressure, safety factor, and the like, are all explicitly calculable, given the resistivity profile. The equilibria found range from tokamak-like ones to paramagnetic pinch configurations. The latter are of some interest in their own right, and exhibit some features that are close to those configurations achieved in reversed field pinches (RFPs) before spontaneous field reversal sets in (Bodin and Newton 1980) – though, in the absence of flow velocity, we do not find field reversal itself among our class of possible steady states.

The main result of this paper may be said to be the explicit determination of axisymmetric resistive steady states without flow in a straight cylinder with a realistic-looking anisotropic tensor resistivity. As will be seen, this turns out to be algebraically more demanding than one might have guessed. As far as we know, such MHD steady states have not been previously reported, though some toroidal states involving anisotropic resistivity have been calculated by van der Woude (2000).

The paper is organized as follows: Sec. 2 is a summary of the mathematical description used and the assumptions made, including a discussion of boundary conditions; Sec. 3 describes in detail the solutions found and their resulting profiles; a brief summary of the conclusions appears in Sec. 4.

2. The model

2.1. The MHD equations

We consider a globally neutral plasma with time-constant uniform density ρ_0, characterized by an isotropic scalar kinematic viscosity ν and a suitable resistivity tensor $\boldsymbol{\eta}$ to be discussed below, in the standard approximation in which the displacement current is neglected in Maxwell's equations, so that the electric field actually behaves as a decoupled quantity. Thus, it is assumed that the plasma (center-of-mass) velocity $\mathbf{v}$ evolves according to the magnetically driven Navier–Stokes (NS) equation, the magnetic field $\mathbf{B}$ evolves according to the Faraday–Ohm (FO) law, while the current density $\mathbf{J}$ is 'defined' by Ampere's law. The equations for the fields $\mathbf{v}$, $\mathbf{B}$, $\mathbf{J}$ and the pressure p (in SI units) are then

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} - \frac{1}{\rho_0}\boldsymbol{\nabla}p + \frac{1}{\rho_0}\mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{v}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times\left(\mathbf{v}\times\mathbf{B} - \frac{M}{\rho_0 e}\mathbf{J}\times\mathbf{B} - \boldsymbol{\eta}\cdot\mathbf{J}\right), \tag{2}$$

$$\mu_0\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{4}$$

where M is the ion mass, $-e$ the electron charge and μ_0 the vacuum magnetic permeability; see for example Woods (1987) or Goldston and Rutherford (1995).

The resistivity tensor $\boldsymbol{\eta}$ appearing in (2) will be assumed to have the form

$$\boldsymbol{\eta} = \eta(r)[\mathbf{I} - (1-\gamma)\mathbf{bb}], \qquad 0 \leqslant \gamma \leqslant 1, \tag{5}$$

where $\mathbf{I}$ is the identity tensor, $\mathbf{b} \equiv \mathbf{B}/|\mathbf{B}|$ denotes the unit vector in the direction of the magnetic field $\mathbf{B}$, while the 'resistivity profile' $\eta(r)$ is assumed to be a positive function of the distance r from the axis of the cylinder modeling the 'chamber' containing the plasma. Such a resistivity profile will turn out to play a relevant role, and its properties will be discussed in the next section.

Actually, we might have taken a resistivity tensor of a little more general form suggested by kinetic theory (Braginskii 1965; Woods 1987; Balescu 1988), namely $\boldsymbol{\eta} = \eta_{\|}(\mathbf{x})\mathbf{bb} + \eta_{\perp}(\mathbf{x})(\mathbf{I} - \mathbf{bb})$, which involves two independent resistivities $\eta_{\|}$ and $\eta_{\perp}$ (parallel and orthogonal to $\mathbf{b}$ respectively) and makes no assumption of cylindrical symmetry. However, the simpler form (5), which is obtained from the latter one by setting $\eta_{\|} = \gamma\eta_{\perp}$ (with $\eta_{\perp} \equiv \eta$) and by assuming cylindrical symmetry, seems to be of a sufficiently general character. So the model is defined by the form of the resistivity profile $\eta(r)$ and the anisotropy parameter γ. Concerning such a parameter $\gamma = \eta_{\|}/\eta_{\perp}$, let us recall that, by kinetic theory, the value $\gamma = 0.5$ is predicted for high-magnetic-field and/or high-temperature regimes, while an isotropic tensor, i.e. $\gamma = 1$, is predicted in the opposite regime. The choice made here, of considering γ as a free parameter, allows us to deal with different physical situations in a simple way. The range of γ suggested by kinetic theory is actually $0.5 \leqslant \gamma \leqslant 1$; we chose instead to consider the enlarged interval $0 < \gamma \leqslant 1$, because small values of γ might be of interest in mimicking noncollisional transport phenomena.

Concerning the FO law (2), notice that no pressure gradient has been included in the right-hand side, due to the assumption of incompressibility. Indeed Ohm's law is nothing but the equation of motion of the electron fluid in which inertia (acceleration) terms have been neglected, namely

$$\mathbf{E} = -\left(\mathbf{v} - \frac{\mathbf{J}}{n_0 e}\right)\times\mathbf{B} - \frac{1}{n_0 e}\boldsymbol{\nabla}p_e + \boldsymbol{\eta}\cdot\mathbf{J}\,. \tag{6}$$

where $\mathbf{v} - \mathbf{J}/n_0 e \equiv \mathbf{v}_e$ is the electron fluid velocity, $n_0 = \rho_0/M$ the number density of ions and/or electrons (we think of hydrogen plasmas, say) and p_e the electron pressure; thus, for a uniform n_0 the pressure term becomes curl-free, making no contribution to Faraday's law $\partial\mathbf{B}/\partial t = -\boldsymbol{\nabla}\times\mathbf{E}$.

Finally, concerning the plasma pressure p, notice that incompressibility implies it

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to be a 'slave' variable. Indeed, by taking the divergence of (1), the Poisson equation

$$\nabla^2 p = \boldsymbol{\nabla} \cdot (-\rho_0 \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} + \mathbf{J} \times \mathbf{B} + \rho_0 \nu \nabla^2 \mathbf{v}) \tag{7}$$

is obtained, from which p is determined, up to a constant, with suitable boundary conditions (e.g. Landau and Lifshitz 1987). A detailed discussion concerning the necessity and the role of boundary conditions in determining pressure can be found in Kress and Montgomery (2000). So the pressure term in the Navier–Stokes equation can be formally rewritten as

$$-\boldsymbol{\nabla} p = -\boldsymbol{\nabla} \Delta^{-1} [\boldsymbol{\nabla} \cdot (-\rho_0 \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} + \mathbf{J} \times \mathbf{B} + \rho_0 \nu \nabla^2 \mathbf{v})], \tag{8}$$

Δ^{-1} denoting the inverse of the Laplacian, and thus eliminated (Doering and Gibbon 1995).

2.2. *Geometry and boundary conditions*

As a model for the 'chamber' containing the plasma, for the sake of simplicity we take the so-called 'straightened torus', namely the cylinder

$$\boldsymbol{\Omega} \equiv \{(r, \theta, z) \mid 0 \leqslant r \leqslant a,\ 0 \leqslant \theta < 2\pi,\ 0 \leqslant z \leqslant L\},$$

with periodic boundary conditions on the variable z, i.e. with

$$\psi(r, \theta, z) = \psi(r, \theta, z + L) \qquad \forall z \in \boldsymbol{\Omega}$$

being assumed to hold for any generic field component ψ. This should be thought of as a mathematically simplified model for an actual physical torus of minor radius a and major radius R if one takes $L = 2\pi R$, an approximation that should be good enough for large-aspect-ratio (R/a) devices. With regard to the boundary $\partial\boldsymbol{\Omega}$ (i.e. the subset of $\boldsymbol{\Omega}$ with $r = a$), as is usual for idealized two-dimensional surfaces modeling physical walls, we will have to deal with cases of functions that possibly have discontinuities through it. In this connection, the convention will be made that for a generic function f the value $f|_{\partial\boldsymbol{\Omega}}(\theta, z)$ denotes the 'internal limit', i.e.

$$f|_{\partial\boldsymbol{\Omega}}(\theta, z) = \lim_{r \to a^-} f(r, \theta, z),$$

while the 'external limit' will be denoted by $f^{\text{ext}}(\theta, z)$.

We first settle down an 'internal' boundary condition for the resistivity profile $\eta(r)$, which is a quantity entering the definition itself of the model, namely

$$\lim_{r \to a^-} \eta(r) = +\infty. \tag{9}$$

This is due to the fact that we want to model a confined plasma, which does not touch the wall of the vessel, so that the temperature at the wall should be negligible with respect to the temperature on the axis; this can be simply modeled by assuming a vanishing temperature at the wall. So (9) follows from Spitzer's law $\eta \sim T^{-3/2}$ (Spitzer 1956). In the discussion to be given below, we will have to make use of a further condition related to the assumed absence of plasma at the wall and involving the electron pressure p_e, namely

$$\mathbf{e}_r \times \boldsymbol{\nabla} p_e|_{\partial\boldsymbol{\Omega}} = 0. \tag{10}$$

We now come to the boundary conditions for the fields entering the MHD equations, namely $\mathbf{v}$, p, $\mathbf{B}$ and $\mathbf{J}$. In fact, a boundary condition is also required for the electric field $\mathbf{E}$, inasmuch as it enters Faraday's law $\partial\mathbf{B}/\partial t = -\boldsymbol{\nabla} \times \mathbf{E}$; indeed, while inside the chamber the electric field $\mathbf{E}$ is assumed to be given by (6), outside the chamber it is instead controlled as an independent quantity, as also are the magnetic field and the currents. Actually, we model the action of the external coils controlling the plasma current and the toroidal magnetic field by means of a constant surface current density $\mathbf{K}$ flowing on the boundary of the cylinder, and of a constant axial electric field $E_0\mathbf{e}_z$ at the wall ($\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ denoting, as usual, the unit vectors in the radial, poloidal and axial directions, respectively). In a real device, the surface current K_θ is provided by the toroidal field coils placed around the chamber in the poloidal direction; if there are N_T such coils, each carrying a current I_T, then a suitable estimate is given by $K_\theta = N_T I_T/L$. The surface current K_z could instead be provided by a set of poloidal field coils rolling up the chamber in the toroidal direction.

The physical situation of interest for laboratory plasmas should take into account what occurs on the outer side of $\partial\boldsymbol{\Omega}$; we assume

$$\mathbf{v}^{\text{ext}} = 0, \qquad p^{\text{ext}} = 0, \qquad \mathbf{J}^{\text{ext}} = 0, \qquad \mathbf{B}^{\text{ext}} = 0, \tag{11}$$

and

$$\mathbf{E}^{\text{ext}} = E_0\mathbf{e}_z, \qquad p_e^{\text{ext}} = 0. \tag{12}$$

Indeed, the first three conditions of (11) are just due to the fact that outside $\boldsymbol{\Omega}$ there is no plasma at all. With regard to $\mathbf{B}^{\text{ext}} = 0$, its radial component, $B_r^{\text{ext}} = 0$, just follows from the standard settings of the external coils, while the vanishing of the tangential components corresponds to considering an ideal, magnetically screened system. Finally, concerning the conditions (12), the first was just discussed above, while the latter corresponds to the absence of plasma outside $\boldsymbol{\Omega}$.

Such external conditions now allow one, through the standard ‘jump’ or continuity conditions, to deduce the relevant ‘internal’ boundary conditions for the fields entering the MHD equations. These turn out to be

$$\mathbf{v}|_{\partial\Omega} = 0, \tag{13}$$

$$B_r|_{\partial\Omega} = 0, \qquad \mathbf{e}_r \times \mathbf{B}|_{\partial\Omega} = -\mu_0\mathbf{K}, \tag{14}$$

$$\boldsymbol{\eta}\cdot\mathbf{J}|_{\partial\Omega} = E_0\mathbf{e}_z, \tag{15}$$

where $\mathbf{K}$ is the surface current density flowing on the wall, which is assumed to be a given constant vector. Notice that, from the proof of (15), to be given below, it also follows that $\mathbf{J}|_{\partial\Omega} = 0$. In turn, from such a condition together with the condition (13), through (1) one obtains for the ‘slave’ variable p the Neumann boundary condition

$$\left.\frac{dp}{dr}\right|_{\partial\Omega} = \rho_0\nu[\nabla^2\mathbf{v}]\cdot\mathbf{e}_r. \tag{16}$$

The proof is as follows. The conditions $v_r|_{\partial\Omega} = 0$ and $B_r|_{\partial\Omega} = 0$ follow, as usual, from the vanishing of the corresponding quantities outside $\boldsymbol{\Omega}$, making use of the solenoidal character of $\mathbf{v}$ and $\mathbf{B}$, while the ‘no-slip’ boundary condition $\mathbf{e}_r \times \mathbf{v}|_{\partial\Omega} = 0$ is the usual one of viscous hydrodynamics. With regard to the tangential component of $\mathbf{B}$ at the wall, i.e. the second equation of (14), this is nothing but the standard jump condition

$$\mathbf{e}_r \times (\mathbf{B}^{\text{ext}} - \mathbf{B}|_{\partial\Omega}) = \mu_0\mathbf{K}, \tag{17}$$

with the above assumption $\mathbf{B}^{\text{ext}} = 0$.

Finally, we come to the last internal condition (15), which follows by taking the limit $r \to a^-$ in the relation (6) defining the electric field inside the chamber. First of all, notice that $J_r|_{\partial\Omega} = 0$ follows from the fact that $\mathbf{J}$ is a solenoidal vector field and $\mathbf{J}^{\text{ext}} = 0$. Moreover, the tangential component too vanishes. Indeed, by the continuity of the tangential component of the electric field $\mathbf{E}$ through $\partial\Omega$, the left-hand side of (6) is known to have a finite limit, namely $E_0\mathbf{e}_z$. On the other hand, η is assumed to diverge at the wall (the condition (9)), so that from the right-hand side one deduces that the tangential component of $\mathbf{J}|_{\partial\Omega}$ vanishes, i.e. $\mathbf{J}|_{\partial\Omega} = 0$. Thus, by (13), the first term on the right-hand side also vanishes at the wall, while the second vanishes by the assumption (10), which was made above in connection with the absence of plasma at the wall.

As a final comment on the geometry and the boundary conditions, we point out that, with regard to the magnetic helicity of the system, the correct quantity according to the prescriptions of Moffatt (1969) and Taylor (1986) is the 'effective' magnetic helicity, which is discussed in Appendix A.

2.3. *Relevant quantities: axial magnetic flux, plasma current, pinch ratio and field parameter*

Given a transverse section Σ_0 of Ω, i.e. the subset defined by $z = z_0$, with boundary $\partial\Sigma_0 \subset \partial\Omega$, the axial magnetic flux

$$\Phi(t) \equiv \int_{\Sigma_0} B_z \, d^2x \tag{18}$$

is defined; by Gauss' theorem, Φ is independent of z_0, and in fact one has $\Phi = \pi a^2 \langle B_z \rangle_\Omega$, where $\langle \cdot \rangle_\Omega$ denotes the volume average in Ω. The flux Φ is easily seen to be an integral of motion, i.e. to satisfy $\Phi(t) = \Phi(0)$. Indeed, from (2) and the boundary condition (15), by Stokes' theorem one obtains

$$\frac{d}{dt}\Phi(t) = \oint_{\partial\Sigma_0} \left(\mathbf{v} \times \mathbf{B} - \frac{M}{\rho_0 e}\mathbf{J} \times \mathbf{B} - E_0\mathbf{e}_z \right) \cdot \mathbf{e}_\theta a \, d\theta,$$

and the assertion follows because the assumed boundary condition (13) $\mathbf{v}|_{\partial\Omega} = 0$ and the property $\mathbf{J}|_{\partial\Omega} = 0$ insure that the 'dynamo' electric field component $-\mathbf{v} \times \mathbf{B}$ and the Hall electric field component $(M/\rho_0 e)\mathbf{J} \times \mathbf{B}$ vanish at the boundary.

The total axial current I_p supported by the plasma is defined as

$$I_p(t) \equiv \int_{\Sigma_0} J_z \, d^2x = \frac{1}{\mu_0} \oint_{\partial\Sigma_0} B_\theta a \, d\theta, \tag{19}$$

the second equality following from Stokes' theorem; by Gauss' theorem I_p too does not depend on z_0.

Notice that, at variance with the axial flux Φ, the value of which is determined by the initial data, the plasma current I_p is determined by the boundary conditions through the value $B_\theta|_{\partial\Omega}$, i.e. of K_z. So, in our case, it turns out to be constant too, but just because constant boundary conditions for B_θ were assumed.

Let us recall that from the three quantities Φ, $\langle B_z \rangle_{\partial\Omega}$, and $\langle B_\theta \rangle_{\partial\Omega}$, two relevant quantities are obtained (Ortolani and Schnack 1993), namely the *pinch ratio*

$$\Theta \equiv \frac{\langle B_\theta \rangle_{\partial\Omega}}{\langle B_z \rangle_\Omega} = \frac{B_\theta|_{\partial\Omega}}{\Phi/\pi a^2} = \frac{\mu_0 a}{2} \, \frac{I_p}{\Phi} \tag{20}$$

and the *field parameter*

$$F \equiv \frac{\langle B_z \rangle_{\partial\Omega}}{\langle B_z \rangle_{\Omega}} = \frac{B_z|_{\partial\Omega}}{\Phi/\pi a^2}. \tag{21}$$

The latter is well known to be a fundamental parameter in fusion plasma physics, since it measures how strong the pinch effect on the toroidal field lines is. For tokamaks, one has $F \simeq 1$, since B_z has an almost-constant, uniform profile, while RFPs are characterized by a negative F. The pinch ratio Θ measures instead how much field is 'produced' by the externally driven current. In RFPs, Θ is of order one, while in tokamaks, in general, it is about one order of magnitude lower.

In our model, the quantities Θ and F turn out to be constant, their values being determined both by the constant boundary data $B_z|_{\partial\Omega}$, $B_\theta|_{\partial\Omega}$ (or I_p) and by the initial data entering the constant of motion Φ.

3. One-dimensional (axisymmetric) steady states

3.1. General properties

Given the equations (1)–(5) with the boundary conditions (13)–(15), let us now look for a zero-flow equilibrium, namely a stationary solution $\mathbf{B}_0$, $\mathbf{J}_0$, p_0, with the plasma at rest, i.e. with $\mathbf{v} = 0$. Then one gets the system

$$\mathbf{J}_0 \times \mathbf{B}_0 = \boldsymbol{\nabla} p_0, \tag{22a}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\eta} \cdot \mathbf{J}_0) = 0, \tag{22b}$$

together with $\mu_0 \mathbf{J}_0 = \boldsymbol{\nabla} \times \mathbf{B}_0$ and $\boldsymbol{\nabla} \cdot \mathbf{B}_0 = 0$. Notice that, at equilibrium, the Hall electric field $(M/\rho_0 e)\mathbf{J}_0 \times \mathbf{B}_0$ make no contribution to the FO equation, since the Lorentz force $\mathbf{J}_0 \times \mathbf{B}_0$ is curl-free. So, *the Hall term makes no contribution in determining zero-flow, Ohmic equilibria*, although, as will be shown in the forthcoming paper, the Hall effect strongly affects the stability properties of such equilibria.

We now look for solutions of the system (22) in axial symmetry, i.e. with each field, as well as η, depending on the coordinate r only. The equation $\boldsymbol{\nabla} \times (\boldsymbol{\eta} \cdot \mathbf{J}_0) = 0$ can be immediately integrated, giving

$$\eta \left[\mathbf{J}_0 - (1-\gamma) \frac{\mathbf{J}_0 \cdot \mathbf{B}_0}{|\mathbf{B}_0|^2} \mathbf{B}_0 \right] = -\mathbf{e}_r \frac{d\phi}{dr} + C\mathbf{e}_z,$$

where C and $\phi(r)$ are a constant and a radial function to be determined, respectively; indeed, under the requirement of axial symmetry, the right-hand side of the above relation represents the most general curl-free expression in cylindrical coordinates. Now, using the boundary condition (15) and remembering that B_{0r} and J_{0r} vanish identically due to the assumed axial symmetry, one gets $\phi = \text{const}$ and $C = E_0$, where E_0 is the axial constant electric field at the wall. So, recalling that

$$J_{0\theta} = -\frac{1}{\mu_0} \frac{dB_{0z}}{dr}, \qquad J_{0z} = \frac{1}{\mu_0 r} \frac{d}{dr}(r B_{0\theta}), \tag{23}$$

(22b) reduces to a system of two first-order differential equations:

$$\frac{dB_{0z}}{dr} = -\frac{\mu_0 E_0}{\eta} \frac{1-\gamma}{\gamma} \frac{B_{0z} B_{0\theta}}{|\mathbf{B}_0|^2}, \tag{24a}$$

$$\frac{1}{r} \frac{d}{dr}(r B_{0\theta}) = \frac{\mu_0 E_0}{\eta} \left(1 + \frac{1-\gamma}{\gamma} \frac{B_{0z}^2}{|\mathbf{B}_0|^2} \right), \tag{24b}$$

with boundary conditions

$$B_{0z}(a) = B_z|_{\partial\Omega}, \qquad B_{0\theta}(a) = B_\theta|_{\partial\Omega}. \tag{25}$$

In axial symmetry, (22a) just reduces to $dp_0/dr = J_{0\theta}B_{0z} - J_{0z}B_{0\theta}$, which, making use of (24), gives an explicit expression for p_0 in terms of $B_{0\theta}$, namely

$$p_0(r) = p_0(a) + \int_r^a \frac{E_0}{\eta(s)} B_{0\theta}(s)\, ds. \tag{26}$$

Thus, in axial symmetry, the equilibrium system (22) just reduces to (24). This constitutes a system of two first-order differential equations that, for a given resistivity profile $\eta(r)$, can in principle be solved, or at least integrated by standard numerical methods. Let us stress two important properties of such a system and of the relation (26):

(i) $p_0(r)$ is a monotonically decreasing function of r.

(ii) In the anisotropic case $\gamma < 1$, $B_{0z}(r)$ is a monotonically decreasing function of r; in particular, $B_{0z}(a) > 0$; for $\gamma = 1$, i.e. for isotropic resistivities, one has $B_{0z}(r) = B_{0z}(a)$, i.e. the equilibrium axial magnetic field has a flat, uniform profile.

The first of these is proved as follows. By integrating (24b) between 0 and r and supposing $B_{0\theta}$ to be regular at $r = 0$ (in fact, it must vanish there, as will be shown below), one gets

$$B_{0\theta}(r) = \frac{1}{r}\int_0^r \frac{\mu_0 E_0}{\eta(s)} \left[1 + \frac{1-\gamma}{\gamma}\frac{B_{0z}^2(s)}{|\mathbf{B}_0(s)|^2}\right] s\, ds,$$

which, by virtue of $\gamma \leqslant 1$, gives $B_{0\theta}(r) > 0$; the assertion then follows from (26). Analogously, (ii) follows by integrating (24a), which gives

$$B_{0z}(r) = B_{0z}(0)\exp\left[-\int_0^r \frac{\mu_0 E_0}{\eta(s)}\frac{1-\gamma}{\gamma}\frac{B_{0\theta}(s)}{|\mathbf{B}_0(s)|^2}\, ds\right].$$

Since the argument of the exponential function is positive for $\gamma < 1$, one concludes that in such a case $B_{0z}(r)$ is a monotonically decreasing function of its argument; in particular $B_{0z}(r)$ can never pass through zero and $B_z(a) > 0$. The relation $B_{0z}(r) = B_{0z}(a)$ for $\gamma = 1$ also follows from the above expression for $B_{0z}(r)$. Thus one sees that for $\gamma < 1$, the system (24) admits solutions only if $B_z|_{\partial\Omega} > 0$, corresponding to a positive field parameter. Under the assumptions stated, there are no RFP solutions without flow; the 'reversed state' found by Taylor (1974), for example, does not satisfy Ohm's law and cannot be made to do so for any value of γ.

Let us recall that an axisymmetric equilibrium satisfying the above two properties with $\gamma < 1$ is a *paramagnetic pinch*, since B_{0z} and p_0 are both monotonically decreasing functions of r and thus the axial magnetic field is higher where the pressure is. The isotropic limit case $\gamma = 1$ presents a vanishing poloidal component of the current density ($J_{0\theta} = 0$); this is a kind of 'resistive Z-pinch' equilibrium configuration that, once supplied with a strong axial uniform magnetic field $B_{0z} = B_z|_{\partial\Omega} > 0$, represents a crude starting point to approach tokamaks.

Notice that, since we are supposing that $\eta(r)$ diverges at the wall with a finite value of $B_{0\theta}(a)$, (26) yields $dp_0/dr(a) = 0$, which is consistent with the Neumann boundary condition (16) in the case $\mathbf{v} = 0$.

3.2. Analytic solution via the self-consistency method

The system (24) is a rather complicated nonlinear one that, if the resistivity profile $\eta(r)$ were given a priori, could be solved by standard numerical methods. But it turns out that physics requires some kind of functional dependence of the resistivity profile on the magnetic field, and so one is confronted with a problem of 'closure'. This problem is dealt with here by taking inspiration from the self-consistency method, which, for example, is applied by Freidberg (1987) to the so-called force-free paramagnetic model.

The method proceeds as follows. In place of η, as an assigned function defining the model, one takes the auxiliary function

$$\psi(x) \equiv \frac{a\mu_0 E_0}{\eta(x)} \frac{B_{0\theta}(x)}{|\mathbf{B}_0(x)|^2}, \qquad x \equiv \frac{r}{a}. \tag{27}$$

which is in one-to-one correspondence with it, once a solution is given (notice that ψ is positive, since $B_{0\theta}$ is). The consideration of such a function $\psi(x)$ is suggested by the form itself of the system (24a), because the change of variables

$$X(x) = B_{0z}(x), \qquad Y(x) = [xB_{0\theta}(x)]^2,$$

brings it into the form

$$X'(x) = -\frac{1-\gamma}{\gamma}\psi(x)X(x), \tag{28a}$$

$$Y'(x) = 2\psi(x)Y(x) + \frac{2}{\gamma}x^2\psi(x)X^2(x) \tag{28b}$$

(the prime denoting the derivative with respect to x), in which the unknown $X(x)$ decouples away and satisfies a linear equation if the function $\psi(x)$ is assumed to be a given one. The equivalent integral form of (28) is

$$X(x) = X_w \exp\left[\frac{1-\gamma}{\gamma}\mathbf{\Psi}(x)\right], \tag{29a}$$

$$Y(x) = \exp[-2\mathbf{\Psi}(x)]\left(Y_w + X_w^2\left\{1 - x^2\exp\left[\frac{2}{\gamma}\mathbf{\Psi}(x)\right] - 2\int_x^1 \exp\left[\frac{2}{\gamma}\mathbf{\Psi}(s)\right] s\,ds\right\}\right), \tag{29b}$$

where

$$\mathbf{\Psi}(x) \equiv \int_x^1 \psi(s)\,ds,$$

and the 'initial data' at $x = 1$ were taken as required by (25), namely $X(1) \equiv X_w$ and $Y(1) \equiv Y_w$; correspondingly, the resistivity profile is then given by the formula

$$\eta(x) = \frac{a\mu_0 E_0}{\psi(x)} \frac{x\sqrt{Y(x)}}{x^2X^2(x) + Y(x)}. \tag{30}$$

Now, the system (29) is completely equivalent to the original one, being nothing but a transcription of it in the form of a system of integral equations, with the function $\mathbf{\Psi}$ still depending on the unknowns X and Y themselves. The idea of the method consists in taking ψ (and thus $\mathbf{\Psi}$) as an assigned function of x, i.e. in letting the resistivity profile η depend on the fields themselves through the relation (30) with an assigned $\psi(x)$; indeed, in such a way (29) no longer appears as an integral equation, but just as an explicit expression of the solution. Obviously, this makes sense only if the functional dependence of η on the fields as given by (30) is reasonable, and the problem is then reduced to that of finding physically reasonable conditions determining the analytical form of the auxliliary function ψ.

To this end, one starts by noticing that the condition $B_{0\theta}(0) = 0$ has to be satisfied, and this entails the constraint

$$\lim_{x\to 0^+} \frac{Y(x)}{x^2} = 0, \tag{31}$$

which requires the necessary condition $Y(0) = 0$. Now, by (29b), the condition $Y(0) = 0$ takes the form

$$\frac{Y_w}{X_w^2} \equiv \frac{B_{0\theta}^2(a)}{B_{0z}^2(a)} = \int_0^1 \exp\left[\frac{2}{\gamma}\Psi(s)\right] d(s^2) - 1; \tag{32}$$

this in turn can always be satisfied because of the relation

$$\int_x^1 s^2 \frac{d}{ds}\left\{\exp\left[\frac{2}{\gamma}\Psi(s)\right]\right\} ds = 1 - x^2 \exp\left[\frac{2}{\gamma}\Psi(x)\right] - \int_x^1 \exp\left[\frac{2}{\gamma}\Psi(s)\right] d(s^2) < 0,$$

which holds since $\psi(s) = -d\Psi/ds$ is a non-negative function. In particular, the relation (32) allows one to rewrite the solution (29) in the more manageable form

$$X(x) = X_w \exp\left[\frac{1-\gamma}{\gamma}\Psi(x)\right], \tag{33a}$$

$$Y(x) = X_w^2 \exp[-2\Psi(x)] \left\{\int_0^x \exp\left[\frac{2}{\gamma}\Psi(s)\right] 2s\, ds - x^2 \exp\left[\frac{2}{\gamma}\Psi(x)\right]\right\}. \tag{33b}$$

In determining the form of the auxiliary function ψ, one should take into account that, as is easily seen, (30) implies for η the two limiting behaviors

$$\eta(0) = \frac{a\mu_0 E_0}{X(0)} \frac{1}{\sqrt{\psi'(0)}}$$

and

$$\eta(x) \sim \frac{1}{\psi(x)} \qquad \text{as } x \to 1.$$

It is thus clear that the auxiliary function ψ should be chosen in such a way that:

(i) the condition (31) is guaranteed;

(ii) $\psi'(0) \neq 0$;

(iii) ψ vanishes at $x = 1$, in such a way that a suitable divergence for η is guaranteed.

A convenient choice turns out to be the function

$$\psi(x) = (2\delta + 2)\alpha x(1 - x^2)^\delta, \qquad \alpha > 0, \qquad \delta > 0, \tag{34}$$

which depends on the two positive parameters α and δ. Correspondingly, one has $\Psi(x) = \alpha(1 - x^2)^{\delta+1}$, and furthermore one finds (see Appendix B)

$$\int_x^1 \exp\left[\frac{2}{\gamma}\Psi(s)\right] 2s\, ds = (1 - x^2)\, K_\delta\left[\frac{2\alpha}{\gamma}(1 - x^2)^{\delta+1}\right] \tag{35}$$

where $K_\delta(z)$ is a shorthand notation for the Kummer confluent hypergeometric function with indices $1/(1+\delta)$ and $(2+\delta)/(1+\delta)$; this is defined as the solution of the second-order differential equation

$$z\frac{d^2K}{dz^2} + \left(\frac{2+\delta}{1+\delta} - z\right)\frac{dK}{dz} - \frac{K}{1+\delta} = 0, \tag{36}$$

which is regular at the origin and satisfies the boundary condition $K(0) = 1$ (Abramowitz and Stegun 1965). We thank Sergio Cacciatori (Milan University) for kindly pointing this out to us.

The regularity condition $Y(0) = 0$ now reads

$$\frac{B_{0\theta}^2(a)}{B_{0z}^2(a)} = \int_0^1 \exp\left[\frac{2\alpha}{\gamma}(1-y)^{3/2}\right] dy - 1 = K_\delta\left(\frac{2\alpha}{\gamma}\right) - 1. \tag{37}$$

This finally allows us to give the analytical solutions for the poloidal and the axial components $B_{0\theta}$ and B_{0z} of the magnetic field, namely

$$B_{0\theta}(r) = \frac{aB_{0z}(a)}{r}\exp\left[-\alpha\left(1-\frac{r^2}{a^2}\right)^{\delta+1}\right]\left\{K_\delta\left(\frac{2\alpha}{\gamma}\right) - \frac{r^2}{a^2}\exp\left[\frac{2\alpha}{\gamma}\left(1-\frac{r^2}{a^2}\right)^{\delta+1}\right] - \left(1-\frac{r^2}{a^2}\right)K_\delta\left[\frac{2\alpha}{\gamma}\left(1-\frac{r^2}{a^2}\right)^{\delta+1}\right]\right\}^{1/2}, \tag{38}$$

$$B_{0z}(r) = B_{0z}(a)\exp\left[\alpha\frac{1-\gamma}{\gamma}\left(1-\frac{r^2}{a^2}\right)^{\delta+1}\right]. \tag{39}$$

As expected, in the isotropic case $\gamma = 1$, one has a uniform axial profile, i.e. $B_{0z}(r) = B_{0z}(a)$, while for $\gamma < 1$, the field B_{0z} presents a monotonically decreasing profile. It can easily be checked that, with the above choice for ψ, one has the asymptotic behaviors $B_{0\theta}(r) \sim r$ for $r/a \ll 1$ and $B_{0\theta}(r) \sim 1/r$ for $r \simeq a$, as should be expected from a physical point of view; in particular, the first of these properties insures that condition (31) is satisfied, since $Y(x)$ is of order x^4 as $x \to 0$. The resistivity profile turns out to be given by

$$\eta(r) = \frac{a\mu_0 E_0}{\psi(r/a)}\frac{B_{0\theta}(r)}{B_{0\theta}^2(r) + B_{0z}^2(r)}, \tag{40}$$

with $B_{0\theta}$ and B_{0z} given by (38) and (39); it is easily seen to present the limiting behaviors

$$\eta(r) \sim \frac{1}{[1-(r/a)]^\delta}, \qquad r \simeq a \tag{41}$$

$$\eta(0) \simeq \frac{a\mu_0 E_0}{B_{0z}(0)}, \qquad r \to 0.$$

As shown in Fig. 3, the resistivity profiles look quite reasonable from a physical point of view, for any value of γ. Having assigned the parameters γ and a entering the model, the parameter E_0 through the boundary conditions, and the parameter Φ_0 depending on the initial data (the value of the pressure at the wall, $p_0(a)$, is arbitrary because for incompressible fluids the pressure is determined up to an additive constant), the equilibrium solution is determined for any choice of the free parameters α and δ defining the auxiliary function ψ given in (34). One then gets

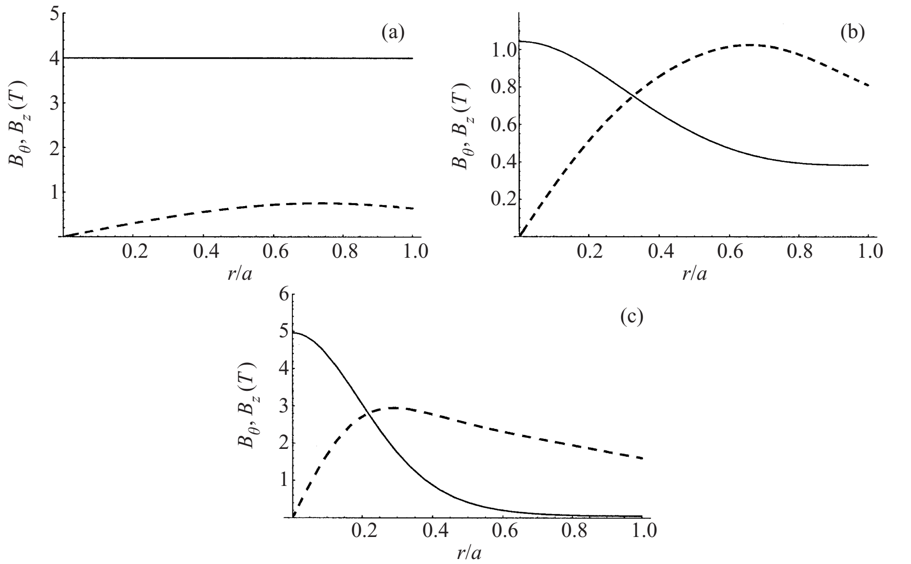

Figure 1. Profiles of the magnetic field components: B_θ (dashed lines) and B_z (solid lines): (a) $\gamma = 1$; (b) $\gamma = 0.5$; (c) $\gamma = 0.1$.

as output the equilibrium values of the field parameter and of the pinch ratio,

$$F_0 = \frac{1}{K_\delta\left(\alpha\dfrac{1-\gamma}{\gamma}\right)}, \qquad \Theta_0 = \frac{\sqrt{K_\delta\left(\dfrac{2\alpha}{\gamma}\right) - 1}}{K_\delta\left(\alpha\dfrac{1-\gamma}{\gamma}\right)}, \tag{42}$$

from which the boundary values of the magnetic field components

$$B_{0\theta}(a) = \frac{\Phi_0}{\pi a^2}\Theta_0, \qquad B_{0z}(a) = \frac{\Phi_0}{\pi a^2}F_0 \tag{43}$$

are computed (as well as the value of the total plasma current, namely $I_{0p} = 2\pi a B_{0\theta}(a)/\mu_0$). So, if one assumes that the boundary values $B_{0\theta}(a)$ and $B_{0z}(a)$ are assigned as well, the free parameters α and δ are to be adjusted in order to match them. Such a procedure was found to work for all the cases that were considered.

3.3. Field profiles

We illustrate now the explicit solutions of the axisymmetric zero-flow equilibrium problem. Three values of the anisotropy parameter γ are considered, namely $\gamma = 1$, corresponding to the isotropic case, $\gamma = 0.5$, which is the value predicted by kinetic theory for fusion plasmas, and $\gamma = 0.1$. Figures 1–6 display, respectively, magnetic field components, current density components, resistivity profile, pressure, alignment cosine and safety factor plotted versus r/a, the normalized radial coordinate. Figure 7 shows the (F, Θ) diagrams. In each figure, three plots are given, corresponding to the three considered values of γ.

We recall that the alignment cosine, denoted by $\cos\{\mathbf{J}, \mathbf{B}\}$ and defined as the cosine of the angle between $\mathbf{J}_0$ and $\mathbf{B}_0$, measures the extent to which and where the equilibrium configuration at hand is approximately force-free; the well-known safety factor is defined as $q \equiv rB_{0z}/RB_{0\theta}$, where $R = L/2\pi$ is the major radius

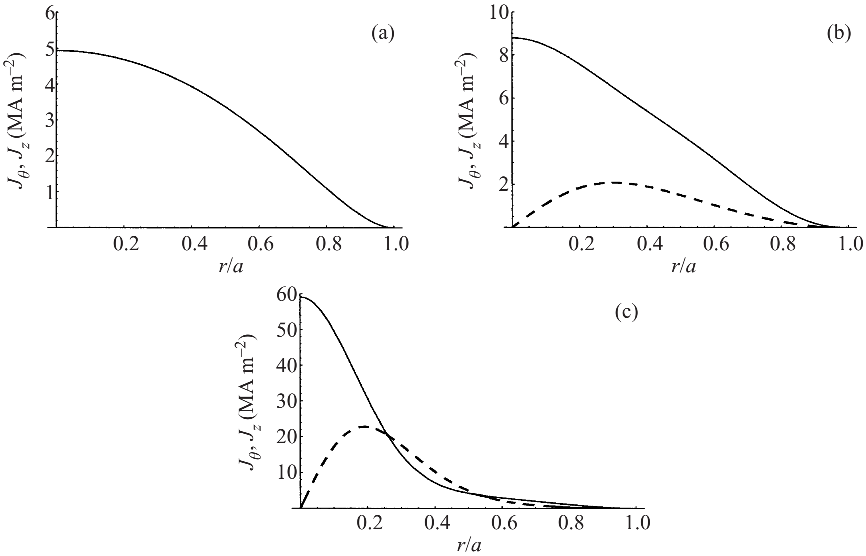

Figure 2. Profiles of the current density components: J_θ (dashed lines) and J_z (solid lines): (a) $\gamma = 1$; (b) $\gamma = 0.5$; (c) $\gamma = 0.1$. Notice that for $\gamma = 1$, one has $J_\theta \equiv 0$.

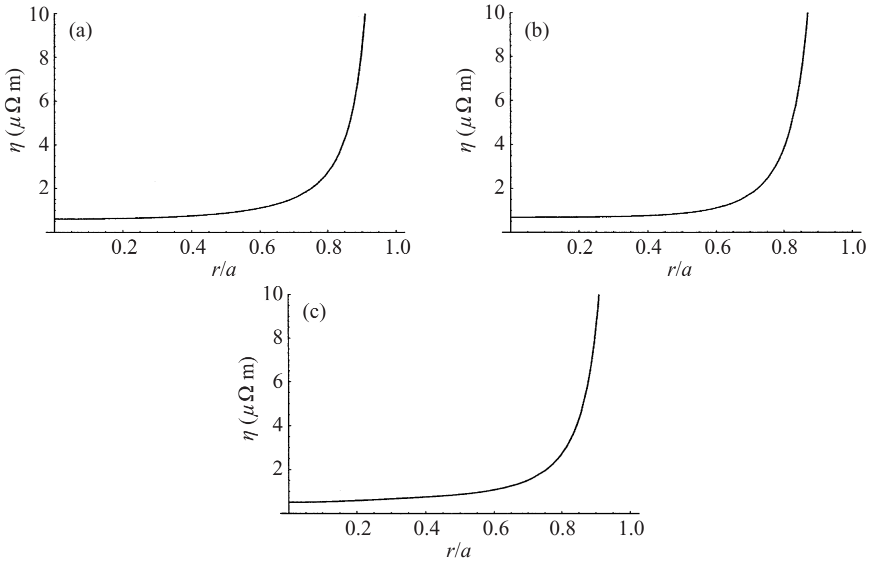

Figure 3. Profiles of the (orthogonal) resistivity: (a) $\gamma = 1$; (b) $\gamma = 0.5$; (c) $\gamma = 0.1$.

of the equivalent straightened torus. The pressure profiles, given by (26), are in practice obtained by numerical integration of the differential equation

$$\frac{d}{dr}p_0(r) = -\frac{E_0}{\eta(r)}B_{0\theta}(r),$$

with the 'arbitrary' boundary condition $p_0(a) = 0$. The (F, Θ) diagrams are obtained from (42) by fixing (in addition to γ) a value for δ, which produces a plain curve with parametric representation $F_0(\alpha)$, $\Theta_0(\alpha)$. It can be shown that the curves

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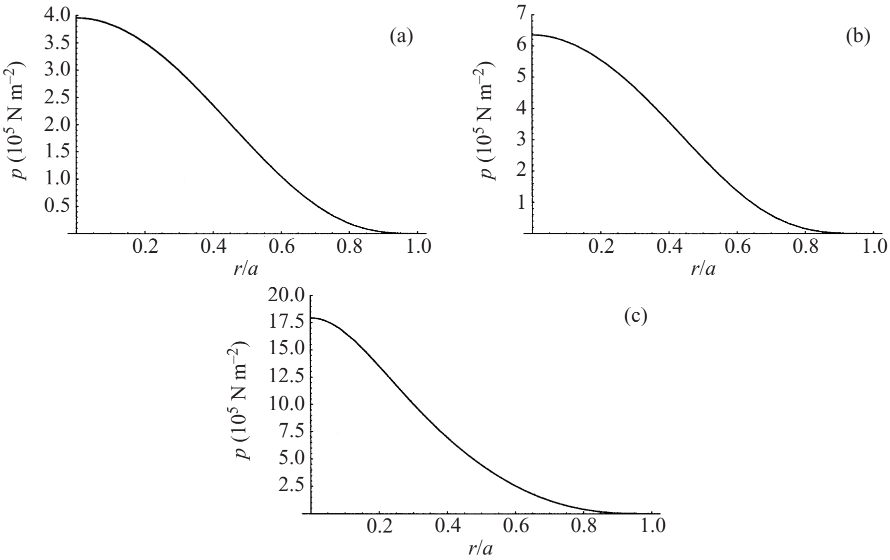

Figure 4. Pressure profiles: (a) $\gamma = 1$; (b) $\gamma = 0.5$; (c) $\gamma = 0.1$.

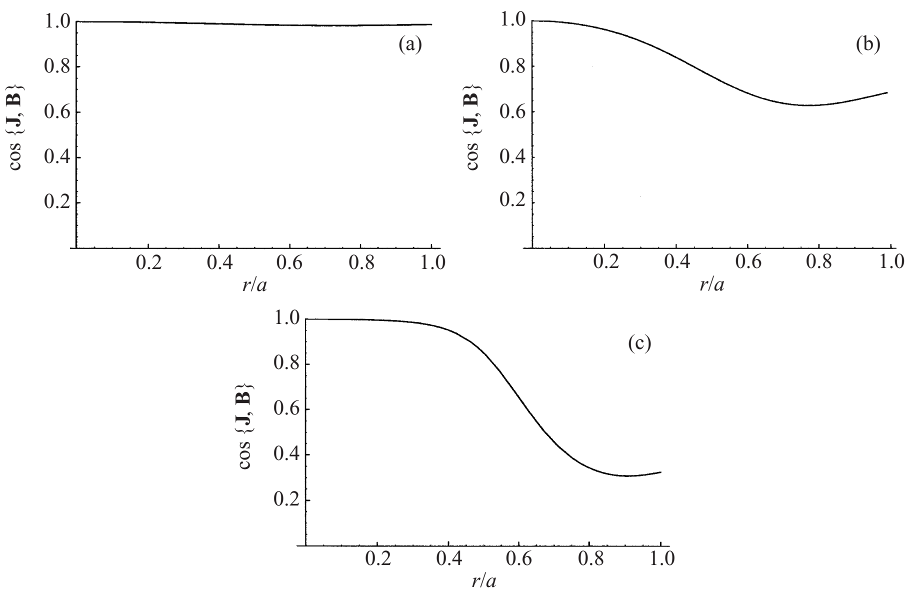

Figure 5. Profiles of the alignment cosine: (a) $\gamma = 1$; (b) $\gamma = 0.5$; (c) $\gamma = 0.1$.

shown in Fig. 7 are nearly independent of the particular value of the parameter δ chosen in a reasonable range.

All the plots are for a cylinder of minor radius $a = 0.5\,\mathrm{m}$ and length $L = 4\pi\,\mathrm{m}$; the externally applied toroidal loop voltage is fixed to $V_{\mathrm{loop}} = 12\pi\,\mathrm{V}$, corresponding to a driving axial field $E_0 = V_{\mathrm{loop}}/L = 3\,\mathrm{V\,m^{-1}}$. Different values of the bias field $\langle B_{0z}\rangle_\Omega = \Phi_0/\pi a^2$ and of the boundary conditions are instead considered for each value of γ. In particular, we have chosen $\langle B_{0z}\rangle_\Omega = 4\,\mathrm{T}$ for $\gamma = 1$ and $\langle B_{0z}\rangle_\Omega = 0.5\,\mathrm{T}$

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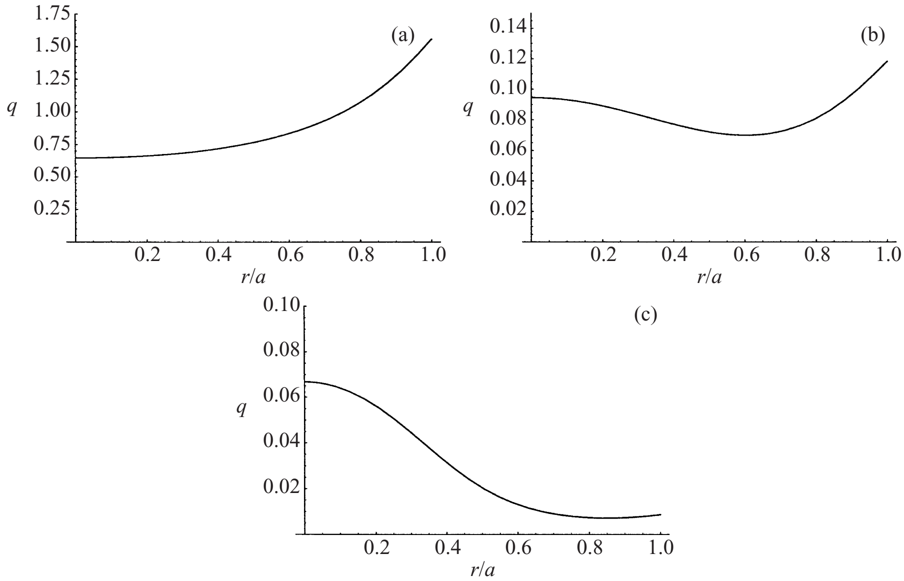

Figure 6. Safety factor profiles: (a) $\gamma = 1$; (b) $\gamma = 0.5$; (c) $\gamma = 0.1$.

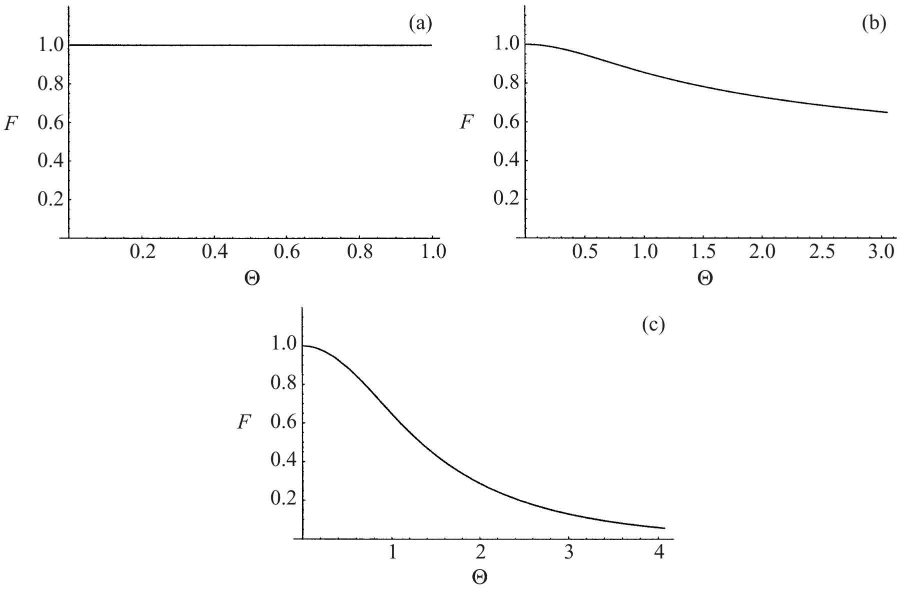

Figure 7. (F, Θ) diagrams: (a) $\gamma = 1$; (b) $\gamma = 0.5$; (c) $\gamma = 0.1$.

for $\gamma = 0.5$ and $\gamma = 0.1$. The different boundary conditions chosen can be seen by inspection from Fig. 1.

The main feature that seems to come out even from a quick inspection of the figures is the relevance of the role of the anisotropy parameter γ. Indeed, while resistivity (Fig. 3) and pressure (Fig. 4) are essentially insensitive to it, the other quantities exhibit a rather strong, physically meaningful dependence. The most paradigmatic case is that of the field B_z (Fig. 1), which is completely flat in the

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isotropic case, while exhibiting a pinch-like effect that increases with anisotropy. Alignment too is seen to be strongly influenced (see Fig. 5): the plasma configuration is everywhere almost force-free inside the chamber for $\gamma = 1$, and it continues to be so only in a central region in the anisotropic case, while at the wall the alignment decreases as the anisotropy increases. The role of γ in connection with the current seems to be particularly interesting (Fig. 2). Indeed, here too the pinch effect is made evident, with the current channel tending to be localized near the axis as the anisotropy increases. This, in fact, was to be expected, because, by the definition of $\gamma = \eta_{\|}/\eta_{\perp}$, it is clear that in the limit case $\gamma \ll 1$ the equilibrium magnetic field lines become nearly perfectly conducting. It is just the relevance of such a limit case that induced us to consider for the anisotropy parameter γ the extended range $0 < \gamma \leqslant 1$.

Concerning the safety factor (q) profiles (Fig. 6), notice that they should not be interpreted in the way familiar in ideal MHD, because the role of q with respect to the stability properties of the equilibria might a priori be different when resistivity is taken into account.

As far as the (F, Θ) diagrams (Fig. 7) are concerned, they display the qualitatively expected behaviour at low Θ, but not at higher values. This makes it clear how different a laboratory RFP state (Bodin and Newton 1980) must be from those obtained in the present paper.

Notice that the main feature pointed out above concerning the dependence of the physically relevant effects (pinch effect and alignment) on anisotropy does not depend on the particular choice made for the bias field $\langle B_{0z}\rangle_\Omega$ and for the boundary conditions. Indeed, the fact that the isotropic case necessarily leads to a flat axial magnetic field profile was explicitly proven in Sec. 3.1, together with the general properties concerning paramagnetic pinches in the anisotropic cases. The choice of the parameters in the figures was just made in order to obtain physically interesting field profiles.

4. Conclusions

One-dimensional resistive MHD equilibria, with field variables depending only upon radius and with no fluid velocity, have been investigated analytically for the periodic cylinder. An attempt has been made to make the enforced (non-ideal) boundary conditions as realistic as possible. The resistivity tensor has been characterized as having proportional, spatially variable, components parallel and perpendicular to the local magnetic field, with an anisotropy parameter that can be varied between wholly isotropic and highly anisotropic limits. The emphasis has been upon determining the effect of an anisotropic resistivity on the allowed profiles.

The principal result of the calculation has been to show that anisotropic resistivity strongly affects the allowed profiles of current, magnetic field, pressure, etc. This has been possible because the modeling of the resistivity has been simple enough to make explicit analytical calculations possible. Several of the features found are not inconsistent with the numerically computed profiles of Goodman (1993, 1998) resulting from a considerably more complicated and less transparent model involving a temperature equation.

A final remark is in order concerning the 'straight-cylinder' approximation used in the present paper. The effects of toroidal curvature on zero-flow equilibria were shown by Montgomery et al. (1997) to be significant for resistive equilbria with only a scalar resistivity. Zero-flow equilibria could be found only by choosing a somewhat artificial, and likely unphysical, spatial profile for that resistivity; otherwise, velocity fields were required. Similar results were found by van der Woude (2000) for the toroidal case using anisotropic resistivity. Whether zero-flow equilibria are possible in the toroidal case purely as a consequence of anisotropic resistivity may still be regarded as an open question worthy of further investigation – one to which we hope to return in a later paper. We also defer until later a consideration of the possible stability of the straight-cylinder equilibria found here.

Appendix A. The effective magnetic helicity

An important quantity to be considered is the total magnetic helicity

$$H_\Omega(t) = \int_\Omega \mathbf{A}\cdot\mathbf{B}\,d^3x; \tag{A 1}$$

this involves the vector potential $\mathbf{A}$, which is defined, up to a gauge transformation $\mathbf{A}\rightarrow\mathbf{A}+\boldsymbol{\nabla}\chi$, by $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, but is in fact gauge-invariant by virtue of the boundary condition $B_r|_{\partial\Omega}=0$.

Let us compute the time derivative of H_Ω. Choosing the Coulomb gauge for the vector potential, i.e. $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, (2) can be integrated, giving

$$\frac{\partial\mathbf{A}}{\partial t} = \mathbf{v}\times\mathbf{B} - \frac{M}{\rho_0 e}\mathbf{J}\times\mathbf{B} - \boldsymbol{\eta}\cdot\mathbf{J} - \boldsymbol{\nabla}\phi, \tag{A 2}$$

where ϕ is a 'gauge' potential, which can be determined by taking the divergence of (A 2) and solving a Poisson equation. One then computes

$$\begin{aligned}\frac{d}{dt}H_\Omega &= \int_\Omega\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial t}+\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)d^3x\\ &= -2\int_\Omega \mathbf{B}\cdot\boldsymbol{\eta}\cdot\mathbf{J}\,d^3x + \int_{\partial\Omega}(\mathbf{A}\times\boldsymbol{\eta}\cdot\mathbf{J})\cdot\mathbf{e}_r\,d^2x,\end{aligned}$$

where standard vector identities have been used. The first term on the right-hand side in the second line represents Ohmic dissipation, and nothing more can be said about it in general. The second one term be evaluated using the boundary conditions (13)–(15); this gives

$$\int_{\partial\Omega}(\mathbf{A}\times\boldsymbol{\eta}\cdot\mathbf{J})\cdot\mathbf{e}_r\,d^2x = \int_{\partial\Omega}A_\theta E_0\,d^2x = (E_0L)a\int_0^{2\pi}A_\theta(a,\theta,z)\,d\theta,$$

the last integral being nothing but the (time-constant) axial magnetic flux Φ. The factor E_0L is also easily interpreted. Indeed, $E_0L = E_0 2\pi R \equiv V_{\text{loop}}$ is nothing but the toroidal loop voltage applied by the external coils to the plasma; in real experiments, it is produced by inducing a variation of the magnetic flux through the hole of the toroidal chamber. Moreover, one easily finds that

$$V_{\text{loop}} = -\frac{d}{dt}\Phi_{\text{hole}}(t),$$

with Φ_{hole} defined by

$$\Phi_{\text{hole}}(t) = \int_0^L A_z(a,\theta,z)\,dz,$$

which is independent of θ. Notice that, by Stokes' theorem, Φ_{hole} represents just the flux induced through the hole of the physical 'curved' torus of major radius $R = L/2\pi$. Notice that the gauge potential ϕ appearing in (A 2) makes no contribution to the time derivative of Φ_{hole}, due to the periodicity in θ and z. So, finally, making use of the explicit expression (5) for the resistivity tensor, one obtains

$$\frac{d}{dt}H_{\text{eff}} = -2\gamma \int_\Omega \eta \mathbf{B}\cdot\mathbf{J} d^3x. \tag{A 3}$$

with the effective magnetic helicity, H_{eff}, being defined by

$$H_{\text{eff}} = H_\Omega + \Phi_{\text{hole}}\Phi. \tag{A 4}$$

This is indeed the interesting quantity to be considered, because it takes into account the additional contribution to the helicity due to the linkage of the magnetic field lines residing inside the chamber (which, being flux-preserving, constitutes a magnetic flux tube) with those passing through the hole of the physical torus.

From (A 3), one sees that the magnetic helicity H_{eff}, (A 4), is an integral of motion not only in the ideal limit (infinite conductivity), but also in the limit case $\gamma = 0$, i.e. for perfectly conducting magnetic field lines.

Appendix B. The integral (35)

Here we want to prove (35). Let us denote by $I(x)$ the integral appearing in the left-hand side of this formula, i.e.

$$I(x) \equiv \int_x^1 \exp\left[\frac{2\alpha}{\gamma}(1-s^2)^{\delta+1}\right] d(s^2);$$

by the change of variables

$$\xi(s) = \frac{2\alpha}{\gamma}(1-s^2)^{\delta+1}, \tag{B 1}$$

one gets

$$I(x) = G[\xi(x)] \equiv \frac{1}{\delta+1}\left(\frac{\gamma}{2\alpha}\right)^{1/(\delta+1)} \int_0^{\xi(x)} e^t t^{-\delta/(\delta+1)}\, dt, \tag{B 2}$$

Now, it is easily shown that the function $G(z)$ defined above is such that

$$\lim_{z\to 0^+} \frac{G(z)}{(z\gamma/2\alpha)^{1/(\delta+1)}} = 1,$$

and, as a consequence, is of the form

$$G(z) = \left(\frac{z\gamma}{2\alpha}\right)^{1/(\delta+1)} H(z), \tag{B 3}$$

with H satisfying $H(0) = 1$. Exploiting the definition (B 2), one easily checks that G satisfies the differential equation

$$\frac{d^2}{dz^2}G(z) = \left(1 - \frac{\delta}{\delta+1}\frac{1}{z}\right)\frac{d}{dz}G(z). \tag{B 4}$$

Taking (B 3) into account, one easily gets from (B 4) the differential equation satisfied by H, namely

$$z\frac{d^2}{dz^2}H(z) + \left(\frac{\delta+2}{\delta+1} - z\right)\frac{d}{dz}H(z) - \frac{1}{\delta+1}H(z) = 0. \tag{B 5}$$

As is well known (see Abramowitz and Stegun 1965), the solution of such an equation, regular at the origin and satisfying the condition $H(0) = 1$, is the Kummer confluent hypergeometric function with indices $1/(\delta+1)$ and $(\delta+2)/(\delta+1)$. Let us denote by $K_\delta(z)$ this function. Thus, making use of (B 2), (B 3) and (B 1), one finally gets

$$I(x) = G[\xi(x)] = \left(\frac{\xi(x)\gamma}{2\alpha}\right)^{1/(\delta+1)} K_\delta[\xi(x)] = (1-x^2)K_\delta\left[\frac{2\alpha}{\gamma}(1-x^2)^{\delta+1}\right],$$

i.e. just the formula (35).

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