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Impulse and conformal mapping of vortex flows

By T. W. G. DE LAAT

Netherlands Defence Academy, PO Box 90 002, 4800 PA Breda, The Netherlands

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The concept of impulse is employed with conformal mapping to yield relatively simple relations for the force exerted on a two-dimensional stationary object by an incompressible irrotational and unsteady flow with moving vortices. An explicit relation for symmetric vortex flows is found, involving the vortex strength and the first and second derivatives of the mapping function evaluated at the vortex position. Furthermore an expression for not-necessarily symmetric vortex flows is derived, containing vortex strength, the first derivative of the mapping function evaluated at the vortex position, and the vortex velocity.

1. Introduction

The evaluation of the force on a body due to an unsteady flow field is generally a laborious process of integration of the unsteady pressure. Such a flow field is especially unsteady when moving vortices are present. From Helmholtz's vorticity laws, e.g. Batchelor (1967, p. 274) or Saffman (1995, p. 10), it is known that in inviscid flow these vortices move with the local flow velocity. In the present paper it is shown that employing the concept of the impulse of a two-dimensional incompressible irrotational and unsteady flow field with (moving) point vortices, combined with conformal mapping, leads to relatively simple relations to calculate the force on a stationary object.

The *impulse* of a velocity field $\boldsymbol{u}$ within volume $\mathscr{V}$ bounded by surface S is defined by the time integral of the impulsive force $\boldsymbol{F}$, which must be applied on the flow during a short time τ to generate the velocity field from rest, e.g. Lighthill (1996, p. 80). The force can be imagined to be the result of an impulsive pressure distribution p applied on the bounding surface. The impulse exerted on the flow from rest equals the volume integral of the momentum of the flow field, e.g. Milne-Thomson (1967, p. 91), for incompressible flow, so we can write

$$\boldsymbol{I} = \int_0^\tau \boldsymbol{F}\,\mathrm{d}t = -\int_0^\tau \int_S p\boldsymbol{n}\,\mathrm{d}S\,\mathrm{d}t = \rho \int_{\mathscr{V}} \boldsymbol{u}\,\mathrm{d}\mathscr{V}, \tag{1.1}$$

with $\boldsymbol{n}$ the unit normal at surface S directed out of the fluid and ρ the density of the fluid. For irrotational flow a velocity potential ϕ can be defined with $\boldsymbol{u} = \nabla\phi$. Substitution in (1.1) yields, after applying the divergence theorem

$$\boldsymbol{I} = \int_0^\tau \boldsymbol{F}\,\mathrm{d}t = -\int_0^\tau \int_S p\boldsymbol{n}\,\mathrm{d}S\,\mathrm{d}t = \rho \int_S \phi\boldsymbol{n}\,\mathrm{d}S. \tag{1.2}$$

From this equation it follows that the integral of the impulsive pressure p over the bounding surface needed to generate the motion from rest in short time τ equals the integral of $-\rho\phi$ over that surface. It is noted by Lamb (1932, p. 161)† that this definition of impulse was first used by Lord Kelvin and that the calculation of the momentum of the system (1.1) generally leads to indeterminate integrals. For rectilinear vortices or point vortices in a plane the impulse can be calculated, however. Lamb (1932) describes the application of the concept of impulse for vortex pairs in two-dimensional flow. In the present paper the impulse of a two-dimensional vortex pair is employed in combination with conformal mapping.

The surface integral of ϕ (1.2) has been applied by various authors to different surfaces with various names. The impulse of part of a fluid system is sometimes called the *Kelvin impulse*, which was the case for the application of the integral of ϕ over the surface of a cavity in a flow by Best & Blake (1994). Saffman (1995, p. 74) named the surface integral of ϕ (1.2) over the surface S of a body moving through an inviscid incompressible fluid, the *virtual momentum* of that body. Lighthill (1996, p. 135) took integral (1.2) over the surface surrounding the entire fluid as the *total impulse* acting on the fluid. This is the definition applicable in the present paper.

We now consider a stationary object. The virtual momentum or Kelvin impulse of such an object remains zero, e.g. Saffman (1995, p. 51). Furthermore, due to the conservation of momentum, the force acting on a stationary object in a flow field, which requires a reaction force to keep the body in place, equals the opposite of the time rate of change of the impulse of the flow field. If the flow contains vortices, which are known to move with the flow from Helmholtz's vortex laws, the motion of the vortices is the source of unsteadiness. The velocity potential ϕ is then implicitly time dependent through the motion of the vortices and from (1.2) it follows that the impulse varies with time, which results in a force on the stationary object.

For two-dimensional applications (1.2) can be calculated in the complex plane, analogous to the derivation of Blasius' theorem, e.g. Milne-Thomson (1967, pp. 173–174). For a complex potential $\chi=\phi+\mathrm{i}\psi$, with ϕ the velocity potential and ψ the streamfunction, we have for the integral along a streamline $\psi=\text{constant}=c(t)$, and consequently $\chi=\phi+\mathrm{i}c(t)$. The integral of ψ on a closed streamline yields zero, so we may write (1.2) in complex form, and defining $\mathscr{I}\equiv I_x+\mathrm{i}I_y$, as

$$\mathscr{I}\equiv I_x+\mathrm{i}I_y=\mathrm{i}\rho\int_C \chi(z)\,\mathrm{d}z, \tag{1.3}$$

with $z=x+\mathrm{i}y$ and C a closed streamline surrounding the entire fluid. Such a streamline surrounding the entire fluid may in general not be feasible. In the case of point vortices which we will be evaluating, integral (1.3), however, only yields non-zero values if a singularity representing a point vortex is enclosed within a contour. Furthermore it is generally possible to divide the fluid into regions bounded by streamlines with and without singularities. The regions with singularities yield non-zero values from Cauchy's residue theorem, e.g. Milne-Thomson (1967, p. 138), whereas the contour integrals of the remaining regions yield zero.

We now consider the flow field of a point vortex of strength $\varGamma$ on $z=z_1$. This flow field has the complex potential

$$\chi(z;z_1)=-\frac{\mathrm{i}\varGamma}{2\pi}\ln(z-z_1). \tag{1.4}$$

† Lamb (1932) used the definition of velocity potential with minus sign: $\boldsymbol{u}=-\nabla\phi$, so he reported the impulsive pressure $+\rho\phi$.

Substitution into (1.3) yields

$$\mathscr{I} \equiv I_x + \mathrm{i}I_y = \rho\frac{\Gamma}{2\pi}\int_C \ln(z - z_1)\,\mathrm{d}z,$$

which we can write as

$$\mathscr{I} \equiv I_x + \mathrm{i}I_y = \rho\frac{\Gamma}{2\pi}\int_C \ln z + \ln\left(1 - \frac{z_1}{z}\right)\mathrm{d}z.$$

For $z \to \infty$ we can apply the series representation for the logarithm, upon which we obtain

$$\mathscr{I} \equiv I_x + \mathrm{i}I_y = \rho\frac{\Gamma}{2\pi}\int_C \ln z - \frac{z_1}{z} - \frac{1}{2}\left(\frac{z_1}{z}\right)^2 - O\left(\left(\frac{z_1}{z}\right)^3\right)\mathrm{d}z.$$

Application of Cauchy's residue theorem yields

$$\mathscr{I} \equiv I_x + \mathrm{i}I_y = \rho\frac{\Gamma}{2\pi}2\pi\mathrm{i}(-z_1) = -\mathrm{i}\rho\Gamma z_1. \tag{1.5}$$

The impulse can also be formulated utilizing the concept of vorticity $\boldsymbol{\omega}$ instead of (1.2). This is treated in e.g. Batchelor (1967, p. 519), Saffman (1995, p. 50) or Lighthill (1996, p. 213), and leads to volume integral

$$\boldsymbol{I} = \frac{1}{2}\rho\int_{\mathscr{V}} \boldsymbol{x} \times \boldsymbol{\omega}\,\mathrm{d}\mathscr{V}, \tag{1.6}$$

with $\boldsymbol{x}$ being the position vector from the origin. This expression is also given in Lamb (1932, p. 215), though written in three separate components. Lamb (1932, p. 229) gives the two-dimensional expressions per unit depth for vortex pairs, where the factor $\frac{1}{2}$ has vanished. The two-dimensional impulse (Lamb 1932, p. 229) per unit depth can be written in complex formulation as two-dimensional surface integral

$$\mathscr{I} \equiv I_x + \mathrm{i}I_y = -\mathrm{i}\rho\int_S z\omega\,\mathrm{d}S. \tag{1.7}$$

If an amount of vorticity is assumed to be concentrated at point $z = z_1$ and we substitute $\omega\,\mathrm{d}S = \Gamma$ into (1.7) we again obtain (1.5).

As we know from Helmholtz's vortex laws, a vortex always appears in a closed filament (or ends on a boundary surface). In two-dimensional flow with the vortices perpendicular to the plane of the flow, we thus generally deal with vortex pairs consisting of two vortices with vortex strength of equal magnitude but opposite direction. The impulse of such a vortex pair is then easily found with equation (1.5).

As stated above, in the case that the body is stationary the impulse of the body itself remains zero, so the integral of the entire flow field then equals the impulse of the outer flow field. The force acting on the stationary body, which we indicate as $\mathscr{F} \equiv F_x + \mathrm{i}F_y$, equals the negative of the time rate of change of the impulse of the outer flow field, which we thus express by

$$\mathscr{F} = -\mathrm{d}\mathscr{I}/\mathrm{d}t. \tag{1.8}$$

We will now first consider the force on a symmetric body in uniform flow with a symmetric vortex pair and subsequently the force on an arbitrary body in uniform flow with a single vortex. Three examples are given to demonstrate the application of the expressions obtained.

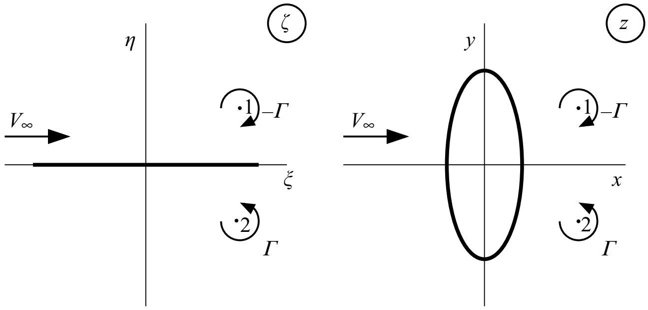

FIGURE 1. Transformation of a vortex pair in uniform flow to a vortex pair near a contour, symmetric with respect to the uniform flow; $\zeta = \xi + \mathrm{i}\eta$ and $z = x + \mathrm{i}y$.

2. The force on a symmetric body in uniform flow with a symmetric vortex pair

We start with the impulse of a vortex pair consisting of two point vortices with vortex strength of equal magnitude but opposite direction, as shown in the ζ-plane of figure 1. The impulse of this vortex pair follows from equation (1.5). As the vortex pair in the ζ-plane of figure 1 consists of vortices with strength $-\Gamma$ at ζ_1 and Γ at $\zeta_2 = \overline{\zeta}_1$, the impulse of that vortex pair equals

$$\mathscr{I} = I_\xi(\zeta_1, \overline{\zeta}_1) = \mathrm{i}\rho\Gamma\{\zeta_1 - \overline{\zeta}_1\}, \tag{2.1}$$

with $\mathscr{I} \equiv I_\xi + \mathrm{i}I_\eta$ and $\overline{\zeta}_1$ indicating the complex conjugate of ζ_1. Due to the symmetry with respect to the ξ-axis the impulse has a component in the ξ-direction only. An expression for the force on the object in the z-plane will now be formulated by employing a conformal transformation to the flow field of this vortex pair in the ζ-plane, and using the implicit time dependence of the flow through the motion of the vortices.

Let $\zeta(z)$ be a conformal mapping function, which transforms the flow field in the ζ-plane, as shown in figure 1, to the flow field in the z-plane, which is symmetric with respect to the x-axis. It is assumed that the conformal mapping function has real coefficients, implying $\overline{\zeta(z)} = \zeta(\overline{z})$, and that it keeps the vortices symmetric and outside the contour. We consider the flow field of figure 1 in the ζ-plane, which is a uniform flow combined with a free vortex pair symmetric with respect to the direction of the uniform flow. The flow in the z-plane is the physical flow, in which we want to calculate the force on the body. It can be obtained from the flow in the ζ-plane through the substitution of the appropriate conformal transformation $\zeta(z)$. The force can be calculated from (1.8), using the implicit time dependence of the impulse resulting from the motion of the vortices, by

$$\mathscr{F} = -\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}(z_1(t), \overline{z}_1(t)) = -\frac{\partial\mathscr{I}}{\partial z_1}\frac{\mathrm{d}z_1}{\mathrm{d}t} - \frac{\partial\mathscr{I}}{\partial\overline{z}_1}\frac{\mathrm{d}\overline{z}_1}{\mathrm{d}t}. \tag{2.2}$$

With conformal transformation $z = z(\zeta)$ we have, introducing χ^* as the transformed potential function,

$$\chi(z; z_1, \overline{z}_1) = \chi(z(\zeta); z_1(\zeta_1), \overline{z}_1(\overline{\zeta}_1)) = \chi^*(\zeta; \zeta_1, \overline{\zeta}_1), \tag{2.3}$$

and thus for the impulse of the entire flow field (1.3):

$$\mathscr{I}(z_1, \overline{z}_1) \equiv I_x + \mathrm{i}I_y = \mathrm{i}\rho\int \chi(z; z_1, \overline{z}_1)\,\mathrm{d}z. \tag{2.4}$$

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With conformal transformation $z=z(\zeta)$ we have, introducing $\mathscr{I}^*$ as the transformed impulse,

$$\mathscr{I}(z_1,\bar{z}_1)=\mathscr{I}(z_1(\zeta_1),\bar{z}_1(\bar{\zeta}_1))=\mathscr{I}^*(\zeta_1,\bar{\zeta}_1). \tag{2.5}$$

The impulse can be calculated in the ζ-plane from

$$\mathscr{I}^*(\zeta_1,\bar{\zeta}_1)\equiv I_\xi+\mathrm{i}I_\eta=\mathrm{i}\rho\int\chi^*(\zeta;\zeta_1,\bar{\zeta}_1)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta. \tag{2.6}$$

We now use the impulse transformation (2.5) to calculate the force on the stationary object in the physical z-plane from the simple flow field in the ζ-plane from the rate of change in time of the impulse (2.2), by

$$\mathscr{F}=-\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t}=-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}^*(\zeta_1(z_1(t)),\bar{\zeta}_1(\bar{z}_1(t)))=-\frac{\partial\mathscr{I}^*}{\partial\zeta_1}\frac{\mathrm{d}\zeta_1}{\mathrm{d}z_1}\frac{\mathrm{d}z_1}{\mathrm{d}t}-\frac{\partial\mathscr{I}^*}{\partial\bar{\zeta}_1}\frac{\mathrm{d}\bar{\zeta}_1}{\mathrm{d}\bar{z}_1}\frac{\mathrm{d}\bar{z}_1}{\mathrm{d}t}. \tag{2.7}$$

Substitution of the impulse of the vortex pair (2.1), which is now indicated as $\mathscr{I}^*(\zeta_1,\bar{\zeta}_1)$, into (2.7) yields

$$\mathscr{F}=-\mathrm{i}\rho\Gamma\left(\frac{\mathrm{d}\zeta_1}{\mathrm{d}z_1}\frac{\mathrm{d}z_1}{\mathrm{d}t}-\frac{\mathrm{d}\bar{\zeta}_1}{\mathrm{d}\bar{z}_1}\frac{\mathrm{d}\bar{z}_1}{\mathrm{d}t}\right). \tag{2.8}$$

For the transformation $\zeta(z)$ with real coefficients, (2.8) can be written as

$$\mathscr{F}=2\rho\Gamma\,\mathrm{Im}\left\{\frac{\mathrm{d}\zeta_1}{\mathrm{d}z_1}\frac{\mathrm{d}z_1}{\mathrm{d}t}\right\}, \tag{2.9}$$

with Im indicating the imaginary part of a complex variable. The vortex velocity in the z-plane, $\mathrm{d}z_1/\mathrm{d}t$, is related to the vortex velocity in the ζ-plane. However, the vortex velocity does not simply transform with the conformal mapping. It changes according to the change of the path function described by Routh's correction, see e.g. Lugt (1996, p. 162), which in integral form is known as Routh's theorem, e.g. Milne-Thomson (1967, p. 372). The vortex velocity in the physical z-plane can be calculated using the complex velocity potential in the ζ-plane, using Routh's correction, by

$$\overline{\frac{\mathrm{d}z_1}{\mathrm{d}t}}=\frac{\mathrm{d}x_1}{\mathrm{d}t}-\mathrm{i}\frac{\mathrm{d}y_1}{\mathrm{d}t}=\left(\frac{\mathrm{d}\chi^*}{\mathrm{d}\zeta}-\frac{\mathrm{i}\Gamma}{2\pi}\frac{1}{\zeta-\zeta_1}\right)_{\zeta=\zeta_1}\zeta_1'+\frac{\mathrm{i}\Gamma}{4\pi}\left(\frac{\zeta_1''}{\zeta_1'}\right), \tag{2.10}$$

with $\zeta_1'\equiv(\mathrm{d}\zeta/\mathrm{d}z)_{z=z_1}$ and $\zeta_1''\equiv(\mathrm{d}^2\zeta/\mathrm{d}z^2)_{z=z_1}$. Substitution of the complex conjugate of (2.10) into (2.9) and utilization of $\mathrm{d}\zeta_1/\mathrm{d}z_1=(\mathrm{d}\zeta/\mathrm{d}z)_{z=z_1}$, yields

$$\mathscr{F}=F_x=2\rho\Gamma\mathrm{Im}\left\{\overline{\left(\frac{\mathrm{d}\chi^*}{\mathrm{d}\zeta}-\frac{\mathrm{i}\Gamma}{2\pi}\frac{1}{\zeta-\zeta_1}\right)}_{\zeta=\zeta_1}|\zeta_1'|^2\right\}-\frac{\rho\Gamma^2}{2\pi}\mathrm{Re}\left\{\frac{\overline{\zeta_1''}}{\overline{\zeta_1'}}\zeta_1'\right\},$$

with Re indicating the real part of a complex variable. The first term is proportional to the imaginary part of the vortex velocity in the ζ-plane, which equals zero, so there only remains

$$\mathscr{F}=F_x=-\frac{\rho\Gamma^2}{2\pi}\mathrm{Re}\left\{\frac{\overline{\zeta_1''}}{\overline{\zeta_1'}}\zeta_1'\right\}. \tag{2.11}$$

Now we have an expression (2.11) for the force on the contour due to the symmetric vortex pair in terms of its strength and the derivatives of the transformation. Compliance with the boundary condition on the contour is assured by the conformal mapping. It is interesting to note that this force is invariant to the strength of the uniform flow V_∞. We will now consider the special case of a circular cylinder in uniform flow with a symmetric vortex pair as in figure 2.

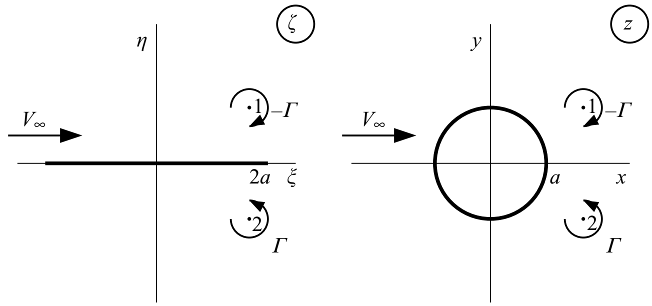

FIGURE 2. Joukowski transformation $\zeta = z + a^2/z$ of a symmetric vortex pair in a uniform flow to a symmetric vortex pair near a circle in uniform flow.

2.1. *Force on a circular cylinder in uniform flow with a symmetric vortex pair*

To demonstrate the application of relation (2.11), the force on a circular cylinder in the flow field of the right-hand side of figure 2 will now be calculated. Equilibrium positions where the vortex velocity equals zero were first calculated by L. Föppl, e.g. Milne-Thomson (1967, p. 370). The force on the cylinder was first reported correctly by Bickley (1928) and Tomotika & Sugawara (1938), carrying out the laborious integration of the unsteady pressure distribution. Sarpkaya (1963) showed that this force is more efficiently evaluated using an extension of Lagally's theorem combined with the vortex velocities. The force on the cylinder as a function of vortex strength and position can, however, more simply be obtained through the use of relation (2.11). Substitution of the Joukowski transformation $\zeta = z + a^2/z$ (figure 2), which gives the contour of a circle with radius a in the z-plane, into (2.11) yields after some simplification

$$F_x = -\frac{\rho \Gamma^2}{\pi} \frac{a^2}{r_1^2} \mathrm{Re}\left\{ \frac{z_1^2 - a^2}{z_1(\bar{z}_1^2 - a^2)} \right\}, \tag{2.12}$$

with $r_1 = |z_1|$. Substitution of polar coordinates $z_1 \equiv r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ yields

$$F_x = \frac{\rho \Gamma^2}{\pi} \frac{a^2}{r_1^3} \cos\theta_1 \left\{ \frac{4r_1^2 \sin^2\theta_1 - \left(r_1 - a^2/r_1\right)^2}{4a^2 \sin^2\theta_1 + \left(r_1 - a^2/r_1\right)^2} \right\}. \tag{2.13}$$

This relation represents the unsteady force (including the $\rho\partial\phi/\partial t$-term) of a symmetric vortex pair moving with the local velocity in the neighbourhood of a circle in a uniform flow, as a function of the vortex position, which was found through a laborious unsteady pressure integration by Bickley (1928) and Tomotika & Sugawara (1938). In de Laat & Coene (2002) this case is further discussed and a contour plot of the force is produced.

3. The force on an arbitrary two-dimensional body in uniform flow with a single vortex

To obtain the force exerted by a single vortex on an arbitrary two-dimensional stationary body in uniform flow, we use the flow field of a vortex near a circular cylinder in uniform flow. Using conformal mapping from this flow field we can obtain an arbitrary non-symmetric flow as depicted in figure 3, as opposed to the symmetric flow fields of § 2. The force can now be obtained by evaluation of the impulse of the vortex pair formed by the free vortex and its well-known mirror image in the circle, e.g. Saffman (1995, p. 42). A vortex may be added in the centre of the circle.

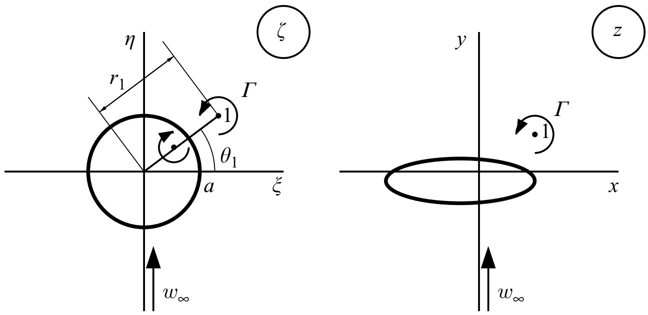

FIGURE 3. Conformal transformation of the non-symmetric flow field with one vortex in uniform flow near a circle to the flow field of a general contour in uniform flow.

Sacks (1955) applied such a conformal mapping to the flow of vortices near a circle to calculate the force on an arbitrary contour. He investigated the development of a vortex along a slender body at an incidence angle in oncoming flow. Evaluating the pressure integrals in the transformed circle plane of the two-dimensional crossflow, he found a proportionality between the lateral force on a wing–body and the impulse of the vortex pair consisting of the free vortex and its mirror-image in the transformed circle plane at the trailing edge of the slender body. He did not evaluate time-dependent flows, though he considered the vortices to change strength and position in the crossflow when travelling in the longitudinal direction from the origin to the trailing edge of a wing. The results can be interpreted as the rate of change (when travelling in the longitudinal direction) of the crossflow vortex strength and position. The crossflow impulse thus has a rate of change (in the longitudinal direction) which yields the force on the three-dimensional object in the crossflow direction. As the impulses of the physical and transformed planes are related via the transformation, the impulse of the free vortices and their mirror images inside the transformed circle result from the pressure integrals.

The concept of impulse as applied in §2 can also be used for the unsteady two-dimensional non-symmetric flow field of a vortex near a stationary object in uniform flow. We now derive the expression for the force using the impulse of the free vortex and its image inside the circle, combined with the conformal transformation function and the vortex velocity in the physical z-plane.

Let $\zeta(z)$ be a conformal mapping function, transforming the flow field around a circle with a vortex in the ζ-plane to a flow field around a contour in the z-plane, see figure 3, while keeping the vortex outside that contour. Equation (2.7) gives the relation between the rate of change in time of the impulse in the transformed ζ-plane and the force in the physical z-plane. The impulse in the transformed ζ-plane of figure 3, of the entire flow field represented by a free vortex at $\zeta=\zeta_1=r_1\,\mathrm{e}^{\mathrm{i}\theta_1}$ and its mirror image, e.g. Saffman (1995, p. 42), can be written, using the relation for the impulse of a vortex pair given in §2, in complex notation as

$$\mathscr{I}^* \equiv I_\xi + \mathrm{i}I_\eta = \rho\Gamma\left(r_1-\frac{a^2}{r_1}\right)\mathrm{e}^{\mathrm{i}(\theta_1-\pi/2)} = -\mathrm{i}\rho\Gamma\left(\zeta_1-\frac{a^2}{\bar{\zeta}_1}\right). \tag{3.1}$$

The impulse of the uniform flow around the circular cylinder is assumed steady, so it does not directly contribute to the rate of change of impulse of the flow. The uniform flow field, however, does contribute to the vortex velocity and consequently has an influence on the rate of change in time of the impulse of the vortex pair. The impulse of the vortex pair which consists of a free vortex and its forced mirror image inside the circle, which is placed there to maintain the boundary condition on the circle, does change in time as a result of the motion of the free vortex and the related motion of the mirror vortex. Substitution of impulse (3.1) into (2.7) yields for the force on the body in the z-plane

$$\mathscr{F} \equiv F_x + \mathrm{i}F_y = \mathrm{i}\rho\Gamma\left\{\frac{\mathrm{d}\zeta_1}{\mathrm{d}z_1}\frac{\mathrm{d}z_1}{\mathrm{d}t} + \frac{a^2}{\overline{\zeta}_1^2}\frac{\mathrm{d}\overline{\zeta}_1}{\mathrm{d}\overline{z}_1}\frac{\mathrm{d}\overline{z}_1}{\mathrm{d}t}\right\}. \tag{3.2}$$

It is noted that the vortex pair in the circle plane of figure 3 could have been obtained from a Möbius transformation from the vortex pair on the left-hand side of figure 2, but the uniform flow would then have been deformed. The flow of the circle plane of figure 3, however, is also simple and therefore a suitable reference flow to be transformed into a flow field with a complicated contour.

We now consider an example of the application of (3.2), after first evaluating the force on the circular cylinder in uniform upwash and a single vortex filament, as shown in the ζ-plane of figure 3.

3.1. *Force of a vortex near a circular cylinder in upwash*

When investigating the force on the body in the z-plane it is of interest to know the force on the circular contour in the ζ-plane, see figure 3. This force is found from (3.2), without transformation, or by putting $\zeta_1 = z_1$. The vortex velocity is obtained from the complex velocity potential by $\mathrm{d}\overline{z}_1/\mathrm{d}t = (\mathrm{d}\chi/\mathrm{d}z)_{z=z_1}$, omitting the infinite self-induced vortex velocity, with the flow field in the ζ-plane being described by the complex potential χ^*:

$$\chi^*(\zeta;\zeta_1,\overline{\zeta}_1) = \frac{\mathrm{i}\Gamma}{2\pi}\ln\left(\frac{\zeta - a^2/\overline{\zeta_1}}{\zeta-\zeta_1}\right) - \mathrm{i}w_\infty\left(\zeta - \frac{a^2}{\zeta}\right). \tag{3.3}$$

The vortex velocity is substituted into (3.2), which yields, with $\zeta_1 = z_1$ and upon making the force dimensionless,

$$C_x + \mathrm{i}C_y \equiv \frac{F_x + \mathrm{i}F_y}{\frac{1}{2}\rho w_\infty^2 2a} = 2\pi G\left\{\frac{a^4}{z_1^2\overline{z}_1^2} + G\frac{a}{\overline{z}_1} - 1\right\}, \tag{3.4}$$

with $G \equiv \Gamma/(2\pi a w_\infty)$. Contour plots of the dimensionless force components in the x- and y-direction for $G = 1$ are depicted in figure 4. The force component C_x has extreme values on the real axis of $C_x = -2\pi G((3/16)(2G)^{4/3} + 1)$ at $x/a = -(4/G)^{1/3}$ and $C_x = 2\pi G^2$ at $x/a = 1$. The component C_y has extreme values on the y-axis of $C_y = 2\pi G^2$ at $y/a = 1$ and $C_y = -2\pi G^2$ at $y/a = -1$. In figure 4(c) a point with $|Cx + \mathrm{i}Cy| = 0$ is indicated, which relates to the equilibrium point, where the vortex velocity equals zero. The positive-force and negative-force areas are interesting features of the flow in view of potential applications of such a vortex flow.

3.2. *Force of a vortex on a wing–body combination in upwash*

A single vortex above a wing–body configuration arises from the slender body approximation of an airplane with a vortex over the wing. Such a vortex may be formed by the forebody, a sharp edge or a control surface, or might come from another aircraft. Flow fields of this kind were studied by Sacks (1955) and Nielsen (1960) for multiple vortex pairs, though without evaluating the $\rho\partial\phi/\partial t$-part. Nielsen (1960, p. 100) furthermore used the approximation that $\mathrm{d}z/\mathrm{d}\zeta = 1$. We will now use conformal mapping to transform a simple wing–body combination to a circle, by combining two transformations as depicted in figure 5. We have the same transformation as in Sacks (1955) and

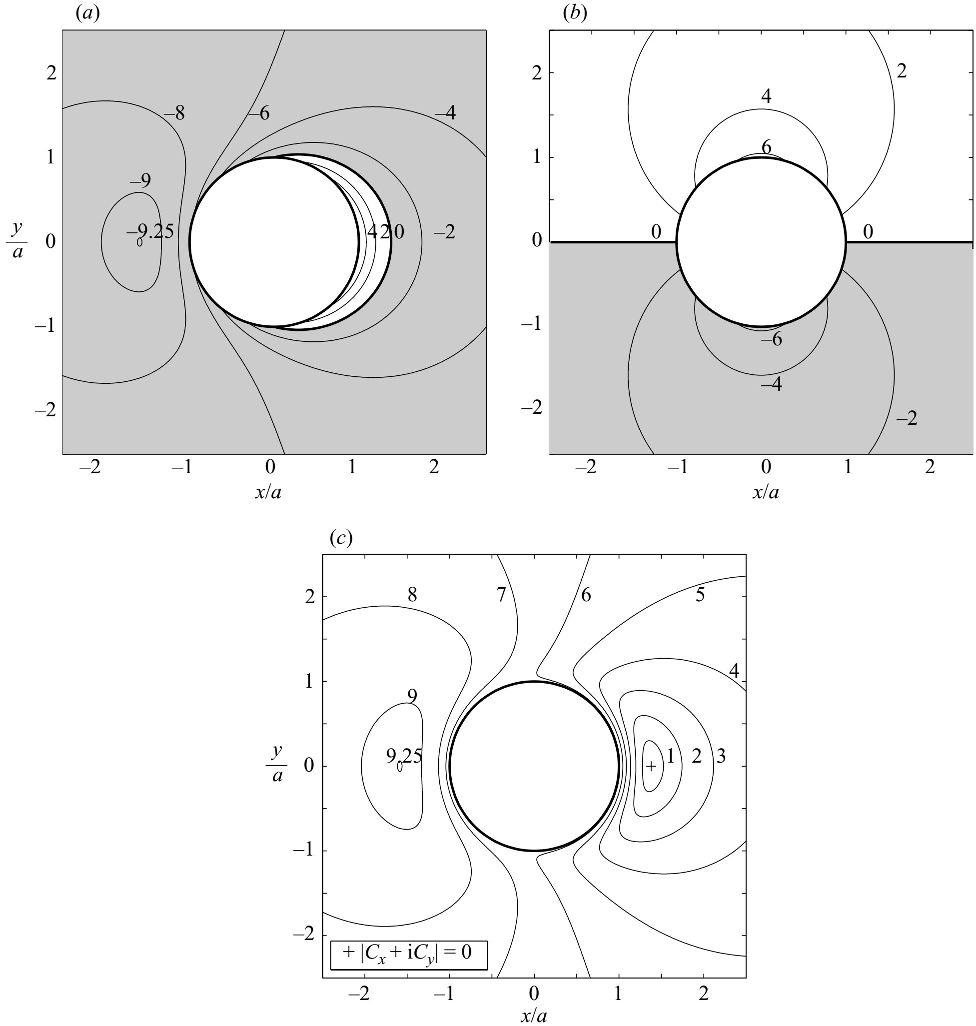

FIGURE 4. Force of a vortex near a circular cylinder in upwash for $G \equiv \Gamma/(2\pi a w_\infty) = 1$, areas in grey have negative values. (*a*) Contour plot of C_x. (*b*) Contour plot of C_y. (*c*) Contour plot of $|C_x + iC_y|$.

de Laat & Coene (1995), with $\zeta^* = \zeta + a^2/\zeta$ and $\zeta^* = z + c^2/z$ (with $\zeta = \xi + i\eta$ and $z = x + iy$). Elimination of ζ^* and expressing ζ in terms of z, we have

$$\zeta = \frac{1}{2}\left(z + \frac{c^2}{z} + \sqrt{\left(z + \frac{c^2}{z}\right)^2 - 4a^2}\right), \tag{3.5}$$

from which it follows that we have semi-span $s = a + \sqrt{a^2 - c^2}$ in the z-plane of figure 5.

To calculate the force using (3.2), the vortex velocities are easily determined with complex potential (3.3), transformation (3.5) and Routh's correction (2.10), with the reverse sign of the subtracted vortex and Routh's correction, as vortex 1 in the ζ-plane of figure 5 has the reverse sign of vortex 1 of the ζ-plane of figure 2. The force components in the x- and y-direction exerted on the wing–body combination are made dimensionless to obtain $C_x \equiv F_x/(\frac{1}{2}\rho w_\infty^2 2c)$ and $C_y \equiv F_y/(\frac{1}{2}\rho w_\infty^2 2c)$. These components

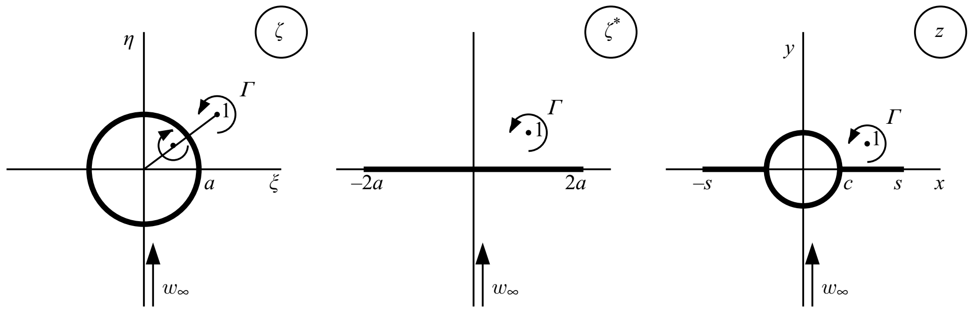

FIGURE 5. Transformation of the non-symmetric flow field with one vortex near a circle in uniform flow to the flow field of a wing–body combination.

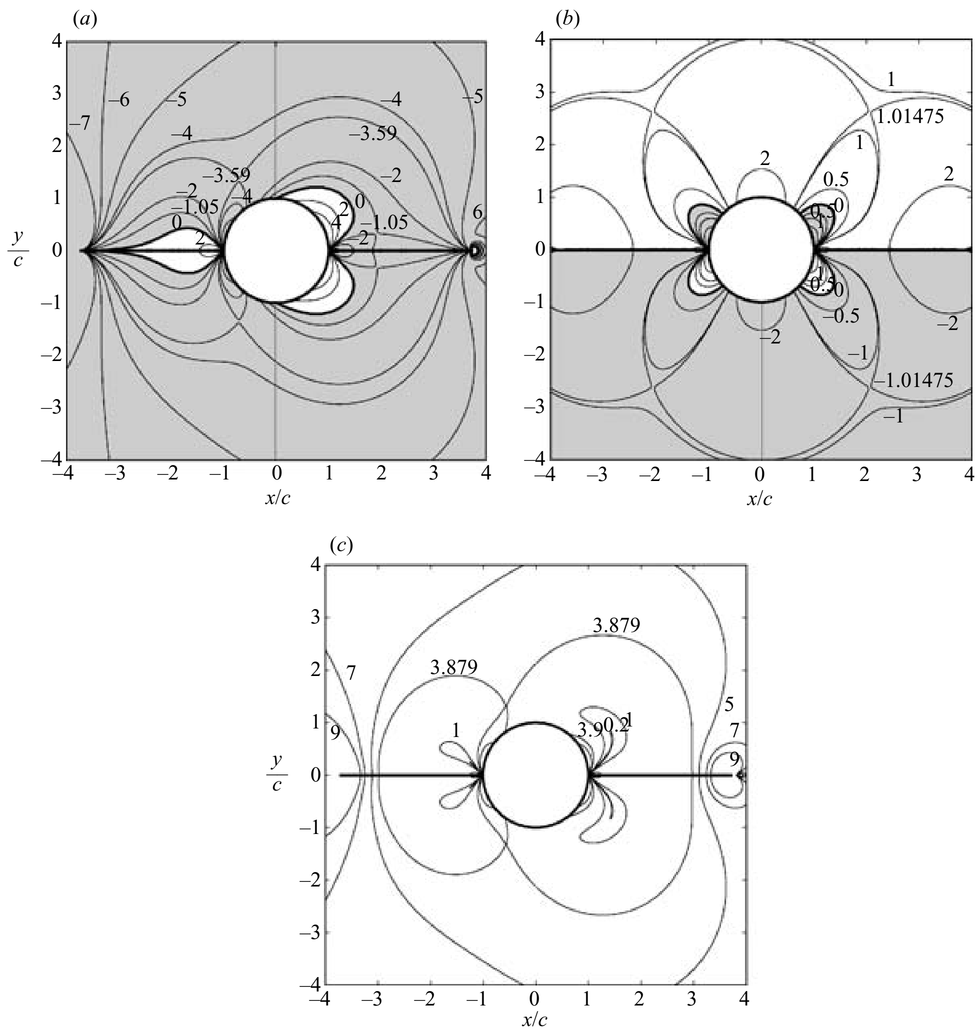

FIGURE 6. Force of a vortex near the wing–body combination of figure 5 for $a/c = 2$ (so $s/c = 3.73$) and $\Gamma/(2\pi c w_\infty) = 1$, areas in grey have negative values. (*a*) Contour plot of C_x. (*b*) Contour plot of C_y. (*c*) Contour plot of $|C_x + \mathrm{i}C_y|$.

are plotted in figure 6 as a function of the position. The regions with negative values are indicated in grey. The magnitude and direction of the force are obviously very important when evaluating flows with vortices near aircraft or missile surfaces. The case with two vortices of opposite strength is discussed in de Laat & Coene (2002).

4. Conclusions

The application of the concept of impulse combined with conformal mapping yields relatively simple relations to calculate the force on a two-dimensional stationary object due to an incompressible irrotational flow field with (moving) vortices. For symmetric flow fields a simple expression is obtained, employing the vortex strength and the first and second derivatives of the transformation function evaluated at the vortex position. It is the result of Routh's correction, which appears in the vortex velocity when applying a conformal transformation. The well-known force of a symmetric vortex pair behind a circular cylinder is simply obtained with this relation. The concept of impulse and conformal mapping is also applied to (non-)symmetric flow fields by transformation of the impulse of a vortex near a circle and its mirror image, yielding a simple relation requiring the strength, position and velocity of the vortex in the physical plane and the first derivative of the mapping function. This relation is applied to a single vortex near a cylinder and a simple wing–body configuration in upwash, thus demonstrating the efficient application of the relations obtained. For two examples, interesting areas of vortex positions are identified, at the boundaries of which the vortex force changes direction.

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