# Inverse tangent series via telescoping sums

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When first learning about infinite series, students typically are shown some examples for which the partial sums can be simplified by taking advantage of telescoping sums. In this paper, we present many examples of such series, all involving the inverse tangent function and most of which involve the Fibonacci and Lucas numbers. Most of the series presented here have appeared in various papers (see the references), but the authors are usually working in an abstract setting which makes it difficult for students to follow the basic ideas. We seek to make these results accessible to a wider audience.

All of our results stem from the basic trigonometric identities

$$\arctan u - \arctan v = \arctan\left(\frac{u - v}{1 + uv}\right) \text{ and } \arctan u + \arctan v = \arctan\left(\frac{u + v}{1 - uv}\right).$$

The first identity is valid for all positive numbers $u$ and $v$, while the second one is valid for all positive numbers $u$ and $v$ that satisfy $uv < 1$. The following result is a simple consequence of the first identity and properties of telescoping sums.

*Theorem* 1: Let $\{a_k\}$ be an unbounded, strictly increasing sequence of positive numbers. Then

$$\sum_{k=1}^{\infty} \arctan\left(\frac{a_{k+1} - a_k}{1 + a_k a_{k+1}}\right) = \arctan\left(\frac{1}{a_1}\right).$$

*Proof*: Using the inverse tangent identity for differences, we find that

$$\begin{aligned}\sum_{k=1}^{\infty} \arctan\left(\frac{a_{k+1} - a_k}{1 + a_k a_{k+1}}\right) &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\arctan a_{k+1} - \arctan a_k\right) \\ &= \lim_{n\to\infty} \left(\arctan a_{n+1} - \arctan a_1\right) \\ &= \frac{\pi}{2} - \arctan a_1 = \arctan \frac{1}{a_1}.\end{aligned}$$

This completes the proof.

To illustrate Theorem 1, we begin with three simple examples.

*Example* 1: (see [1]) Letting $a_k = 2k - 1$, we find that

$$\sum_{k=1}^{\infty} \arctan\left(\frac{(2k + 1) - (2k - 1)}{1 + (2k - 1)(2k + 1)}\right) = \sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \arctan \frac{1}{1} = \frac{\pi}{4}.$$

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*Example* 2: (see [1]) Letting $a_k = 2k$, we find that

$$\sum_{k=1}^{\infty} \arctan\left(\frac{(2k+2) - 2k}{1 + 2k(2k+2)}\right) = \sum_{k=1}^{\infty} \arctan \frac{2}{(2k+1)^2} = \arctan \frac{1}{2};$$

$$\sum_{k=1}^{\infty} \arctan \frac{2}{(2k-1)^2} = \frac{\pi}{2}.$$

Note the use of the basic fact that $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ for positive numbers $x$.

*Example* 3: Letting $b > 1$ be a fixed number and $a_k = b^{k-1}$, we find that

$$\sum_{k=1}^{\infty} \arctan \frac{b^k - b^{k-1}}{1 + b^{2k-1}} = \sum_{k=1}^{\infty} \arctan \frac{b - 1}{b^k + b^{1-k}} = \arctan 1 = \frac{\pi}{4}.$$

It is interesting to note that the sums of these series are independent of the value of $b$.

Many of our series results involve the sequence $\{f_n\}$ of Fibonacci numbers and the sequence $\{\ell_n\}$ of Lucas numbers, where

$$\{f_n\}_{n=0}^{\infty} = 0,1,1,2,3,5,8,13,21,\ldots \text{ and } \{\ell_n\}_{n=0}^{\infty} = 2,1,3,4,7,11,18,29,47,\ldots.$$

To establish some common notation for these numbers, let $\alpha = \phi$ and $\beta = -\frac{1}{\phi}$, where $\phi$ is the golden ratio. Recall that $\alpha$ and $\beta$ are the two solutions to the equation $x^2 = x + 1$ and thus satisfy the equations $\alpha\beta = -1$, $\alpha + \beta = 1$ and $\alpha - \beta = \sqrt{5}$. By the Binet formulas for these numbers, it is known that $\ell_n = \alpha^n + \beta^n$ and $\sqrt{5} f_n = \alpha^n - \beta^n$ for all integers $n \geqslant 0$. There are many relationships between these numbers, such as

$$f_{k+1} + f_{k-1} = \ell_k \qquad \text{and} \qquad \ell_{k+1} + \ell_{k-1} = 5f_k.$$

We assume the reader is familiar with the basic properties of these numbers, but we include details for some of the less familiar identities when they are needed.

*Example* 4: (see [2, 3, 4]) Using the identity

$$\begin{aligned} 1 + f_{2k} f_{2k+2} &= \frac{1}{5}\left(5 + \left(\alpha^{2k} - \beta^{2k}\right)\left(\alpha^{2k+2} - \beta^{2k+2}\right)\right) \\ &= \frac{1}{5}\left(5 + \alpha^{4k+2} - \beta^2 - \alpha^2 + \beta^{4k+2}\right) \\ &= \frac{1}{5}\left(\alpha^{4k+2} + 2 + \beta^{4k+2}\right) = \frac{1}{5}\left(\alpha^{2k+1} - \beta^{2k+1}\right)^2 = f_{2k+1}^2, \end{aligned}$$

and letting $a_k = f_{2k}$, we find that

$$\frac{a_{k+1} - a_k}{1 + a_k a_{k+1}} = \frac{f_{2k+2} - f_{2k}}{1 + f_{2k} f_{2k+2}} = \frac{f_{2k+1}}{f_{2k+1}^2} = \frac{1}{f_{2k+1}}.$$

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It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{f_{2k+1}} = \arctan \frac{1}{f_2} = \frac{\pi}{4} \qquad \text{and thus} \qquad \sum_{k=0}^{\infty} \arctan \frac{1}{f_{2k+1}} = \frac{\pi}{2}.$$

*Example* 5: (see [3, 4]) Letting $a_k = \alpha^{2k-1}$, we have

$$\frac{a_{k+1} - a_k}{1 + a_k a_{k+1}} = \frac{\alpha^{2k+1} - \alpha^{2k-1}}{1 + \alpha^{4k}} = \frac{\alpha - \alpha^{-1}}{\beta^{2k} + \alpha^{2k}} = \frac{\alpha + \beta}{\alpha^{2k} + \beta^{2k}} = \frac{1}{\ell_{2k}}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{\ell_{2k}} = \arctan \frac{1}{\alpha}.$$

Making note of the fact that

$$\arctan \alpha - \arctan \frac{1}{\alpha} = \arctan \frac{\alpha - \frac{1}{\alpha}}{1 + 1} = \arctan \frac{\alpha + \beta}{2} = \arctan \frac{1}{2},$$

we see that

$$\sum_{k=0}^{\infty} \arctan \frac{1}{\ell_{2k}} = \arctan \frac{1}{2} + \arctan \frac{1}{\alpha} = \arctan \alpha.$$

*Example* 6: (see [5]) Letting $a_k = \alpha^{2k-2}$, we have

$$\frac{a_{k+1} - a_k}{1 + a_k a_{k+1}} = \frac{\alpha^{2k} - \alpha^{2k-2}}{1 + \alpha^{4k-2}} = \frac{-\alpha - \beta}{\beta^{2k-1} - \alpha^{2k-1}} = \frac{\alpha + \beta}{\alpha^{2k-1} - \beta^{2k-1}} = \frac{1}{\sqrt{5} f_{2k-1}}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{\sqrt{5} f_{2k-1}} = \arctan \frac{1}{\alpha^0} = \frac{\pi}{4}.$$

This result is the particular case of $b = \alpha^2$ in Example 3.

*Example* 7: (see [6]) Using the identity

$$\begin{aligned} 1 + \ell_{2k-1}\ell_{2k+1} &= 1 + \left(\alpha^{2k-1} + \beta^{2k-1}\right)\left(\alpha^{2k+1} + \beta^{2k+1}\right) \\ &= 1 + \alpha^{4k} - \beta^2 - \alpha^2 + \beta^{4k} \\ &= \alpha^{4k} - 2 + \beta^{4k} = \left(\alpha^{2k} - \beta^{2k}\right)^2 = 5f_{2k}^2 \end{aligned}$$

and letting $a_k = \ell_{2k-1}$, we find that

$$\frac{a_{k+1} - a_k}{1 + a_k a_{k+1}} = \frac{\ell_{2k+1} - \ell_{2k-1}}{1 + \ell_{2k-1}\ell_{2k+1}} = \frac{\ell_{2k}}{5f_{2k}^2}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{\ell_{2k}}{5f_{2k}^2} = \arctan \frac{1}{\ell_1} = \frac{\pi}{4}.$$

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*Example* 8: (see [6]) Referring to the identity in Example 4, we see that

$$1 + 5f_{2k}f_{2k+2} = \alpha^{4k+2} - 2 + \beta^{4k+2} = \left(\alpha^{2k+1} + \beta^{2k+1}\right)^2 = \ell_{2k+1}^2.$$

Letting $a_k = \sqrt{5}f_{2k}$, we find that

$$\frac{a^{k+1} - a^k}{1 + \alpha_k a_{k+1}} = \frac{\sqrt{5}f_{2k+2} - \sqrt{5}f_{2k}}{1 + 5f_{2k}f_{2k+2}} = \frac{\sqrt{5}f_{2k+1}}{\ell_{2k+1}^2}$$

and thus obtain the following two series results:

$$\sum_{k=1}^{\infty} \arctan \frac{\sqrt{5}f_{2k+1}}{\ell_{2k+1}^2} = \arctan \frac{1}{\sqrt{5}} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \arctan \frac{\sqrt{5}f_{2k-1}}{\ell_{2k-1}^2} = \frac{\pi}{2}.$$

*Example* 9: (see [6]) Referring to the identity in Example 7, we see that

$$3 + \ell_{2k-1}\ell_{2k+1} = \alpha^{4k} + \beta^{4k} = \ell_{4k}.$$

Letting $a_k = \dfrac{\ell_{2k-1}}{\sqrt{3}}$, we find that

$$\frac{3a_{k+1} - 3a_k}{3 + 3a_k a_{k+1}} = \frac{\sqrt{3}\ell_{2k+1} - \sqrt{3}\ell_{2k-1}}{3 + \ell_{2k-1}\ell_{2k+1}} = \frac{\sqrt{3}\ell_{2k}}{\ell_{4k}}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{\sqrt{3}\ell_{2k}}{\ell_{4k}} = \arctan \sqrt{3} = \frac{\pi}{3}.$$

*Example* 10: (see [6]) Using the identity

$$\begin{aligned} 9 + \ell_{4k-3}\ell_{4k+1} &= 9 + \left(\alpha^{4k-3} + \beta^{4k-3}\right)\left(\alpha^{4k+1} + \beta^{4k+1}\right) \\ &= 9 + \alpha^{8k-2} - \beta^4 - \alpha^4 + \beta^{8k-2} \\ &= \alpha^{8k-2} + 2 + \beta^{8k-2} = \left(\alpha^{4k-1} - \beta^{4k-1}\right)^2 = 5f_{4k-1}^2, \end{aligned}$$

and letting $a_k = \frac{1}{3}\ell_{4k-1}$, we find that

$$\frac{9a_{k+1} - 9a_k}{9 + 9a_k a_{k+1}} = \frac{3\ell_{4k+1} - 3\ell_{4k-3}}{9 + \ell_{4k-3}\ell_{4k+1}} = \frac{3(\ell_{4k} + \ell_{4k-2})}{5f_{4k-1}^2} = \frac{15f_{4k-1}}{5f_{4k-1}^2} = \frac{3}{f_{4k-1}}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{3}{f_{4k-1}} = \arctan 3.$$

The reader is encouraged to find the related series result corresponding to $a_k = \frac{1}{3}\ell_{4k-1}$.

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*Example* 11: (see [6]) Using the identity

$$\begin{aligned} 9 + 5f_{4k-2}f_{4k+2} &= 9 + \left(\alpha^{4k-2} - \beta^{4k-2}\right)\left(\alpha^{4k+2} - \beta^{4k+2}\right) \\ &= 9 + \alpha^{8k} - \beta^4 - \alpha^4 + \beta^{8k} \\ &= \alpha^{8k} + 2 + \beta^{8k} = \left(\alpha^{4k} + \beta^{4k}\right)^2 = \ell_{4k}^2, \end{aligned}$$

and letting $a_k = \frac{\sqrt{5}}{3}f_{4k-2}$, we find that

$$\frac{9a_{k+1} - 9a_k}{9 + 9a_k a_{k+1}} = \frac{3\sqrt{5}f_{4k+2} - 3\sqrt{5}f_{4k-2}}{9 + 5f_{4k-2}f_{4k+2}} = \frac{3\sqrt{5}(f_{4k+1} + f_{4k-1})}{\ell_{4k}^2} = \frac{3\sqrt{5}\ell_{4k}}{\ell_{4k}^2} = \frac{3\sqrt{5}}{\ell_{4k}}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan\frac{3\sqrt{5}}{\ell_{4k}} = \arctan\frac{3}{\sqrt{5}}.$$

We next present a slightly modified version of Theorem 1 that can be used to find further sums.

*Theorem* 2: Let $\{a_k\}_{k=0}^{\infty}$ be an unbounded, strictly increasing sequence of non-negative numbers. Then

$$\sum_{k=1}^{\infty} \arctan\left(\frac{a_{k+1} - a_{k-1}}{1 + a_{k-1}a_{k+1}}\right) = \pi - \arctan a_0 - \arctan a_1;$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan\left(\frac{a_{k+1} - a_{k-1}}{1 + a_{k-1}a_{k+1}}\right) = \arctan a_1 - \arctan a_0.$$

*Proof*: Using the inverse tangent identity for differences, we find that

$$\begin{aligned} &\sum_{k=1}^{\infty} \arctan\left(\frac{a_{k+1} - a_{k-1}}{1 + a_{k-1}a_{k+1}}\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\arctan a_{k+1} - \arctan a_{k-1}\right) \\ &= \lim_{n\to\infty} \left(\left(\arctan a_n + \arctan a_{n+1}\right) - \left(\arctan a_0 + \arctan a_1\right)\right) \\ &= \pi - \arctan a_0 - \arctan a_1. \end{aligned}$$

This establishes the first series result. For the second one, we have

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan\frac{a_{k+1} - a_{k-1}}{1 + a_{k-1}a_{k+1}}$$

$$\begin{aligned} &= \lim_{n\to\infty} \sum_{k=1}^{n} \left((-1)^{k+1} \arctan a_{k+1} - (-1)^{k-1} \arctan a_{k-1}\right) \\ &= \lim_{n\to\infty} \left((-1)^n\left(\arctan a_n - \arctan a_{n+1}\right) - \left(\arctan a_0 - \arctan a_1\right)\right) \\ &= \arctan a_1 - \arctan a_0. \end{aligned}$$

This completes the proof.

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The next examples present series for which Theorem 2 easily determines the sums.

*Example* 12: (see [1, 7, 8]) Letting $a_k = k$, we have

$$\frac{a_{k+1} - a_{k-1}}{1 + a_{k-1}a_{k+1}} = \frac{(k+1) - (k-1)}{1 + (k-1)(k+1)} = \frac{2}{k^2}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{2}{k^2} = \pi - \arctan 0 - \arctan 1 = \frac{3\pi}{4};$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{2}{k^2} = \arctan 1 - \arctan 0 = \frac{\pi}{4}.$$

The reader may note that these sums also follow from the series in Examples 1 and 2.

*Example* 13: (see [8]) Letting $a_k = \dfrac{k}{\sqrt{3}}$, we have

$$\frac{3a_{k+1} - 3a_{k-1}}{3 + 3a_{k-1}a_{k+1}} = \frac{\sqrt{3}(k+1) - \sqrt{3}(k-1)}{3 + (k+1)(k-1)} = \frac{2\sqrt{3}}{k^2 + 2}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{2\sqrt{3}}{k^2 + 2} = \pi - \arctan 0 - \arctan \frac{1}{\sqrt{3}} = \frac{5\pi}{6};$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{2\sqrt{3}}{k^2 + 2} = \arctan \frac{1}{\sqrt{3}} - \arctan 0 = \frac{\pi}{6}.$$

*Example* 14: (see [8]) Letting $b > 1$ be a fixed number and $a_k = b^k$, we find that

$$\sum_{k=1}^{\infty} \arctan \frac{b^{k+1} - b^{k-1}}{1 + b^{2k}} = \sum_{k=1}^{\infty} \arctan \frac{b - b^{-1}}{b^k + b^{-k}} = \pi - \arctan 1 - \arctan b$$

$$= \arctan 1 + \arctan \frac{1}{b} = \arctan \frac{b+1}{b-1};$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{b - b^{-1}}{b^k + b^{-k}} = \arctan b - \arctan 1 = \arctan \frac{b-1}{b+1}.$$

Letting $b = e^r$ and using the hyperbolic trigonometric functions, these two results can be written as

$$\sum_{k=1}^{\infty} \arctan \frac{\sinh r}{\cosh rk} = \frac{3\pi}{4} - \arctan e^r;$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{\sinh r}{\cosh rk} = \arctan e^r - \frac{\pi}{4}.$$

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*Example* 15: (see [5, 6]) Letting $a_k = \alpha^{2k}$, we have

$$\frac{a_{k+1} - a_{k-1}}{1 + a_{k-1}a_{k+1}} = \frac{\alpha^{2k+2} - \alpha^{2k-2}}{1 + \alpha^{4k}} = \frac{\alpha^2 - \alpha^{-2}}{\beta^{2k} + \alpha^{2k}} = \frac{\alpha^2 - \beta^2}{\alpha^{2k} + \beta^{2k}} = \frac{\sqrt{5}}{\ell_{2k}}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan\frac{\sqrt{5}}{\ell_{2k}} = \pi - \arctan 1 - \arctan\alpha^2 = \arctan 1 + \arctan\beta^2$$

$$= \arctan\frac{1+\beta^2}{1-\beta^2} = \arctan\frac{\alpha-\beta}{\alpha+\beta} = \arctan\sqrt{5};$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan\frac{\sqrt{5}}{\ell_{2k}} = \arctan\alpha^2 - \arctan 1 = \arctan\frac{\alpha^2 - 1}{1 + \alpha^2}$$

$$= \arctan\frac{\alpha + \beta}{\alpha - \beta} = \arctan\frac{1}{\sqrt{5}}.$$

Note that these results follow from Example 14 with $b = \alpha^2$.

*Example* 16: (see [8]) Using the identity

$$1 + \ell_{2k-2}\ell_{2k+2} = 1 + \left(\alpha^{2k-2} + \beta^{2k-2}\right)\left(\alpha^{2k+2} + \beta^{2k+2}\right)$$

$$= 1 + \alpha^{4k} + \beta^4 + \alpha^4 + \beta^{4k}$$

$$= \ell_{4k} + 8$$

and letting $a_k = \ell_{2k}$, we have

$$\frac{a_{k+1} - a_{k-1}}{1 + a_{k-1}a_{k+1}} = \frac{\ell_{2k+2} - \ell_{2k-2}}{1 + \ell_{2k-2}\ell_{2k+2}} = \frac{\ell_{2k+1} + \ell_{2k-1}}{\ell_{4k} + 8} = \frac{5f_{2k}}{\ell_{4k} + 8}.$$

It follows that

$$\sum_{k=1}^{\infty} \arctan\frac{5f_{2k}}{\ell_{4k}+8} = \pi - \arctan 2 - \arctan 3 = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

$$= \arctan 1 = \frac{\pi}{4};$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan\frac{5f_{2k}}{\ell_{4k}+8} = \arctan 3 - \arctan 2 = \arctan\frac{1}{7}.$$

*Example* 17: (see [8]) Using the identity in Example 16 and letting $a_k = \frac{1}{2}\ell_{2k}$, we have

$$\frac{4a_{k+1} - 4a_{k-1}}{4 + 4a_{k-1}a_{k+1}} = \frac{2\ell_{2k+2} - 2\ell_{2k-2}}{4 + \ell_{2k-2}\ell_{2k+2}} = \frac{2(\ell_{2k+1} + \ell_{2k-1})}{\ell_{4k} + 11} = \frac{10f_{2k}}{\ell_{4k} + 11}.$$

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It follows that

$$\sum_{k=1}^{\infty} \arctan \frac{10f_{2k}}{\ell_{4k} + 11} = \pi - \arctan 1 - \arctan \frac{3}{2} = \arctan 1 + \arctan \frac{2}{3} = \arctan 5;$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{10f_{2k}}{\ell_{4k} + 11} = \arctan \frac{3}{2} - \arctan 1 = \arctan \frac{1}{5}.$$

It is possible to find the sums of more complicated inverse tangent series using telescoping sums. The next theorem lays the groundwork for a few such results. The equalities listed in the theorem are presented in more general form in [5].

*Theorem* 3: The following equalities are valid for all positive integers $k$:

$$\arctan \frac{1}{f_{2k}} = \arctan \alpha^{1-2k} + \arctan \alpha^{-1-2k}; \tag{1}$$

$$\arctan \frac{1}{f_{2k-1}} = \arctan \alpha^{3-2k} - \arctan \alpha^{-1-2k}; \tag{2}$$

$$\arctan \frac{1}{\ell_{2k}} = \arctan \alpha^{1-2k} - \arctan \alpha^{-1-2k}; \tag{3}$$

$$\arctan \frac{2}{\ell_{2k-1}} = 2 \arctan \alpha^{1-2k}; \tag{4}$$

$$\arctan \frac{3}{\ell_{2k-1}} = \arctan \alpha^{3-2k} + \arctan \alpha^{-1-2k}. \tag{5}$$

*Proof*: Noting that (using the relevant equation numbers)

$$\frac{\alpha^{1-2k} + \alpha^{-1-2k}}{1 - \alpha^{-4k}} = \frac{\alpha + \alpha^{-1}}{\alpha^{2k} - \alpha^{-2k}} = \frac{\alpha - \beta}{\alpha^{2k} - \beta^{2k}} = \frac{1}{f_{2k}}; \tag{1}$$

$$\frac{\alpha^{3-2k} - \alpha^{-1-2k}}{1 + \alpha^{2-4k}} = \frac{\alpha^2 - \alpha^{-2}}{\alpha^{2k-1} + \alpha^{1-2k}} = \frac{\alpha^2 - \beta^2}{\alpha^{2k-1} - \beta^{2k-1}} = \frac{1}{f_{2k-1}}; \tag{2}$$

$$\frac{\alpha^{1-2k} - \alpha^{-1-2k}}{1 + \alpha^{-4k}} = \frac{\alpha - \alpha^{-1}}{\alpha^{2k} + \alpha^{-2k}} = \frac{\alpha + \beta}{\alpha^{2k} + \beta^{2k}} = \frac{1}{\ell_{2k}}; \tag{3}$$

$$\frac{2\alpha^{1-2k}}{1 - \alpha^{2-4k}} = \frac{2}{\alpha^{2k-1} - \alpha^{1-2k}} = \frac{2}{\alpha^{2k-1} + \beta^{1-2k}} = \frac{2}{\ell_{2k-1}}; \tag{4}$$

$$\frac{\alpha^{3-2k} + \alpha^{-1-2k}}{1 - \alpha^{2-4k}} = \frac{\alpha^2 + \alpha^{-2}}{\alpha^{2k-1} - \alpha^{1-2k}} = \frac{\alpha^2 + \beta^2}{\alpha^{2k-1} + \beta^{2k-1}} = \frac{3}{\ell_{2k-1}}; \tag{5}$$

for all positive integers $k$, each of the equalities in the Theorem follows from the appropriate inverse tangent identity.

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To simplify the notation in the examples that follow, we let $t_k = \arctan \alpha^{1-2k}$ for every integer $k$. We also interpret $\arctan \frac{1}{0}$ as $\frac{\pi}{2}$; this makes equation (1) in Theorem 3 valid for $k = 0$.

*Example* 18: (see [5]) Using (1) and (2) from Theorem 3, we find that the partial sum

$$\sum_{k=1}^{n} \arctan \frac{1}{f_{2k-1}} \left( \arctan \frac{1}{f_{2k}} + \arctan \frac{1}{f_{2k-2}} \right)$$

is equal to

$$\begin{aligned}\sum_{k=1}^{n} (t_{k-1} - t_{k+1})\big((t_k + t_{k+1}) + (t_{k-1} + t_k)\big) &= \sum_{k=1}^{n} \big((t_{k-1}^2 - t_{k+1}^2) + 2t_k(t_{k-1} - t_{k+1})\big) \\ &= \sum_{k=1}^{n} \big((t_{k-1} + t_k)^2 - (t_k + t_{k+1})^2\big) \\ &= (t_0 + t_1)^2 - (t_n + t_{n+1})^2.\end{aligned}$$

The sum of the corresponding series is thus

$$\lim_{n \to \infty} \big((t_0 + t_1)^2 - (t_n + t_{n+1})^2\big) = (t_0 + t_1)^2 = \left(\arctan \alpha + \arctan \frac{1}{\alpha}\right)^2 = \frac{\pi^2}{4}.$$

Splitting the following sum into its even and odd terms, we find that

$$\begin{aligned}&\sum_{k=1}^{\infty} \arctan \frac{1}{f_k} \arctan \frac{1}{f_{k+1}} \\ &= \sum_{k=2}^{\infty} \arctan \frac{1}{f_{2k-2}} \arctan \frac{1}{f_{2k-1}} + \sum_{k=1}^{\infty} \arctan \frac{1}{f_{2k-1}} \arctan \frac{1}{f_{2k}} \\ &= \sum_{k=2}^{\infty} \arctan \frac{1}{f_{2k-1}} \left( \arctan \frac{1}{f_{2k}} + \arctan \frac{1}{f_{2k-2}} \right) + \frac{\pi}{4} \cdot \frac{\pi}{4} \\ &= \frac{\pi^2}{4} - \frac{\pi}{4}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) + \frac{\pi^2}{16} = \frac{\pi^2}{8}.\end{aligned}$$

This is a rather interesting result and gives the unexpected equality

$$\sum_{k=1}^{\infty} \arctan \frac{1}{f_k} \arctan \frac{1}{f_{k+1}} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

*Example* 19: (see [5]) Using (1) and (3) from Theorem 3, we find that

$$\begin{aligned}\sum_{k=1}^{\infty} \arctan \frac{1}{f_{2k}} \arctan \frac{1}{\ell_{2k}} &= \lim_{n \to \infty} \sum_{k=1}^{n} (t_k + t_{k+1})(t_k - t_{k+1}) \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} (t_k^2 - t_{k+1}^2)\end{aligned}$$

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$$= \lim_{n \to \infty} \left(t_1^2 - t_{n+1}^2\right)$$

$$= t_1^2 = \left(\arctan \frac{1}{\alpha}\right)^2 = \left(\frac{\arctan 2}{2}\right)^2.$$

(See the end of Example 5 for a hint to prove of the last equality.)

*Example* 20: (see [5]) Using (2) and (4) from Theorem 3, we find that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{f_{2k-1}} \arctan \frac{2}{\ell_{2k-1}} = \lim_{n \to \infty} \sum_{k=1}^{n} \left(t_{k-1} - t_{k+1}\right)\left(2t_k\right)$$

$$= \lim_{n \to \infty} 2 \sum_{k=1}^{n} \left(t_{k-1}t_k - t_k t_{k+1}\right)$$

$$= 2 \lim_{n \to \infty} \left(t_0 t_1 - t_n t_{n+1}\right)$$

$$= 2t_0 t_1 = 2 \arctan \alpha \arctan \frac{1}{\alpha}$$

$$= \arctan \alpha \arctan 2.$$

*Example* 21: (see [5]) Using (2) and (5) from Theorem 3, we find that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{f_{2k-1}} \arctan \frac{3}{\ell_{2k-1}} = \lim_{n \to \infty} \sum_{k=1}^{n} \left(t_{k-1} - t_{k+1}\right)\left(t_{k-1} + t_{k+1}\right)$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left(t_{k-1}^2 - t_{k+1}^2\right)$$

$$= \lim_{n \to \infty} \left(t_0^2 + t_1^2 - t_n^2 - t_{n+1}^2\right)$$

$$= t_0^2 + t_1^2$$

$$= (\arctan \alpha)^2 + \left(\arctan \frac{1}{\alpha}\right)^2.$$

We present one final example, omitting the simple proofs of the needed identities.

*Example* 22: (see [4, 7, 9, 10]) Noting that

$$\frac{\dfrac{f_k}{f_{k+1}} - \dfrac{f_{k-1}}{f_k}}{1 + \dfrac{f_{k-1}f_k}{f_k f_{k+1}}} = \frac{f_k^2 - f_{k-1}f_{k+1}}{f_k\left(f_{k+1} + f_{k-1}\right)} = \frac{(-1)^{k+1}}{f_k \ell_k} = \frac{(-1)^{k+1}}{f_{2k}},$$

we find that

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{1}{f_{2k}} = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\arctan \frac{f_k}{f_{k+1}} - \arctan \frac{f_{k-1}}{f_k}\right)$$

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$$= \lim_{n\to\infty}\left(\arctan\frac{f_n}{f_{n+1}} - \arctan\frac{f_0}{f_1}\right)$$

$$= \arctan\frac{1}{\alpha}.$$

We hope that the examples provided here spark the reader's curiosity for these types of series. The references listed below include many more such series, along with various generalisations. We end the paper with an exercise and an open question.

*Exercise*: Show that $\sum_{k=1}^{\infty} \arctan\frac{2p^2}{k^2} = p\pi - \frac{\pi}{4}$ for each positive integer $p$.

*Open question*: Find a simple expression for the sum of the series

$$\sum_{k=1}^{\infty} \arctan\frac{1}{f_{2k}}.$$

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