COMPOSITIO MATHEMATICA

Mirabolic Satake equivalence and supergroups

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To our friend Sasha Shen on the occasion of his 60th birthday

Abstract

We construct a mirabolic analogue of the geometric Satake equivalence. We also prove an equivalence that relates representations of a supergroup to the category of $\mathrm{GL}(N-1,\mathbb{C}[\![t]\!])$-equivariant perverse sheaves on the affine Grassmannian of GL_N. We explain how our equivalences fit into a more general framework of conjectures due to Gaiotto and to Ben-Zvi, Sakellaridis and Venkatesh.

Contents

1. Introduction

1.1 Reminder on geometric Satake

Let $\mathbf{F} = \mathbb{C}((t)) \supset \mathbb{C}[[t]] = \mathbf{O}$. Throughout the paper, we fix an integer $N \geq 1$, set $\mathbf{G_F} = \mathrm{GL}(N, \mathbf{F})$ and $\mathbf{G_O} = \mathrm{GL}(N, \mathbf{O})$, and let $\mathbf{Gr} = \mathbf{G_F}/\mathbf{G_O}$ be the affine Grassmannian of GL_N. This is an ind-scheme equipped with a natural action of the group $\mathbf{G_O} \rtimes \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts by loop rotation. Let $D_{\mathbf{G_O}}(\mathbf{Gr})$ (respectively, $D_{\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr})$) be the $\mathbf{G_O}$-equivariant (respectively, $\mathbf{G_O} \rtimes \mathbb{C}^\times$-equivariant) constructible derived category of $\mathbf{Gr}$.[1] This is a monoidal category with respect to convolution (which coincides with fusion); cf. [MV07].

Let $\mathfrak{gl}_N$ be the complex vector space of $N \times N$ matrices and let GL_N act on $\mathfrak{gl}_N$ by conjugation. Write $\mathrm{Sym}^\bullet(\mathfrak{gl}_N[-2])$ for the symmetric algebra of $\mathfrak{gl}_N$ viewed as a dg-algebra such that the space $\mathfrak{gl}_N$, of generators, is placed in degree 2 and the differential is equal to zero. Let $D_{\mathrm{perf}}^{\mathrm{GL}_N}(\mathrm{Sym}^\bullet(\mathfrak{gl}_N[-2]))$ be the triangulated category of perfect GL_N-equivariant dg-modules over $\mathrm{Sym}^\bullet(\mathfrak{gl}_N[-2])$, localized with respect to quasi-isomorphisms. The tensor product of dg-modules gives this category a monoidal structure. One of the versions of derived Satake equivalence proved in [BF08] states that there is an equivalence $D_{\mathrm{perf}}^{\mathrm{GL}_N}(\mathrm{Sym}^\bullet(\mathfrak{gl}_N[-2])) \xrightarrow{\sim} D_{\mathbf{G_O}}(\mathbf{Gr})$ of triangulated monoidal categories.

It will be convenient for us to reformulate the above result as follows. Let $T^*\mathrm{GL}_N[2]$ be the shifted cotangent bundle on GL_N, viewed as a dg-scheme equipped with zero differential. The action of GL_N on itself by left and right translations induces a $\mathrm{GL}_N \times \mathrm{GL}_N$-action on $T^*\mathrm{GL}_N[2]$. Let $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(T^*\mathrm{GL}_N[2])$ be the triangulated category of $\mathrm{GL}_N \times \mathrm{GL}_N$-equivariant perfect complexes of $\mathcal{O}_{T^*\mathrm{GL}_N[2]}$-modules on $T^*\mathrm{GL}_N[2]$. The fiber of $T^*\mathrm{GL}_N[2]$ over $1 \in \mathrm{GL}_N$ may (and will) be identified with $(\mathfrak{gl}_N[-2])^* = \mathfrak{gl}_N^*[2]$. Restriction to this fiber induces a monoidal equivalence $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(T^*\mathrm{GL}_N[2]) \xrightarrow{\sim} D_{\mathrm{perf}}^{\mathrm{GL}_N}(\mathrm{Sym}^\bullet(\mathfrak{gl}_N[-2]))$, where the category on the right is identified with the triangulated category of GL_N-equivariant perfect complexes on $(\mathfrak{gl}_N[-2])^*$.

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[1] Throughout the paper, we consider the sheaves with complex coefficients (with a few technical exceptions).

Thus, the derived Satake equivalence stated above may be interpreted as an equivalence $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(T^* \mathrm{GL}_N[2]) \xrightarrow{\sim} D_{\mathbf{G_O}}(\mathbf{Gr})$.

There is also a natural 'quantum' counterpart of the latter equivalence, where the category $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(T^* \mathrm{GL}_N[2])$ is replaced by an appropriately defined category of asymptotic shifted weakly equivariant D-modules on GL_N, and the category $D_{\mathbf{G_O}}(\mathbf{Gr})$ is replaced by $D_{\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr})$; see [BF08].

1.2 Mirabolic Satake category

In the present paper we will be interested in a mirabolic analogue of the above setting. To explain this, fix an N-dimensional vector space V and put $\mathbf{V} = V \otimes \mathbf{F}$, $\mathbf{V}_0 = V \otimes \mathbf{O}$ and $\mathbf{\mathring{V}} = \mathbf{V} \smallsetminus \{0\}$. We identify $\mathbf{G} = \mathrm{GL}_N$ with $\mathrm{GL}(V)$, so that $\mathbf{G_F}$ acts on $\mathbf{V}$ and $\mathbf{G_O}$ is the stabilizer of $\mathbf{V}_0$. Following [FGT09], the *mirabolic affine Grassmannian* is defined as $\mathbf{Gr} \times \mathbf{V}$. We let $\mathbf{G_O}$ act on $\mathbf{Gr} \times \mathbf{V}$ diagonally. The orbits of $\mathbf{G_O}$ in $\mathbf{Gr} \times \mathbf{\mathring{V}}$ are, unlike the case of $\mathbf{G_O}$-orbits in $\mathbf{Gr}$, not finite-dimensional. Heuristically, these orbits are of semi-infinite type in the sense that the 'closure' of an orbit projects onto a (finite dimensional) Schubert variety in $\mathbf{Gr}$ and onto a lattice in $\mathbf{V}$.

In view of the above, defining the correct mirabolic analogue of the equivariant derived Satake category requires some care. According to our definition, an object of this category is supported on the product of a finite-dimensional Schubert variety in $\mathbf{Gr}$ and a lattice inside the Tate vector space $\mathbf{V}$; moreover, this object is pulled back from a finite-dimensional quotient of this lattice. According to three possible choices (!-, ∗-, and !∗-) of pullback, one gets the three versions $D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ and $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ of $\mathbf{G_O}$-equivariant constructible derived categories on $\mathbf{Gr} \times \mathbf{V}$. These categories are related to each other by certain renormalization equivalences.

We equip the above categories with monoidal structures given by various types of convolution operation. The convolution along the Grassmannian, the first factor in $\mathbf{Gr} \times \mathbf{V}$, is defined similarly to the case of the usual Satake category. The convolution along the second factor depends on the choice of category. Specifically, the convolution operation $\overset{!}{\circledast}$ in $D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ involves the !-tensor product of constructible sheaves on $\mathbf{V}$. The convolution operation $\overset{*}{*}$ on $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ involves a ∗-pushforward along $+ \colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$, the map given by addition. These two types of convolution are related to each other via Fourier transform (along $\mathbf{V}$). Finally, we define a monoidal structure $\star$ on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ via the fusion operation on a mirabolic analogue of the Beilinson–Drinfeld Grassmannian.

The categories above have natural counterparts involving the action of $\mathbb{C}^\times$ on $\mathbf{Gr}$ by loop rotation. These are $\mathbb{C}[\hbar]$-linear categories where $\mathbb{C}[\hbar] = H_{\mathbb{C}^\times}^\bullet(\mathrm{pt})$. Thus, there is a category $D_{!\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V})$ (respectively, $D_{*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V})$) equipped with a similarly defined monoidal structure $\overset{!}{\circledast}$ (respectively, $\overset{*}{*}$). The fusion operation $\star$ on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ has *no* $\mathbf{G_O} \rtimes \mathbb{C}^\times$-equivariant counterpart, however.

1.3 Mirabolic Satake equivalence

The category on the 'other side' of mirabolic Satake equivalence is an appropriately defined triangulated category of 'equivariant asymptotic shifted D-modules' on the vector space $\mathfrak{gl}_N$ of $N \times N$ matrices. More formally, we introduce a $\mathbb{C}[\hbar]$-algebra $\mathfrak{D}^\bullet$, a graded version of the algebra of differential operators on $\mathfrak{gl}_N$ (where $\deg \hbar = 2$). The action of GL_N on $\mathfrak{gl}_N$ by left and right multiplication induces a $\mathrm{GL}_N \times \mathrm{GL}_N$-action on $\mathfrak{D}^\bullet$. The relevant category is then

defined to be the derived category of weakly $\mathrm{GL}_N \times \mathrm{GL}_N$-equivariant perfect dg-modules over $\mathfrak{D}^\bullet$, where $\mathfrak{D}^\bullet$ is viewed as a dg-algebra with zero differential. Similarly to the constructible story, we actually define three versions, $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{D}_{1,1}^\bullet)$, $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{D}_{0,2}^\bullet)$ and $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{D}_{2,0}^\bullet)$, of such a category that correspond to three different choices of grading on our algebra. The algebra of asymptotic differential operators specializes at $\hbar = 0$ to the algebra $\mathbb{C}[T^*\mathfrak{gl}_N]$. We denote the specialization of $\mathfrak{D}_{i,2-i}^\bullet$ at $\hbar = 0$ by $\mathfrak{S}_{i,2-i}^\bullet$. Accordingly, the above-defined $\mathbb{C}[\hbar]$-linear categories specialize at $\hbar = 0$ to various versions of the derived category of $\mathrm{GL}_N \times \mathrm{GL}_N$-equivariant coherent sheaves on the shifted cotangent bundle $T^*\mathfrak{gl}_N$.

Next, we equip the above-defined categories with monoidal structures. The monoidal structure $\overset{A}{*}$ on $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{D}_{2,0}^\bullet)$ is defined as a convolution operation on D-modules associated with the map $\mathfrak{gl}_N \times \mathfrak{gl}_N \to \mathfrak{gl}_N$, given by matrix multiplication. The monoidal structure $\overset{B}{*}$ on $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{D}_{0,2}^\bullet)$ is defined as $\mathsf{F}^{-1} \circ (\overset{A}{*}) \circ (\mathsf{F} \times \mathsf{F})$, where F is the functor of Fourier transform on D-modules. Each of these monoidal structures has a quasiclassical limit at $\hbar = 0$, defined as a convolution of coherent sheaves arising from a certain Lagrangian correspondence. Finally, the tensor product of coherent sheaves, that is, the functor $\mathcal{M}, \mathcal{M}' \mapsto \mathcal{M} \otimes_{\mathfrak{S}_{1,1}^\bullet} \mathcal{M}'$, gives a monoidal structure on $D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{S}_{1,1}^\bullet)$. This monoidal structure has no counterpart for D-modules (i.e. for $\hbar \neq 0$)z.

One of the main results of the present paper (see Theorems 3.6.1 and 5.1.1) states that one has the following equivalences of triangulated monoidal categories, called mirabolic Satake equivalences:

$$\left(D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{D}_{2,0}^\bullet),\ \overset{A}{*}\right) \ \xrightarrow[\cong]{\Phi_\hbar^{2,0}} \ \left(D_{!\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}),\ \overset{!}{\circledast}\right),$$

$$\left(D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{D}_{0,2}^\bullet),\ \overset{B}{*}\right) \ \xrightarrow[\cong]{\Phi_\hbar^{0,2}} \ \left(D_{*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}),\ \overset{*}{*}\right),$$

$$\left(D_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{S}_{1,1}^\bullet),\ \otimes_{\mathfrak{S}_{1,1}^\bullet}\right) \ \xrightarrow[\cong]{\Phi^{1,1}} \ \left(D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}),\ \star\right).$$

Furthermore, it turns out that the triangulated category $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ is equivalent to the bounded derived category of the abelian category $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ of $\mathbf{G_O}$-equivariant perverse sheaves on $\mathbf{Gr} \times \mathbf{V}$, the heart of the perverse t-structure on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

Remark 1.3.1. The counterpart of the last statement in the case of the usual Satake category is false: the triangulated category $D_{!*\mathbf{G_O}}(\mathbf{Gr})$ is not equivalent to the derived category of the category $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr})$, which is well known to be a semisimple abelian category.

Remark 1.3.2. One can view the group GL_N as a Zariski open subset of the vector space $\mathfrak{gl}_N$ of all $N \times N$ matrices. Associated with the open embedding $\mathrm{GL}_N \hookrightarrow \mathfrak{gl}_N$, one has a restriction functor on D-modules. It turns out that the counterpart of this functor for constructible derived categories is a suitably defined version of restriction with respect to the 'zero section' $\mathbf{Gr} \times \{0\} \hookrightarrow \mathbf{Gr} \times \mathbf{V}$. That is, one has a natural functor $D_{\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{\mathbf{G_O}}(\mathbf{Gr})$. We will show that the latter functor makes the derived Satake category $D_{\mathbf{G_O}}(\mathbf{Gr})$ a localization of the mirabolic derived

Satake category $D_{\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. Moreover, the mirabolic Satake equivalence is compatible with the standard Satake equivalence in the sense that there is a commutative diagram of functors

$$
\begin{array}{ccc}
\begin{array}{c}\text{asymptotic equivariant} \\ D\text{-modules on } \mathfrak{gl}_N\end{array} & \xrightarrow[\cong]{\text{mirabolic derived Satake}} & D_{\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}) \\
\downarrow{\scriptstyle\text{restriction}} & & \downarrow{\scriptstyle\text{localization}} \\
\begin{array}{c}\text{asymptotic equivariant} \\ D\text{-modules on } \mathrm{GL}_N\end{array} & \xrightarrow[\cong]{\text{derived Satake}} & D_{\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr})
\end{array}
$$

1.4 Conjectural Iwahori-equivariant version

Let $\mathbf{I} \subset \mathbf{G_O}$ be an Iwahori subgroup and $\mathbf{Fl} := \mathbf{G_F}/\mathbf{I}$ the affine flag variety. In [Bez16], Bezrukavnikov established an equivalence of $D_{\mathbf{I}}(\mathbf{Fl})$, the $\mathbf{I}$-equivariant constructible derived category of $\mathbf{Fl}$, and the derived category of GL_N-equivariant coherent sheaves on an appropriate dg-version of the Steinberg variety. Motivated by this result, we expect that there is a mirabolic counterpart of this equivalence.

To explain this, fix a pair V_1, V_2, of N-dimensional vector spaces and let $\mathcal{F}\ell_i$, $i = 1, 2$, denote the variety of complete flags in V_i. Further, consider a dg-scheme with zero differential

$$\mathrm{H}_{\mathrm{mir}} := \mathrm{Hom}(V_1, V_2)[1] \times \mathrm{Hom}(V_2, V_1)[1] \times \mathcal{F}\ell_1 \times \mathcal{F}\ell_2.[2]$$

Write A (respectively, B) for an element of $\mathrm{Hom}(V_1, V_2)[1]$ (respectively, $\mathrm{Hom}(V_2, V_1)[1]$), and $F_i = (F_i^{(1)} \subset F_i^{(2)} \subset \cdots \subset F_i^{(N)} = V_i)$ for an element of $\mathcal{F}\ell_i$.

We define the *mirabolic Steinberg scheme* to be a dg-subscheme $\mathrm{St}_{\mathrm{mir}}$ of $\mathrm{H}_{\mathrm{mir}}$ cut out by the equations saying that the flag F_2 is stable under the composition AB and the flag F_1 is stable under the composition BA. Thus the mirabolic Steinberg scheme is a shifted variety of quadruples:

$$\mathrm{St}_{\mathrm{mir}} = \{(A, B, F_1, F_2) \in \mathrm{H}_{\mathrm{mir}} \mid AB(F_2^{(j)}) \subseteq F_2^{(j)} \text{ and } BA(F_1^{(j)}) \subseteq F_1^{(j)}, \ \forall j \in [1, N]\}.$$

Let $D_{\mathbf{I}}(\mathbf{Fl} \times \mathbf{V})$ be the $\mathbf{I}$-equivariant constructible derived category of $\mathbf{Fl} \times \mathbf{V}$. We propose the following conjecture.

CONJECTURE 1.4.1. *There exists an equivalence of triangulated categories*

$$D^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)} \mathrm{Coh}(\mathrm{St}_{\mathrm{mir}}) \cong D_{\mathbf{I}}(\mathbf{Fl} \times \mathbf{V}).$$

This conjecture would explain, in particular, the appearance of the same polynomials, called the Kostka–Shoji polynomials, in two different problems. On the one hand, it was proved in [FGT09] that these polynomials are equal to the Poincaré polynomials of the stalks of $\mathbf{G_O}$-equivariant intersection cohomology (IC) sheaves on the mirabolic affine Grassmannian. On the other hand, it was proved in [FI18] that the Kostka–Shoji polynomials are equal to the Poincaré polynomials of graded multiplicities of the natural $\mathrm{GL}_N \times \mathrm{GL}_N$-action on the space of global sections of line bundles on a convolution diagram of the cyclic $\tilde{A}_1$-quiver.

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[2] Here we view both $\mathrm{Hom}(V_1, V_2)$ and $\mathrm{Hom}(V_2, V_1)$ as *odd* vector spaces, so that the functions on $\mathrm{Hom}(V_1, V_2)[1] \times \mathrm{Hom}(V_2, V_1)[1]$ (with grading disregarded) really form a symmetric (infinite-dimensional) algebra, not an exterior algebra.

1.5 Satake equivalence for some Lie supergroups

We consider the Lie superalgebra $\mathfrak{gl}(M|N)$ of endomorphisms of a super vector space $\mathbb{C}^{M|N}$, and the corresponding Lie supergroup $\mathrm{GL}(M|N) = \mathrm{Aut}(\mathbb{C}^{M|N})$. We also consider a degenerate version $\underline{\mathfrak{gl}}(M|N)$ where the supercommutator of the even elements (with even or odd elements) is the same as in $\mathfrak{gl}(M|N)$, while the supercommutator of any two odd elements is set to zero. In other words, the even part $\underline{\mathfrak{gl}}(M|N)_{\bar{0}} = \mathfrak{gl}_M \oplus \mathfrak{gl}_N$ acts naturally on the odd part $\underline{\mathfrak{gl}}(M|N)_{\bar{1}} = \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}^N, \overline{\mathbb{C}^M})$, but the supercommutator $\underline{\mathfrak{gl}}(M|N)_{\bar{1}} \times \underline{\mathfrak{gl}}(M|N)_{\bar{1}} \to \underline{\mathfrak{gl}}(M|N)_{\bar{0}}$ equals zero.

The category of finite-dimensional representations of the corresponding supergroup $\underline{\mathrm{GL}}(M|N)$ (in vector superspaces) is denoted by $\mathrm{Rep}(\underline{\mathrm{GL}}(M|N))$, and its bounded derived category is denoted by $SD(\underline{\mathrm{GL}}(M|N))$.

There is a Koszul equivalence $\varkappa \colon SD(\underline{\mathrm{GL}}(N|N)) \xrightarrow{\sim} SD_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{S}_{1,1}^\bullet)$ (equivariant perfect dg-*super*modules over dg-*super*algebra $\mathfrak{S}_{1,1}^\bullet$); see, for example, [MR10]. It intertwines the usual tensor product of $\underline{\mathrm{GL}}(N|N)$-modules with the tensor product $\otimes_{\mathfrak{S}_{1,1}^\bullet}$ on $SD_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{S}_{1,1}^\bullet)$. Composing the Koszul equivalence $\varkappa$ with the mirabolic Satake equivalence $\Phi^{1,1} \colon SD_{\mathrm{perf}}^{\mathrm{GL}_N \times \mathrm{GL}_N}(\mathfrak{S}_{1,1}^\bullet) \xrightarrow{\sim} SD_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ (constructible sheaves of *super*vector spaces) we obtain a super Satake equivalence $SD(\underline{\mathrm{GL}}(N|N)) \xrightarrow{\sim} SD_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ that intertwines the usual tensor product of $\underline{\mathrm{GL}}(N|N)$-modules with the fusion product on $SD_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. Moreover, this derived equivalence is exact with respect to the tautological t-structure on $SD(\underline{\mathrm{GL}}(N|N))$ with the heart $\mathrm{Rep}(\underline{\mathrm{GL}}(N|N))$ (respectively, the perverse t-structure on $SD_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ with the heart $S\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$).

Similarly, we construct equivalences

$$SD(\underline{\mathrm{GL}}(N-1|N)) \xrightarrow[\bar{\varkappa}]{\sim} SD_{\mathrm{perf}}^{\mathrm{GL}_{N-1} \times \mathrm{GL}_N}(\bar{\mathfrak{S}}_{1,1}^\bullet) \xrightarrow[\bar{\Phi}]{\sim} SD_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N}),$$

where $\bar{\mathfrak{S}}_{1,1}^\bullet = \mathrm{Sym}^\bullet\big(\mathrm{Hom}(\mathbb{C}^{N-1}, \mathbb{C}^N)[-1] \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{N-1})[-1]\big)$.[3] The composition is again t-exact with respect to the tautological t-structure on $SD(\underline{\mathrm{GL}}(N-1|N))$ with the heart $\mathrm{Rep}(\underline{\mathrm{GL}}(N-1|N))$ (respectively, the perverse t-structure on $SD_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N})$ with the heart $S\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N})$). Moreover, the composition intertwines the usual tensor product of $\underline{\mathrm{GL}}(N-1|N)$-modules with the fusion product on $S\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N})$. Similarly to §1.3, the functor $\bar{\Phi}$ can be extended to an equivalence $\bar{\Phi}_\hbar \colon SD_{\mathrm{perf}}^{\mathrm{GL}_{N-1} \times \mathrm{GL}_N}(\bar{\mathfrak{D}}^\bullet) \xrightarrow{\sim} SD_{\mathrm{GL}(N-1,\mathbf{O}) \rtimes \mathbb{C}^\times}(\mathbf{Gr}_{\mathrm{GL}_N})$, where $\bar{\mathfrak{D}}^\bullet$ is the graded Weyl algebra of shifted differential operators on $\mathrm{Hom}(\mathbb{C}^{N-1}, \mathbb{C}^N)$ (with $\deg \hbar = 2$ and all the other generators in degree 1).

1.6 Gaiotto conjectures

One may wonder if there is a geometric realization of categories of representations of *non-degenerate* supergroups $\mathrm{GL}(N|N)$, $\mathrm{GL}(N-1|N)$. It turns out that such a realization exists (conjecturally) for the categories of integrable representations of *quantized* algebras $U_q(\mathfrak{gl}(N|N))$, $U_q(\mathfrak{gl}(N-1|N))$. First of all, similarly to the classical Kazhdan–Lusztig equivalence, it is expected that $U_q(\mathfrak{gl}(M|N))$-mod $\cong \mathrm{KL}_c(\widehat{\mathfrak{gl}}(M|N))$, where $q = \exp(\pi\sqrt{-1}/c)$, and $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(M|N))$ stands for the derived category of $\mathrm{GL}(M, \mathbf{O}) \times \mathrm{GL}(N, \mathbf{O})$-equivariant $\widehat{\mathfrak{gl}}(M|N)$-modules at the level corresponding to the invariant bilinear form

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[3] Here we view both $\mathrm{Hom}(\mathbb{C}^{N-1}, \mathbb{C}^N)$ and $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{N-1})$ as *odd* vector spaces, so that $\mathrm{Sym}^\bullet\big(\mathrm{Hom}(\mathbb{C}^{N-1}, \mathbb{C}^N)[-1] \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{N-1})[-1]\big)$ (with grading disregarded) is really a symmetric (infinite-dimensional) algebra, not an exterior algebra.

$(X, Y) = c \cdot \mathrm{sTr}(XY) - \frac{1}{2} \mathrm{Killing}_{\mathfrak{gl}(M|N)}(X, Y)$ on $\mathfrak{gl}(M|N)$. Second, it is expected that the category $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(N-1|N))$ is equivalent to the q-monodromic $\mathrm{GL}(N-1, \mathbf{O})$-equivariant derived constructible category of the complement $\mathcal{L}^\bullet$ of the zero section of the determinant line bundle on $\mathbf{Gr}_{\mathrm{GL}_N}$, and this equivalence takes the standard t-structure of $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(N-1|N))$ to the perverse t-structure.

Further, it is expected that $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(N|N))$ is equivalent to the q-monodromic $\mathrm{GL}(N, \mathbf{O})$-equivariant derived constructible category of $\mathcal{L}^\bullet \times \mathbf{V}$, and this equivalence takes the standard t-structure of $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(N|N))$ to the perverse t-structure. For $M < N-1$ it is expected that $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(M|N))$ is equivalent to the q-monodromic $\mathrm{GL}(M, \mathbf{O})$-equivariant derived constructible category of $\mathcal{L}^\bullet$ with certain Whittaker conditions; cf. § 2.6 for more details. In particular, the special case $M = 0$ of this conjecture follows from the fundamental local equivalence [Gai08, Gai16, Gai20] of the geometric Langlands program.

There are similar expectations for other classical (i.e. orthosymplectic) Lie superalgebras; the interested reader may try to find them in [GW09].

1.7 Conjectures of Ben-Zvi, Sakellaridis and Venkatesh

In an ongoing project of D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh, the authors propose the Periods–L-functions duality conjectures. Their conjectures predict, among other things, that, given a reductive group G and its spherical homogeneous variety $X = \mathrm{G}/\mathrm{H}$, there is a subgroup $\mathrm{G}_X^\vee \subset \mathrm{G}^\vee$, its graded representation $V_X^\vee = \bigoplus_{i \in \mathbb{Z}} V_{X,i}^\vee[i]$, and an equivalence $D\mathsf{Coh}(V_X^\vee/\mathrm{G}_X^\vee) = D\mathsf{Coh}\big((\bigoplus_{i \in \mathbb{Z}} V_{X,i}^\vee[i])/\mathrm{G}_X^\vee\big) \simeq D_{\mathrm{G}(\mathbf{O})}(X(\mathbf{F}))$. For a partial list of examples, see the table at the end of [Sak13]. The relevant representations $V_X^\vee$ (constructed in terms of the Luna diagram of X) can be read off from the fourth column of the table.

It turns out that the equivalences discussed in § 1.5 fit into the general setting outlined in the previous paragraph. Thus the case of [Sak13, Example 13] corresponds to the equivalence $\bar{\bar{\Phi}} \colon D_{\mathrm{perf}}^{\mathrm{GL}_{N-1} \times \mathrm{GL}_N}(\bar{\mathfrak{S}}_{1,1}^\bullet) \xrightarrow{\sim} D_{\mathrm{GL}(N-1, \mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N})$ of § 1.5. To explain this, let $\mathrm{G} := \mathrm{GL}_{N-1} \times \mathrm{GL}_N$ and $\mathrm{H} := \mathrm{GL}_{N-1}$. We view H as a block-diagonal subgroup of G and put $X = \mathrm{G}/\mathrm{H}$. Then, loosely speaking, we have $D_{\mathrm{GL}(N-1, \mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N}) \simeq D\big(\mathrm{GL}(N-1, \mathbf{O})\backslash \mathrm{GL}(N, \mathbf{F})/\mathrm{GL}(N, \mathbf{O})\big) \simeq D\big(\mathrm{G}(\mathbf{O})\backslash \mathrm{G}(\mathbf{F})/\mathrm{H}(\mathbf{F})\big) \simeq D\big(\mathrm{G}(\mathbf{O})\backslash X(\mathbf{F})\big) \simeq D_{\mathrm{G}(\mathbf{O})}(X(\mathbf{F}))$. On the other hand, we consider a graded $\mathrm{G}^\vee$-module $V_X^\vee := \mathrm{Hom}(\mathbb{C}^{N-1}, \mathbb{C}^N)[1] \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{N-1})[1]$ (similarly to the footnote in § 1.5, we view $V_X^\vee$ as an *odd* vector space placed in cohomological degree -1; note also that $\mathrm{G}^\vee \simeq \mathrm{G} = \mathrm{GL}_{N-1} \times \mathrm{GL}_N$). Hence, the equivalence $\bar{\bar{\Phi}}$ of § 1.5 takes the form $D\mathsf{Coh}(V_X^\vee/\mathrm{G}^\vee) \simeq D_{\mathrm{G}(\mathbf{O})}(X(\mathbf{F}))$.

Similarly, [Sak13, Example 14] matches the Gaiotto conjecture of § 1.6 for an orthosymplectic Lie superalgebra.

1.8 Organization of the paper

In § 2 (which is not necessary for understanding subsequent sections) we formulate the Gaiotto conjectures and explain their relation to the geometric Langlands program and, in particular, with the fundamental local equivalence. Section 3 is the technical core of the paper. In this section we establish a coherent description of the spherical mirabolic affine Hecke category $D_{\mathrm{GL}(N, \mathbf{O})}(\mathbf{Gr} \times \mathbf{V})$ with its three monoidal structures. Also, as preparation for § 4 we give a coherent description of the restriction functor $D_{\mathrm{GL}(N, \mathbf{O})}(\mathbf{Gr} \times \mathbf{V}) \to D_{\mathrm{GL}(N, \mathbf{O})}(\mathbf{Gr} \times \mathbf{V}_0)$.[4] In § 4 we establish a coherent description of the category $D_{\mathrm{GL}(N-1, \mathbf{O})}(\mathbf{Gr})$ along with its fusion

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[4] Recall that $\mathbf{V}_0 = V[\![t]\!]$ is the standard lattice in the Tate vector space $\mathbf{V}$.

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monoidal structure. In §5 we prove the quantum analogues, that is, counterparts for categories equivariant with respect to the loop rotations, of the results above.

2. Gaiotto conjectures

The purpose of the present section is to put the main results of this paper into a wider framework which has to do with the local geometric Langlands correspondence. This section will be used for motivation purposes only. It can be safely skipped by readers who are not interested in the geometric Langlands correspondence. The ideas of this section are due to D. Gaiotto (private communication). Gaiotto informed us that his ideas were motivated, to a large extent, by his discussions with P. Yoo as well as by [MW15].

2.1 Reminder on strong actions on categories

Let G be a connected reductive group over $\mathbb{C}$ and let κ denote an invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G (when G is simple, the vector space of such bilinear forms is one-dimensional, so we can think of κ as an element of $\mathbb{C}$). Then there is a notion of *strong action of the group $G_{\mathbf{F}}$ on a category $\mathcal{C}$ of level κ*. We refer the reader to [Gai16] for details of the definition. It will be important for us later that this definition is, in some sense, invariant under integral shifts. Specifically, any category $\mathcal{C}$ endowed with a $G_{\mathbf{F}}$-action of level κ has a natural $G_{\mathbf{F}}$-action of level $\kappa + \kappa'$, where κ' is another form as above which is, moreover, integral in the sense that the corresponding quadratic form is integral and even on elements of the coweight lattice of G. Here are two important examples.

(1) Let $\hat{\mathfrak{g}}_{\kappa}$ denote the central extension of $\mathfrak{g}_{\mathbf{F}}$ associated with the form κ. Let $\hat{\mathfrak{g}}_{\kappa}$-mod be the category of continuous (with respect to t-adic topology) modules over $\hat{\mathfrak{g}}_{\kappa}$ such that the element 1 of the center acts on the module as the identity. Then the adjoint action of $G_{\mathbf{F}}$ on $\hat{\mathfrak{g}}_{\kappa}$ has a natural lift to a strong action of $G_{\mathbf{F}}$ on $\hat{\mathfrak{g}}_{\kappa}$-mod of level κ. More generally, let $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ be a Lie superalgebra. Make the following assumptions.

(i) G acts on $\mathfrak{a}$.
(ii) We are given a map $\iota \colon \mathfrak{g} \to \mathfrak{a}_{\bar{0}}$ such that the corresponding adjoint action of $\mathfrak{g}$ on $\mathfrak{a}$ is equal to the derivative of the G-action from (i).
(iii) The algebra $\mathfrak{a}$ is equipped with an invariant symmetric (in the super sense) bilinear form $\kappa_{\mathfrak{a}}$. Let $\kappa_{\mathfrak{g}}$ be its pullback to $\mathfrak{g}$.

Associated with the form $\kappa_{\mathfrak{a}}$, there is a canonical Kac–Moody extension $\hat{\mathfrak{a}}$ of $\mathfrak{a}_{\mathbf{F}}$. As before, we denote by $\hat{\mathfrak{a}}$-mod the category of continuous modules M over $\hat{\mathfrak{a}}$ such that the element 1 of the center acts on M by the identity. This category comes equipped with a $G_{\mathbf{F}}$-action of level $\kappa_{\mathfrak{g}}$.

We will mostly be interested in the following special case of the above construction. Fix a pair M, N, of non-negative integers. Let $\mathfrak{a} = \mathfrak{gl}(M|N)$, $G = \mathrm{GL}_M \times \mathrm{GL}_N$. Let $\kappa_{\mathfrak{a}}(x, y) = c \cdot \mathrm{sTr}(xy)$, where sTr stands for 'super-trace' and $c \in \mathbb{C}$.

(2) Assume that the form κ is integral and even. Then this form gives rise to a central extension $\hat{G}$ of $G_{\mathbf{F}}$. Let X be an ind-scheme equipped with a $\hat{G}$-equivariant line bundle $\mathcal{L}$. Then, for any $c \in \mathbb{C}$, one has the category $D_c(X)$-mod of c-twisted D-modules on X. This category has a natural strong $G_{\mathbf{F}}$-action of level $c \cdot \kappa$.

2.2 Digression on the local (quantum) geometric Langlands correspondence

Let G and κ be as above and assume in addition that the form κ is non-degenerate. Let $G^\vee$ denote the Langlands dual group. Since κ is non-degenerate it gives a similar form $\kappa^\vee$ for $G^\vee$. Further, put $\kappa_{\mathrm{crit}} = -\frac{1}{2}\mathrm{Killing}_{\mathfrak{g}}$, where $\mathrm{Killing}_{\mathfrak{g}}$ stands for the Killing form on the Lie algebra $\mathfrak{g}$ of G.

The local quantum geometric Langlands conjecture is, roughly speaking, an equivalence of 2-categories

$$\{\text{categories with strong } G_{\mathbf{F}}\text{-action of level } \kappa + \kappa_{\mathrm{crit}}\}$$

and

$$\{\text{categories with strong } G_{\mathbf{F}}^\vee\text{-action of level } -\kappa^\vee - \kappa_{\mathrm{crit}}^\vee\}.$$

The form κ_{crit} being integral, the shift by κ_{crit} in the above formulation is not essential. However, it is convenient for many applications to make this shift.

To give a rigorous meaning to the above conjecture one needs, first of all, to replace all 'categories' by suitable 'dg-categories'. This upgrades each side of the equivalence to an $(\infty, 2)$-category. Then, for generic (i.e. non-rational) κ, the above equivalence is expected to hold as stated. For general κ, more corrections are necessary but we will not discuss this here since the local geometric Langlands correspondence will only serve as a guiding principle.

There is a 'limiting version' of the above conjecture for $\kappa = 0$. To explain this, we use the notion of a 'category over a stack $\mathcal{S}$'; cf. [Gai16, § 6] and references therein. Write $\mathcal{D}^\circ = \mathrm{Spec}(\mathbf{F})$ and let $\mathrm{LocSys}_{G^\vee}(\mathcal{D}^\circ)$ be the classifying stack of principal $G^\vee$-bundles on $\mathcal{D}^\circ$ equipped with a connection. The 'classical' local geometric Langlands conjecture predicts a close relationship between

$$\{\text{categories with strong } G_{\mathbf{F}}\text{-action of level } \kappa_{\mathrm{crit}}\}$$

and

$$\{\text{categories over } \mathrm{LocSys}_{G^\vee}(\mathcal{D}^\circ)\}.$$

Here, κ_{crit} can be replaced by 0, since κ_{crit} is integral. Again, it is possible to make the informal relation above a rigorous mathematical conjecture; cf. again [Gai16, § 6] for a more detailed discussion.

In the remainder of this section we will pretend that both quantum and classical geometric Langlands conjectures hold as stated. We will write $\mathcal{C} \mapsto \mathcal{C}^\vee$ for the resulting correspondence.

An important example of such a correspondence is as follows. Fix a non-degenerate form κ and let $\mathcal{C} = D_{\kappa + \kappa_{\mathrm{crit}}}(\mathbf{Gr}_G)$-mod be the category of $(\kappa - \kappa_{\mathrm{crit}})$-twisted D-modules on the affine Grassmannian $\mathbf{Gr}_G$ of G. It is expected that in this case the category $\mathcal{C}^\vee$ is the category $D_{-\kappa^\vee - \kappa_{\mathrm{crit}}^\vee}(\mathbf{Gr}_{G^\vee})$-mod. In the limiting case $\kappa = 0$ category $\mathcal{C}^\vee$ is expected to be a pushforward of category $\mathrm{QCoh}(\mathrm{pt}/G^\vee)$ under the natural map $\mathrm{pt}/G^\vee \to \mathrm{LocSys}_{G^\vee}(\mathcal{D}^\circ)$ induced by the embedding of the trivial local system. These expectations would have the following implication.

 (i) If κ is non-degenerate, then $\mathcal{C}^{G_{\mathbf{O}}} \simeq (\mathcal{C}^\vee)^{G_{\mathbf{O}}^\vee}$ (here $\mathcal{C}^{G_{\mathbf{O}}}$ denotes the category of $G_{\mathbf{O}}$-equivariant objects in $\mathcal{C}$).
 (ii) If $\kappa = 0$ then $\mathcal{C}^{G_{\mathbf{O}}}$ is equivalent to the pullback of the category $\mathcal{C}^\vee$ under the map $\mathrm{pt}/G^\vee \to \mathrm{LocSys}_{G^\vee}(\mathcal{D}^\circ)$ corresponding to the trivial local system.

2.3 Whittaker category and the fundamental local equivalence

In this subsection we discuss another important example of Langlands dual categories. We refer to [Gai16, Gai20] for details.

Let U be a maximal unipotent subgroup of G and let $\chi_0 \colon U \to \mathbb{G}_a$ be a non-degenerate character. We define a character $\chi \colon U_{\mathbf{F}} \to \mathbb{G}_a$ by

$$\chi(u(t)) = \operatorname{Res}_{t=0} \chi_0(u(t)) \, dt.$$

Given a (cocomplete, dg) category $\mathcal{C}$ with a strong $G_{\mathbf{F}}$-action of a fixed level, one can define a category $\operatorname{Whit}(\mathcal{C})$ of $(U_{\mathbf{F}}, \chi)$-equivariant objects in $\mathcal{C}$.

We consider the category $D_\kappa(G_{\mathbf{F}})$-mod of κ-twisted D-modules on $G_{\mathbf{F}}$. According to [AG02], this category has a natural action of $G_{\mathbf{F}}$ of level κ that comes from left translations, and also another action of $G_{\mathbf{F}}$ of level $-\kappa + 2\kappa_{\mathrm{crit}}$ that comes from right translations.

Let $\operatorname{Whit}^r_\kappa(G_{\mathbf{F}})$ denote its Whittaker category with respect to the right action. This category inherits the left action of level κ. One could ask what is its Langlands dual category.

For simplicity, below we will only consider either the case where κ is not rational (i.e. the value of the corresponding quadratic form on any coroot is not a rational number) or the case $\kappa = 0$.

Assume first that κ is non-degenerate and not rational. Then it is expected (cf. [Gai16]) that

$$\operatorname{Whit}^r_{\kappa - \kappa_{\mathrm{crit}}}(G_{\mathbf{F}})^\vee \text{ is the category } \widehat{\mathfrak{g}^\vee}_{-\kappa^\vee + \kappa^\vee_{\mathrm{crit}}}\text{-mod.} \tag{2.3.1}$$

In the case $\kappa = 0$ the expected answer is the category $\operatorname{QCoh}(\operatorname{LocSys}_{G^\vee}(\mathcal{D}^\circ))$.

Let $\operatorname{Whit}_\kappa(\mathbf{Gr}_G)$ denote the category of κ-twisted D-modules on $\mathbf{Gr}_G$ and let $\operatorname{KL}_\kappa(\mathfrak{g}) = (\hat{\mathfrak{g}}_\kappa\text{-mod})^{G_{\mathbf{O}}}$. Then, statements (i) and (ii) of § 2.2 imply the following result, which has been proved rigorously in [Gai20, Gai08] (it goes under the name 'fundamental local equivalence').

THEOREM 2.3.1. *Assume that κ is non-degenerate and not rational. Then*

$$\operatorname{Whit}_{\kappa + \kappa_{\mathrm{crit}}}(\mathbf{Gr}_G) \simeq \operatorname{KL}_{\kappa^\vee + \kappa^\vee_{\mathrm{crit}}}(\mathfrak{g}^\vee), \tag{2.3.2}$$

$$\operatorname{Whit}(\mathbf{Gr}_G) \simeq \operatorname{Rep}(G^\vee). \tag{2.3.3}$$

Moreover, these equivalences hold at the level of abelian categories.

Remarks 2.3.2. (1) The critical shifts in (2.3.2) are not important for irrational κ if one only cares about both sides as abstract categories. However, we still prefer to keep them, since in this way one can also extend the statement to rational κ; in addition, the shifts are important if we keep track of some natural structures on these categories (cf. (4)) below.

(2) Note the absence of a negative sign before $\kappa^\vee + \kappa^\vee_{\mathrm{crit}}$ on the right-hand side of (2.3.2). This has to do with the fact that $\operatorname{Whit}_{\kappa + \kappa_{\mathrm{crit}}}(\mathbf{Gr}_G)$ is actually the category of $G_{\mathbf{O}}$-equivariant objects with respect to the *left* action of $G_{\mathbf{O}}$ on the category of D-modules on $G_{\mathbf{F}}$ which are Whittaker *on the right*. This change from right to left is what is responsible for the change of sign.

(3) The fact that the above equivalences respect the natural t-structures does not follow (to the best of our knowledge) from any geometric Langlands considerations. In fact, at the level of (unbounded) derived categories the statement holds for all κ, but when κ is positive rational it is very far from an abelian equivalence.

(4) The above equivalences are in fact not only equivalences of abstract categories, but also equivalences of categories with *factorization structure*, a notion closely related to the notion of

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a braided monoidal category (it is also worthwhile to note that as an abstract category, $\mathrm{KL}_\kappa(\hat{\mathfrak{g}})$ is independent of κ if κ is irrational; but it is not so if we take into account the factorization structure).

2.4 Gaiotto conjectures: geometric Langlands form in the case $N > M$

There is one more series of examples which are relevant to the subject of this paper. Fix two non-negative integers N and M such that $N \geq M$ and set $G_{M,N} = \mathrm{GL}_M \times \mathrm{GL}_N$. Note that $G_{M,N}$ is isomorphic to $G_{M,N}^\vee$.

We are going to produce an example of Langlands dual categories for $G_{M,N}$; for $M = 0$ we will recover (2.3.1). First, we describe the analogue of the right-hand side. Let $c \in \mathbb{C}$. Then the category in question will be the category $\widehat{\mathfrak{gl}}(M|N)_c$-mod of modules over the affine Lie superalgebra $\widehat{\mathfrak{gl}}(M|N)$ of level $c \cdot \kappa_{M,N} + \frac{1}{2}\mathrm{Killing}_{M,N}$, where

(1) $\kappa_{M,N}(x,y) = \mathrm{sTr}(xy)$,
(2) $\mathrm{Killing}_{M,N}$ is the restriction of the Killing form of $\mathfrak{gl}(M|N)$ to the even part (note that it is degenerate if $M = N$).

This category has a natural action of the group $G_{M,N}(\mathbf{F})$ of a certain level which is an integral shift of $(c \cdot \kappa_M, -c \cdot \kappa_N)$ (here κ_N denotes the standard invariant bilinear form on the Lie algebra $\mathfrak{gl}_N$ equal to $\mathrm{Tr}(X \cdot Y)$). As has been explained above, one can twist the action of $G_{M,N}(\mathbf{F})$ on this category so that the twist becomes equal to $(c \cdot \kappa_M, -c \cdot \kappa_N) - \kappa_{\mathrm{crit}}$ (here by κ_{crit} we mean the critical bilinear form for the Lie algebra $\mathfrak{g}_{M,N}$). Hence it makes sense to consider its Langlands dual. This should be a category with a strong action of $G_{M,N}(\mathbf{F})$ of level $c^{-1} \cdot (\kappa_M, -\kappa_N) - \kappa_{\mathrm{crit}}$. However, we can again twist the action and think of it as a category with an action of $G_{M,N}(\mathbf{F})$ of level $c^{-1} \cdot (\kappa_M, -\kappa_N)$. Let us give a conjectural description of the Langlands dual category according to a prediction of D. Gaiotto.

Next, let $M < N$. We define a unipotent subgroup $U_{M,N}$ of GL_N as follows. If $M = N - 1$ this subgroup is trivial, and if $M = 0$ it is the group U_N of unipotent upper-triangular matrices. In the general case, $U_{M,N}$ is a subgroup of U_N defined as follows.

Let $e_{M,N} \in \mathfrak{gl}_N$ be the standard upper-triangular Jordan block of size $N - M$, that is,

$$e_{M,N} = \sum_{i=1}^{N-M-1} E_{i,i+1},$$

where E_{ij} stands for the matrix whose (i,j)th entry is equal to 1 and all other entries are equal to 0. The element $e_{M,N}$ is part of an $\mathfrak{sl}_2$-triple $(e_{M,N}, h_{M,N}, f_{M,N})$. Here $f_{M,N} = \sum_{i=1}^{N-M-1} i(N-M-i)E_{i+1,i}$ and $h_{M,N}$ is the diagonal matrix which has diagonal entries $(N-M-1, N-M-3, \ldots, -N+M+1, 0, \ldots, 0)$. For any integer l we let $\mathfrak{g}_l$ denote the l-eigenspace of the adjoint action of $h_{M,N}$ on $\mathfrak{gl}_N$. We set

$$\mathfrak{u}_{M,N} = \bigoplus_{l \geq 2} \mathfrak{g}_l \oplus \mathfrak{g}_1^+,$$

where $\mathfrak{g}_1^+$ is the intersection of $\mathfrak{g}_1$ with the Lie algebra of upper-triangular matrices. We define a Lie algebra homomorphism $\chi_{M,N}^0 \colon \mathfrak{u}_{M,N} \to \mathbb{C}$ by sending a matrix (u_{ij}) to $\sum_{i=1}^{N-M-1} u_{i,i+1}$. Let $U_{M,N}$ be a unipotent subgroup of GL_N with Lie algebra $\mathfrak{u}_{M,N}$ and let $U_{M,N} \to \mathbb{G}_a$ be the homomorphism induced by $\chi_{M,N}^0$ (which we denote by the same symbol $\chi_{M,N}^0$).

We embed the group GL_M into the centralizer of the element $h_{M,N}$ in GL_N (this is the block-diagonal embedding corresponding to rows $N - M + 1, \ldots, N$). It is easy to see that the group GL_M normalizes the subgroup $U_{M,N}$, and the homomorphism $\chi^0_{M,N}$ is fixed by the GL_M-action on $U_{M,N}$ by conjugation.

Remark 2.4.1. $U_{M,N}$ is conjugate to the subgroup $U'_{M,N}$ formed by the block-upper-triangular matrices

$$\begin{pmatrix} U_r & * & * \\ 0 & 1_{M+1} & * \\ 0 & 0 & U_s \end{pmatrix}, \quad \text{where } r = \lceil (N - M - 1)/2 \rceil, \ s = \lfloor (N - M - 1)/2 \rfloor;$$

in particular, $r + s = N - M - 1$. Here U_p stands for an arbitrary unipotent upper-triangular matrix in GL_p, and the notation '$*$' is used for arbitrary matrices of an appropriate size. Moreover, the conjugation can be chosen so that the character $\chi^0_{M,N}$ corresponds to the character $\chi^{(r,s)}_{M,N}$ on $\mathfrak{u}^{(r,s)}_{M,N} := \operatorname{Lie} U^{(r,s)}_{M,N}$ given by $(u_{ij}) \mapsto \sum_{i=1}^{r-1} u_{i,i+1} + u_{rk} + u_{k,N-s+1} + \sum_{i=N-s+1}^{N-1} u_{i,i+1}$ for any choice of $k \in \{r+1, \ldots, N-s\}$. The subgroup $U^{(r,s)}_{M,N} \subset GL_N$ and the character $\chi^{(r,s)}_{M,N}$ are defined for an arbitrary pair $(r,s) \in \mathbb{N}^2 := \mathbb{Z}^2_{\geq 0}$ with $r + s = N - M - 1$. (In the two extreme cases $\{r,s\} = \{0, N-M-1\}$, one of the middle terms in the formula for $\chi^{(r,s)}_{M,N}$ is undefined and should be omitted.) Moreover, $\chi^{(r,s)}_{M,N}$ can be replaced by an arbitrary representative of the open $N_{GL_N}(U^{(r,s)}_{M,N})$-orbit in $\mathfrak{u}^{(r,s)*}_{M,N}$.

Remark 2.4.2. $U^{(0,N-M-1)}_{M,N}$ is also conjugate to the unipotent radical $U_{(M+1,1,\ldots,1)}$ of the standard parabolic subgroup $P_{(M+1,1,\ldots,1)}$ of GL_N corresponding to the partition $(M+1, 1, \ldots, 1)$ of N. The character $\chi^0_{M,N}$ is conjugate to the restriction of the regular character $u \mapsto \sum_{i=1}^{N-1} u_{i,i+1}$ of the upper-triangular subgroup to $U_{(M+1,1,\ldots,1)}$; cf. [JPS83, §(2.11)] and [Cog04, beginning of § 2 of Lecture 5].

As before, we define a homomorphism $\chi_{M,N} \colon U_{M,N}(\mathbf{F}) \to \mathbb{G}_a$ to be $\operatorname{Res}_{t=0} \chi^0_{M,N}$.

For any $c' \in \mathbb{C}$, we now consider the category $D_{c'}(GL(N, \mathbf{F}))^{U_{M,N}(\mathbf{F}), \chi_{M,N}}$. By definition this is the derived category of D-modules on $GL(N, \mathbf{F})$ twisted by $c' \cdot \kappa_N$ that are equivariant *on the left* with respect to $(U_{M,N}(\mathbf{F}), \chi_{M,N})$. This category has a natural action of $GL(M, \mathbf{F})$ of level $c' \cdot \kappa_M$ coming from left multiplication and an action of $GL(N, \mathbf{F})$ of level $-c' \cdot \kappa_M - \operatorname{Killing}_{\mathfrak{gl}_N}$ coming from right multiplication (recall that $\operatorname{Killing}_{\mathfrak{gl}_N}$ denotes the Killing form on $\mathfrak{gl}_N$). As before, we can twist the second action to make it an action of level $-c' \cdot \kappa_M$.

Remark 2.4.3. In fact, using the Fourier transform for D-modules, one can show that replacing (r,s) in Remark 2.4.1 with any pair of non-negative integers whose sum equals $N - M - 1$ produces a category equivalent to $D_{c'}(GL(N, \mathbf{F}))^{U_{M,N}(\mathbf{F}), \chi_{M,N}}$.

We now take $c' = 1/c$. Then, according to a conjecture of D. Gaiotto, the category $D_{1/c}(GL(N, \mathbf{F}))^{U_{M,N}(\mathbf{F}), \chi_{M,N}}$ is Langlands dual to $\widehat{\mathfrak{gl}}(M|N)_c$-mod.

We can also consider the limit $c \to \infty$. To simplify the discussion we will not do it now, but we will discuss it later when we turn to $G_{\mathbf{O}}$-equivariant objects.

2.5 Gaiotto conjectures: geometric Langlands form for $N = M$

Let us also discuss the case $N = M$. In this case on the left we again take the same category $\widehat{\mathfrak{gl}}(N|N)_c$-mod of modules over the affine Lie superalgebra $\widehat{\mathfrak{gl}}(N|N)$ of level $c \cdot \kappa_{N,N} - \frac{1}{2}\operatorname{Killing}_{N,N}$

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(note that the Killing form is zero for $N = M$). The Langlands dual category (according to Gaiotto) is the derived category $D_{1/c}(\mathrm{GL}(N, \mathbf{F}) \times \mathbf{V})$ of $1/c$-twisted D-modules on $\mathrm{GL}(N, \mathbf{F}) \times \mathbf{V}$ where

(1) the twisting is with respect to the first factor,
(2) the two actions of $\mathrm{GL}(N, \mathbf{F})$ come from the diagonal action coming from the left action of $\mathrm{GL}(N, \mathbf{F})$ on the first factor and the natural action of $\mathrm{GL}(N, \mathbf{F})$ on the second factor, and the right multiplication action $\mathrm{GL}(N, \mathbf{F})$ on the first factor.

2.6 Gaiotto conjectures: 'simple-minded' form

The above statements are not really well-formulated mathematical conjectures, since the local geometric Langlands duality is not known at present. However, we can turn them into precise conjectures by using (i) at the end of § 2.2, that is, we are going to take $G_{M,N}(\mathbf{O})$-invariants on both sides. We get the following conjectures.

CONJECTURE 2.6.1. *Assume that $N > M$ and assume that $c \neq 0$ is not a rational number. The categories $D_{1/c}^{(\mathrm{GL}(M,\mathbf{O}) \ltimes U_{M,N}(\mathbf{F}), \chi_{M,N})}(\mathbf{Gr}_{\mathrm{GL}_N})$ and $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(M|N))$ are equivalent as factorization categories. Here $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(M|N))$ is the category of $G_{M,N}(\mathbf{O})$-equivariant objects in the category $\widehat{\mathfrak{gl}}(M|N)_c$-mod.*

Similarly, the category $D_{1/c}^{\mathrm{GL}(N,\mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N} \times \mathbf{V})$ is equivalent to $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(N|N))$.

We now want to take the limit $c \to \infty$. In this case $1/c$ goes to 0 and the category of $1/c$-twisted D-modules in Conjecture 2.6.1 just becomes the category of usual D-modules. The $c \to \infty$ limit of the category $\mathrm{KL}_c(\widehat{\mathfrak{gl}}(M|N))$ is not canonically defined: one has to choose some nice extension of the corresponding family of categories from $\mathbb{A}^1$ to $\mathbb{P}^1$; cf. [Zha17, §6]. Naively, one might think that the correct extension is just the category of representations of the supergroup $\mathrm{GL}(M|N)$. However, it turns out that this is not the right choice. Instead, one needs to consider the category of representations of the group $\underline{\mathrm{GL}}(M|N)$) defined in §1.5. With these conventions one gets the following conjecture.

CONJECTURE 2.6.2. *Assume that $N > M$. Then the category $D^{(\mathrm{GL}(M,\mathbf{O}) \ltimes U_{M,N}(\mathbf{F}), \chi_{M,N})}(\mathbf{Gr}_{\mathrm{GL}_N})$ is equivalent to the category of modules over the group $\underline{\mathrm{GL}}(M|N)$).*

Similarly, for $N = M$, the category $D^{\mathrm{GL}(N,\mathbf{O})}(\mathbf{Gr}_{\mathrm{GL}_N} \times \mathbf{V})$ is equivalent to the category of modules over the group $\underline{\mathrm{GL}}(N|N)$).

These equivalences should hold for both derived and abelian categories.

In the present work we prove Conjecture 2.6.2 for $M = N$ and $M = N - 1$.

3. A coherent realization of $D_{\mathbf{GL(N,O)}}(\mathbf{Gr} \times \mathbf{V})$

3.1 Setup and notation

We follow the notation of [FGT09]. Recall that $\mathbf{Gr} = \mathbf{Gr}_{\mathrm{GL}_N} = \mathbf{G_F}/\mathbf{G_O} = \mathrm{GL}(N, \mathbf{F})/\mathrm{GL}(N, \mathbf{O})$, where $\mathbf{F} = \mathbb{C}((t)) \supset \mathbb{C}[[t]] = \mathbf{O}$. We consider a complex vector space V with a basis $e_1, \ldots, e_N$. We set $\mathbf{V} = V \otimes \mathbf{F} \supset V \otimes \mathbf{O} = \mathbf{V}_0$.

Recall that the $\mathbf{G_F}$-orbits in $\mathbf{Gr} \times \mathbf{Gr}$ (respectively, in $\mathbf{Gr} \times \mathbf{Gr} \times \overset{\circ}{\mathbf{V}}$) are numbered in [FGT09, §3.1] by signatures[5] (respectively, by bisignatures, i.e. pairs of signatures) in such a

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[5] Sequences of integers $\boldsymbol{\nu} = (\nu_1 \geq \cdots \geq \nu_N)$, following the terminology of H. Weyl.

way that the $\mathbf{G_F}$-orbits on $\mathbf{Gr} \times \mathbf{Gr}$ numbered by partitions $\boldsymbol{\nu} = (\nu_1 \geq \cdots \geq \nu_N \geq 0)$ correspond to the pairs of lattices $L^1 \subset L^2$. More precisely, the orbit corresponding to a bisignature $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ contains a point

$$\left(L^1 = \mathbf{O}\langle e_1, e_2, \ldots, e_N \rangle, \; L^2 = \mathbf{O}\langle t^{-\lambda_1 - \mu_1}e_1, t^{-\lambda_2 - \mu_2}e_2, \ldots, t^{-\lambda_N - \mu_N}e_N \rangle, \; v = \sum_{i=1}^{N} t^{-\lambda_i}e_i \right).$$

Irreducible representations of $\mathrm{GL}_N = \mathrm{GL}(V)$ are also numbered by the signatures, so that, for example, the determinant character $\det V$ corresponds to (1^N). To a signature $\boldsymbol{\nu} = (\nu_1 \geq \cdots \geq \nu_N)$ we associate an irreducible representation $V_{\boldsymbol{\nu}}$ with the highest weight $\boldsymbol{\nu}$. The geometric Satake equivalence takes the irreducible perverse sheaf $\mathrm{IC}_{\boldsymbol{\nu}}$ to the irreducible representation $V_{\boldsymbol{\nu}}^*$. Thus if $\boldsymbol{\nu}$ is a partition (respectively, a negative partition $(0 \geq \nu_1 \geq \cdots \geq \nu_N)$), then $V_{\boldsymbol{\nu}}^*$ is an antipolynomial (respectively, polynomial) representation of GL_N (a polynomial functor in V^*, V, respectively).

A word of apology for our weird convention is in order. The numbering of $\mathbf{G_O}$-orbits in $\mathbf{Gr}$ such that the orbits of *sub*lattices $L \subset \mathbf{V}_0$ are numbered by partitions goes back at least to [Lus81]. We choose the numbering such that the orbits of sublattices are numbered by negative partitions since under this numbering the adjacency order of $\mathbf{G_O}$-orbits in $\mathbf{Gr} \times \overset{\circ}{\mathbf{V}}$ goes to Shoji's order ([Sho04], [FGT09, Proposition 12]) on the set of bisignatures. Furthermore, we choose the Satake equivalence $\mathrm{IC}_{\boldsymbol{\nu}} \mapsto V_{\boldsymbol{\nu}}^*$ (as opposed to $\mathrm{IC}_{\boldsymbol{\nu}} \mapsto V_{\boldsymbol{\nu}}$) since it makes the statement of our main result Theorem 3.6.1 neater.

3.2 Constructible mirabolic category and convolutions

The triangulated category $D_{\mathbf{G_O}}(\mathbf{Gr} \times \overset{\circ}{\mathbf{V}})$ is defined as in [FGT09, § 2.6]. We will denote it by $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. Recall that an object $\mathcal{F}$ of $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ is supported on $\mathbf{Gr} \times t^m \mathbf{V}_0$ for certain $m \in \mathbb{Z}$, and there exist $n > m$ and a $\mathbf{G_O}$-equivariant sheaf $\mathcal{F}_n$ on $\mathbf{Gr} \times (t^m \mathbf{V}_0 / t^n \mathbf{V}_0)$ such that $\mathcal{F} = p_n^* \mathcal{F}_n$, where

$$p_n \colon \mathbf{Gr} \times t^m \mathbf{V}_0 \to \mathbf{Gr} \times (t^m \mathbf{V}_0 / t^n \mathbf{V}_0)$$

is the natural projection. In other words, $\mathcal{F}$ is a collection of $\mathbf{G_O}$-equivariant sheaves $\mathcal{F}_{n'}$ on $\mathbf{Gr} \times (t^m \mathbf{V}_0 / t^{n'} \mathbf{V}_0)$ for $n' \geq n$ along with a compatible system of isomorphisms $p_{n''/n'}^* \mathcal{F}_{n'} \xrightarrow{\sim} \mathcal{F}_{n''}$ for $n'' \geq n'$, where

$$p_{n''/n'} \colon \mathbf{Gr} \times t^m \mathbf{V}_0 / t^{n''} \mathbf{V}_0 \twoheadrightarrow \mathbf{Gr} \times t^m \mathbf{V}_0 / t^{n'} \mathbf{V}_0$$

are the natural projections.

If in the above definition we replace p_n^* by $p_n^!$ and $p_{n''/n'}^*$ by $p_{n''/n'}^!$, then we obtain a triangulated category $D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. Note that $p_{n''/n'}$ is a smooth morphism of relative dimension $N(n'' - n')$, so we have a canonical isomorphism $p_{n''/n'}^! \cong p_{n''/n'}^*[2N(n'' - n')]$. We also consider the intermediate version

$$p_{n''/n'}^{!*} := p_{n''/n'}^*[N(n'' - n')] = p_{n''/n'}^![-N(n'' - n')],$$

exact for the perverse t-structure. The corresponding triangulated category will be denoted $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

We will make use of an identification $\mathbf{Gr} \times \mathbf{V} = \mathbf{G_F} \overset{\mathbf{G_O}}{\times} \mathbf{V} = (\mathbf{G_F} \times \mathbf{V})/\mathbf{G_O}$ (quotient with respect to the diagonal right-left action). We will denote the orbit of $(g, v) \in \mathbf{G_F} \times \mathbf{V}$ by $[g, v]$.

Note that the left diagonal action of $\mathbf{G_O}$ on $\mathbf{Gr} \times \mathbf{V}$ in terms of the above identification is $h \cdot [g, v] = [hg, v]$. We consider the following convolution diagram:

$$
\begin{array}{ccc}
\mathbf{G_F} \times (\mathbf{Gr} \times \mathbf{V}) & \xrightarrow{\ \mathbf{q}\ } & \mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V}) \\
\mathbf{p} \downarrow & & \mathbf{m} \downarrow \\
(\mathbf{Gr} \times \mathbf{V}) \times (\mathbf{Gr} \times \mathbf{V}) & & \mathbf{Gr} \times \mathbf{V},
\end{array}
\tag{3.2.1}
$$

$$([g_1, g_2 v], [g_2, v]) \xleftarrow{\ \mathbf{p}\ } (g_1, [g_2, v]) \xrightarrow{\ \mathbf{q}\ } [g_1, [g_2, v]] \xrightarrow{\ \mathbf{m}\ } [g_1 g_2, v].$$

Given $\mathcal{F}_1, \mathcal{F}_2 \in D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we define $\mathcal{F}_1 \overset{!}{\circledast} \mathcal{F}_2 := \mathbf{m}_*(\mathcal{F}_1 \widetilde{\boxtimes}^! \mathcal{F}_2)$, where $\mathcal{F}_1 \widetilde{\boxtimes}^! \mathcal{F}_2 \in D_{!\mathbf{G_O}}\big(\mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V})\big)$ is the canonical descent of $\mathbf{p}^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ along $\mathbf{q}$. Similarly, given $\mathcal{F}_1, \mathcal{F}_2 \in D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we define $\mathcal{F}_1 \overset{*}{\circledast} \mathcal{F}_2 := \mathbf{m}_*(\mathcal{F}_1 \widetilde{\boxtimes}^* \mathcal{F}_2)$, where $\mathcal{F}_1 \widetilde{\boxtimes}^* \mathcal{F}_2 \in D_{*\mathbf{G_O}}\big(\mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V})\big)$ is the canonical descent of $\mathbf{p}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ along $\mathbf{q}$, a unique sheaf such that $\mathbf{p}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = \mathbf{q}^*(\mathcal{F}_1 \widetilde{\boxtimes}^* \mathcal{F}_2)$.

We also consider another convolution diagram:

$$
\begin{array}{ccc}
\mathbf{G_F} \times \mathbf{V} \times (\mathbf{Gr} \times \mathbf{V}) & \xrightarrow{\ \mathfrak{q}\ } & \mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V} \times \mathbf{V}) \\
\mathfrak{p} \downarrow & & \mathfrak{m} \downarrow \\
(\mathbf{Gr} \times \mathbf{V}) \times (\mathbf{Gr} \times \mathbf{V}) & & \mathbf{Gr} \times \mathbf{V},
\end{array}
\tag{3.2.2}
$$

$$([g_1, v_1], [g_2, v_2]) \xleftarrow{\ \mathfrak{p}\ } (g_1, v_1, [g_2, v_2]) \xrightarrow{\ \mathfrak{q}\ } [g_1, [g_2, g_2^{-1} v_1, v_2]] \xrightarrow{\ \mathfrak{m}\ } [g_1 g_2, g_2^{-1} v_1 + v_2].$$

Given $\mathcal{F}_1, \mathcal{F}_2 \in D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we define $\mathcal{F}_1 \overset{!}{*} \mathcal{F}_2 := \mathfrak{m}_!(\mathcal{F}_1 \widetilde{\boxtimes}^! \mathcal{F}_2)$, where $\mathcal{F}_1 \widetilde{\boxtimes}^! \mathcal{F}_2 \in D_{!\mathbf{G_O}}\big(\mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V} \times \mathbf{V})\big)$ is the canonical descent of $\mathfrak{p}^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ along $\mathfrak{q}$. Similarly, given $\mathcal{F}_1, \mathcal{F}_2 \in D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we define $\mathcal{F}_1 \overset{*}{*} \mathcal{F}_2 := \mathfrak{m}_*(\mathcal{F}_1 \widetilde{\boxtimes}^* \mathcal{F}_2)$, where $\mathcal{F}_1 \widetilde{\boxtimes}^* \mathcal{F}_2 \in D_{*\mathbf{G_O}}\big(\mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V} \times \mathbf{V})\big)$ is the canonical descent of $\mathfrak{p}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ along $\mathfrak{q}$.

If we formally put $v_1 = 0$ in (3.2.2), we obtain the convolution diagram

$$
\begin{array}{ccc}
\mathbf{G_F} \times \mathbf{Gr} \times \mathbf{V} & \xrightarrow{\ \mathfrak{q}\ } & \mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V}) \\
\mathfrak{p}_{\text{left}} \downarrow & & \mathfrak{m} \downarrow \\
\mathbf{Gr} \times (\mathbf{Gr} \times \mathbf{V}) & & \mathbf{Gr} \times \mathbf{V},
\end{array}
\tag{3.2.3}
$$

$$([g_1], [g_2, v]) \xleftarrow{\ \mathfrak{p}_{\text{left}}\ } (g_1, [g_2, v]) \xrightarrow{\ \mathfrak{q}\ } [g_1, [g_2, v]] \xrightarrow{\ \mathfrak{m}\ } [g_1 g_2, v],$$

and if we put $v_2 = 0$, we obtain

$$
\begin{array}{ccc}
\mathbf{G_F} \times \mathbf{V} \times \mathbf{Gr} & \xrightarrow{\ \mathfrak{q}\ } & \mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V}) \\
\mathfrak{p}_{\text{right}} \downarrow & & \mathfrak{m} \downarrow \\
(\mathbf{Gr} \times \mathbf{V}) \times \mathbf{Gr} & & \mathbf{Gr} \times \mathbf{V},
\end{array}
\tag{3.2.4}
$$

$$([g_1, v], [g_2]) \xleftarrow{\ \mathfrak{p}_{\text{right}}\ } (g_1, v, [g_2]) \xrightarrow{\ \mathfrak{q}\ } [g_1, [g_2, g_2^{-1} v]] \xrightarrow{\ \mathfrak{m}\ } [g_1 g_2, g_2^{-1} v].$$

Given $\mathcal{P} \in D_{\mathbf{G_O}}(\mathbf{Gr})$ and $\mathcal{F} \in D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ (where $? =!, *, !*$), we define $\mathcal{P} * \mathcal{F} := \mathfrak{m}_*(\mathcal{P} \widetilde{\boxtimes} \mathcal{F})$, where $\mathcal{P} \widetilde{\boxtimes} \mathcal{F} \in D_{?\mathbf{G_O}}(\mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V}))$ is the canonical descent of $\mathfrak{p}_{\mathrm{left}}^*(\mathcal{P} \boxtimes \mathcal{F})$ along $\mathfrak{q}$. We also define $\mathcal{F} * \mathcal{P} := \mathfrak{m}_*(\mathcal{F} \widetilde{\boxtimes} \mathcal{P})$, where $\mathcal{F} \widetilde{\boxtimes} \mathcal{P} \in D_{?\mathbf{G_O}}(\mathbf{G_F} \overset{\mathbf{G_O}}{\times} (\mathbf{Gr} \times \mathbf{V}))$ is the canonical descent of $\mathfrak{p}_{\mathrm{right}}^*(\mathcal{F} \boxtimes \mathcal{P})$ along $\mathfrak{q}$. Both the left and right convolutions are biexact for the perverse t-structures on $D_{\mathbf{G_O}}(\mathbf{Gr})$ and $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$; see [FGT09, § 3.9].

3.3 Fusion

Let X be a smooth curve. For any integer $k > 0$, and a collection $x = (x_i)_{i=1}^k$ of S-points of X, we denote by $\mathcal{D}_x$ the formal neighborhood of the union of graphs $|x| := \bigcup_{i=1}^k \Gamma_{x_i} \subset S \times X$, and we denote by $\mathcal{D}_x^\circ := \mathcal{D}_x \smallsetminus |x|$ the punctured formal neighborhood. The mirabolic version of the Beilinson–Drinfeld Grassmannian is the ind-scheme $\mathbf{Gr}_{BD,k}^{\mathrm{mir}}$ over X^k parametrizing the following collections of data:

$$(x_i)_{i=1}^k, \ \mathcal{E}, \ \phi \colon \mathcal{E}_{\mathrm{triv}}|_{\mathcal{D}_x^\circ} \overset{\sim}{\longrightarrow} \mathcal{E}|_{\mathcal{D}_x^\circ}, \ v \in \Gamma(\mathcal{D}_x^\circ, \mathcal{E}),$$

where $\mathcal{E}$ is a rank-N vector bundle on $\mathcal{D}_x$. If $X = \mathbb{A}^1$, we have over the complement to the diagonals a canonical isomorphism

$$(\mathbb{A}^k \smallsetminus \Delta) \times_{\mathbb{A}^k} \mathbf{Gr}_{BD,k}^{\mathrm{mir}} \cong (\mathbb{A}^k \smallsetminus \Delta) \times (\mathbf{Gr} \times \mathbf{V})^k.$$

We denote the projection $(\mathbb{A}^k \smallsetminus \Delta) \times (\mathbf{Gr} \times \mathbf{V})^k \to (\mathbf{Gr} \times \mathbf{V})^k$ by pr_2. Given $\mathcal{F}_1, \mathcal{F}_2 \in D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ (where $? = !, *, !*$) we take $k = 2$ and define the fusion

$$\mathcal{F}_1 \star \mathcal{F}_2 := \mathrm{pr}_{2*} \psi_{x-y} \, \mathrm{pr}_2^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)[1],$$

where x, y are coordinates on $\mathbb{A}^2$ (so that $x - y = 0$ is the equation of the diagonal $\Delta \subset \mathbb{A}^2$), and ψ_{x-y} is the nearby cycles functor for the pullback of the function $x - y$ to $\mathbf{Gr}_{BD,2}^{\mathrm{mir}}$, normalized so as to preserve the perverse t-structure. Note that the leftmost occurrence of pr_2 in the above definition projects $\mathbb{A}^1 \times \mathbf{Gr} \times \mathbf{V}$ to $\mathbf{Gr} \times \mathbf{V}$, while the rightmost occurrence of pr_2 projects $(\mathbb{A}^2 \smallsetminus \Delta) \times (\mathbf{Gr} \times \mathbf{V})^2$ to $(\mathbf{Gr} \times \mathbf{V})^2$.

3.4 Coherent mirabolic category and convolutions

We write ΠE for an odd vector space obtained from a vector space E by reversing the parity.

We fix a pair of N-dimensional vector spaces $V_1 \simeq \mathbb{C}^N \simeq V_2$. We consider the Lie superalgebra $\mathfrak{gl}(N|N) = \mathfrak{gl}(V_1 \oplus \Pi V_2)$. We have $\mathfrak{gl}(N|N) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{1}} = \Pi \mathrm{Hom}(V_1, V_2) \oplus \Pi \mathrm{Hom}(V_2, V_1)$, and $\mathfrak{g}_{\bar{0}} = \mathrm{End}(V_1) \oplus \mathrm{End}(V_2)$. We set $G_{\bar{0}} = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. We consider the dg-algebra[6] $\mathfrak{G}_{1,1}^\bullet = \mathrm{Sym}(\mathfrak{g}_{\bar{1}}[-1])$ with zero differential, and the triangulated category $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ obtained by localization (with respect to quasi-isomorphisms) of the category of perfect $G_{\bar{0}}$-equivariant dg-$\mathfrak{G}_{1,1}^\bullet$-modules. The category $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ is monoidal with respect to $\mathcal{M}, \mathcal{M}' \mapsto \mathcal{M} \otimes_{\mathfrak{G}_{1,1}^\bullet} \mathcal{M}'$; see § 3.7 below.

We will also need two more versions of $\mathfrak{G}_{1,1}^\bullet$, namely

$$\mathfrak{G}_{0,2}^\bullet = \mathrm{Sym}(\mathrm{Hom}(V_1, V_2)) \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)[-2]),$$

$$\mathfrak{G}_{2,0}^\bullet = \mathrm{Sym}(\mathrm{Hom}(V_1, V_2)[-2]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)),$$

---

[6] We view $\mathfrak{g}_{\bar{1}}$ as an *odd* vector space, so that $\mathrm{Sym}(\mathfrak{g}_{\bar{1}}[-1])$ (with grading disregarded) is really a symmetric (infinite-dimensional) algebra, not an exterior algebra.

and the corresponding triangulated categories $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet})$ and $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet})$. We will now define the monoidal structures on $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet})$ and $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet})$.

We consider the variety $\mathcal{Q}^A$ (respectively, $\mathcal{Q}^B$) of sextuples

$$A \in \mathrm{Hom}(V_1, V_2), \quad B \in \mathrm{Hom}(V_2, V_1), \quad A' \in \mathrm{Hom}(V_1', V_2),$$
$$B' \in \mathrm{Hom}(V_2, V_1'), \quad A'' \in \mathrm{Hom}(V_1, V_1'), \quad B'' \in \mathrm{Hom}(V_1', V_1),$$

such that

$$A = A'A'', \ B' = A''B, \ B'' = BA' \ (\text{respectively}, \ B = B''B', \ A' = AB'', \ A'' = B'A)$$

(here V_1' is a copy of V_1). Clearly, $\mathcal{Q}^A \simeq \mathrm{Hom}(V_1, V_1') \times \mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1)$, and $\mathcal{Q}^B \simeq \mathrm{Hom}(V_1', V_1) \times \mathrm{Hom}(V_2, V_1') \times \mathrm{Hom}(V_1, V_2)$. We have

$$
\begin{array}{c}
V_1 \xleftarrow{\quad B \quad} \\
\Big\uparrow A'' \ \Big\| B'' \quad \xrightarrow{A} \quad V_2 \\
V_1' \xleftarrow{\quad A' \quad} \quad \xrightarrow{B'}
\end{array}
\tag{3.4.1}
$$

We denote $\mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1) \times \mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1') \times \mathrm{Hom}(V_1, V_1') \times \mathrm{Hom}(V_1', V_1)$ by $\mathcal{H}$. We have the natural projections

$$\mathrm{pr}_{12} \colon \mathcal{H} \to \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1), \quad \mathrm{pr}_{1'2} \colon \mathcal{H} \to \mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1'),$$
$$\mathrm{pr}_{11'} \colon \mathcal{H} \to \mathrm{Hom}(V_1, V_1') \times \mathrm{Hom}(V_1', V_1).$$

The group $G_{\mathcal{Q}} := \mathrm{GL}(V_1) \times \mathrm{GL}(V_1') \times \mathrm{GL}(V_2)$ naturally acts on $\mathcal{H}$:

$$(g_1, g_1', g_2)(A, A', A'', B, B', B'') = (g_2 A g_1^{-1}, g_2 A' g_1'^{-1}, g_1' A'' g_1^{-1}, g_1 B g_2^{-1}, g_1' B' g_2^{-1}, g_1 B'' g_1'^{-1}).$$

The projections $\mathrm{pr}_{12}, \mathrm{pr}_{1'2}, \mathrm{pr}_{11'}$ are equivariant with respect to the same-named projections from $G_{\mathcal{Q}}$ to $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$, $\mathrm{GL}(V_1') \times \mathrm{GL}(V_2)$, $\mathrm{GL}(V_1) \times \mathrm{GL}(V_1')$.

Given $\mathcal{M}_{1'2} \in \mathsf{Coh}^{\mathrm{GL}(V_1') \times \mathrm{GL}(V_2)}\big(\mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1')\big)$ and $\mathcal{M}_{11'} \in \mathsf{Coh}^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_1')}\big(\mathrm{Hom}(V_1, V_1') \times \mathrm{Hom}(V_1', V_1)\big)$, we set

$$\mathcal{M}_{11'} \overset{A}{*} \mathcal{M}_{1'2} := \mathrm{pr}_{12*}(\mathrm{pr}_{11'}^* \mathcal{M}_{11'} \otimes_{\mathbb{C}[\mathcal{H}]} \mathbb{C}[\mathcal{Q}^A] \otimes_{\mathbb{C}[\mathcal{H}]} \mathrm{pr}_{1'2}^* \mathcal{M}_{1'2})^{\mathrm{GL}(V_1')}$$
$$\in \mathsf{Coh}^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)}\big(\mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1)\big),$$

$$\mathcal{M}_{11'} \overset{B}{*} \mathcal{M}_{1'2} := \mathrm{pr}_{12*}(\mathrm{pr}_{11'}^* \mathcal{M}_{11'} \otimes_{\mathbb{C}[\mathcal{H}]} \mathbb{C}[\mathcal{Q}^B] \otimes_{\mathbb{C}[\mathcal{H}]} \mathrm{pr}_{1'2}^* \mathcal{M}_{1'2})^{\mathrm{GL}(V_1')}$$
$$\in \mathsf{Coh}^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)}\big(\mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1)\big).$$

We will actually need the following modifications of these functors:

$$\overset{B}{*} \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet}) \times D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet}) \to D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet}), \quad \overset{A}{*} \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet}) \times D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet}) \to D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet}),$$

obtained using the dg-algebras with trivial differentials $\mathbb{C}[\mathcal{H}]^\bullet_{0,2}$ and $\mathbb{C}[\mathcal{H}]^\bullet_{2,0}$ respectively, where

$$\mathbb{C}[\mathcal{H}]^\bullet_{0,2} = \mathrm{Sym}(\mathrm{Hom}(V_1, V_2)) \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)[-2]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1', V_2))$$
$$\otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1')[-2]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1, V_1')) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1', V_1)[-2]),$$
$$\mathbb{C}[\mathcal{H}]^\bullet_{2,0} = \mathrm{Sym}(\mathrm{Hom}(V_1, V_2)[-2]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1', V_2)[-2])$$
$$\otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1')) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1, V_1')[-2]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1', V_1)).$$

(we identify $\mathbb{C}[\mathrm{Hom}(U, W)]$ with $\mathrm{Sym}\,\mathrm{Hom}(W, U)$).

3.5 Localization, coherent

We identify

$$\mathrm{Hom}(V_1, V_2) = \mathrm{Hom}(V_2, V_1)^*, \quad \mathrm{Hom}(V_2, V_1) = \mathrm{Hom}(V_1, V_2)^*,$$

so that

$$\mathrm{Sym}(\mathrm{Hom}(V_1, V_2)) = \mathbb{C}[\mathrm{Hom}(V_2, V_1)], \quad \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)) = \mathbb{C}[\mathrm{Hom}(V_1, V_2)].$$

We have an open subvariety $\mathrm{Isom}(V_2, V_1) \subset \mathrm{Hom}(V_2, V_1)$, so that $\mathbb{C}[\mathrm{Hom}(V_2, V_1)] \subset \mathbb{C}[\mathrm{Isom}(V_2, V_1)]$. We set

$$\mathfrak{B}^\bullet := \mathbb{C}[\mathrm{Isom}(V_2, V_1)] \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)[-2])$$

(a dg-algebra with trivial differential). Similarly, we define

$$\mathfrak{A}^\bullet := \mathrm{Sym}(\mathrm{Hom}(V_1, V_2)[-2]) \otimes \mathbb{C}[\mathrm{Isom}(V_1, V_2)].$$

An equivalent formulation of [BF08, Theorem 5] is an existence of a monoidal equivalence $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{A}^\bullet) \cong D_{\mathbf{G_O}}(\mathbf{Gr})$ (and, changing the roles of V_1, V_2, a monoidal equivalence $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{B}^\bullet) \cong D_{\mathbf{G_O}}(\mathbf{Gr})$).

Since $\mathfrak{A}^\bullet$ (respectively, $\mathfrak{B}^\bullet$) is a localization of $\mathfrak{G}^\bullet_{2,0}$ (respectively, of $\mathfrak{G}^\bullet_{0,2}$), we have the restriction of scalars functors

$$\mathrm{Res}_A \colon D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{A}^\bullet) \to \hat{D}^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0}), \quad \mathrm{Res}_B \colon D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{B}^\bullet) \to \hat{D}^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{0,2}),$$

where $\hat{D}^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{?,?})$ stands for the Ind-completion of $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{?,?})$.

However, one can check that, for $\mathcal{N} \in D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{A}^\bullet)$ and $\mathcal{M} \in D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0})$, both convolutions $\mathrm{Res}_A(\mathcal{N}) \overset{A}{*} \mathcal{M}$ and $\mathcal{M} \overset{A}{*} \mathrm{Res}_A(\mathcal{N})$ lie in $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0}) \subset \hat{D}^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0})$. Thus we have the left and right convolution actions

$$\overset{A}{*} \colon D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{A}^\bullet) \times D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0}) \to D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0}), \quad D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0}) \times D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{A}^\bullet) \to D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0}),$$

and similarly,

$$\overset{B}{*} \colon D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{B}^\bullet) \times D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{0,2}) \to D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{0,2}), \quad D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{0,2}) \times D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{B}^\bullet) \to D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{0,2}).$$

3.6 Renormalizations

The action of the center $Z(\mathrm{GL}(V_1)) \cong \mathbb{G}_m$ on an object $\mathcal{M} \in D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{1,1})$ defines a grading, and the corresponding degrees will be denoted by $\deg_1$. Similarly, the action of $Z(\mathrm{GL}(V_2)) \cong \mathbb{G}_m$

gives rise to another grading with degrees denoted by $\deg_2$. The cohomological degrees will be denoted simply by deg. Clearly, the degrees of the generators are as follows:

$$\deg_2(\mathrm{Hom}(V_1, V_2)) = \deg_1(\mathrm{Hom}(V_2, V_1)) = 1,$$
$$\deg_1(\mathrm{Hom}(V_1, V_2)) = \deg_2(\mathrm{Hom}(V_1, V_2)) = -1.$$

Hence changing cohomological degrees by the formula $\deg \rightsquigarrow \deg + \deg_1$ (respectively, $\deg \rightsquigarrow \deg - \deg_2$) yields equivalences

$$D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet}) \xrightarrow{\varrho_{\mathrm{right}}} D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}) \xrightarrow{\varrho_{\mathrm{right}}} D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet}),$$
$$D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet}) \xrightarrow{\varrho_{\mathrm{left}}} D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}) \xrightarrow{\varrho_{\mathrm{left}}} D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet}),$$

respectively. The notation is due to the fact that ϱ_{left} commutes with the left action of $\mathrm{Rep}(\mathrm{GL}(V_1))$ on our categories, while ϱ_{right} commutes with the right action of $\mathrm{Rep}(\mathrm{GL}(V_2))$ on our categories.

Now recall the notation in the definition of categories $D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ (where $? = !, *, !*$) of §3.1. Given $\mathcal{F} = (\mathcal{F}_n)_{n>m} \in D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we define $\varrho_{\mathrm{right}}\mathcal{F} := (\mathcal{F}_n[-Nn])_{n>m} \in D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. Similarly, given $\mathcal{F} = (\mathcal{F}_n)_{n>m} \in D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ we define $\varrho_{\mathrm{right}}\mathcal{F} := (\mathcal{F}_n[-Nn])_{n>m} \in D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. The functors ϱ_{right} commute with the action of the monoidal category $D_{\mathbf{G_O}}(\mathbf{Gr})$ by right convolutions. Recall also that the affine Grassmannian is a union of connected components $\mathbf{Gr} = \bigsqcup_{k \in \mathbb{Z}} \mathbf{Gr}^{(k)}$, where $\mathbf{Gr}^{(k)}$ parametrizes the lattices of virtual dimension k (e.g. $\dim(t\mathbf{V}_0) = -N$). For $\mathcal{F}$ supported on $\mathbf{Gr}^{(k)} \times \mathbf{V}$ we set $\varrho_{\mathrm{left}}(\mathcal{F}) := \varrho_{\mathrm{right}}(\mathcal{F})[-k]$. Then the functors

$$\varrho_{\mathrm{left}} \colon D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \ D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$$

commute with the action of the monoidal category $D_{\mathbf{G_O}}(\mathbf{Gr})$ by the left convolutions.

Our goal is the following theorem.

THEOREM 3.6.1. *There exist monoidal[7] equivalences of triangulated categories*

$$
\begin{array}{ccc}
\left(D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^{\bullet}), \overset{A}{*}\right) & \xrightarrow[\Phi^{2,0}]{\sim} & \left(D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \overset{!}{\circledast}\right) \\
\wr \downarrow \varrho_{\mathrm{right}} & & \wr \downarrow \varrho_{\mathrm{right}} \\
\left(D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}), \otimes_{\mathfrak{G}_{1,1}^{\bullet}}\right) & \xrightarrow[\Phi^{1,1}]{\sim} & \left(D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \star\right) \\
\wr \downarrow \varrho_{\mathrm{right}} & & \wr \downarrow \varrho_{\mathrm{right}} \\
\left(D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^{\bullet}), \overset{B}{*}\right) & \xrightarrow[\Phi^{0,2}]{\sim} & \left(D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \overset{*}{*}\right)
\end{array}
\tag{3.6.1}
$$

(the vertical equivalences are not monoidal). The squares are commutative. The horizontal equivalences commute with the actions of the monoidal spherical Hecke category $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr}) \cong \mathrm{Rep}(\mathrm{GL}_N)$ *by the left and right convolutions.*

The proof will be given in §3.16 after some necessary preparation.

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[7] See §3.7 for the definition of the left middle monoidal structure.

3.7 Super

Strictly speaking, in accordance with the footnote at the beginning of §3.4, we should consider the category $SD_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$ of super dg-modules over the superalgebra $\mathfrak{G}_{1,1}^{\bullet} = \mathrm{Sym}(\mathfrak{g}_{\bar{1}}[-1])$. The latter superalgebra is supercommutative, hence we have a symmetric monoidal structure $\otimes_{\mathfrak{G}_{1,1}^{\bullet}}$ on the category $SD_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$. The equivalence $\Phi^{1,1}$ of Theorem 3.6.1 can be upgraded to a monoidal equivalence $S\Phi^{1,1}\colon \left(SD_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}),\ \otimes_{\mathfrak{G}_{1,1}^{\bullet}}\right) \xrightarrow{\sim} \left(SD_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}),\ \star\right)$ to the derived category of sheaves with coefficients in *super* vector spaces.

However, the action of the central element $(\mathrm{Id}_{V_1}, -\mathrm{Id}_{V_2}) \in G_{\bar{0}}$ on an object of $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$ equips this object with an extra $\mathbb{Z}/2\mathbb{Z}$-grading, and thus defines a fully faithful functor $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}) \to SD_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$ of a 'superization', such that its essential image is closed under the monoidal structure $\otimes_{\mathfrak{G}_{1,1}^{\bullet}}$. This defines the desired monoidal structure $\otimes_{\mathfrak{G}_{1,1}^{\bullet}}$ on the category $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$.

3.8 Koszul equivalence

We consider the following complex $H^{\bullet}$ of *odd* vector spaces living in degrees $0, 1$: $\mathfrak{g}_{\bar{1}} \xrightarrow{\mathrm{Id}} \mathfrak{g}_{\bar{1}}$. We define the Koszul complex $K^{\bullet}$ as the symmetric algebra $\mathrm{Sym}(H^{\bullet})$. The degree-0 part

$$K^0 = \Lambda\big(\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)\big) =: \Lambda$$

(as a vector space, with a superstructure disregarded). We turn $K^{\bullet}$ into a dg-$\mathfrak{G}_{1,1}^{\bullet} - \Lambda$-bimodule by letting $\mathfrak{G}_{1,1}^{\bullet}$ act by multiplication, and Λ by differentiation. Note that $K^{\bullet}$ is quasi-isomorphic to $\mathbb{C}$ in degree 0 as a complex of vector spaces, but *not* as a dg-$\mathfrak{G}_{1,1}^{\bullet} - \Lambda$-bimodule. We consider the derived category $D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ of finite-dimensional complexes of $G_{\bar{0}} \ltimes \Lambda$-modules. If we remember the superstructure of Λ, we obtain the corresponding category of super dg-modules $SD_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$. We have the Koszul equivalence functors

$$\varkappa\colon D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda) \xrightarrow{\sim} D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}), \quad SD_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda) \xrightarrow{\sim} SD_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}), \quad \mathcal{N} \mapsto K^{\bullet} \otimes_{\Lambda} \mathcal{N}.$$

Here is an equivalent definition of the category $SD_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$. We consider the following degeneration $\underline{\mathfrak{gl}}(N|N)$ of the Lie superalgebra $\mathfrak{gl}(N|N)$: the supercommutator of the even elements (with even or odd elements) remains intact, but the supercommutator of any two odd elements is set to zero. Let $SD_{\mathrm{int}}(\underline{\mathfrak{gl}}(N|N))$ denote the derived category of bounded complexes of integrable $\underline{\mathfrak{gl}}(N|N)$-modules (note that the even part of $\underline{\mathfrak{gl}}(N|N)$ is just $\mathfrak{g}_{\bar{0}}$, and the integrability is nothing but $\mathfrak{g}_{\bar{0}}$-integrability, i.e. $G_{\bar{0}}$-equivariance). Then $SD_{\mathrm{int}}(\underline{\mathfrak{gl}}(N|N)) \cong SD_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ tautologically. The resulting Koszul equivalence $\varkappa\colon SD_{\mathrm{int}}(\underline{\mathfrak{gl}}(N|N)) \xrightarrow{\sim} SD_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$ is monoidal with respect to the usual tensor structure on the left-hand side and $\otimes_{\mathfrak{G}_{1,1}^{\bullet}}$ on the right-hand side.

As in §3.7, the action of $(\mathrm{Id}_{V_1}, -\mathrm{Id}_{V_2}) \in G_{\bar{0}}$ gives rise to a 'superization' fully faithful functor $D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda) \to SD_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda) \cong SD_{\mathrm{int}}(\underline{\mathfrak{gl}}(N|N))$ with the essential image closed under the tensor structure. This defines the tensor structure on $D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ such that the Koszul equivalence $\varkappa\colon D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda) \xrightarrow{\sim} D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$ is monoidal.

COROLLARY 3.8.1 (of Theorem 3.6.1). (a) *The composed equivalence*

$$\Phi^{1,1} \circ \varkappa\colon D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda) \xrightarrow{\sim} D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$$

is exact with respect to the tautological *t*-structure on $D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ and the perverse *t*-structure on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

(b) *This equivalence is monoidal with respect to the tensor structure on* $D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ *and the fusion* $\star$ *on* $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

(c) *The equivariant derived category* $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ *is equivalent to the bounded derived category of the abelian category* $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

Proof. We consider an irreducible $G_{\bar{0}}$-module $V_{1,\boldsymbol{\lambda}} \otimes V_{2,\boldsymbol{\mu}}$ as a $G_{\bar{0}} \ltimes \Lambda$-module with the trivial action of Λ. Then $\varkappa(V_{1,\boldsymbol{\lambda}} \otimes V_{2,\boldsymbol{\mu}}) = V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^{\bullet} \otimes V_{2,\boldsymbol{\mu}}$, and $\Phi^{1,1}(V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^{\bullet} \otimes V_{2,\boldsymbol{\mu}}) = \mathrm{IC}_{(\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)}$ by construction of $\Phi^{1,1}$. Here, for a signature $\boldsymbol{\nu} = (\nu_1 \geq \cdots \geq \nu_N)$, we set $\boldsymbol{\nu}^* := (-\nu_N \geq \cdots \geq -\nu_1)$.

Finally, since $D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ is equivalent to the bounded derived category of its heart $\mathrm{Rep}(G_{\bar{0}} \ltimes \Lambda)$, (c) follows from (a). $\square$

3.9 De-equivariantized Ext algebra

Recall from [FGT09, Proposition 8] that the $\mathbf{G_O}$-orbits in $\mathbf{Gr} \times \overset{\circ}{\mathbf{V}}$ are numbered by bisignatures $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ where both $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have length N. The IC extension of the constant one-dimensional local system on such an orbit is denoted by $\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}$. In particular, $\mathrm{IC}_{(0^N,0^N)}$ is the constant sheaf on $\mathbf{Gr}^0 \times \mathbf{V}_0$, to be denoted by E_0 for short. Also recall from [FGT09, §3] that the left and right actions of the monoidal Satake category $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr}) \cong \mathrm{Rep}(\mathrm{GL}_N)$ on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ by convolutions respect the perverse *t*-structure with the heart $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \subset D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. As has been mentioned in §3.6, the right actions of $D_{\mathbf{G_O}}(\mathbf{Gr})$ on $D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ commute with the equivalences ϱ_{right}, but the left actions only commute with ϱ_{right} up to cohomological shifts depending on the connected components of $\mathbf{Gr}$.

We restrict the left and right actions of $D_{\mathbf{G_O}}(\mathbf{Gr})$ on $D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ to the left and right actions of $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr}) \cong \mathrm{Rep}(\mathrm{GL}_N)$. Thus we obtain the action of $\mathrm{Rep}(\mathsf{G})$ for $\mathsf{G} = \mathrm{GL}_N \times \mathrm{GL}_N$. Let $D_{?\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})$ denote the corresponding de-equivariantized category (see [AG03] in the setting of abelian categories and [Gai15] in the setting of dg-categories). Recall that to construct $D_{?\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})$, we first consider objects of the ind-completion of $D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ endowed with an action of R_{G} where R_{G} is the regular representation of G considered as a ring object in $\mathrm{Rep}(\mathsf{G})$. Then we take compact objects of the resulting category. More explicitly, an object of $D_{?\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})$ is an object $\mathcal{F}$ of $D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ together with a system of isomorphisms $\mathrm{IC}_{\boldsymbol{\lambda}} * \mathcal{F} * \mathrm{IC}_{\boldsymbol{\mu}} \overset{\sim}{\longrightarrow} V_{\boldsymbol{\lambda}}^* \otimes V_{\boldsymbol{\mu}}^* \otimes \mathcal{F}$ (recall that the geometric Satake equivalence takes $\mathrm{IC}_{\boldsymbol{\lambda}}$ to $V_{\boldsymbol{\lambda}}^*$) for any signatures $\boldsymbol{\lambda}, \boldsymbol{\mu}$, satisfying some natural compatibilities with respect to direct sums and tensor products. By definition we have a natural forgetful functor $D_{?\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V}) \to D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. This functor admits a left adjoint that sends an object $\mathcal{F}$ to $R_{\mathsf{G}} * \mathcal{F} = \bigoplus_{\boldsymbol{\lambda},\boldsymbol{\mu}} V_{\boldsymbol{\lambda}} \otimes V_{\boldsymbol{\mu}} \otimes \mathrm{IC}_{\boldsymbol{\lambda}} * \mathcal{F} * \mathrm{IC}_{\boldsymbol{\mu}}$.

Thus, given $\mathcal{F}_1, \mathcal{F}_2 \in D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we denote the corresponding objects of the de-equivariantized category by the same symbols, and we have

$$\mathrm{RHom}_{D_{?\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}(\mathcal{F}_1, \mathcal{F}_2) = \bigoplus_{\boldsymbol{\lambda},\boldsymbol{\mu}} \mathrm{RHom}_{D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(\mathcal{F}_1, \mathrm{IC}_{\boldsymbol{\lambda}} * \mathcal{F}_2 * \mathrm{IC}_{\boldsymbol{\mu}}) \otimes V_{\boldsymbol{\lambda}} \otimes V_{\boldsymbol{\mu}}. \quad (3.9.1)$$

LEMMA 3.9.1. *The dg-algebra* $\mathrm{RHom}_{D_{!*\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}(E_0, E_0)$ *is formal, that is, it is quasi-isomorphic to the graded algebra* $\mathrm{Ext}_{D_{!*\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}^{\bullet}(E_0, E_0)$ *with trivial differential.*

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Proof. We change the setting to the base field $\mathbb{F}_q$. Then a $\mathbf{G_O}$-equivariant irreducible perverse sheaf $\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}$ on $\mathbf{Gr} \times \mathbf{V}$ carries a natural Weil structure (see [FGT09, §2.6]) and $\mathrm{Ext}^\bullet_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0, \mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})})$ is pure. Indeed, if $\bar{\imath}_0$ denotes the closed embedding $\mathbf{V}_0 \cong \mathbf{Gr}^0 \times \mathbf{V}_0 \hookrightarrow \mathbf{Gr} \times \mathbf{V}$, then $\mathrm{Ext}^\bullet_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0, \mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}) = H^\bullet_{\mathbf{G_O}}(\mathbf{V}_0, \bar{\imath}_0^! \mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})})$. Let us consider the loop rotation action $\mathbb{G}_m \curvearrowright (\mathbf{Gr} \times \mathbf{V})$. We have an embedding $\mathbb{A}^1 \hookrightarrow \mathbf{V}$ (constant Laurent series), and $\mathbf{Gr}^0 \times \mathbb{A}^1$ is a fixed point component of the $\mathbb{G}_m$-action, so that $H^\bullet_{\mathbf{G_O}}(\mathbf{V}_0, \bar{\imath}_0^! \mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})})$ is the hyperbolic restriction to this component (more precisely, the hyperbolic restriction is a geometrically constant complex on $\mathbb{A}^1$ with the above stalks) [Bra03, DG14]. But the hyperbolic restriction preserves purity.

Now, given the purity of $\mathrm{Ext}^\bullet_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0, \mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})})$, the desired result follows by an application of [BF08, Lemma 15] (and then of [BF08, Proposition 5]). $\qquad\square$

We denote the dg-algebra $\mathrm{Ext}^\bullet_{D^{\mathrm{deeq}}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0, E_0)$ (with trivial differential) by $\mathfrak{E}^\bullet$. Since it is an Ext-algebra in the de-equivariantized category between objects induced from the original category, it is automatically equipped with an action of $\mathrm{GL}_N \times \mathrm{GL}_N = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) = G_{\bar{0}}$, and we can consider the corresponding triangulated category $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{E}^\bullet)$.

LEMMA 3.9.2. *There is a canonical equivalence* $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{E}^\bullet) \xrightarrow{\sim} D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

Proof. The desired functor is constructed as in [BF08, §6.5, Propositions 5,6]. Since E_0 generates the triangulated category $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ (with respect to the left and right actions of the Satake category), the claim follows from Lemma 3.9.1. $\qquad\square$

We also consider the left and right actions of the monoidal Satake category $\mathrm{Perv}_{\mathbf{G_O}}(\mathbf{Gr}) \cong \mathrm{Rep}(\mathrm{GL}_N)$ on $D_{\mathbf{G_O}}(\mathbf{Gr})$ by convolutions. Let $D^{\mathrm{deeq}}_{\mathbf{G_O}}(\mathbf{Gr})$ denote the corresponding de-equivariantized category. Then the dg-algebra $\mathrm{RHom}_{D^{\mathrm{deeq}}_{\mathbf{G_O}}(\mathbf{Gr})}(\mathrm{IC}_0, \mathrm{IC}_0)$ is formal, that is, it is quasi-isomorphic to the graded algebra $\mathrm{Ext}^\bullet_{D^{\mathrm{deeq}}_{\mathbf{G_O}}(\mathbf{Gr})}(\mathrm{IC}_0, \mathrm{IC}_0)$ with trivial differential. Furthermore, it follows from [BF08, Theorem 5] that there is a natural isomorphism $\mathrm{Ext}^\bullet_{D^{\mathrm{deeq}}_{\mathbf{G_O}}(\mathbf{Gr})}(\mathrm{IC}_0, \mathrm{IC}_0) \cong \mathfrak{A}^\bullet$ (in the notation of §3.5).

3.10 Localization, constructible

We have an automorphism $\alpha \colon \mathbf{Gr} \times \mathbf{V} \xrightarrow{\sim} \mathbf{Gr} \times \mathbf{V}$, $(L, v) \mapsto (L, tv)$. We have a morphism of endofunctors $\alpha^* \to \mathrm{Id} \colon D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ constructed as follows. We consider a family of automorphisms $\boldsymbol{\alpha} \colon \mathbb{A}^1 \times \mathbf{Gr} \times \mathbf{V} \hookrightarrow \mathbf{Gr} \times \mathbf{V}$, $(c, L, v) \mapsto (L, (c + t)v)$, so that $\boldsymbol{\alpha}_0 = \alpha$. Note that $\boldsymbol{\alpha}_c^* \cong \mathrm{Id}$ on $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ for $c \neq 0$. Now the desired morphism $\alpha^* \to \mathrm{Id}$ is just the cospecialization[8] morphism from the stalk at $0 \in \mathbb{A}^1$ to the nearby stalk. For $\mathcal{F}_1, \mathcal{F}_2 \in D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we have an inductive system

$$\cdots \to \mathrm{RHom}_{D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}((\alpha^{n-1})^* \mathcal{F}_1, \mathcal{F}_2) \to \mathrm{RHom}_{D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}((\alpha^n)^* \mathcal{F}_1, \mathcal{F}_2) \to \cdots.$$

Note that it stabilizes since, by definition of $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, the restriction of $(\alpha^n)^* \mathcal{F}_1$ to the support of $\mathcal{F}_2$ becomes the pullback of an appropriate sheaf in $D_{\mathbf{G_O}}(\mathbf{Gr})$ for $n \gg 0$.

---

[8] Terminology of [Sch19, 6.2.7].

We define the localized category $D^{\mathrm{loc}}_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ as the category with the same objects as $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, and with morphisms

$$\mathrm{RHom}_{D^{\mathrm{loc}}_{*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\mathcal{F}_1, \mathcal{F}_2) := \varinjlim \mathrm{RHom}_{D_{*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}((\alpha^n)^*\mathcal{F}_1, \mathcal{F}_2).$$

The tautological functor $D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D^{\mathrm{loc}}_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ is denoted by u_0^* ('restriction to $v \approx 0$').

We also have the Verdier dual (to the above $\alpha^* \to \mathrm{Id}$) morphism of endofunctors $\mathrm{Id} \to \alpha^! \colon D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. This gives rise to an inductive system for $\mathcal{F}_1, \mathcal{F}_2 \in D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$,

$$\cdots \to \mathrm{RHom}_{D_{!\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\mathcal{F}_1, (\alpha^{n-1})^!\mathcal{F}_2) \to \mathrm{RHom}_{D_{!\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\mathcal{F}_1, (\alpha^n)^!\mathcal{F}_2) \to \cdots,$$

stabilizing for reasons similar to those above. We define the localized category $D^{\mathrm{loc}}_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ as the category with the same objects as $D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, and with morphisms

$$\mathrm{RHom}_{D^{\mathrm{loc}}_{!\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\mathcal{F}_1, \mathcal{F}_2) := \varinjlim \mathrm{RHom}_{D_{!\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\mathcal{F}_1, (\alpha^n)^!\mathcal{F}_2).$$

The tautological functor $D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D^{\mathrm{loc}}_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ is denoted by $u_0^!$ ('corestriction to $v \approx 0$').

We also have the projection $\mathrm{pr}\colon \mathbf{Gr} \times \mathbf{V}_0 \to \mathbf{Gr}$, and the corresponding pullbacks $\mathrm{pr}^*\colon D_{\mathbf{G_O}}(\mathbf{Gr}) \to D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ and $\mathrm{pr}^!\colon D_{\mathbf{G_O}}(\mathbf{Gr}) \to D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ (with the essential images supported on $\mathbf{Gr} \times \mathbf{V}_0 \subset \mathbf{Gr} \times \mathbf{V}$).

LEMMA 3.10.1. *The compositions*

$$u_0^* \circ \mathrm{pr}^*\colon D_{\mathbf{G_O}}(\mathbf{Gr}) \to D^{\mathrm{loc}}_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \quad u_0^! \circ \mathrm{pr}^!\colon D_{\mathbf{G_O}}(\mathbf{Gr}) \to D^{\mathrm{loc}}_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$$

are equivalences of categories sending IC_0 *to* $\varrho_{\mathrm{right}}E_0$, $\varrho_{\mathrm{right}}^{-1}E_0$, *respectively.*

Proof. Clear. $\qquad\square$

The localizations of the de-equivariantized categories $D^{\mathrm{deeq}}_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ and $D^{\mathrm{deeq}}_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ will be denoted by $D^{\mathrm{loc,deeq}}_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ and $D^{\mathrm{loc,deeq}}_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, respectively.

Recall the dg-algebras $\mathfrak{A}^\bullet, \mathfrak{B}^\bullet$ introduced in § 3.5.

COROLLARY 3.10.2. *There are canonical isomorphisms*

(a) $\mathrm{Ext}^\bullet_{D^{\mathrm{loc,deeq}}_{!\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\varrho_{\mathrm{right}}^{-1}E_0, \varrho_{\mathrm{right}}^{-1}E_0) \cong \mathfrak{A}^\bullet$,

(b) $\mathrm{Ext}^\bullet_{D^{\mathrm{loc,deeq}}_{*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\varrho_{\mathrm{right}}E_0, \varrho_{\mathrm{right}}E_0) \cong \mathfrak{B}^\bullet$.

Note that the dg-algebra $\mathrm{RHom}_{D^{\mathrm{deeq}}_{*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\varrho_{\mathrm{right}}E_0, \varrho_{\mathrm{right}}E_0)$ (respectively, $\mathrm{RHom}_{D^{\mathrm{deeq}}_{!\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\varrho_{\mathrm{right}}^{-1}E_0, \varrho_{\mathrm{right}}^{-1}E_0)$) is also formal, that is, it is quasi-isomorphic to $\mathrm{Ext}^\bullet_{D^{\mathrm{deeq}}_{*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\varrho_{\mathrm{right}}E_0, \varrho_{\mathrm{right}}E_0)$ (respectively, $\mathrm{Ext}^\bullet_{D^{\mathrm{deeq}}_{!\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(\varrho_{\mathrm{right}}^{-1}E_0, \varrho_{\mathrm{right}}^{-1}E_0)$) with trivial differential. If we disregard their gradings, they are both isomorphic to $\mathrm{Ext}^\bullet_{D^{\mathrm{deeq}}_{!*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(E_0, E_0)$. We will denote the algebra $\mathrm{Ext}^\bullet_{D^{\mathrm{deeq}}_{!*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(E_0, E_0)$ with grading forgotten by $\mathrm{Ext}_{D^{\mathrm{deeq}}_{!*\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})}(E_0, E_0)$; the

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same applies to $\mathfrak{A}^\bullet, \mathfrak{B}^\bullet, \mathfrak{E}^\bullet, \mathfrak{G}^\bullet$. Thus we have equalities $\mathrm{Ext}_{D_{*\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}(\varrho_{\mathrm{right}}E_0, \varrho_{\mathrm{right}}E_0) = \mathrm{Ext}_{D_{!*\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}(E_0, E_0) = \mathrm{Ext}_{D_{!\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}(\varrho_{\mathrm{right}}^{-1}E_0, \varrho_{\mathrm{right}}^{-1}E_0)$.

LEMMA 3.10.3. *The natural morphism*

$$\mathrm{Ext}^\bullet_{D_{!\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}(\varrho_{\mathrm{right}}^{-1}E_0, \varrho_{\mathrm{right}}^{-1}E_0) \to \mathrm{Ext}^\bullet_{D_{!\mathbf{G_O}}^{\mathrm{loc,deeq}}(\mathbf{Gr} \times \mathbf{V})}(\varrho_{\mathrm{right}}^{-1}E_0, \varrho_{\mathrm{right}}^{-1}E_0)$$

is injective.

Proof. We have $\alpha^! \varrho_{\mathrm{right}}^{-1}\mathrm{IC}_{(-1^N,1^N)}[N] \simeq \varrho_{\mathrm{right}}^{-1}E_0$, and we need to check that, for any $\boldsymbol{\lambda}, \boldsymbol{\mu}$, the natural morphism

$$\begin{aligned}
\vartheta \colon \mathrm{RHom}_{D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}&(\varrho_{\mathrm{right}}^{-1}E_0, \varrho_{\mathrm{right}}^{-1}\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}) \\
&\to \mathrm{RHom}_{D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(\varrho_{\mathrm{right}}^{-1}\mathrm{IC}_{(-1^N,1^N)}[-N], \varrho_{\mathrm{right}}^{-1}\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})})
\end{aligned} \tag{3.10.1}$$

is injective. Indeed, since $\alpha^!$ is a Hecke transformation endofunctor (namely, it is given by convolution $\mathrm{IC}_{(1^N)} * ? * \mathrm{IC}_{(-1)^N}$ with invertible objects of the Satake category), the morphism ϑ is an endomorphism of the identity endofunctor of the de-equivariantized category $D_{!\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})$. In other words, ϑ is an element of the center of this category. But the natural morphism from the center to its localization is injective if and only if the multiplication by ϑ is injective.

We change the setting to the base field $\mathbb{F}_q$ as in the proof of Lemma 3.9.1. Then all the IC sheaves in question carry a natural Weil structure, and it was proved in the proof of Lemma 3.9.1 that both the left- and right-hand side of (3.10.1) are pure; it is immediate to see that they are pure of the same weight w. The cone of ϑ is $H^\bullet_{\mathbf{G_O}}(\Omega_{(0^N,0^N)}, \imath_0^! \varrho_{\mathrm{right}}^{-1}\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})})$, where $\imath_0$ stands for the locally closed embedding of the $\mathbf{G_O}$-orbit $\Omega_{(0^N,0^N)} = \mathbf{Gr}^0 \times (\mathbf{V}_0 \smallsetminus t\mathbf{V}_0) \hookrightarrow \mathbf{Gr} \times \mathbf{V}$. Due to the pointwise purity of $\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}$ [FGT09, § 3], $\imath_0^!\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}$ is a pure local system on $\Omega_{(0^N,0^N)}$; hence $H^\bullet_{\mathbf{G_O}}(\Omega_{(0^N,0^N)}, \imath_0^!\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})})$ is also pure of weight w. It follows that the kernel of ϑ vanishes, and ϑ is injective. $\square$

COROLLARY 3.10.4. *The algebra* $\mathfrak{E} = \mathrm{Ext}_{D_{!*\mathbf{G_O}}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}(E_0, E_0)$ *is a commutative integrally closed domain.*

3.11 Calculation of the Ext algebra

Recall that the first fundamental coweight of GL_N is $\omega_1 = (1, 0, \ldots, 0)$, and $\omega_1^* = (0, \ldots, 0, -1)$. We have $\mathrm{RHom}_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(\mathrm{IC}_{\omega_1} * E_0, E_0 * \mathrm{IC}_{\omega_1}) = \mathrm{RHom}_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0, \mathrm{IC}_{\omega_1^*} * E_0 * \mathrm{IC}_{\omega_1})$, $\mathrm{RHom}_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0 * \mathrm{IC}_{\omega_1}, \mathrm{IC}_{\omega_1} * E_0) = \mathrm{RHom}_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0, \mathrm{IC}_{\omega_1} * E_0 * \mathrm{IC}_{\omega_1^*})$. Now $\mathrm{IC}_{\omega_1} * E_0$ is the constant IC sheaf of the stratum closure formed by all the pairs (L, v) such that the lattice L contains $\mathbf{V}_0$ as a hyperplane, and $v \in L$. Furthermore, $E_0 * \mathrm{IC}_{\omega_1}$ is the constant IC sheaf of the stratum closure formed by all the pairs (L, v) such that the lattice L contains $\mathbf{V}_0$ as a hyperplane, and $v \in \mathbf{V}_0$. In particular, the latter stratum closure is a smooth divisor in the former stratum closure, so we have canonical elements $h \in \mathrm{Ext}^1_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(\mathrm{IC}_{\omega_1} * E_0, E_0 * \mathrm{IC}_{\omega_1})$ and $h^* \in \mathrm{Ext}^1_{D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})}(E_0 * \mathrm{IC}_{\omega_1}, \mathrm{IC}_{\omega_1} * E_0)$. Hence we obtain the subspaces $h \otimes V_{\omega_1^*} \otimes V_{\omega_1} \subset \mathfrak{E}^1$ and $h^* \otimes V_{\omega_1} \otimes V_{\omega_1^*} \subset \mathfrak{E}^1$ (see § 3.9 for the definition of $\mathfrak{E}^\bullet$ and (3.9.1)). We identify the former

subspace with $\mathrm{Hom}(V_1, V_2)$ and the latter one with $\mathrm{Hom}(V_2, V_1)$. Thus we obtain a homomorphism $\phi^\bullet\colon \mathrm{Sym}\big(\Pi\mathrm{Hom}(V_1,V_2)[-1] \oplus \Pi\mathrm{Hom}(V_2,V_1)[-1]\big) = \mathrm{Sym}(\mathfrak{g}_{\bar{1}}[-1]) = \mathfrak{E}^\bullet_{1,1} \to \mathfrak{E}^\bullet$ (due to commutativity of $\mathfrak{E}$).

LEMMA 3.11.1. $\phi^\bullet$ *is an isomorphism.*

Proof. We can and will disregard the grading. The morphism ϕ induces the morphism $\phi^*\colon \mathrm{Spec}\mathfrak{E} \to \mathfrak{g}_{\bar{1}}^*$ that is an isomorphism over the open subset $\mathrm{Isom}(V_2,V_1) \times \mathrm{Hom}(V_1,V_2) \subset \mathfrak{g}_{\bar{1}}^*$ due to Corollary 3.10.2(a). Similarly, ϕ^* is an isomorphism over the open subset $\mathrm{Hom}(V_2,V_1) \times \mathrm{Isom}(V_1,V_2)$ due to Corollary 3.10.2(b).

Since the complement to the union of these two open subsets has codimension 2 in $\mathfrak{g}_{\bar{1}}^*$, we can apply Lemma 3.11.2 below. Note that the irreducibility of $\mathrm{Spec}\mathfrak{E}$ is guaranteed by Corollary 3.10.4. It remains only to check that the ratio of the above isomorphisms is the identity birational isomorphism between the varieties $\mathrm{Isom}(V_2,V_1) \times \mathrm{Hom}(V_1,V_2)$ and $\mathrm{Hom}(V_2,V_1) \times \mathrm{Isom}(V_1,V_2)$.

The composition $h \circ h^* \in \mathrm{Ext}^2_{D_{!*}\mathbf{G_O}}(\mathbf{Gr}\times\mathbf{V})(E_0 * \mathrm{IC}_{\omega_1}, E_0 * \mathrm{IC}_{\omega_1})$ is multiplication by the first Chern class of the normal line bundle $\mathcal{N}$ to the divisor $\mathrm{supp}(E_0 * \mathrm{IC}_{\omega_1})$ in $\mathrm{supp}(\mathrm{IC}_{\omega_1} * E_0)$. Recall that $\mathbf{Gr}^{\omega_1} \simeq \mathbb{P}^{N-1}$. The line bundle $\mathcal{N}$ is pulled back from the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{N-1} \simeq \mathbf{Gr}^{\omega_1}$. Recall also that the restriction of the determinant line bundle $\mathcal{L}$ from $\mathbf{Gr}$ to $\mathbf{Gr}^{\omega_1}$ is also isomorphic to $\mathcal{O}(1)$. We conclude that $h \circ h^* = c_1(\mathcal{L})$.

On the other hand, in the equivariant Satake category

$$D_{\mathbf{G_O}}(\mathbf{Gr}) \cong D^{\mathrm{GL}_N}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_N[-2])) \cong D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{A}^\bullet),$$

the first Chern class $c_1(\mathcal{L}) \in \mathrm{Ext}^2_{D_{\mathbf{G_O}}(\mathbf{Gr})}(\mathrm{IC}_0 * \mathrm{IC}_{\omega_1}, \mathrm{IC}_0 * \mathrm{IC}_{\omega_1}) \subset \mathfrak{A}^2$ corresponds to the identity element (shifted by 2) $\mathrm{Id} \in \mathrm{Hom}(V_2,V_1)^* \otimes \mathrm{Hom}(V_2,V_1)$.

The lemma is proved. $\square$

LEMMA 3.11.2. *Let* $\pi\colon X \to \mathbb{A}^n$ *be a morphism from an irreducible affine algebraic variety to an affine space. Let* $f,g \in \mathbb{C}[\mathbb{A}^n]$ *be such that the codimension of the closed subvariety* $\mathbb{A}^n \setminus (U_f \cup U_g)$ *in* $\mathbb{A}^n$ *is at least 2, where* $U_f = \{u \in \mathbb{A}^n \mid f(u) \neq 0\}$. *Assume, moreover, that each of the morphisms* $\pi^{-1}(U_f) \to U_f$ *and* $\pi^{-1}(U_g) \to U_g$, *induced by* π, *is an isomorphism. Then* π *is an isomorphism.*

Proof. Let $X_f = \pi^{-1}(U_f)$ (respectively, $X_g = \pi^{-1}(U_g)$), and write $j\colon U_f \cup U_g \hookrightarrow \mathbb{A}^n$ (respectively, $j_X\colon X_f \cup X_g \hookrightarrow X$) for the open embedding. We have the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}[\mathbb{A}^n] \stackrel{j^*}{\hookrightarrow} \Gamma(U_f \cup U_g, \mathcal{O}_{U_f \cup U_g}) & = & \{u \oplus v \in \mathbb{C}[U_f] \oplus \mathbb{C}[U_g] \mid u|_{U_f \cap U_g} = v|_{U_f \cap U_g}\} \\
\downarrow \pi^* & & \downarrow (\pi|_{X_f})^* \oplus (\pi|_{X_g})^* \\
\mathbb{C}[X] \stackrel{j_X^*}{\hookrightarrow} \Gamma(X_f \cup X_g, \mathcal{O}_{X_f \cup X_g}) & = & \{\tilde{u} \oplus \tilde{v} \in \mathbb{C}[X_f] \oplus \mathbb{C}[X_g] \mid \tilde{u}|_{X_f \cap X_g} = \tilde{v}|_{X_f \cap X_g}\}
\end{array}$$

The map j^* in this diagram is an isomorphism by the assumption of codimension at least 2. The map j_X^* is injective since X is irreducible. The assumptions imply also that the vertical map $(\pi|_{X_f})^* \oplus (\pi|_{X_g})^*$ on the right is an isomorphism. It follows that the vertical map π^* on the left must be an isomorphism, as required. $\square$

3.12 Restriction to $\mathbf{Gr} \times \mathbf{V}_0$

The existence of the desired equivalence $\Phi^{1,1}$ of Theorem 3.6.1 follows from Lemmas 3.9.2 and 3.11.1. The equivalences $\Phi^{0,2}$ and $\Phi^{2,0}$ are obtained by conjugating with $\varrho^{\pm 1}_{\mathrm{right}}$. It remains to check their compatibility with monoidal structures.

We denote by $\bar{\jmath}_0$ (respectively, $\bar{\jmath}_{-1}$) the closed embedding $\mathbf{Gr} \times \mathbf{V}_0 \hookrightarrow \mathbf{Gr} \times \mathbf{V}$ (respectively, $\mathbf{Gr} \times t\mathbf{V}_0 \hookrightarrow \mathbf{Gr} \times \mathbf{V}$). We also denote by $\jmath_0$ the locally closed embedding $\mathbf{Gr} \times (\mathbf{V}_0 \smallsetminus t\mathbf{V}_0) \hookrightarrow \mathbf{Gr} \times \mathbf{V}$. Our goal in this section is a description in terms of $\Phi^{1,1}$ of the endofunctors $\bar{\jmath}_{0*}\bar{\jmath}_0^!$, $\jmath_{0*}\jmath_0^!\colon D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

Recall the setup and notation of §3.4. We have the natural morphisms

$$p\colon \mathcal{Q}^A \to \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1) = \Pi\mathfrak{g}_{\bar{1}}^*,$$

$$q\colon \mathcal{Q}^A \to \mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1') = \Pi\mathfrak{g}_{\bar{1}}^*;$$

$$p(A, A', A'', B, B', B'') = (A, B), \quad q(A, A', A'', B, B', B'') = (A', B').$$

We also have the natural morphisms

$$p, q\colon G_{\mathcal{Q}} \to G_{\bar{0}}, \quad p(g_1, g_1', g_2) = (g_1, g_2), \quad q(g_1, g_1', g_2) = (g_1', g_2).$$

Clearly, $p, q\colon \mathcal{Q}^A \to \Pi\mathfrak{g}_{\bar{1}}^*$ are equivariant with respect to $p, q\colon G_{\mathcal{Q}} \to G_{\bar{0}}$. Hence we have the convolution functor

$$p_*q^*\colon \mathsf{Coh}^{G_{\bar{0}}}(\Pi\mathfrak{g}_{\bar{1}}^*) = \mathsf{Coh}(G_{\bar{0}}\backslash\Pi\mathfrak{g}_{\bar{1}}^*) \xrightarrow{q^*} \mathsf{Coh}(G_{\mathcal{Q}}\backslash\mathcal{Q}^A) \xrightarrow{p_*} \mathsf{Coh}(G_{\bar{0}}\backslash\Pi\mathfrak{g}_{\bar{1}}^*) = \mathsf{Coh}^{G_{\bar{0}}}(\Pi\mathfrak{g}_{\bar{1}}^*)$$

(in particular, p_* involves taking $\mathrm{GL}(V_1')$-invariants). We will actually need the same-named functor $p_*q^*\colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet) \to D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ defined similarly using the dg-algebra with trivial differential $\mathfrak{G}_{1,1}^\bullet \otimes \mathbb{C}[\mathrm{Hom}(V_1, V_1')]$ (the grading on $\mathbb{C}[\mathrm{Hom}(V_1, V_1')]$ is trivial, and if we disregard the grading, then $\mathfrak{G}_{1,1}^\bullet \otimes \mathbb{C}[\mathrm{Hom}(V_1, V_1')] \simeq \mathbb{C}[\mathcal{Q}^A]$).

We also have a $G_{\mathcal{Q}}$-invariant subvariety $\mathcal{Q}_0^A \subset \mathcal{Q}^A$ given by the equation that A'' is non-invertible. The restriction of p, q to $\mathcal{Q}_0^A$ will be denoted by p_0, q_0. As above, we obtain the functor $p_{0*}q_0^*\colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet) \to D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$.

PROPOSITION 3.12.1. (a) *There is an isomorphism of functors*

$$\bar{\jmath}_{0*}\bar{\jmath}_0^! \circ \Phi^{1,1} \simeq \Phi^{1,1} \circ p_*q^*\colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}).$$

(b) *The isomorphism in (a) can be extended to the following commutative diagram of morphisms:*

$$\bar{\jmath}_{-1*}\bar{\jmath}_{-1}^! \circ \Phi^{1,1} \xrightarrow{\sim} \Phi^{1,1} \circ \big(\det V_1 \otimes (p_*q^*) \circ (\det{}^{-1}V_1' \otimes -)\big)$$

(c) *There is an isomorphism of functors*

$$\jmath_{0*}\jmath_0^! \circ \Phi^{1,1} \simeq \Phi^{1,1} \circ p_{0*}q_0^*\colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}).$$

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Proof. (a) The support of $\mathrm{IC}_{(\boldsymbol{\nu},\boldsymbol{\mu})}$ lies in $\mathbf{Gr} \times \mathbf{V}_0$ if and only if $\boldsymbol{\nu}$ is a negative partition ($0 \geq \nu_1 \geq \cdots \geq \nu_N$) (see [FGT09, Proof of Proposition 8]). So $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}_0)$ is generated by the collection of objects $\mathrm{IC}_{(\boldsymbol{\nu},\boldsymbol{\mu})}$ where $\boldsymbol{\nu}$ is a negative partition. Thus we see that $\bar{\jmath}_{0*}$ is a fully faithful functor whose image is generated by $\{\mathrm{IC}_{(\boldsymbol{\nu},\boldsymbol{\mu})} \mid \boldsymbol{\nu} \leq 0\}$ and $\bar{\jmath}_0^!$ is the right adjoint of $\bar{\jmath}_{0*}$. Recall that $V_{\boldsymbol{\nu}}^*$ denotes an irreducible representation of $\mathrm{GL}(V)$ obtained by applying the corresponding Schur functor to V^*. The result of application of the same Schur functor to V_1^* (respectively, $V_1'^*, V_2^*$) will be denoted by $V_{1,\boldsymbol{\nu}}^*$ (respectively, $V_{1,\boldsymbol{\nu}}'^*, V_{2,\boldsymbol{\nu}}^*$). Since

$$\mathrm{IC}_{(\boldsymbol{\nu},\boldsymbol{\mu})} = \Phi^{1,1}(V_{1,\boldsymbol{\nu}}^* \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}}^*) = \Phi^{1,1}(V_{1,\boldsymbol{\nu}^*} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}^*})$$

(where $\boldsymbol{\nu}^* = (-\nu_N, -\nu_{N-1}, \ldots, -\nu_2, -\nu_1)$ for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$), we have to show that $p_* q^*$ lands in the subcategory $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet) \subset D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ generated by objects $\{V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}} \mid \boldsymbol{\lambda} \geq 0\}$ and to construct the adjunction isomorphism

$$\mathrm{Hom}_{D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)}(V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}}, V_{1,\boldsymbol{\lambda}'} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}'})$$

$$\xrightarrow{\sim} \mathrm{Hom}_{D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)}(V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}}, p_* q^*(V_{1,\boldsymbol{\lambda}'}' \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}'}))$$

for a partition $\boldsymbol{\lambda}$. Equivalently, we have to construct an isomorphism

$$\left((V_{1,\boldsymbol{\lambda}}'^* \otimes V_{1,\boldsymbol{\lambda}'}') \otimes \mathbb{C}[\mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1')] \otimes (V_{2,\boldsymbol{\mu}}^* \otimes V_{2,\boldsymbol{\mu}'})\right)^{\mathrm{GL}(V_1') \times \mathrm{GL}(V_2)}$$

$$\xrightarrow{\sim} \left(V_{1,\boldsymbol{\lambda}}^* \otimes p_* q^*(V_{1,\boldsymbol{\lambda}'}' \otimes \mathbb{C}[\mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1')] \otimes V_{2,\boldsymbol{\mu}'}) \otimes V_{2,\boldsymbol{\mu}}^*\right)^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)}$$

$$:= \left(V_{1,\boldsymbol{\lambda}}^* \otimes V_{1,\boldsymbol{\lambda}'}' \otimes \mathbb{C}[\mathcal{Q}^A] \otimes (V_{2,\boldsymbol{\mu}}^* \otimes V_{2,\boldsymbol{\mu}'})\right)^{G_\mathrm{Q}}.$$

Recall that $\mathcal{Q}^A = \mathrm{Hom}(V_1, V_1') \times \mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_2, V_1)$. We apply Lemma 3.13.1(c) below to $U_1 = V_2$, $U_2 = V_1$, $U_3 = V_1'$, $\boldsymbol{\nu} = \boldsymbol{\lambda}$ (in the notation of §3.13) to obtain an isomorphism

$$V_{1,\boldsymbol{\lambda}}'^* \otimes \mathbb{C}[\mathrm{Hom}(V_2, V_1')] \xrightarrow{\sim} \left(\mathbb{C}[\mathrm{Hom}(V_2, V_1) \otimes V_{1,\boldsymbol{\lambda}}^* \otimes \mathbb{C}[\mathrm{Hom}(V_1, V_1')]\right)^{\mathrm{GL}(V_1)}$$

whose inverse induces the desired adjunction isomorphism.

We still have to check that the essential image of $p_* q^*$ lies in the subcategory $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet) \subset D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ generated by $\{V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}} \mid \boldsymbol{\lambda} \geq 0\}$. We consider the homomorphism $\mathfrak{G}_{1,1}^\bullet \to \mathbb{C}$ killing all the generators, and for $\mathcal{M} \in D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ we set $z_0^* \mathcal{M} := \mathbb{C} \otimes_{\mathfrak{G}_{1,1}^\bullet}^L \mathcal{M} \in D^{G_{\bar{0}}}(\mathbb{C})$ ('the fiber at $0 \in \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1)$'). Note that $z_0^* p_* q^*$ lands in the category generated by $\{V_{1,\boldsymbol{\lambda}} \otimes V_{2,\boldsymbol{\mu}} \mid \boldsymbol{\lambda} \geq 0\}$, that is, the category of modules with polynomial action of $\mathrm{GL}(V_1)$. Indeed, recall that G_Q acts on $\mathcal{Q}^A$ via

$$(g_1, g_1', g_2)(A', A'', B) = (g_2 A' g_1'^{-1}, g_1' A'' g_1^{-1}, g_1 B g_2^{-1}).$$

Since we impose the conditions $B = 0 = A := A'A''$, the action of $\mathrm{GL}(V_1)$ on $z_0^* p_* q^* \mathcal{N}$ (for a free dg-$\mathfrak{G}_{1,1}^\bullet$-module $\mathcal{N}$) comes from its action on functions of A'', and the latter action is polynomial.

Finally, we claim that if the action of $\mathrm{GL}(V_1)$ on $z_0^* \mathcal{M}$ is polynomial, then $\mathcal{M}$ lies in the subcategory generated by $\{V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}} \mid \boldsymbol{\lambda} \geq 0\}$. To this end we apply the Koszul equivalence $\varkappa\colon D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda) \xrightarrow{\sim} D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ of §3.8. It is easy to see that if the action of $\mathrm{GL}(V_1)$ on the total cohomology of $\mathcal{K} \in D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ is polynomial, then $\mathcal{K}$ lies in the subcategory $\mathrm{Pol} \subset D_{\mathrm{fd}}^{G_{\bar{0}}}(\Lambda)$ generated by the $G_{\bar{0}} \ltimes \Lambda$-modules with polynomial action of $\mathrm{GL}(V_1)$ and trivial action of Λ.

Now $\varkappa(\mathrm{Pol})$ is the subcategory $D^{G_{\bar 0},\geq}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{1,1}) \subset D^{G_{\bar 0}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{1,1})$. And if the action of $\mathrm{GL}(V_1)$ on $z_0^* \mathcal{M}$ is polynomial, then $\mathcal{M} \simeq \varkappa(\mathcal{K})$, where the action of $\mathrm{GL}(V_1)$ on the total cohomology of $\mathcal{K}$ is polynomial. This completes the proof of (a).

(b) As $\alpha^* \bar\jmath_{-1*} \bar\jmath^!_{-1} \simeq \bar\jmath_{0*} \bar\jmath^!_0 \alpha^*$ (in the notation of §3.10), we deduce an isomorphism of functors

$$\bar\jmath_{-1*}\bar\jmath^!_{-1} \circ \Phi^{1,1} \simeq \Phi^{1,1} \circ \big(\det V_1 \otimes (p_* q^*) \circ (\det^{-1} V_1' \otimes -) \big) : D^{G_{\bar 0}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{1,1}) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}).$$

Thus the upper and lower rectangles of the diagram in (b) are commutative by construction. We have to prove the commutativity of the big curved quadrangle. The endofunctors $\bar\jmath_{0*}\bar\jmath^!_0$, $\bar\jmath_{-1*}\bar\jmath^!_{-1}$ of $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ are equipped with the structure of idempotent comonads. The desired commutativity follows from the fact that, given two idempotent comonads $T_0, T_1 \colon \mathcal{C} \to \mathcal{C}$, there is at most one morphism of functors $\chi \colon T_1 \to T_0$ such that $\varepsilon_1 = \varepsilon_0 \circ \chi$ for the counits $\varepsilon_i \colon T_i \to \mathrm{Id}_{\mathcal{C}}$. Indeed, $\chi = T_0(\varepsilon_1) \circ \chi_1$, where χ_1 is defined as the composition

$$T_1 \cong T_1 \circ T_1 \xrightarrow{\chi \circ T_1} T_0 \circ T_1.$$

We claim that χ_1 is an isomorphism uniquely defined as the inverse to the morphism $\varepsilon_0 \circ T_1 \colon T_0 \circ T_1 \xrightarrow{\sim} T_1$. In effect, the composition

$$T_1 \cong T_1 \circ T_1 \xrightarrow{\chi \circ T_1} T_0 \circ T_1 \xrightarrow{\varepsilon_0 \circ T_1} T_1$$

equals the composition

$$T_1 \cong T_1 \circ T_1 \xrightarrow{\varepsilon_1 \circ T_1} \mathrm{Id}_{\mathcal{C}} \circ T_1 = T_1$$

which in turn equals Id_{T_1}. Conversely, the composition

$$T_0 \circ T_1 \xrightarrow{\varepsilon_0 \circ T_1} T_1 \cong T_1 \circ T_1 \xrightarrow{\chi \circ T_1} T_0 \circ T_1$$

equals the composition

$$T_0 \circ T_1 \cong T_0 \circ T_1 \circ T_1 \xrightarrow{T_0 \circ \chi \circ T_1} T_0 \circ T_0 \circ T_1 \cong T_0 \circ T_1$$

which in turn equals $\mathrm{Id}_{T_0 \circ T_1}$.

This completes the proof of (b), but we would like to give one more independent argument that will prove useful later on.

Recall that we have to prove the commutativity of the big curved quadrangle. To this end we change the setting to the base field $\mathbb{F}_q$ as in the proof of Lemma 3.9.1. That is, we replace $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ by the equivalent equivariant derived category of sheaves on $(\mathbf{Gr} \times \mathbf{V})_{\bar{\mathbb{F}}_q}$ as in [BF08, Proposition 5] (in particular, choosing an isomorphism $\mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$). However, we preserve the notation $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ for this category in order not to overload our notation (anyway, it will only be used during the current proof). All the irreducible perverse sheaves $\mathrm{IC}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}$ carry a natural Tate Weil structure by [FGT09, Proposition 11]. They (along with their Tate twists) generate a subcategory[9] $\hat{D}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ of the mixed version $D^{\mathrm{mix}}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ of $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. We will use a particular dg-model of $\hat{D}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ viewed as a category enriched over complexes equipped with an action of the Frobenius automorphism Fr. Note that the absolute values of the eigenvalues of Fr lie in $\sqrt{q}^{\mathbb{Z}}$, and hence our complexes carry an additional grading according to the absolute values of the eigenvalues of Fr. If we forget the mixed structure and remember only

---

[9] Closed with respect to taking cones and direct summands.

this additional grading, we obtain a category $\underline{\tilde{D}}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ enriched over complexes equipped with an additional grading. Its localization with respect to quasi-isomorphisms will be denoted $\tilde{D}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

On the other hand, we consider the category $\underline{D}_{\mathrm{perf}}^{G_{\bar{0}} \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet)$ of perfect $G_{\bar{0}} \times \mathbb{C}^\times$-equivariant dg-$\mathfrak{G}_{1,1}^\bullet$-modules and its localization (with respect to quasi-isomorphisms) $D_{\mathrm{perf}}^{G_{\bar{0}} \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet)$. Here all the generators of $\mathfrak{G}_{1,1}^\bullet$ have weight 1 with respect to the action of $\mathbb{C}^\times$. Then the standard modification of our construction of the equivalence $\Phi^{1,1}\colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet) \xrightarrow{\sim} D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ produces a functor $\underline{\tilde{\Phi}}^{1,1}\colon \underline{D}_{\mathrm{perf}}^{G_{\bar{0}} \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet) \to \underline{\tilde{D}}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ and its localization $\tilde{\Phi}^{1,1}\colon D_{\mathrm{perf}}^{G_{\bar{0}} \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet) \xrightarrow{\sim} \tilde{D}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$.

We have the similar diagram of morphisms of functors $D_{\mathrm{perf}}^{G_{\bar{0}} \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet) \to \tilde{D}_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ with commutative upper and lower rectangles

$$
\begin{array}{ccc}
\bar{\jmath}_{-1*}\bar{\jmath}_{-1}^! \circ \tilde{\Phi}^{1,1} & \xrightarrow{\sim} & \tilde{\Phi}^{1,1} \circ \left(\det V_1 \otimes (p_* q^*) \circ (\det^{-1} V_1' \otimes -) \right) \\
\downarrow & & \downarrow \\
\tilde{\Phi}^{1,1} & \xrightarrow{\mathrm{Id}} & \tilde{\Phi}^{1,1} \\
\uparrow & & \uparrow \\
\bar{\jmath}_{0*}\bar{\jmath}_0^! \circ \tilde{\Phi}^{1,1} & \xrightarrow{\sim} & \tilde{\Phi}^{1,1} \circ p_* q^*
\end{array}
\quad \cdot \det A''
$$

and we have to prove the commutativity of the big curved quadrangle. The endofunctors $\det V_1 \otimes (p_* q^*) \circ (\det^{-1} V_1' \otimes -)$, Id, $p_* q^*$ of $D_{\mathrm{perf}}^{G_{\bar{0}} \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet)$ are given by their respective kernels K_1, K_2, K_3 in $\underline{D}_{\mathrm{perf}}^{G_{\bar{0}}^2 \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet \otimes \mathfrak{G}_{1,1}^\bullet)$. Note that the equivariance with respect to $\mathbb{C}^\times$ (as opposed to $(\mathbb{C}^\times)^2$) suffices since all the three functors under consideration commute with the shifts of the additional grading. Similarly, all the three functors on the constructible side commute with the Tate twists. All three kernels are pure of weight 0, that is, their additional gradings coincide with their cohomological gradings. The category of pure weight 0 objects in $D_{\mathrm{perf}}^{G_{\bar{0}}^2 \times \mathbb{C}^\times}(\mathfrak{G}_{1,1}^\bullet \otimes \mathfrak{G}_{1,1}^\bullet)$ is equivalent to the abelian category of $G_{\bar{0}}^2 \times \mathbb{C}^\times$-equivariant $\mathfrak{G}_{1,1} \otimes \mathfrak{G}_{1,1}$-modules (the equivalence being obtained by taking cohomology). Therefore, the morphisms of functors $(\det V_1 \otimes -) \circ (p_* q^*) \circ (\det^{-1} V_1' \otimes -) \to \mathrm{Id}$ and $p_* q^* \to \mathrm{Id}$ arise from the morphisms between the respective kernels that are injective as morphisms of $\mathfrak{G}_{1,1} \otimes \mathfrak{G}_{1,1}$-modules. We have to compare certain morphisms to $\underline{\tilde{\Phi}}^{1,1} K_3$, and we know that their compositions with the monomorphism $\underline{\tilde{\Phi}}^{1,1} K_3 \to \underline{\tilde{\Phi}}^{1,1} K_2$ coincide, hence the desired equality of morphisms. This completes our second proof of (b).

(c) This follows from the comparison of the distinguished triangles

$$
(\bar{\jmath}_{-1*}\bar{\jmath}_{-1}^! \to \bar{\jmath}_{0*}\bar{\jmath}_0^! \to \jmath_{0*}\jmath_0^!) \circ \Phi^{1,1}
$$

and

$$
\Phi^{1,1} \circ \left(\det V_1 \otimes (p_* q^*) \circ (\det^{-1} V_1' \otimes -) \to p_* q^* \to p_{0*} q_0^* \right).
$$

The proposition is proved. □

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3.13 Some invariant theory

Let U_1, U_2, U_3 be vector spaces of dimensions n_1, n_2, n_3. The irreducible polynomial (respectively, antipolynomial) representations of $\mathrm{GL}(U_i)$ are realized in the Schur spaces $\mathbb{S}_{\boldsymbol{\lambda}} U_i$ (respectively, $\mathbb{S}_{\boldsymbol{\lambda}} U_i^*$), where $\boldsymbol{\lambda}$ is a partition with $\ell(\boldsymbol{\lambda}) \leq n_i$. We will also write $\mathbb{S}_{\boldsymbol{\lambda}^*} U_i$ for $\mathbb{S}_{\boldsymbol{\lambda}} U_i^*$, where

$$\boldsymbol{\lambda}^* = -w_0 \boldsymbol{\lambda} = (-\lambda_{n_i}, -\lambda_{n_i-1}, \ldots, -\lambda_2, -\lambda_1)$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n_i})$. We set $\mathbb{S}_{\boldsymbol{\lambda}} U_i = 0 = \mathbb{S}_{\boldsymbol{\lambda}^*} U_i$ for $\ell(\boldsymbol{\lambda}) > n_i$. We denote by $\mathrm{Hom}_{\leq n_2}(U_1, U_3) \subset \mathrm{Hom}(U_1, U_3)$ the subvariety formed by all the homomorphisms of rank at most n_2.

LEMMA 3.13.1. *The composition of homomorphisms induces the following isomorphisms of* $\mathrm{GL}(U_1) \times \mathrm{GL}(U_3)$-*modules.*

(a) $\mathbb{C}[\mathrm{Hom}_{\leq n_2}(U_1, U_3)] \xrightarrow{\sim} \left(\mathbb{C}[\mathrm{Hom}(U_1, U_2)] \otimes \mathbb{C}[\mathrm{Hom}(U_2, U_3)] \right)^{\mathrm{GL}(U_2)}$.

(b) *If* $n_1 \leq n_2$, *for a partition* $\boldsymbol{\nu}$,

$$\mathbb{S}_{\boldsymbol{\nu}} U_1 \otimes \mathbb{C}[\mathrm{Hom}(U_1, U_3)] \xrightarrow{\sim} \left(\mathbb{C}[\mathrm{Hom}(U_1, U_2)] \otimes \mathbb{S}_{\boldsymbol{\nu}} U_2 \otimes \mathbb{C}[\mathrm{Hom}(U_2, U_3)] \right)^{\mathrm{GL}(U_2)}.$$

(c) *If* $n_3 \leq n_2$, *for a partition* $\boldsymbol{\nu}$,

$$\mathbb{C}[\mathrm{Hom}(U_1, U_3)] \otimes \mathbb{S}_{\boldsymbol{\nu}} U_3^* \xrightarrow{\sim} \left(\mathbb{C}[\mathrm{Hom}(U_1, U_2)] \otimes \mathbb{S}_{\boldsymbol{\nu}} U_2^* \otimes \mathbb{C}[\mathrm{Hom}(U_2, U_3)] \right)^{\mathrm{GL}(U_2)}.$$

Proof. (a) We have $\mathbb{C}[\mathrm{Hom}(U_i, U_j)] = \bigoplus_{m \geq 0} \mathrm{Sym}^m(U_j^* \otimes U_i) = \bigoplus_{\boldsymbol{\lambda}} \mathbb{S}_{\boldsymbol{\lambda}} U_i \otimes \mathbb{S}_{\boldsymbol{\lambda}^*} U_j$ as a $\mathrm{GL}(U_i) \times \mathrm{GL}(U_j)$-module. Also, $\mathbb{C}[\mathrm{Hom}_{\leq r}(U_i, U_j)] = \bigoplus_{\ell(\boldsymbol{\lambda}) \leq r} \mathbb{S}_{\boldsymbol{\lambda}} U_i \otimes \mathbb{S}_{\boldsymbol{\lambda}^*} U_j$ as a $\mathrm{GL}(U_i) \times \mathrm{GL}(U_j)$-module. Clearly, $(\mathbb{S}_{\boldsymbol{\lambda}^*} U_2 \otimes \mathbb{S}_{\boldsymbol{\mu}} U_2)^{\mathrm{GL}(U_2)} = \mathbb{C}^{\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}}$. So the two sides of (a) are isomorphic as $\mathrm{GL}(U_1) \times \mathrm{GL}(U_3)$-modules. On the other hand, the morphism in question is injective since the composition morphism $\mathrm{Hom}(U_1, U_2) \times \mathrm{Hom}(U_2, U_3) \to \mathrm{Hom}_{\leq n_2}(U_1, U_3)$ is dominant. Hence the morphism in question is an isomorphism.

(b) We consider a copy U_2' of U_2, we tensor both sides of (b) with $\mathbb{S}_{\boldsymbol{\nu}^*} U_2'$, and take the direct sum over all partitions $\boldsymbol{\nu}$ with $\ell(\boldsymbol{\nu}) \leq n_2$. Then we have to prove that the morphism

$$\gamma \colon \mathbb{C}[\mathrm{Hom}(U_1, U_2')] \otimes \mathbb{C}[\mathrm{Hom}(U_1, U_3)]$$
$$\to \left(\mathbb{C}[\mathrm{Hom}(U_1, U_2)] \otimes \mathbb{C}[\mathrm{Hom}(U_2, U_2')] \otimes \mathbb{C}[\mathrm{Hom}(U_2, U_3)] \right)^{\mathrm{GL}(U_2)},$$

induced by the composition of arrows of the D_4-quiver in

(3.13.1)
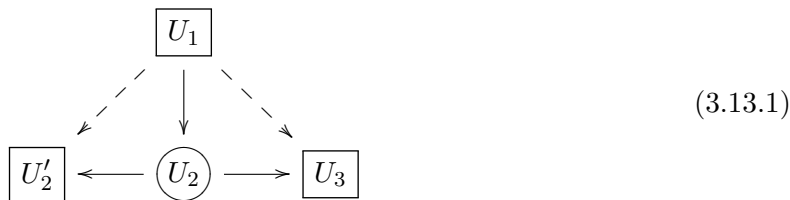
is an isomorphism. Now the statement can be reduced to (a) using the substitution $U_3 \rightsquigarrow U_3 \oplus U_2'$. Alternatively, the condition $n_1 \leq n_2$ guarantees that the morphism from the representation space of the D_4-quiver to the representation space of the dashed A_3-quiver is dominant. Hence γ is injective. The surjectivity of γ follows, for example, from [LP90, Theorem 1].

(c) This is dual to (b). $\square$

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3.14 The monoidal property of $\Phi^{2,0}$

Recall the notation of § 3.4. The monoidal structure $\overset{A}{*}$ on $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0})$ is defined via the kernel $\mathbb{C}[\mathcal{Q}^A]^\bullet_{2,0}$: a $G_{\mathcal{Q}}$-equivariant dg-$\mathbb{C}[\mathcal{H}]^\bullet_{2,0}$-module. The monoidal structure $\overset{!}{\circledast}$ on $D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ transferred to $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0})$ via the equivalence $\Phi^{2,0}$ is also defined via a kernel $\mathcal{K}^\bullet$ (a $G_{\mathcal{Q}}$-equivariant dg-$\mathbb{C}[\mathcal{H}]^\bullet_{2,0}$-module). We have to construct an isomorphism of $G_{\mathcal{Q}}$-equivariant dg-$\mathbb{C}[\mathcal{H}]^\bullet_{2,0}$-modules $\mathbb{C}[\mathcal{Q}^A]^\bullet_{2,0} \overset{\sim}{\longrightarrow} \mathcal{K}^\bullet$.

We denote by $\mathbb{C}[\mathcal{H}_{\mathrm{loc}}]^\bullet_{2,0}$ the localization of $\mathbb{C}[\mathcal{H}]^\bullet_{2,0}$ defined as $\mathbb{C}[\mathcal{H}]^\bullet_{2,0}[\det^{-1}A, \det^{-1}A', \det^{-1}A'']$. We define $\mathcal{K}^\bullet_{\mathrm{loc}} := \mathbb{C}[\mathcal{H}_{\mathrm{loc}}]^\bullet_{2,0} \otimes_{\mathbb{C}[\mathcal{H}]^\bullet_{2,0}} \mathcal{K}^\bullet$ and $\mathbb{C}[\mathcal{Q}^A_{\mathrm{loc}}]^\bullet_{2,0} := \mathbb{C}[\mathcal{H}_{\mathrm{loc}}]^\bullet_{2,0} \otimes_{\mathbb{C}[\mathcal{H}]^\bullet_{2,0}} \mathbb{C}[\mathcal{Q}^A]^\bullet_{2,0}$.

We have $\Phi^{2,0}\mathfrak{G}^\bullet_{2,0} \simeq \varrho^{-1}_{\mathrm{right}}E_0$. Also, for $\mathcal{F} \in D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, we have $\varrho^{-1}_{\mathrm{right}}E_0 \overset{!}{\circledast} \mathcal{F} = \bar{\jmath}_{0*}\bar{\jmath}^!_0\mathcal{F}$. Thus Proposition 3.12.1(a) yields an isomorphism of functors $\Phi^{2,0}(\mathfrak{G}^\bullet_{2,0} \overset{A}{*} -) \overset{\sim}{\longrightarrow} (\Phi^{2,0}\mathfrak{G}^\bullet_{2,0}) \overset{!}{\circledast} \Phi^{2,0}-$. This isomorphism yields in turn an isomorphism of kernels $\mathbb{C}[\mathcal{Q}^A_{\mathrm{forg}}]^\bullet_{2,0} \overset{\sim}{\longrightarrow} \mathcal{K}^\bullet_{\mathrm{forg}}$, where the subscript $_{\mathrm{forg}}$ denotes the restriction of the dg-module structure from $\mathbb{C}[\mathcal{H}]^\bullet_{2,0}$ to $\mathbb{C}[\mathcal{H}_{\mathrm{forg}}]^\bullet_{2,0} := \mathrm{Sym}(\mathrm{Hom}(V_1, V_2)[-2]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)) \otimes \mathrm{Sym}(\mathrm{Hom}(V'_1, V_2)[-2]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V'_1))$.

According to Proposition 3.12.1(b), the following diagram of functors $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{2,0}) \to D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ commutes:

$$
\begin{array}{ccc}
\bar{\jmath}_{-1*}\bar{\jmath}^!_{-1} \circ \Phi^{2,0} & \overset{\sim}{\longrightarrow} & \Phi^{2,0} \circ \big(\det V_1 \otimes (p_*q^*) \circ (\det^{-1}V'_1 \otimes -) \big) \\
\downarrow & & \downarrow {\scriptstyle \cdot \det A''} \\
\bar{\jmath}_{0*}\bar{\jmath}^!_0 \circ \Phi^{2,0} & \overset{\sim}{\longrightarrow} & \Phi^{2,0} \circ p_*q^*
\end{array}
$$

Hence the diagram

$$
\begin{array}{ccc}
\mathbb{C}[\mathcal{Q}^A_{\mathrm{forg}}]^\bullet_{2,0} & \overset{\sim}{\longrightarrow} & \mathcal{K}^\bullet_{\mathrm{forg}} \\
\downarrow {\scriptstyle \cdot \det A''} & & \downarrow {\scriptstyle \cdot \det A''} \\
\mathbb{C}[\mathcal{Q}^A_{\mathrm{forg}}]^\bullet_{2,0} & \overset{\sim}{\longrightarrow} & \mathcal{K}^\bullet_{\mathrm{forg}}
\end{array}
$$

commutes as well, and in particular the multiplication by $\det A''$ is injective on $\mathcal{K}^\bullet$, and hence $\mathcal{K}^\bullet \hookrightarrow \mathcal{K}^\bullet_{\mathrm{loc}}$.

Now since $\Phi^{2,0}_{\mathrm{loc}} \colon D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{A}^\bullet) \overset{\sim}{\longrightarrow} D_{\mathbf{G_O}}(\mathbf{Gr}) \cong D^{\mathrm{loc}}_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ (see Lemma 3.10.1 and Corollary 3.10.2) coincides with the equivariant Satake equivalence, and the latter is monoidal, we obtain an isomorphism of localized kernels $\mathbb{C}[\mathcal{Q}^A_{\mathrm{loc}}]^\bullet_{2,0} \overset{\sim}{\longrightarrow} \mathcal{K}^\bullet_{\mathrm{loc}}$ as $G_{\mathcal{Q}}$-equivariant $\mathbb{C}[\mathcal{H}_{\mathrm{loc}}]^\bullet_{2,0}$-modules. By the argument in the second proof of Proposition 3.12.1(b) (using the additional grading and purity of $\mathcal{K}^\bullet$), it remains to verify that this isomorphism restricts to the desired isomorphism from $\mathbb{C}[\mathcal{Q}^A]^\bullet_{2,0} \subset \mathbb{C}[\mathcal{Q}^A_{\mathrm{loc}}]^\bullet_{2,0}$ to $\mathcal{K}^\bullet \subset \mathcal{K}^\bullet_{\mathrm{loc}}$. For this verification it suffices to restrict the scalars to $\mathbb{C}[\mathcal{H}_{\mathrm{forg}}]^\bullet_{2,0}$. But we have already seen that over $\mathbb{C}[\mathcal{H}_{\mathrm{forg}}]^\bullet_{2,0}$ we obtain an isomorphism $\mathbb{C}[\mathcal{Q}^A_{\mathrm{forg}}]^\bullet_{2,0} \overset{\sim}{\longrightarrow} \mathcal{K}^\bullet_{\mathrm{forg}}$.

This completes the proof of the monoidal property of $\Phi^{2,0}$.

3.15 Fourier transform

We have the Fourier transform functors (along $\mathbf{V}$)

$$\text{FT} \colon D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \quad \text{FT} \colon D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}),$$

$$\text{FT} \colon D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}).$$

Strictly speaking, the Fourier transform goes not to $D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$, but to $D_{?\mathbf{G_O}}(\mathbf{Gr} \times (V^* \otimes \mathbf{F}))$. However, we identify V^* with V using our choice of (self-dual) basis $e_1, \ldots, e_N$, and accordingly change the action of $\mathbf{G_O} = \text{GL}(N, \mathbf{O})$ by composing it with the automorphism $g \mapsto {}^t g^{-1}$. Note that the resulting Fourier transform to $D_{?\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ is independent of the choice of basis in V.

To describe the effect of FT on the coherent side, we identify $V_1 \cong V_1^*$ and $V_2 \cong V_2^*$ using our bases. Furthermore, we identify

$$\text{Hom}(V_1, V_2) \cong \text{Hom}(V_2^*, V_1^*) \cong \text{Hom}(V_2, V_1), \quad A \mapsto B := -{}^t A,$$

$$\text{Hom}(V_2, V_1) \cong \text{Hom}(V_1^*, V_2^*) \cong \text{Hom}(V_1, V_2), \quad B \mapsto A := {}^t B.$$

Thus we obtain an ι-equivariant transposition isomorphism

$$\tau \colon \mathfrak{G}_{2,0}^\bullet = \text{Sym}(\text{Hom}(V_1, V_2)) \otimes \text{Sym}(\text{Hom}(V_2, V_1)[-2])$$

$$\to \text{Sym}(\text{Hom}(V_1, V_2)[-2]) \otimes \text{Sym}(\text{Hom}(V_2, V_1)) = \mathfrak{G}_{0,2}^\bullet,$$

where $\iota \colon G_{\bar{0}} \to G_{\bar{0}}$ is an automorphism $(g_1, g_2) \mapsto ({}^t g_1^{-1}, {}^t g_2^{-1})$. We denote the extension of scalars via τ by $\varphi\tau \colon D_{\text{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^\bullet) \to D_{\text{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^\bullet)$. Clearly, the functor $\varphi\tau \colon \left(D_{\text{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^\bullet), \overset{A}{*}\right) \to \left(D_{\text{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{0,2}^\bullet), \overset{B}{*}\right)$ is monoidal. Also, by the standard properties of the Fourier transform, the functor

$$\text{FT} \colon \left(D_{!\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \overset{!}{\circledast}\right) \to \left(D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}), \overset{*}{*}\right)$$

is monoidal. Thus the monoidal property of $\Phi^{0,2}$ is a corollary of the following proposition.

PROPOSITION 3.15.1. *The functors* $\text{FT} \circ \Phi^{2,0} \colon D_{\text{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{2,0}^\bullet) \to D_{*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ *and* $\Phi^{0,2} \circ \varphi\tau$ *are isomorphic.*

Proof. Going over the construction of equivalences of $\Phi^{0,2}, \Phi^{2,0}$, we see that it suffices to construct the isomorphisms $\text{FT}(E_0) \xrightarrow{\sim} E_0$ (evident), and

$$\text{FT}(-_{\text{left}} * \bullet * -_{\text{right}}) \xrightarrow{\sim} \iota(-_{\text{left}}) * \text{FT}(\bullet) * \iota(-_{\text{right}}) \tag{3.15.1}$$

(left and right convolution functors $D_{\mathbf{G_O}}(\mathbf{Gr}) \times D_{!*\mathbf{G_O}}(\mathbf{Gr} \times V) \times D_{\mathbf{G_O}}(\mathbf{Gr}) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times V)$). Here $\iota \colon D_{\mathbf{G_O}}(\mathbf{Gr}) \to D_{\mathbf{G_O}}(\mathbf{Gr})$ is a monoidal autoequivalence induced by the automorphism $\iota \colon g \mapsto {}^t g^{-1}$ of $\mathbf{G_F}$. Note that the Satake equivalence intertwines $\iota \colon D_{\mathbf{G_O}}(\mathbf{Gr}) \to D_{\mathbf{G_O}}(\mathbf{Gr})$ with the same-named autoequivalence of $\text{Rep}(\text{GL}_N)$. Also note that $\iota \colon D_{\mathbf{G_O}}(\mathbf{Gr}) \to D_{\mathbf{G_O}}(\mathbf{Gr})$ is induced by the automorphism $L \mapsto L^\perp$ of $\mathbf{Gr}$. Here $L^\perp := \{v \in \mathbf{V} : (v, L) \in \mathbf{O}\}$, and $(\cdot, \cdot)$ stands for the $\mathbf{F}$-bilinear pairing on $\mathbf{V}$ such that $(e_i, e_j) = \delta_{ij}$. Now the existence of the desired isomorphism (3.15.1) follows from the definitions of FT and convolutions (3.2.3) and (3.2.4). $\quad\square$

3.16 The monoidal property of $\Phi^{1,1}$

The argument is very similar to that of § 3.14. We introduce

$$\mathbb{C}[\mathcal{H}]^\bullet_{1,1} = \mathrm{Sym}(\mathrm{Hom}(V_1, V_2)[-1]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1)[-1]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1', V_2)[-1])$$

$$\otimes \mathrm{Sym}(\mathrm{Hom}(V_2, V_1')[-1]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1, V_1')[-1]) \otimes \mathrm{Sym}(\mathrm{Hom}(V_1', V_1)[-1]).$$

Then the monoidal structure $\otimes_{\mathfrak{G}^\bullet_{1,1}}$ on $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{1,1})$ is defined via the kernel $\mathbb{C}[\Delta]^\bullet_{1,1}$: the diagonal $G_{\bar{0}}$-equivariant dg-$\mathfrak{G}^\bullet_{1,1}$-trimodule. The fusion monoidal structure $\star$ on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ transferred to $D^{G_{\bar{0}}}_{\mathrm{perf}}(\mathfrak{G}^\bullet_{1,1})$ via the equivalence $\Phi^{1,1}$ is also defined via a kernel $\mathsf{K}^\bullet$ (a $G_{\bar{0}}$-equivariant dg-$\mathfrak{G}^\bullet_{1,1}$-trimodule). Note that we have a different equivariant structure than for $\mathcal{K}^\bullet$ (in the notation of § 3.14) because of the different compatibility with the Hecke action. We have to construct an isomorphism of $G_{\bar{0}}$-equivariant dg-$\mathfrak{G}^\bullet_{1,1}$-trimodules $\mathbb{C}[\Delta]^\bullet_{1,1} \xrightarrow{\sim} \mathsf{K}^\bullet$.

We have $\Phi^{1,1}\mathfrak{G}^\bullet_{1,1} \simeq E_0$. Also, we have an isomorphism of endofunctors $E_0 \star - \cong \mathrm{Id} \colon D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}) \to D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. Thus we obtain an isomorphism of functors $\Phi^{1,1}(\mathfrak{G}^\bullet_{1,1} \otimes_{\mathfrak{G}^\bullet_{1,1}} -) \xrightarrow{\sim} (\Phi^{1,1}\mathfrak{G}^\bullet_{1,1}) \star \Phi^{1,1}-$. This isomorphism yields an isomorphism of kernels $\mathbb{C}[\Delta_{\mathrm{forg}}]^\bullet_{1,1} \xrightarrow{\sim} \mathsf{K}^\bullet_{\mathrm{forg}}$ (the notation is explained in § 3.14).

The following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{IC}_{(-1^N, 1^N)} \star \Phi^{1,1} & \xrightarrow{\sim} & \Phi^{1,1} \circ \left(\det V_1 \otimes - \otimes \det^{-1} V_1' \right) \\
\downarrow & & \downarrow {\scriptstyle \cdot \det A''} \\
E_0 \star \Phi^{1,1} & \xrightarrow{\hspace{2cm}\sim\hspace{2cm}} & \Phi^{1,1}
\end{array}
$$

Hence the diagram

$$
\begin{array}{ccc}
\mathbb{C}[\Delta]^\bullet_{1,1} & \xrightarrow{\sim} & \mathsf{K}^\bullet_{\mathrm{forg}} \\
\downarrow {\scriptstyle \cdot \det A''} & & \downarrow {\scriptstyle \cdot \det A''} \\
\mathbb{C}[\Delta]^\bullet_{1,1} & \xrightarrow{\sim} & \mathsf{K}^\bullet_{\mathrm{forg}}
\end{array}
$$

commutes as well, and the rest of the argument proceeds just as in § 3.14.

This completes the proof of the monoidal property of $\Phi^{1,1}$ along with Theorem 3.6.1.

4. A coherent realization of $D_{\mathbf{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$

4.1 Notation

We consider a complex vector space $\bar{V}_1$ with a basis $e_2, e_3, \ldots, e_N$. We consider the Lie superalgebra $\mathfrak{gl}(N-1|N) = \mathfrak{gl}(\bar{V}_1 \oplus \Pi V_2)$. We have a decomposition $\mathfrak{gl}(N-1|N) = \bar{\mathfrak{g}}_{\bar{0}} \oplus \bar{\mathfrak{g}}_{\bar{1}}$, where $\bar{\mathfrak{g}}_{\bar{1}} = \Pi\mathrm{Hom}(\bar{V}_1, V_2) \oplus \Pi\mathrm{Hom}(V_2, \bar{V}_1)$, and $\bar{\mathfrak{g}}_{\bar{0}} = \mathrm{End}(\bar{V}_1) \oplus \mathrm{End}(V_2)$. We set $\bar{G}_{\bar{0}} = \mathrm{GL}(\bar{V}_1) \times \mathrm{GL}(V_2)$. We consider the dg-algebra $\bar{\mathfrak{G}}^\bullet = \mathrm{Sym}(\bar{\mathfrak{g}}_{\bar{1}}[-1])$ with trivial differential, and the triangulated category $D^{\bar{G}_{\bar{0}}}_{\mathrm{perf}}(\bar{\mathfrak{G}}^\bullet)$ obtained by localization (with respect to quasi-isomorphisms) of the category of perfect $\bar{G}_{\bar{0}}$-equivariant dg-$\bar{\mathfrak{G}}^\bullet$-modules. Finally, we set $\bar{\mathbf{G}}_{\mathbf{O}} = \mathrm{GL}(N-1, \mathbf{O})$.

THEOREM 4.1.1. *There exists an equivalence of triangulated categories* $\bar{\Phi} \colon D^{\bar{G}_{\bar{0}}}_{\mathrm{perf}}(\bar{\mathfrak{G}}^\bullet) \xrightarrow{\sim} D_{\bar{\mathbf{G}}_{\mathbf{O}}}(\mathbf{Gr})$ *commuting with the left convolution action of the monoidal spherical Hecke category* $\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{N-1}) \cong \mathrm{Rep}(\mathrm{GL}_{N-1})$ *and with the right convolution action of the monoidal spherical Hecke category* $\mathrm{Perv}_{\mathrm{GL}(N,\mathbf{O})}(\mathbf{Gr}_N) \cong \mathrm{Rep}(\mathrm{GL}_N)$.

The proof will be given in §4.4 after some preparations in §§4.2 and 4.3.

Similarly to §3.8, we define $\bar{\Lambda} := \Lambda\big(\mathrm{Hom}(\bar{V}_1, V_2) \oplus \mathrm{Hom}(V_2, \bar{V}_1)\big)$. We consider the derived category $D_{\mathrm{fd}}^{\bar{G}_{\bar{0}}}(\bar{\Lambda})$ of finite-dimensional complexes of $\bar{G}_{\bar{0}} \ltimes \bar{\Lambda}$-modules. We have the Koszul equivalence functors

$$\bar{\varkappa}\colon D_{\mathrm{fd}}^{\bar{G}_{\bar{0}}}(\bar{\Lambda}) \xrightarrow{\sim} D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^\bullet), \quad SD_{\mathrm{fd}}^{\bar{G}_{\bar{0}}}(\bar{\Lambda}) \xrightarrow{\sim} SD_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^\bullet).$$

We also consider the degeneration $\underline{\mathfrak{gl}}(N-1|N)$ of the Lie superalgebra $\mathfrak{gl}(N-1|N)$ (defined as in §1.5), and the derived category of bounded complexes of integrable $\underline{\mathfrak{gl}}(N-1|N)$-modules $SD_{\mathrm{int}}(\underline{\mathfrak{gl}}(N-1|N)) \cong SD_{\mathrm{fd}}^{\bar{G}_{\bar{0}}}(\bar{\Lambda})$. The following corollary of Theorem 4.1.1 is proved just like Corollary 3.8.1.

COROLLARY 4.1.2. (a) *The composed equivalence*

$$\bar{\Phi} \circ \bar{\varkappa}\colon D_{\mathrm{fd}}^{\bar{G}_{\bar{0}}}(\bar{\Lambda}) \xrightarrow{\sim} D_{\bar{\mathbf{G}}_{\mathbf{O}}}(\mathbf{Gr})$$

is exact with respect to the tautological t-structure on $D_{\mathrm{fd}}^{\bar{G}_{\bar{0}}}(\bar{\Lambda})$ and the perverse t-structure on $D_{\bar{\mathbf{G}}_{\mathbf{O}}}(\mathbf{Gr})$.

(b) *This equivalence is monoidal with respect to the tensor structure on $D_{\mathrm{fd}}^{\bar{G}_{\bar{0}}}(\bar{\Lambda})$ and the fusion $\star$ on $D_{\bar{\mathbf{G}}_{\mathbf{O}}}(\mathbf{Gr})$.*

(c) *The equivariant derived category $D_{\bar{\mathbf{G}}_{\mathbf{O}}}(\mathbf{Gr})$ is equivalent to the bounded derived category of the abelian category $\mathrm{Perv}_{\bar{\mathbf{G}}_{\mathbf{O}}}(\mathbf{Gr})$.*

In case $N = 2$, both Theorem 4.1.1 and Corollary 4.1.2(a) were proved in [BF18] by a rather different argument.

4.2 Constructible mirabolic restriction

Clearly, $\mathbf{V}_0 \smallsetminus t\mathbf{V}_0$ is a single $\mathbf{G}_{\mathbf{O}}$-orbit, and the stabilizer of the vector $e_1 \in \mathbf{V}_0 \smallsetminus t\mathbf{V}_0$ is the mirabolic subgroup $\mathbf{M}_{\mathbf{O}} \subset \mathbf{G}_{\mathbf{O}}$. Hence $D_{\mathbf{G}_{\mathbf{O}}}(\mathbf{Gr} \times (\mathbf{V}_0 \smallsetminus t\mathbf{V}_0)) \cong D_{\mathbf{M}_{\mathbf{O}}}(\mathbf{Gr})$. We will denote $\mathbf{Gr} = \mathbf{Gr}_{\mathrm{GL}_N}$ by $\mathbf{Gr}_N$ to distinguish it from $\mathbf{Gr}_{N-1} := \mathbf{Gr}_{\mathrm{GL}_{N-1}}$. We will also denote by $\mathbf{Gr}_N^{\subset} \subset \mathbf{Gr}_N$ (respectively, $\mathbf{Gr}_{N-1}^{\subset} \subset \mathbf{Gr}_{N-1}$) the closed subvariety classifying the sublattices in the standard one $\mathbf{V}_0$ (respectively, in $\bar{\mathbf{V}}_0 := \mathbf{O}e_2 \oplus \mathbf{O}e_3 \oplus \cdots \oplus \mathbf{O}e_N$). It is a union of the Schubert varieties numbered by the *negative* partitions of length $\leq N$ (respectively, $\leq N-1$). The category $\mathrm{Perv}_{\mathrm{GL}(N,\mathbf{O})}(\mathbf{Gr}_N^{\subset})$ (respectively, $\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{N-1}^{\subset})$) is monoidal and the Satake equivalence takes it to the monoidal category of polynomial representations of GL_N (respectively, GL_{N-1}).

We have a closed embedding $\varsigma\colon \mathbf{Gr}_{N-1}^{\subset} \hookrightarrow \mathbf{Gr}_N^{\subset}$, $(\bar{L} \subset \bar{\mathbf{V}}_0) \mapsto (\mathbf{O}e_1 \oplus \bar{L} \subset \mathbf{V}_0)$.

LEMMA 4.2.1. (a) *The functor $\varsigma^!\colon D_{\mathrm{GL}(N,\mathbf{O})}(\mathbf{Gr}_N^{\subset}) \to D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{N-1}^{\subset})$ is monoidal.*

(b) *For a negative partition $\boldsymbol{\lambda}$ we have $\varsigma^!\mathrm{IC}_{\boldsymbol{\lambda}} = \mathrm{IC}_{\boldsymbol{\lambda}}[|\boldsymbol{\lambda}|]$ in the notation of §3.13 (i.e. if $\lambda_1 < 0$, then $\varsigma^!\mathrm{IC}_{\boldsymbol{\lambda}} = 0$, and if $\lambda_1 = 0$, then $\varsigma^!\mathrm{IC}_{\boldsymbol{\lambda}} = \mathrm{IC}_{(\lambda_2 \geq \cdots \geq \lambda_N)}[\lambda_2 + \cdots + \lambda_N])$.*

Proof. (a) The Grassmannian of sublattices $\mathbf{Gr}_N^{\subset}$ (respectively, $\mathbf{Gr}_{N-1}^{\subset}$) is the union of connected components $\bigsqcup_{n \in \mathbb{N}} \mathbf{Gr}_N^{\subset,(-n)}$ (respectively, $\bigsqcup_{n \in \mathbb{N}} \mathbf{Gr}_{N-1}^{\subset,(-n)}$) parametrizing sublattices of codimension n. Clearly, $\varsigma(\mathbf{Gr}_{N-1}^{\subset,(-n)}) \subset \mathbf{Gr}_N^{\subset,(-n)}$. Moreover, $\varsigma(\mathbf{Gr}_{N-1}^{\subset,(-n)})$ is a connected component of the intersection of the Levi Grassmannian $\mathbf{Gr}_{\mathrm{GL}_1 \times \mathrm{GL}_{N-1}} \subset \mathbf{Gr}_N$ with $\mathbf{Gr}_N^{\subset,(-n)}$. We have the monoidal functor of hyperbolic restriction to the Levi Grassmannian; see for example,

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[BD, 5.3.28 at p. 213]. Since $(\mathbf{M_F} \cdot \varsigma(\mathbf{Gr}_{N-1}^{\subsetneq,(-n)})) \cap \mathbf{Gr}_N^{\subsetneq,(-n)} = \varsigma(\mathbf{Gr}_{N-1}^{\subsetneq,(-n)})$, the hyperbolic restriction to the component $\varsigma(\mathbf{Gr}_{N-1}^{\subsetneq,(-n)})$ coincides with the corestriction $\varsigma^!$.

(b) We have $(\mathbb{S}_{\boldsymbol{\lambda}} V_1^*)^{\mathrm{GL}_1} = \mathbb{S}_{\boldsymbol{\lambda}} \bar{V}_1^*$. Now the desired claim follows from [BD, Proposition 5.3.29]. $\qquad\square$

Recall that the monoidal Hecke category $D_{\mathrm{GL}(N,\mathbf{O})}(\mathbf{Gr}_N) = D_{\mathbf{G_O}}(\mathbf{Gr})$ acts by left convolution on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. In particular, the monoidal subcategory $D_{\mathbf{G_O}}(\mathbf{Gr}_N^{\subsetneq}) \subset D_{\mathbf{G_O}}(\mathbf{Gr})$ acts on $D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$. We define the action of $D_{\mathbf{G_O}}(\mathbf{Gr}_N^{\subsetneq})$ by the left convolution on $D_{\mathbf{G_O}}(\mathbf{Gr} \times (\mathbf{V}_0 \smallsetminus t\mathbf{V}_0))$ as follows: $\mathcal{A} * \mathcal{F} := \jmath_0^!(\mathcal{A} * \jmath_{0*}\mathcal{F})$ (in the notation of §3.12). Also, the monoidal Hecke category $D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_{N-1})$ acts by left convolution on $D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_N)$. Finally, we denote by $\mathrm{Res}_{\mathbf{M_O}}^{\mathrm{GL}(N-1,\mathbf{O})} \colon D_{\mathbf{M_O}}(\mathbf{Gr}_N) \to D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}_N)$ the functor of restriction of equivariance under $\mathrm{GL}(N-1,\mathbf{O}) \hookrightarrow \mathbf{M_O}$.

LEMMA 4.2.2. *We have a compatible system of isomorphisms for all* $\mathcal{A} \in D_{\mathrm{GL}(N,\mathbf{O})}(\mathbf{Gr}_N^{\subsetneq})$ *and* $\mathcal{F} \in D_{\mathbf{G_O}}(\mathbf{Gr} \times (\mathbf{V}_0 \smallsetminus t\mathbf{V}_0)) \cong D_{\mathbf{M_O}}(\mathbf{Gr})$*:*

$$\mathrm{Res}_{\mathbf{M_O}}^{\mathrm{GL}(N-1,\mathbf{O})}(\mathcal{A} * \mathcal{F}) \xrightarrow{\sim} \varsigma^! \mathcal{A} * \mathrm{Res}_{\mathbf{M_O}}^{\mathrm{GL}(N-1,\mathbf{O})}(\mathcal{F}).$$

Proof. Comparison of definitions. $\qquad\square$

4.3 Coherent mirabolic restriction

Recall the subcategory $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet) \subset D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet)$ generated by $\{V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}} \mid \boldsymbol{\lambda} \geq 0\}$ introduced in the proof of Proposition 3.12.1(a). The equivalence $\Phi^{1,1} \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^\bullet) \xrightarrow{\sim} D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V})$ restricts to the same-named equivalence $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet) \xrightarrow{\sim} D_{!*\mathbf{G_O}}(\mathbf{Gr} \times \mathbf{V}_0)$, and the functor $p_* q^* \colon D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet) \to D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet)$ (in the notation of §3.12) is isomorphic to Id. Thus we have a natural morphism $\mathrm{Id} \cong p_* q^* \to p_{0*} q_0^*$ (again in the notation of §3.12) of endofunctors of $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet)$. Composing with another copy of $p_{0*} q_0^*$, we obtain a morphism $p_{0*} q_0^* \to p_{0*} q_0^* \circ p_{0*} q_0^*$ of endofunctors of $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet)$ that is easily seen to be an isomorphism. Indeed, according to Proposition 3.12.1(c), $\Phi^{1,1}$ takes $p_{0*} q_0^*$ to $\jmath_{0*} \jmath_0^!$, and $\jmath_{0*} \jmath_0^! \xrightarrow{\sim} \jmath_{0*} \jmath_0^! \circ \jmath_{0*} \jmath_0^!$. Inverting the isomorphism $p_{0*} q_0^* \xrightarrow{\sim} p_{0*} q_0^* \circ p_{0*} q_0^*$, we obtain an isomorphism $p_{0*} q_0^* \circ p_{0*} q_0^* \xrightarrow{\sim} p_{0*} q_0^*$ that, together with the morphism $\mathrm{Id} \to p_{0*} q_0^*$, equips $p_{0*} q_0^*$ with a structure of (idempotent) monad in $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet)$.

We denote the dg-algebra with trivial differential $\mathfrak{G}_{1,1}^\bullet \otimes \mathbb{C}[\mathrm{Hom}_{\leq N-1}(V_1, V_1')]$ (the grading on $\mathbb{C}[\mathrm{Hom}_{\leq N-1}(V_1, V_1')]$ is trivial) by $\mathfrak{F}^\bullet$ (here $\mathrm{Hom}_{\leq N-1}(V_1, V_1') \subset \mathrm{Hom}(V_1, V_1')$ stands for the subvariety formed by the non-invertible homomorphisms). It is acted upon by $G_{\mathbb{Q}}$. By Theorem 3.6.1 and Proposition 3.12.1, we have an equivalence of categories Φ' from the Kleisli category (see [Mac98, VI.5]) $D(p_{0*} q_0^*)$ of the monad $p_{0*} q_0^*$ in $D_{\mathrm{perf}}^{G_{\bar{0}},\geq}(\mathfrak{G}_{1,1}^\bullet)$ to $D_{\mathbf{G_O}}(\mathbf{Gr} \times (\mathbf{V}_0 \smallsetminus t\mathbf{V}_0))$. For the modules $V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}}$, $V_{1,\boldsymbol{\lambda}'} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}'}$ over the monad $p_{0*} q_0^*$ (here both $\boldsymbol{\lambda}, \boldsymbol{\lambda}'$ are partitions), we have

$$\mathrm{Hom}_{D(p_{0*} q_0^*)}(V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}}, V_{1,\boldsymbol{\lambda}'} \otimes \mathfrak{G}_{1,1}^\bullet \otimes V_{2,\boldsymbol{\mu}'}) = \left(V_{1,\boldsymbol{\lambda}}^* \otimes V_{1,\boldsymbol{\lambda}'}' \otimes \mathfrak{F}^\bullet \otimes (V_{2,\boldsymbol{\mu}}^* \otimes V_{2,\boldsymbol{\mu}'})\right)^{G_{\mathbb{Q}}}.$$

Recall the equivalence $D_{\mathbf{G_O}}(\mathbf{Gr} \times (\mathbf{V}_0 \smallsetminus t\mathbf{V}_0)) \cong D_{\mathbf{M_O}}(\mathbf{Gr})$. By an abuse of notation, we will denote the composed equivalence $D(p_{0*} q_0^*) \xrightarrow{\sim} D_{\mathbf{M_O}}(\mathbf{Gr})$ also by Φ'.

On the other hand, we consider a full subcategory $D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^{\bullet}) \subset D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^{\bullet})$ generated by the free modules $\bar{V}_{1,\boldsymbol{\lambda}} \otimes \bar{\mathfrak{G}}^{\bullet} \otimes V_{2,\boldsymbol{\mu}}$ for partitions $\boldsymbol{\lambda}$ (of length $\leq N-1$). We construct an equivalence $\Phi \colon D(p_{0*}q_0^*) \xrightarrow{\sim} D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^{\bullet})$ as follows. We consider the variety $\mathcal{P}$ of sextuples

$$A \in \mathrm{Hom}(V_1, V_2), \quad B \in \mathrm{Hom}(V_2, V_1), \quad \bar{A} \in \mathrm{Hom}(\bar{V}_1, V_2),$$

$$\bar{B} \in \mathrm{Hom}(V_2, \bar{V}_1), \quad b \in \mathrm{Hom}(V_1, \bar{V}_1), \quad a \in \mathrm{Hom}(\bar{V}_1, V_1),$$

such that $A = \bar{A}b$, $\bar{B} = bB$, $a = B\bar{A}$. Clearly,

$$\mathcal{P} \simeq \mathrm{Hom}(V_1, \bar{V}_1) \times \mathrm{Hom}(\bar{V}_1, V_2) \times \mathrm{Hom}(V_2, V_1).$$
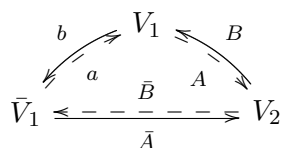
We have the natural morphisms

$$\mathsf{p} \colon \mathcal{P} \to \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1) = \Pi\mathfrak{g}_{\bar{1}}^*, \quad \mathsf{q} \colon \mathcal{P} \to \mathrm{Hom}(\bar{V}_1, V_2) \times \mathrm{Hom}(V_2, \bar{V}_1) = \Pi\bar{\mathfrak{g}}_{\bar{1}}^*;$$

$$\mathsf{p}(A, \bar{A}, a, b, B, \bar{B}) = (A, B), \quad \mathsf{q}(A, \bar{A}, a, b, B, \bar{B}) = (\bar{A}, \bar{B}).$$

The variety $\mathcal{P}$ is acted upon by $G_{\mathcal{P}} := \mathrm{GL}(V_1) \times \mathrm{GL}(\bar{V}_1) \times \mathrm{GL}(V_2)$:

$$(g_1, \bar{g}_1, g_2)(A, \bar{A}, a, b, B, \bar{B}) = (g_2 A g_1^{-1}, g_2 \bar{A} \bar{g}_1^{-1}, g_1 a \bar{g}_1^{-1}, \bar{g}_1 b g_1^{-1}, g_1 B g_2^{-1}, \bar{g}_1 \bar{B} g_2^{-1}).$$

We have the natural morphisms

$$\mathsf{p} \colon G_{\mathcal{P}} \to G_{\bar{0}}, \quad \mathsf{p}(g_1, \bar{g}_1, g_2) = (g_1, g_2); \quad \mathsf{q} \colon G_{\mathcal{P}} \to \bar{G}_{\bar{0}}, \quad \mathsf{q}(g_1, \bar{g}_1, g_2) = (\bar{g}_1, g_2).$$

Clearly, the morphisms $\mathsf{p} \colon \mathcal{P} \to \Pi\mathfrak{g}_{\bar{1}}^*$, $\mathsf{q} \colon \mathcal{P} \to \Pi\bar{\mathfrak{g}}_{\bar{1}}^*$ are equivariant with respect to $\mathsf{p} \colon G_{\mathcal{P}} \to G_{\bar{0}}$, $\mathsf{q} \colon G_{\mathcal{P}} \to \bar{G}_{\bar{0}}$. Hence we have the convolution functor

$$\mathsf{q}_* \mathsf{p}^* \colon \mathsf{Coh}^{G_{\bar{0}}}(\Pi\mathfrak{g}_{\bar{1}}^*) = \mathsf{Coh}(G_{\bar{0}} \backslash \Pi\mathfrak{g}_{\bar{1}}^*) \xrightarrow{\mathsf{p}^*} \mathsf{Coh}(G_{\mathcal{P}} \backslash \mathcal{P}) \xrightarrow{\mathsf{q}_*} \mathsf{Coh}(\bar{G}_{\bar{0}} \backslash \Pi\bar{\mathfrak{g}}_{\bar{1}}^*) = \mathsf{Coh}^{\bar{G}_{\bar{0}}}(\Pi\bar{\mathfrak{g}}_{\bar{1}}^*)$$

(in particular, q_* involves taking $\mathrm{GL}(V_1)$-invariants). We will actually need the same-named functor $\mathsf{q}_* \mathsf{p}^* \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet}) \to D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^{\bullet})$ defined similarly using the dg-algebra with trivial differential

$$\mathrm{Sym}\big(\mathrm{Hom}(V_1, V_2)[-1] \oplus \mathrm{Hom}(\bar{V}_1, V_1)[-1]\big) \otimes \mathrm{SymHom}(\bar{V}_1, V_1)$$

(the grading on $\mathrm{SymHom}(\bar{V}_1, V_1)$ is trivial, and if we disregard the grading, then $\mathrm{Sym}\big(\mathrm{Hom}(V_1, V_2)[-1] \oplus \mathrm{Hom}(\bar{V}_1, V_1)[-1]\big) \otimes \mathrm{SymHom}(\bar{V}_1, V_1) \simeq \mathbb{C}[\mathcal{P}]$).

Now recall that $D(p_{0*}q_0^*)$ is a full subcategory of $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$. The desired functor $\Phi \colon D(p_{0*}q_0^*) \to D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^{\bullet})$ is nothing but the restriction of $\mathsf{q}_* \mathsf{p}^*$ to the full subcategory $D(p_{0*}q_0^*) \subset D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{G}_{1,1}^{\bullet})$. The full subcategory $D(p_{0*}q_0^*)$ is generated by the objects $p_{0*}q_0^*(V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^{\bullet} \otimes V_{2,\boldsymbol{\mu}}) \cong_{D(p_{0*}q_0^*)} V_{1,\boldsymbol{\lambda}} \otimes \mathfrak{G}_{1,1}^{\bullet} \otimes V_{2,\boldsymbol{\mu}}$ (the left- and right-hand sides are isomorphic in the Kleisli category $D(p_{0*}q_0^*)$) with $\boldsymbol{\lambda}$ running through the set of partitions. For a

partition $\boldsymbol{\lambda}$ we have

$$\Phi(V_{1,\boldsymbol{\lambda}}\otimes\mathfrak{G}_{1,1}^{\bullet}\otimes V_{2,\boldsymbol{\mu}})=\bar{V}_{1,\boldsymbol{\lambda}}\otimes\bar{\mathfrak{G}}^{\bullet}\otimes V_{2,\boldsymbol{\mu}}$$

in the notation of §3.13 (i.e. if $\boldsymbol{\lambda}=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N\geq 0)$ and $\lambda_N>0$, then $\Phi(V_{1,\boldsymbol{\lambda}}\otimes\mathfrak{G}_{1,1}^{\bullet}\otimes V_{2,\boldsymbol{\mu}})=0$, and if $\lambda_N=0$, then $\Phi(V_{1,\boldsymbol{\lambda}}\otimes\mathfrak{G}_{1,1}^{\bullet}\otimes V_{2,\boldsymbol{\mu}})=\bar{V}_{1,(\lambda_1\geq\cdots\geq\lambda_{N-1})}\otimes\bar{\mathfrak{G}}^{\bullet}\otimes V_{2,\boldsymbol{\mu}}$). Hence $\Phi(V_{1,\boldsymbol{\lambda}}\otimes\mathfrak{G}_{1,1}^{\bullet}\otimes V_{2,\boldsymbol{\mu}})$ actually lies in $D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^{\bullet})\subset D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^{\bullet})$. It remains to check that Φ is fully faithful.

For partitions $\boldsymbol{\lambda},\boldsymbol{\lambda}'$, we have to check that the following morphism is an isomorphism:

$$\big(V_{1,\boldsymbol{\lambda}}^*\otimes p_{0*}q_0^*(V_{1,\boldsymbol{\lambda}'}'\otimes\mathbb{C}[\mathrm{Hom}(V_1',V_2)\times\mathrm{Hom}(V_2,V_1')]\otimes V_{2,\boldsymbol{\mu}'})\otimes V_{2,\boldsymbol{\mu}}^*\big)^{\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)}$$

$$:=\big(V_{1,\boldsymbol{\lambda}}^*\otimes V_{1,\boldsymbol{\lambda}'}'\otimes\mathbb{C}[\mathcal{Q}_0]\otimes(V_{2,\boldsymbol{\mu}}^*\otimes V_{2,\boldsymbol{\mu}'})\big)^{G_{\mathcal{Q}}}=\big(\mathbb{S}_{\boldsymbol{\lambda}}V_1^*\otimes\mathbb{S}_{\boldsymbol{\lambda}'}V_1'\otimes\mathbb{C}[\mathcal{Q}_0]\otimes(V_{2,\boldsymbol{\mu}}^*\otimes V_{2,\boldsymbol{\mu}'})\big)^{G_{\mathcal{Q}}}$$

$$\xrightarrow{\sim}\big(\mathbb{S}_{\boldsymbol{\lambda}}\bar{V}_1^*\otimes\mathbb{S}_{\boldsymbol{\lambda}'}\bar{V}_1\otimes\mathbb{C}[\mathrm{Hom}(\bar{V}_1,V_2)\times\mathrm{Hom}(V_2,\bar{V}_1)]\otimes(V_{2,\boldsymbol{\mu}}^*\otimes V_{2,\boldsymbol{\mu}'})\big)^{\mathrm{GL}(\bar{V}_1)\times\mathrm{GL}(V_2)}.$$
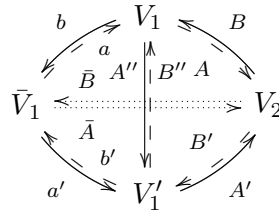
The desired isomorphism is equal to the composition of the following three. The first is induced by

$$\mathbb{C}[\mathrm{Hom}_{\leq N-1}(V_1,V_1')]\xrightarrow{\sim}\big(\mathbb{C}[\mathrm{Hom}(V_1,\bar{V}_1)]\otimes\mathbb{C}[\mathrm{Hom}(\bar{V}_1,V_1')]\big)^{\mathrm{GL}(\bar{V}_1)};$$

see Lemma 3.13.1(a). The second is induced by the inverse of

$$\mathbb{S}_{\boldsymbol{\lambda}'}\bar{V}_1\otimes\mathbb{C}[\mathrm{Hom}(\bar{V}_1,V_2)]\xrightarrow{\sim}\big(\mathbb{C}[\mathrm{Hom}(\bar{V}_1,V_1')]\otimes\mathbb{S}_{\boldsymbol{\lambda}'}V_1'\otimes\mathbb{C}[\mathrm{Hom}(V_1',V_2)]\big)^{\mathrm{GL}(V_1')};$$

see Lemma 3.13.1(b). The third is induced by the inverse of

$$\mathbb{C}[\mathrm{Hom}(V_2,\bar{V}_1)]\otimes\mathbb{S}_{\boldsymbol{\lambda}^*}\bar{V}_1\xrightarrow{\sim}\big(\mathbb{C}[\mathrm{Hom}(V_2,V_1)]\otimes\mathbb{S}_{\boldsymbol{\lambda}^*}V_1\otimes\mathbb{C}[\mathrm{Hom}(V_1,\bar{V}_1)]\big)^{\mathrm{GL}(V_1)};$$

see Lemma 3.13.1(c).

4.4 Proof of Theorem 4.1.1

We consider an $\mathbf{F}$-linear automorphism ξ of $\mathbf{V}$: $\xi(e_1)=e_1$, $\xi(e_i)=te_i$ for $i=2,\ldots,N$, and the same-named induced automorphism of $\mathbf{Gr}$. It is given by the action of an element $\mathrm{diag}(1,t,\ldots,t)$ of the diagonal Cartan torus $\mathbf{T_F}$. Note that $\mathrm{diag}(1,t,\ldots,t)\mathbf{M_O}\,\mathrm{diag}(1,t^{-1},\ldots,t^{-1})\supset\mathbf{M_O}$. Hence ξ_* acts on the equivariant category $D_{\mathbf{M_O}}(\mathbf{Gr})$ (by restricting equivariance from $\mathrm{diag}(1,t,\ldots,t)\mathbf{M_O}\,\mathrm{diag}(1,t^{-1},\ldots,t^{-1})$ to $\mathbf{M_O}$). Clearly, ξ_* acts on the equivariant category $D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$ as well. Recall that $\mathrm{Res}_{\mathbf{M_O}}^{\mathrm{GL}(N-1,\mathbf{O})}\colon D_{\mathbf{M_O}}(\mathbf{Gr})\to D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$ denotes the functor of restriction of equivariance under $\mathrm{GL}(N-1,\mathbf{O})\hookrightarrow\mathbf{M_O}$. We denote by $\mathrm{Av}_{\mathrm{GL}(N-1,\mathbf{O})}^{\mathbf{M_O}}\colon D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})\to D_{\mathbf{M_O}}(\mathbf{Gr})$ the corresponding right adjoint $*$-averaging functor.

LEMMA 4.4.1. (a) *Given* $\mathcal{F}\in D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$, *for* $n\gg 0$ *the canonical morphism* $\mathrm{Res}_{\mathbf{M_O}}^{\mathrm{GL}(N-1,\mathbf{O})}$ $\mathrm{Av}_{\mathrm{GL}(N-1,\mathbf{O})}^{\mathbf{M_O}}\xi_*^n\mathcal{F}\to\xi_*^n\mathcal{F}$ *is an isomorphism.*

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(b) ξ_* is an auto-equivalence of $D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$, and hence the natural functor from $D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$ to $\mathrm{colim}_{\xi_*} D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$ is an equivalence.

(c) The restriction of equivariance functor $\mathrm{Res}_{\mathbf{M_O}}^{\mathrm{GL}(N-1,\mathbf{O})} \colon D_{\mathbf{M_O}}(\mathbf{Gr}) \to D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$ induces the same-named equivalence of the colimits $\mathrm{Res}_{\mathbf{M_O}}^{\mathrm{GL}(N-1,\mathbf{O})} \colon \mathrm{colim}_{\xi_*} D_{\mathbf{M_O}}(\mathbf{Gr}) \to \mathrm{colim}_{\xi_*} D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr}) \cong D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$.

Proof. It suffices to prove (a) for an irreducible perverse $\mathcal{F} \in D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$. This in turn follows from the fact that, for any $\mathrm{GL}(N-1,\mathbf{O})$-orbit $\mathbb{O} \subset \mathbf{Gr}$ and $n \gg 0$, the shift $\xi^{-n}\mathbb{O}$ becomes $\mathbf{M_O}$-invariant. Indeed, $\mathbf{M_O}$ is generated by its radical $\mathbf{U_O}$ and $\mathrm{GL}(N-1,\mathbf{O})$. As n grows, $\mathbf{U_O}^{(n)} := \mathrm{diag}(1, t^{-n}, \ldots, t^{-n}) \mathbf{U_O} \, \mathrm{diag}(1, t^n, \ldots, t^n)$ forms a system of shrinking subgroups of $\mathbf{U_O}$, and we take n big enough so that the action of $\mathbf{U_O}^{(n)}$ on $\mathbb{O}$ is trivial (recall that the action of $\mathbf{U_O}$ on any Schubert subvariety of $\mathbf{Gr}$ factors through a finite-dimensional quotient group).

(b) is evident, and (c) follows from (a) and (b). $\qquad\square$

On the coherent side we consider an endofunctor η of $D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^\bullet)$ obtained by tensoring with the polynomial representation $\det(\bar{V}_1)$ of $\mathrm{GL}(\bar{V}_1)$. The full embedding $D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^\bullet) \hookrightarrow D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^\bullet)$ gives rise to an equivalence of colimits $\mathrm{colim}_\eta D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^\bullet) \xrightarrow{\sim} \mathrm{colim}_\eta D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^\bullet) \cong D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^\bullet)$.

Composing the inverse of Φ with Φ' (in the notation of § 4.3), we obtain an equivalence $\bar{\Phi}' \colon D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^\bullet) \xrightarrow{\sim} D_{\mathbf{M_O}}(\mathbf{Gr})$. According to Lemma 4.2.2, the equivalence $\bar{\Phi}' \colon D_{\mathrm{perf}}^{\bar{G}_{\bar{0}},\geq 0}(\bar{\mathfrak{G}}^\bullet) \xrightarrow{\sim} D_{\mathbf{M_O}}(\mathbf{Gr})$ intertwines the endofunctors η and ξ_*, and hence induces the desired equivalence $\bar{\Phi} \colon D_{\mathrm{perf}}^{\bar{G}_{\bar{0}}}(\bar{\mathfrak{G}}^\bullet) \xrightarrow{\sim} D_{\mathrm{GL}(N-1,\mathbf{O})}(\mathbf{Gr})$ between the colimits. Theorem 4.1.1 is proved.

5. Loop rotation and quantization

5.1 Graded differential operators and convolutions

We have $H_{\mathbb{G}_m}^\bullet(\mathrm{pt}) = \mathbb{C}[\hbar]$. We consider the algebra $\mathfrak{D}$ of 'graded differential operators' on $\mathrm{Hom}(V_2, V_1)$: a $\mathbb{C}[\hbar]$-algebra generated by $\mathrm{Hom}(V_2, V_1)$ and $\mathrm{Hom}(V_1, V_2)$ with relations $[h, h'] = [f, f'] = 0$, $[h, f] = \langle h, f \rangle \hbar$ for $h, h' \in \mathrm{Hom}(V_2, V_1)$, $f, f' \in \mathrm{Hom}(V_1, V_2)$. It is equipped with the grading $\deg f = \deg h = 1$, $\deg \hbar = 2$. We denote by $\mathfrak{D}_{1,1}^\bullet$ this graded algebra viewed as a dg-algebra with trivial differential. We will also need two more versions of $\mathfrak{D}_{1,1}^\bullet$ differing by the gradings of generators: in $\mathfrak{D}_{0,2}^\bullet$ we set $\deg f = 0$ and $\deg h = 2$, while in $\mathfrak{D}_{2,0}^\bullet$ we set $\deg f = 2$ and $\deg h = 0$.

Recall the setup and notation of § 3.4. So we have another copy V_1' of V_1, we rebaptize the algebra $\mathfrak{D}$ of 'graded differential operators' on $\mathrm{Hom}(V_2, V_1)$ as $\mathfrak{D}_{21}$, and along with it we consider $\mathfrak{D}_{21'}$ ('graded differential operators' on $\mathrm{Hom}(V_2, V_1')$) and $\mathfrak{D}_{1'1}$ ('graded differential operators' on $\mathrm{Hom}(V_1', V_1)$). Note that $\mathfrak{D}_{21} \cong \mathfrak{D}_{12}$ ('graded differential operators' on $\mathrm{Hom}(V_1, V_2)$), and similarly $\mathfrak{D}_{21'} \cong \mathfrak{D}_{1'2}$, and $\mathfrak{D}_{1'1} \cong \mathfrak{D}_{11'}$.

We have morphisms

$$\mathsf{m}^A \colon \mathrm{Hom}(V_1', V_2) \times \mathrm{Hom}(V_1, V_1') \to \mathrm{Hom}(V_1, V_2), \quad (A', A'') \mapsto A'A'',$$

$$\mathsf{m}^B \colon \mathrm{Hom}(V_1', V_1) \times \mathrm{Hom}(V_2, V_1') \to \mathrm{Hom}(V_2, V_1), \quad (B'', B') \mapsto B''B'.$$

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They give rise to the functors

$$D_{\mathrm{perf}}^{\mathrm{GL}(V_1') \times \mathrm{GL}(V_2)}(\mathfrak{D}_{1'2}) \times D_{\mathrm{perf}}^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_1')}(\mathfrak{D}_{11'}) \to D_{\mathrm{perf}}^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)}(\mathfrak{D}_{12}),$$

$$(\mathcal{M}_{1'2}, \mathcal{M}_{11'}) \mapsto \left(\mathsf{m}_*^A(\mathcal{M}_{1'2} \boxtimes \mathcal{M}_{11'})\right)^{\mathrm{GL}(V_1')},$$

$$D_{\mathrm{perf}}^{\mathrm{GL}(V_1') \times \mathrm{GL}(V_1)}(\mathfrak{D}_{1'1}) \times D_{\mathrm{perf}}^{\mathrm{GL}(V_2) \times \mathrm{GL}(V_1')}(\mathfrak{D}_{21'}) \to D_{\mathrm{perf}}^{\mathrm{GL}(V_2) \times \mathrm{GL}(V_1)}(\mathfrak{D}_{21}),$$

$$(\mathcal{M}_{1'1}, \mathcal{M}_{21'}) \mapsto \left(\mathsf{m}_*^B(\mathcal{M}_{1'1} \boxtimes \mathcal{M}_{21'})\right)^{\mathrm{GL}(V_1')}$$

(here m_*^A and m_*^B stand for the direct images in the category of $\mathfrak{D}$-modules).

We will actually need the corresponding functors on the categories of dg-modules

$$\overset{A}{*} \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{2,0}^\bullet) \times D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{2,0}^\bullet) \to D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{2,0}^\bullet), \quad \overset{B}{*} \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{0,2}^\bullet) \times D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{0,2}^\bullet) \to D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{0,2}^\bullet).$$

The multiplicative group $\mathbb{G}_m$ acts on $\mathbf{Gr} \times \mathbf{V}$ as loop rotations. The goal of this section is the following theorem.

THEOREM 5.1.1. *There exist monoidal equivalences of triangulated categories*

$$\begin{array}{ccc}
\left(D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{2,0}^\bullet), \overset{A}{*}\right) & \xrightarrow[\Phi_\hbar^{2,0}]{\sim} & \left(D_{!\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}), \overset{!}{\circledast}\right) \\
\wr \downarrow \varrho_{\mathrm{right}} & & \wr \downarrow \varrho_{\mathrm{right}} \\
D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet) & \xrightarrow[\Phi_\hbar^{1,1}]{\sim} & D_{!*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}) \\
\wr \downarrow \varrho_{\mathrm{right}} & & \wr \downarrow \varrho_{\mathrm{right}} \\
\left(D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{0,2}^\bullet), \overset{B}{*}\right) & \xrightarrow[\Phi_\hbar^{0,2}]{\sim} & \left(D_{*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}), \overset{*}{\circledast}\right)
\end{array} \qquad (5.1.1)$$

(the vertical and middle row equivalences are not monoidal). The horizontal equivalences commute with the actions of the monoidal spherical Hecke category $\mathrm{Perv}_{\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr}) \cong \mathrm{Rep}(\mathrm{GL}_N)$ *by left and right convolutions.*

The proof will be given in § 5.3.

5.2 Construction of equivalences

We set $\mathfrak{E}_\hbar^\bullet := \mathrm{Ext}_{D_{!*\mathbf{G_O} \rtimes \mathbb{G}_m}^{\mathrm{deeq}}(\mathbf{Gr} \times \mathbf{V})}^\bullet (E_0, E_0)$ (a dg-algebra with trivial differential). Since it is an Ext-algebra in the de-equivariantized category, it is automatically equipped with an action of $\mathrm{GL}_N \times \mathrm{GL}_N = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) = G_{\bar{0}}$, and we can consider the corresponding triangulated category $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{E}_\hbar^\bullet)$. Similarly to Lemma 3.9.2, there is a canonical equivalence $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{E}_\hbar^\bullet) \xrightarrow{\sim} D_{!*\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathbf{Gr} \times \mathbf{V})$. It remains to construct an isomorphism $\phi_\hbar^\bullet \colon \mathfrak{D}_{1,1}^\bullet \xrightarrow{\sim} \mathfrak{E}_\hbar^\bullet$.

Note that $\mathfrak{E}_\hbar^\bullet$ is a $\mathbb{C}[\hbar]$-algebra, and $\mathfrak{E}_\hbar^\bullet/(\hbar = 0) = \mathfrak{E}^\bullet = \mathfrak{E}_{1,1}^\bullet = \mathrm{Sym}(\mathfrak{g}_{\bar{1}}[-1])$, so that $\mathfrak{E}^\bullet$ acquires a Poisson bracket from this deformation. We claim that this Poisson bracket arises from the canonical symplectic form on $\mathfrak{g}_{\bar{1}}^*$. Indeed, the isomorphism ϕ^* (see the proof of Lemma 3.11.1) over $\mathrm{Isom}(V_2, V_1) \times \mathrm{Hom}(V_1, V_2)$ and $\mathrm{Hom}(V_2, V_1) \times \mathrm{Isom}(V_1, V_2)$ is Poisson. Here the Poisson structure on these open subsets arises from the deformations $\mathfrak{A}_\hbar^\bullet, \mathfrak{B}_\hbar^\bullet$ which in turn arise from

the loop-rotation-equivariant Satake category $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathbf{Gr})$. The corresponding Poisson brackets are the standard ones on the cotangent bundles $T^*\mathrm{Isom}(V_2, V_1), T^*\mathrm{Isom}(V_1, V_2)$, as follows from [BF08, Theorem 5].

Now $\mathfrak{D}_{1,1}^\bullet$ is a unique graded $\mathbb{C}[\hbar]$-algebra with $\mathfrak{D}_{1,1}^\bullet/(\hbar = 0) = \mathrm{Sym}(\mathfrak{g}_{\bar{1}}[-1])$ such that the corresponding Poisson bracket on $\mathrm{Sym}(\mathfrak{g}_{\bar{1}}[-1])$ is the standard one. Thus $\phi_\hbar^\bullet$ and $\Phi_\hbar^{1,1}$ are constructed.

5.3 Restriction to $\mathbf{Gr} \times \mathbf{V_0}$ with loop rotation

Similarly to §3.12, our goal in this section is a description in terms of equivalence of Theorem 5.1.1 of the endofunctors $\bar{\jmath}_{0*}\bar{\jmath}_0^!, \jmath_{0*}\jmath_0^! \colon D_{!*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}) \to D_{!*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V})$. To this end we will need the algebra $\mathfrak{H}$ of 'graded differential operators' on $\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_1, V_1') \oplus \mathrm{Hom}(V_1', V_2)$ defined similarly to $\mathfrak{D}$ of §5.1. It is equipped with the grading where the generators from $\mathrm{Hom}(V_2, V_1), \mathrm{Hom}(V_1, V_2), \mathrm{Hom}(V_1', V_2)$ and $\mathrm{Hom}(V_2, V_1')$ have degree 1, while the generators from $\mathrm{Hom}(V_1, V_1')$ and $\mathrm{Hom}(V_1', V_1)$ have degrees 2 and 0 respectively, and $\deg \hbar = 2$. We denote by $\mathfrak{H}^\bullet$ this graded algebra viewed as a dg-algebra with trivial differential. If we denote the similar dg-algebras of 'graded differential operators' on $\mathrm{Hom}(V_1, V_2), \mathrm{Hom}(V_1, V_1')$ and $\mathrm{Hom}(V_1', V_2)$ by $\mathfrak{D}_{12}^\bullet, \mathfrak{D}_{11'}^\bullet$ and $\mathfrak{D}_{1'2}^\bullet$, then $\mathfrak{H}^\bullet = \mathfrak{D}_{12}^\bullet \otimes \mathfrak{D}_{11'}^\bullet \otimes \mathfrak{D}_{1'2}^\bullet$. Since the canonical line bundle on an affine space carries a canonical flat connection, the algebras $\mathfrak{D}_{ij}^\bullet$ admit $(\mathrm{GL}(V_i) \times \mathrm{GL}(V_j))$-equivariant anti-involutions. Thus we can identify $\mathfrak{H}^\bullet \xrightarrow{\sim} \mathfrak{D}_{12}^\bullet \otimes (\mathfrak{D}_{11'}^\bullet \otimes \mathfrak{D}_{1'2}^\bullet)^{\mathrm{op}}$.

The algebra $\mathfrak{H}^\bullet$ has a cyclic holonomic left dg-module $\mathfrak{Q}^\bullet$ of 'delta-functions along the subvariety cut out by the equation $A = A'A''$'; see (3.4.1). We also consider the $\mathrm{GL}(V_1) \times \mathrm{GL}(V_1')$-equivariant cyclic $\mathfrak{D}_{11'}^\bullet$-module $\mathfrak{D}_{11'0}^\bullet$ corresponding to the $\mathcal{D}_{11'} := \mathcal{D}_{\mathrm{Hom}(V_1, V_1')}$-module given by

$$\mathcal{D}_{11'}/(\mathcal{D}_{11'} \otimes \det V_1 \otimes \det^{-1} V_1')(\det A'') = \mathrm{Ind}_{\mathcal{O}_{\mathrm{Hom}(V_1, V_1')}}^{\mathcal{D}_{11'}}(\mathcal{O}_{\mathrm{Hom}_{\leq N-1}(V_1, V_1')}).$$

We define the following endofunctors of $D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet)$:

$$D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet) = D_{\mathrm{perf}}^{\mathrm{GL}(V_1') \times \mathrm{GL}(V_2)}(\mathfrak{D}_{1'2}^\bullet) \ni M \mapsto p_* q^* M$$
$$:= \left(\mathfrak{Q}^\bullet \otimes_{\mathfrak{D}_{11'}^\bullet \otimes \mathfrak{D}_{1'2}^\bullet} (\mathfrak{D}_{11'}^\bullet \otimes M)\right)^{\mathrm{GL}(V_1')} \in D_{\mathrm{perf}}^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)}(\mathfrak{D}_{12}^\bullet) = D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet),$$

$$D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet) = D_{\mathrm{perf}}^{\mathrm{GL}(V_1') \times \mathrm{GL}(V_2)}(\mathfrak{D}_{1'2}^\bullet) \ni M \mapsto p_{0*} q_0^* M$$
$$:= \left(\mathfrak{Q}^\bullet \otimes_{\mathfrak{D}_{11'}^\bullet \otimes \mathfrak{D}_{1'2}^\bullet} (\mathfrak{D}_{11'0}^\bullet \otimes M)\right)^{\mathrm{GL}(V_1')} \in D_{\mathrm{perf}}^{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)}(\mathfrak{D}_{12}^\bullet) = D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet).$$

Then, similarly to Proposition 3.12.1, one proves the following proposition.

PROPOSITION 5.3.1. (a) *There is an isomorphism of functors*

$$\bar{\jmath}_{0*}\bar{\jmath}_0^! \circ \Phi_\hbar^{1,1} \simeq \Phi_\hbar^{1,1} \circ p_* q^* \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet) \to D_{!*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}).$$

(b) *There is an isomorphism of functors*

$$\jmath_{0*}\jmath_0^! \circ \Phi_\hbar^{1,1} \simeq \Phi_\hbar^{1,1} \circ p_{0*} q_0^* \colon D_{\mathrm{perf}}^{G_{\bar{0}}}(\mathfrak{D}_{1,1}^\bullet) \to D_{!*\mathbf{G_O} \rtimes \mathbb{C}^\times}(\mathbf{Gr} \times \mathbf{V}).$$

Now, using Proposition 5.3.1 in place of Proposition 3.12.1, one checks the monoidal properties of $\Phi_\hbar^{2,0}$ and $\Phi_\hbar^{0,2}$ similarly to the monoidal properties of $\Phi^{2,0}$ and $\Phi^{0,2}$ checked in §§3.14 and 3.15, respectively. This completes the proof of Theorem 5.1.1.

5.4 $\mathfrak{D}$-modules and $D_{\mathbf{GL}(N-1,\mathbf{O}) \rtimes \mathbb{C}^\times}(\mathbf{Gr})$

Similarly to § 5.1, we consider the dg-algebra (with trivial differential) $\bar{\mathfrak{D}}^\bullet$ of 'graded differential operators' on $\mathrm{Hom}(V_2, \bar{V}_1)$. Then, similarly to Theorem 4.1.1, one proves the following theorem.

THEOREM 5.4.1. *There exists an equivalence of triangulated categories* $\bar{\Phi}_\hbar \colon D^{\bar{G}_{\bar{0}}}_{\mathrm{perf}}(\bar{\mathfrak{D}}^\bullet) \xrightarrow{\sim} D_{\bar{\mathbf{G}}_{\mathbf{O}} \rtimes \mathbb{C}^\times}(\mathbf{Gr})$ *commuting with the left convolution action of the monoidal spherical Hecke category* $\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O}) \rtimes \mathbb{C}^\times}(\mathbf{Gr}_{N-1}) \cong \mathrm{Rep}(\mathrm{GL}_{N-1})$ *and with the right convolution action of the monoidal spherical Hecke category* $\mathrm{Perv}_{\mathrm{GL}(N,\mathbf{O}) \rtimes \mathbb{C}^\times}(\mathbf{Gr}_N) \cong \mathrm{Rep}(\mathrm{GL}_N)$.

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