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INCORPORATING OPERATIONAL CHARACTERISTICS AND START-UP COSTS IN OPTION-BASED VALUATION OF POWER GENERATION CAPACITY

Shi-Jie Deng

School of Industrial & Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332
E-mail: deng@isye.gatech.edu

Shmuel S. Oren

Industrial Engineering and Operations Research
University of California at Berkeley
Berkeley, CA 94720
E-mail: oren@ieor.berkeley.edu

We describe a stochastic dynamic programming approach for "real option"-based valuation of electricity generation capacity incorporating operational constraints and start-up costs. Stochastic prices of electricity and fuel are represented by recombining multinomial trees. Generators are modeled as a strip of cross-commodity call options with a delay and a cost imposed on each option exercise. We illustrate implications of operational characteristics on the valuation of generation assets under different modeling assumptions about the energy commodity prices. We find that the impacts of operational constraints on real asset valuation are dependent on both the model specification and the nature of operating characteristics.

1. INTRODUCTION

The restructuring of the electric power industry has become a global trend since the early 1990s. As a result, electricity markets emerged in many regions and countries. In the United States, for instance, electricity wholesale markets have been estab-

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lished in California, Pennsylvania, New Jersey, Maryland (PJM), New York, and New England. In the emerging power markets, one of the crucial issues is the determination of the market-based value of generation capacity in a competitive market environment with volatile electricity prices. The importance of capacity valuation is underscored by the needs of many large utility companies required to divest generation assets in order to ensure competition. Such a valuation is also essential for investors and market participants contemplating investment in or acquisition of new generation assets.

Under the traditional regulatory regime, electricity prices were set by the regulators based on cost of service. Investments in generation capacity by the utilities were subject to approval by the regulators based on integrated resource planning and, upon approval, were allowed to earn a fixed return on investment through regulated electricity tariffs. The economic viability of such investment opportunities could be determined by means of a discounted cash flow (DCF) method. Under this approach, the DCF analysis is coupled with a production simulation model that produces the projected cash flow of the generation unit under consideration given the resource portfolio and the forecasted load. However, this paradigm is being changed by the restructuring of the electricity supply industries. Electricity prices in many regions, at least at the wholesale level, are no longer set by policy makers but rather by market forces. It has also been recognized in literature (e.g., Dixit and Pindyck [5]) that the traditional DCF method tends to undervalue assets in the presence of uncertainty since that approach tends to ignore the value of real options, such as turning off a plant when the price is too low. In the presence of well-developed financial and physical markets for electricity, the payoffs of an electric power plant can be modeled in terms of a financial instruments on electricity. Financial methods can be applied to value the financial instruments and, thus, the power plant. In Deng, Johnson, and Sogomonian [4], a real options approach is proposed to value electricity generation assets. In particular, they construct a spark-spread option-based valuation for fossil-fuel power plants. They demonstrate that the option-based valuation provides a much better approximation to the observed market valuation than does DCF valuation. However, some operational characteristics, such as start-up costs, ramp-up constraints, and operating-level-dependent heat rate, are not explicitly taken into consideration in their work.

Although it is important to identify and account for the embedded real options in valuing generation assets, it is of equal importance to recognize that physical operating characteristics of a real asset often impose restrictions on exercising these embedded options. The constraints on exercising the real options translate into transaction costs borne by the asset owner, thus reducing the asset value. Ignoring operating characteristics in the valuation of a real asset would almost certainly lead to overvaluation. In a typical power asset sales transaction such as the one completed in April 1999 between Pacific Gas & Electric and Southern Energy which totaled $801 million, even a 1% overvaluation would cause a loss of millions of dollars from a purchaser's point of view. It is therefore important to account for operational constraints when applying financial option pricing methodology to value real assets.

In this article, we explicitly incorporate operational characteristics associated with a power plant into the real options valuation approach. The methodology that we employ is to formulate a stochastic dynamic program (SDP) for the asset valuation problem based on a discrete-time lattice price model. This approach has its root in the binomial option pricing model developed by Cox, Ross, and Rubinstein [2]. Tseng and Barz [12] have pursued, independently, a similar approach that focuses on the short-term-generation asset valuation problem. They simulate power prices and solve a unit commitment problem with constraints such as start-up and shutdown costs, minimum run time, and maximum ramp rate over a relatively short time horizon. That approach, however, is computationally infeasible for the long-term asset valuation problem, which we address, with a time horizon of years and granularity of days.

Another task of this article is to investigate the interaction between different modeling assumptions concerning the commodity price models and the effects of operational characteristics in valuing real assets. We take the classic geometric Brownian motion price model and examine the asset valuation problem with operational constraints and then compare the valuation results with those obtained under mean-reversion price models. We find that the significance of overvaluation resulting from ignoring operational characteristics varies under different assumptions regarding the price processes for electricity and for the generating fuel.

The remainder of the article is organized as follows. We first describe an asset valuation problem for a fossil-fuel power-generating asset incorporating operating characteristics in a deregulated electric power industry in Section 2. We highlight several key characteristics that we take into consideration in the asset valuation problem. In Section 3, we construct approximations to two different continuous-time price models for electricity and the generating fuel by using discrete-time multinomial lattice processes. We then develop a stochastic dynamic programming model based on the lattice price processes to incorporate operational constraints into the valuation problem and prove some structural properties of the solutions to the SDP. In Section 4, we present results from numerical experiments to illustrate how significant each of the operational characteristics of a power plant is in terms of affecting the valuation result at different operating efficiency levels. We further demonstrate that the significance of such impacts on power asset valuation by operating characteristics is sensitive to the assumptions on price dynamics of electricity and the generating fuel. Finally, we conclude with observations and remarks.

2. PROBLEM DESCRIPTION

With electricity markets established in more and more regions and countries, market force urges participants of power markets to develop market-based approaches for the valuation of power assets, such as generation and transmission assets. Whereas financial economic theories provide useful tools for capturing the embedded option value of such assets, we note that physical assets differ from financial assets in several important aspects. First, although providing similar benefits to the owner, a

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physical asset usually involves more significant transaction costs than does a financial asset. Second, the value of the optionality associated with operating a physical asset at different time epochs is often interrelated through intertemporal operational constraints. This fact makes the closed-form financial option pricing formulas overly simplistic approximations of the operational option values. Therefore, it is important for us to explicitly take into account the operational characteristics when constructing an option-value-based approach for valuing real assets.

In the context of a deregulated power industry, financial option pricing theory recently has been applied in the valuation of fossil-fuel electricity generation assets. A fossil-fuel power plant converts a generating fuel into electricity at a certain conversion rate, which is termed *heat rate*. Roughly speaking, heat rate measures the number of units of the fuel needed for generating one unit of electricity. The owner of a merchant power plant (i.e., a power plant sells its output into at least one spot market) has the right but not the obligation to generate electricity by burning fuel at any point in time during the lifetime of the power plant. Upon executing such operational rights over time, the owner receives the spot price of electricity less the heat-rate-adjusted generating fuel cost by selling/purchasing electricity/fuel at spot market prices, respectively. A rational power plant owner would only exercise the operational right at time t when the electricity price less generating fuel cost is positive at that time. Recall that a *spark spread call option* is an option that yields its holder the positive part of electricity price less the "strike" heat-rate-adjusted fuel price at its maturity time. Therefore, the payoff obtainable to a rational merchant power plant owner at time t is the same as that of a properly structured *spark spread call option* with strike heat rate being set at the operating heat rate level of the power plant. This observation leads to a spark spread option-based valuation of a fossil-fuel power plant which values the underlying plant by summing up the value of the corresponding set of spark spread call options with maturity time spanning the lifetime of the plant. It is demonstrated that such a spark spread option-based valuation provides a much better approximate to the observed market valuation than does DCF valuation (e.g., Deng et al. [4]).

The financial-option-based valuation approach makes simplifying assumptions regarding the operational characteristics of a power plant. It assumes that a power plant can be instantly turned on or shut down, there are no fixed operating costs but only variable production costs involved in the operations of a power plant, and the operating efficiency of a power plant is at a constant level. However, these assumptions are not very realistic. In operating a fossil-fuel power plant, many operational characteristics can potentially affect the flexibility (viz. optionality) of the power plant (e.g., Wood and Wollenberg [13]). We elaborate on three of them. First, fixed costs are usually incurred whenever a power plant is turned on from the "off" state. For a steam-generating unit, for instance, water in the boiler needs to be boiled before the unit can generate electricity, and the amount of fuel required to boil the water often depends on how long the unit has been shut down; that is, *start-up costs* are involved in the process of turning a power-generating unit on and the costs could be time dependent. Sometimes, there are also costs associated with the process of shutting down a power plant which are called *shutdown costs*. Second, upon turning on a power-generating unit (in general, a power plant often has several generating units, but for the ease of exposition, we assume that a power plant only has one generating unit), we usually do not get the output electricity immediately because a short period of time (e.g., the time for boiling water in the boiler) is needed for the generating unit to start from the "off" state and reach certain operating output levels. This time is often called the *ramp-up time*. Third, regarding the operating efficiency of a power plant, the converting rate at which a power plant transforms the generating fuel into electricity indeed differs with output levels. This converting rate is called *operating heat rate*. The power plant is more efficient when being operated at the rated full-capacity level than at a low-output level. Thus, the operating heat rate of a power plant is a function of its output level. We will explicitly incorporate these operational characteristics of a fossil-fuel power plant into its valuation and explore the effects of them on the valuation.

In principle, one can formulate the operation of a power plant incorporating all operational characteristics as a full-fledged dynamic programming problem. However, the computational complexity makes such an approach prohibitively difficult to implement. What we choose to do is to model the above characteristics under simplifying assumptions. Specifically, we model the start-up/shutdown cost, ramp-up time, and output-dependent operating heat rate as follows:

- **Start-up/shutdown cost:** We assume that fixed costs c_{start} and c_{down} are incurred each time a power plant is turned on and off, respectively. The cost to start up a generating unit depends on how long the unit has been turned off (i.e., the longer the unit is off, the more heat is dissipated from its boiler, thus a higher cost would be incurred when reheating the water); we simplify this effect assuming that c_{start} is a constant.
- **Ramp-up time:** Similar to the case of start-up cost, the length of the ramp-up time also depends on how long the power plant has been off. To reflect this aspect to first order, we approximate the ramp-up time by assuming that whenever a power plant is turned on from the "off" state, there is a fixed delay time of length D between that turn-on point and the time point at which usable electricity is generated. Moreover, during the ramp-up period, there is a cost incurred at a rate of c_r dollars per unit time, which is generally a function of the cost of the fuel burned to ramp up the plant.
- **Output-dependent operating heat rate:** While a power plant is in operation, its operating efficiency measured by its operating heat rate varies with the output level; namely the operating heat rate is output dependent rather than a constant over time. When operated at its rated maximum capacity level, the power plant is very efficient (i.e., operating heat rate is at the low end of the heat rate range); when operated at its rated minimum capacity level, the power plant is very inefficient (i.e., operating heat rate is at the high end of the heat rate range). The operating heat rate of a generating unit is often modeled as a quadratic function of the electricity output quantity (e.g., see Wood and

Wollenberg [13]). To approximate this dependency, we make a simplifying assumption on output level and the operating heat rate. Specifically, we assume that a power plant has only two possible output levels (this can be easily generalized to the case with n possible output levels): one being the rated capacity level $\bar{Q}$ per unit of time, called maximum output level, with an operating heat rate of $\overline{Hr}$; the other one being the minimum capacity level $\underline{Q}$ ($\underline{Q} < \bar{Q}$) per unit of time (i.e., the minimum output level possible in order to keep a power plant being operational) with a corresponding heat rate of $\underline{Hr}$. We make $0 < \overline{Hr} \le \underline{Hr}$ to reflect the fact that a fossil-fuel power plant is more efficient when operated in a high-output level than in a low-output level. We also assume that the switching between the maximum capacity level and the minimum capacity level is instantaneous and costless.

With the above assumptions, we proceed to the formulation of a stochastic dynamic programming problem for the valuation of power generation capacity.

3. A STOCHASTIC DYNAMIC PROGRAMMING FORMULATION

As a common feature in almost all commodity prices, mean reversion appears in energy prices as well (e.g., Schwartz [10]). In addition to mean reversion, electricity prices also exhibit phenomena such as jumps, spikes, and stochastic volatility (e.g., Deng [3]). However, in this article, we model the mean-reversion aspect of the electricity price only. More specifically, we investigate the effects of operational characteristics on valuation of generation capacity under the assumption of mean-reverting electricity price similar to those made in Deng et al. [4].

Let the state space be in $\mathbb{R}^2$, representing the logarithm of the prices of the two underlying commodities. Let $\widetilde{X}_t$ and $\widetilde{Y}_t$ denote the natural logarithm of the prices of electricity and the generating fuel ($\ln S_t^e$, $\ln S_t^g$), respectively. From here on, we use natural gas as one example of the generating fuel, but the assumptions on the generating fuel price are also applicable to other fossil fuels such as coal. We assume that $\widetilde{X}_t$ and $\widetilde{Y}_t$ evolve according to two correlated continuous-time stochastic processes defined by the following stochastic differential equations (SDEs):

$$\begin{aligned} d\widetilde{X}_t &= \kappa_1(t)(\theta_1(t) - \widetilde{X}_t)dt + \sigma_1(t)dW_t^1, \\ d\widetilde{Y}_t &= \kappa_2(t)(\theta_2(t) - \widetilde{Y}_t)dt + \sigma_2(t)dW_t^2, \end{aligned} \tag{1}$$

where $\kappa_i(t)$ $(i = 1,2)$ is the mean-reversion coefficient, $\theta_i(t)$ $(i = 1,2)$ is the long-term mean function, $\sigma_i(t)$ $(i = 1,2)$ is the instantaneous volatility function, and W_t^1 and W_t^2 are two correlated standard Brownian motions with instantaneous correlation $\rho(t)$ and $\rho(t)dt = \mathrm{Cov}(dW_t^1, dW_t^2)$.

One can formulate the asset valuation problem as a stochastic dynamic program based on the continuous-time stochastic price processes $\{\widetilde{X}_t, \widetilde{Y}_t : t \ge 0\}$ and take into consideration the operational constraints. However, such an approach would encounter difficulty when trying to solve the Hamilton–Jaccobi–Bellman equations because of the operational constraints and the fact that action space (which will be

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defined in Sect. 3.2) is a discrete set rather than a continuous set. We choose the approach of discretizing the continuous-time price processes $\{(\tilde{X}_t, \tilde{Y}_t) : t \geq 0\}$ into a recombining lattice process denoted by $\{(X_t, Y_t) : t = t_0, t_1, t_2, t_3, \ldots\}$ with $t_0 = 0$. The size of the state space for $\{X_t, Y_t\}$ grows only as a polynomial function of the number of time steps. We then formulate the valuation problem of a generation asset as a discrete-time stochastic dynamic programming problem incorporating operational constraints involving start-up costs, ramping-up time, and different heat rates under different output levels.

We start with the construction of the discrete-time price processes and then present the model formulation.

3.1. Construction of the Discrete Price Processes

As one of our goals is to investigate the effects of price process assumption on asset valuation, we construct discrete-time log-price processes for two types of continuous-time price processes: a Brownian motion process and a simple mean-reverting process. A multitude of existing literature in finance has addressed the issue of discretizing two or several correlated geometric Brownian motions (e.g., Boyle [1] and He [7]). Li and Kouvelis [8] present a discretization of one mean-reverting process. We provide an extension in Section 3.1.2 to discretize two correlated mean-reverting processes.

We consider a time horizon which starts at 0 and ends at time T. We divide the interval $[0, T]$ into N subintervals, $[0, t_1], (t_1, t_2], \ldots, (t_{N-1}, t_N \equiv T]$ of equal length $\Delta t \equiv T/N$. We assume that the states of the price processes change value only at t_i $(i = 1, 2, \ldots, N)$ and the state vector (X_t, Y_t) takes on a finite set of values. With the understanding that (X_i, Y_i) denotes (X_{t_i}, Y_{t_i}) $(i = 0, 1, 2, \ldots, N)$, we rewrite the processes $\{(X_t, Y_t) : t = t_0, t_1, \ldots, t_N\}$ as $\{(X_i, Y_i) : i = 0, 1, \ldots, N\}$. By properly defining the states and the state transition probabilities, we are able to show that the corresponding discrete-time Markov process $\{(X_t, Y_t)\}$ converges in distribution to either the geometric Brownian motion processes [Eqs. (2)] or the mean-reverting processes [Eqs. (1)].

3.1.1. Brownian motion process. Suppose $\{(\tilde{X}_t, \tilde{Y}_t) : t \geq 0\}$ are two correlated Brownian motions with constant coefficients for mean (μ_1, μ_2) and volatility (σ_1, σ_2), and correlation ρ as defined in the following SDEs:

$$\begin{aligned} d\tilde{X}_t &= \mu_1\, dt + \sigma_1\, dW_t^1, \\ d\tilde{Y}_t &= \mu_2\, dt + \sigma_2\, dW_t^2, \end{aligned} \tag{2}$$

where W_t^1 and W_t^2 are two correlated standard Brownian motions with an instantaneous correlation ρ [viz. $\rho\, dt = \mathrm{Cov}(dW_t^1, dW_t^2)$].

We construct a discrete-time Markov vector process as a recombining trinomial lattice; that is, starting from each log-price state vector (X_t, Y_t) at time t $(t = 0, 1, 2, \ldots, N-1)$, there are three possible states to reach at time $t+1$ as illustrated

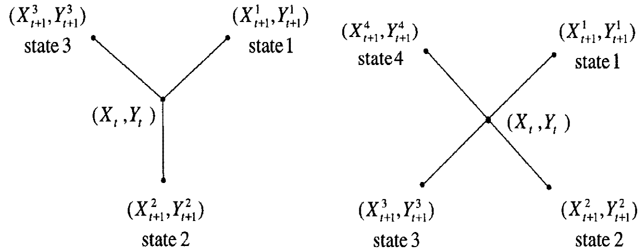

FIGURE 1. Lattice price models: trinomial tree (left panel) versus quadrinomial tree (right panel).

in the left panel of Figure 1. The values of the three possible states (X^j_{t+1}, Y^j_{t+1}) $(j = 1,2,3)$ are given as follows:

$$X^j_{t+1} = \begin{cases} X_t + \mu_1 \Delta t + \sigma_1 \sqrt{\frac{3}{2}} \sqrt{\Delta t} & (j=1) \\ X_t + \mu_1 \Delta t & (j=2) \\ X_t + \mu_1 \Delta t - \sigma_1 \sqrt{\frac{3}{2}} \sqrt{\Delta t} & (j=3), \end{cases}$$

$$Y^j_{t+1} = \begin{cases} Y_t + \mu_2 \Delta t + \rho\sigma_2 \sqrt{\frac{3}{2}} \sqrt{\Delta t} + \sigma_2 \sqrt{1-\rho^2} \sqrt{\frac{1}{2}} \sqrt{\Delta t} & (j=1) \\ Y_t + \mu_2 \Delta t - \sigma_2 \sqrt{1-\rho^2} \frac{2}{\sqrt{2}} \sqrt{\Delta t} & (j=2) \\ Y_t + \mu_2 \Delta t - \rho\sigma_2 \sqrt{\frac{3}{2}} \sqrt{\Delta t} + \sigma_2 \sqrt{1-\rho^2} \sqrt{\frac{1}{2}} \sqrt{\Delta t} & (j=3), \end{cases} \tag{3}$$

where μ_1, μ_2, σ_1, σ_2, and ρ are parameters in (2). Define the state transition probability $P_t \equiv (p^1_t, p^2_t, p^3_t)$ from state (X_t, Y_t) to state (X_{t+1}, Y_{t+1}) as follows:

$$p^1_t = p^2_t = p^3_t = \frac{1}{3}, \qquad t = 0,1,2,\ldots,N-1 \tag{4}$$

where p^j_t is the probability of going from state (X_t, Y_t) to state (X^j_{t+1}, Y^j_{t+1}) $(j = 1,2,3)$.

Let Λ_n denote the time-n state space of the Markov process $\{(X_n, Y_n) : n = 0,1,\ldots,N\}$. Then Λ_n is given by

$$\Lambda_n \equiv \bigcup_{i=0}^{n} \left\{ (X_{i,j}, Y_{i,j}) : \begin{cases} X_{i,j} = X_0 + n\mu_1 \Delta t - (i-2j)\Delta_X \\ Y_{i,j} = Y_0 + n\mu_2 \Delta t - \left(\frac{2n-3i}{\sqrt{2}} \sqrt{1-\rho^2} + \frac{(i-2j)\sqrt{3}}{\sqrt{2}} \rho \right) \Delta_Y \end{cases} j = 0,1,\ldots,i \right\} \tag{5}$$

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where $\Delta_X = \sigma_1\sqrt{\frac{3}{2}}\sqrt{\Delta t}$ and $\Delta_Y = \sigma_2\sqrt{\Delta t}$. It has been shown [7] that the processes $\{(X_n,Y_n): n = 0,1,\ldots,N\}$ defined converge in distribution to the Brownian motion processes (2) with initial condition $(\widetilde{X}_t,\widetilde{Y}_t) = (X_0,Y_0)$ as $N \to \infty$.

3.1.2. Mean-reverting process. Similar to the Brownian motion case, we use a recombining quadrinomial lattice process to approximate a mean-reverting process. For the ease of exposition, we start with taking parameters $\kappa_1(t)$, $\kappa_2(t)$, $\theta_1(t), \theta_2(t), \sigma_1(t), \sigma_2(t)$, and $\rho(t)$ in (1) to be constants. (X_t,Y_t) denotes the log-price state vector at time t $(t = 0,1,\ldots,N)$. Following (X_t,Y_t), there are four possible states (X^j_{t+1},Y^j_{t+1}) $(j = 1,2,3,4)$ at time $t+1$ $(t = 0,1,\ldots,N-1)$ as shown in the right panel of Figure 1. The values of the four states (X^j_{t+1},Y^j_{t+1}) $(j = 1,2,3,4)$ are

$$(X_{t+1},Y_{t+1}) = \begin{cases} (X^1_{t+1},Y^1_{t+1}) = (X_t + \sigma_1\sqrt{\Delta t},\, Y_t + \sigma_2\sqrt{\Delta t}) \\ (X^2_{t+1},Y^2_{t+1}) = (X_t + \sigma_1\sqrt{\Delta t},\, Y_t - \sigma_2\sqrt{\Delta t}) \\ (X^3_{t+1},Y^3_{t+1}) = (X_t - \sigma_1\sqrt{\Delta t},\, Y_t - \sigma_2\sqrt{\Delta t}) \\ (X^4_{t+1},Y^4_{t+1}) = (X_t - \sigma_1\sqrt{\Delta t},\, Y_t + \sigma_2\sqrt{\Delta t}). \end{cases} \quad \textbf{(6)}$$

The state transition probabilities $\{p^1_t,p^2_t,p^3_t,p^4_t\}$ from state (X_t,Y_t) to state (X_{t+1},Y_{t+1}) are chosen to match the local first and second moments of (1), namely

$$\begin{aligned} E[X_{t+1} - X_t|(X_t,Y_t)] &= \kappa_1(\theta_1 - X_t)\Delta t, \\ E[Y_{t+1} - Y_t|(X_t,Y_t)] &= \kappa_2(\theta_2 - Y_t)\Delta t, \\ E[(X_{t+1} - X_t)^2|(X_t,Y_t)] &= \sigma_1^2\Delta t, \\ E[(Y_{t+1} - Y_t)^2|(X_t,Y_t)] &= \sigma_2^2\Delta t, \\ E[(X_{t+1} - X_t)(Y_{t+1} - Y_t)|(X_t,Y_t)] &= \rho\sigma_1\sigma_2\Delta t + o(\Delta t), \end{aligned} \quad \textbf{(7)}$$

where κ_1, κ_2, θ_1, θ_2, σ_1, σ_2, and ρ are parameters in (1). Specifically, $\{p^j_t : j = 1,2,3,4\}$ solves the following system of equations, where p^j_t is the probability of moving from (X_t,Y_t) to (X^j_{t+1},Y^j_{t+1}) $(j = 1,2,3,4)$:

$$\begin{aligned} p^1_t + p^2_t + p^3_t + p^4_t &= 1, \\ (p^1_t + p^2_t - p^3_t - p^4_t)\sigma_1\sqrt{\Delta t} &= \kappa_1(\theta_1 - X_t)\Delta t, \\ (p^1_t - p^2_t - p^3_t + p^4_t)\sigma_2\sqrt{\Delta t} &= \kappa_2(\theta_2 - Y_t)\Delta t, \\ (p^1_t + p^2_t + p^3_t + p^4_t)\sigma_1^2\Delta t &= \sigma_1^2\Delta t, \\ (p^1_t + p^2_t + p^3_t + p^4_t)\sigma_2^2\Delta t &= \sigma_2^2\Delta t, \\ (p^1_t - p^2_t + p^3_t - p^4_t)\sigma_1\sigma_2\Delta t &= \rho\sigma_1\sigma_2\Delta t + o(\Delta t), \end{aligned} \quad \textbf{(8)}$$

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where $o(\Delta t) = [\kappa_1(\theta_1 - X_t)\Delta t][\kappa_2(\theta_2 - Y_t)\Delta t]$. The solution to (8) is

$$\begin{aligned} p_t^1 &= \frac{1+\rho}{4} + \left[\frac{\kappa_1(\theta_1 - X_t)}{4\sigma_1} + \frac{\kappa_2(\theta_2 - Y_t)}{4\sigma_2}\right]\sqrt{\Delta t} + \frac{\kappa_1(\theta_1 - X_t)\kappa_2(\theta_2 - Y_t)}{4\sigma_1\sigma_2}\,\Delta t, \\ p_t^2 &= \frac{1-\rho}{4} + \left[\frac{\kappa_1(\theta_1 - X_t)}{4\sigma_1} - \frac{\kappa_2(\theta_2 - Y_t)}{4\sigma_2}\right]\sqrt{\Delta t} - \frac{\kappa_1(\theta_1 - X_t)\kappa_2(\theta_2 - Y_t)}{4\sigma_1\sigma_2}\,\Delta t, \\ p_t^3 &= \frac{1+\rho}{4} - \left[\frac{\kappa_1(\theta_1 - X_t)}{4\sigma_1} + \frac{\kappa_2(\theta_2 - Y_t)}{4\sigma_2}\right]\sqrt{\Delta t} + \frac{\kappa_1(\theta_1 - X_t)\kappa_2(\theta_2 - Y_t)}{4\sigma_1\sigma_2}\,\Delta t, \\ p_t^4 &= \frac{1-\rho}{4} - \left[\frac{\kappa_1(\theta_1 - X_t)}{4\sigma_1} - \frac{\kappa_2(\theta_2 - Y_t)}{4\sigma_2}\right]\sqrt{\Delta t} - \frac{\kappa_1(\theta_1 - X_t)\kappa_2(\theta_2 - Y_t)}{4\sigma_1\sigma_2}\,\Delta t. \end{aligned} \tag{9}$$

The state space of (X_t, Y_t) is a subset of $\{(X_0 + m\sigma_1\sqrt{\Delta t}, Y_0 + n\sigma_2\sqrt{\Delta t}) : m, n = -t, -t+2, -t+4, \ldots, t-4, t-2, t\}$. We next need to determine the range of m and n for which the components in solution (9) are all between 0 and 1. When N is sufficiently large, $p_t^j \in (0,1)$ for $j = 1,2,3,4$ is equivalent to

$$\begin{aligned} 0 &\le \frac{\rho}{4} + \frac{1}{4}\left(1 - \frac{m\kappa_1 T}{N}\right)\left(1 - \frac{n\kappa_2 T}{N}\right) \le 1, \\ 0 &\le -\frac{\rho}{4} + \frac{1}{4}\left(1 - \frac{m\kappa_1 T}{N}\right)\left(1 + \frac{n\kappa_2 T}{N}\right) \le 1, \\ 0 &\le \frac{\rho}{4} + \frac{1}{4}\left(1 + \frac{m\kappa_1 T}{N}\right)\left(1 + \frac{n\kappa_2 T}{N}\right) \le 1, \\ 0 &\le -\frac{\rho}{4} + \frac{1}{4}\left(1 + \frac{m\kappa_1 T}{N}\right)\left(1 - \frac{n\kappa_2 T}{N}\right) \le 1. \end{aligned} \tag{10}$$

A sufficient set of conditions for (10) to hold is $|m| \le [(1 - \sqrt{|\rho|})/\kappa_1 T]N$ and $|n| \le [(1 - \sqrt{|\rho|})/\kappa_2 T]N$. Let $\overline{m}_t$ and $\bar{n}_t$ denote the integer parts of $\min(t, [(1 - \sqrt{|\rho|})/\kappa_1 T]N)$ and $\min(t, [(1 - \sqrt{|\rho|})/\kappa_2 T]N)$, respectively.

For all $t = 0,1,\ldots,N$, the state space of (X_t, Y_t) at time t, denoted by Λ_t, is given by the following set:

$$\Lambda_t \equiv \bigcup_{n \in \{-\bar{n}_t : 2 : \bar{n}_t\}} \{(X_0 + m\sigma_1\sqrt{\Delta t}, Y_0 + n\sigma_2\sqrt{\Delta t}) : m = -\overline{m}_t, -\overline{m}_t + 2, \ldots, \overline{m}_t - 2, \overline{m}_t\}, \tag{11}$$

where $\{-\bar{n}_t : 2 : \bar{n}_t\}$ represents the sequence of $\{-\bar{n}_t, -\bar{n}_t + 2, -\bar{n}_t + 4, \ldots, \bar{n}_t - 2, \bar{n}_t\}$.

When $t \le \min(\overline{m}_t, \bar{n}_t)$, we set the states $\{(X_{t+1}^j, Y_{t+1}^j)\}$ at $t+1$ reachable from (X_t, Y_t) according to (6) and the transition probabilities P_t according to (9) for all $(X_t, Y_t) \in M_t$. When $t > \min(\overline{m}_t, \bar{n}_t)$, if (X_t, Y_t) is in the interior of the mesh Λ_t [i.e., $X_t \in (X_0 - \overline{m}_t\sigma_1\sqrt{\Delta t}, X_0 + \overline{m}_t\sigma_1\sqrt{\Delta t})$ and $Y_t \in (Y_0 - \bar{n}_t\sigma_2\sqrt{\Delta t}, Y_0 + \bar{n}_t\sigma_2\sqrt{\Delta t})$], then we define the subsequent states $\{(X_{t+1}^j, Y_{t+1}^j)\}$ at stage $t+1$ reachable from

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(X_t, Y_t) according to (6) and the transition probabilities according to (9); if (X_t, Y_t) is on the boundary of Λ_t, then we need to increase the number of states emanating from (X_t, Y_t) and choose the corresponding set of probabilities $\{p_t^j\}$ so that (7) holds true. For instance, when $X_t = X_0 - \bar{m}_t \sigma_1 \sqrt{\Delta t}$ or $X_t = X_0 + \bar{m}_t \sigma_1 \sqrt{\Delta t}$, we increase the number of subsequent transition states (X_{t+1}^j, Y_{t+1}^j) from four to six; let

$$\begin{aligned} X_{t+1}^{2j-1} &= X_{t+1}^{2j} = X_t - \operatorname{sgn}(m)(2j-1)\sigma_1\sqrt{\Delta t} \qquad (j = 1,2,3), \\ Y_{t+1}^{2j-1} &= Y_t + \sigma_2\sqrt{\Delta t} \text{ and } Y_{t+1}^{2j-1} = Y_t - \sigma_2\sqrt{\Delta t} \quad (j = 1,2,3). \end{aligned} \tag{12}$$

The transition probabilities $\{p_t^j : j = 1,2,\ldots,6\}$ are given by the solution of

$$\begin{aligned} p_t^1 + p_t^2 + p_t^3 + p_t^4 + p_t^5 + p_t^6 &= 1, \\ (-p_t^1 - p_t^2 - 3p_t^3 - 3p_t^4 - 5p_t^5 - 5p_t^6)\sigma_1\sqrt{\Delta t} &= \kappa_1(\theta_1 - X_t)\Delta t, \\ (p_t^1 - p_t^2 + p_t^3 - p_t^4 + p_t^5 - p_t^6)\sigma_2\sqrt{\Delta t} &= \kappa_2(\theta_2 - Y_t)\Delta t, \\ (p_t^1 + p_t^2 + 9p_t^3 + 9p_t^4 + 25p_t^5 + 25p_t^6)\sigma_1^2\Delta t &= \sigma_1^2\Delta t, \\ (p_t^1 + p_t^2 + p_t^3 + p_t^4 + p_t^5 + p_t^6)\sigma_2^2\Delta t &= \sigma_2^2\Delta t, \\ (-p_t^1 + p_t^2 - 3p_t^3 + 3p_t^4 - 5p_t^5 + 5p_t^6)\sigma_1\sigma_2\Delta t &= \rho\sigma_1\sigma_2\Delta t + o(\Delta t). \end{aligned} \tag{13}$$

Notice that we can manage to have all p_t^j's to be between 0 and 1 since there are six unknowns and five nonidentical equations in (13).We construct the states $\{(X_{t+1}^j, Y_{t+1}^j)\}$ and the probabilities $P_t \equiv \{p_t^j : j = 1,2,\ldots,J\}$ in the same manner when Y_t takes the boundary values of Λ_t. Through this construction, we obtain a Markov chain $\{(X_t, Y_t) : t = 0,1,2,\ldots,N\}$ with transition probability $\{P_t : t = 0,1,2,\ldots,N\}$ satisfying (7) in every state (X_t, Y_t) for all t.

Proposition 3.1 provides a set of sufficient conditions for a continuous-time Markov chain, which has sample paths being right-continuous with left limit (RCLL), to converge in distribution to the strong solution of a system of SDEs.

Suppose $\bar{X}_t$ is the strong solution of SDEs $d\bar{X}_t = b(\bar{X}_t)dt + \sigma(\bar{X}_t)dW_t$ with $\bar{X}_0 = \bar{x}_0 \in R^n$, where W_t is a standard Brownian motion in R^n; $b(\bar{X}_t)$ and $\sigma(\bar{X}_t)$ are $n \times 1$ and $n \times n$ matrices, respectively. Let $a \equiv (a^{ij}(\bar{X}_t))_{n\times n}$ denote the matrix $\sigma(\bar{X}_t)\sigma(\bar{X}_t)^T$. For any $h > 0$, define a Markov chain $\{\bar{Y}_{mh}^h, m = 0,1,2,\ldots\}$, taking values in $S_h \subset R^d$, with $\Pi_h(\bar{x}, d\bar{y})$ being its sequence of transition probabilities; that is,

$$P(\bar{Y}_{(m+1)h}^h \in A | \bar{Y}_{mh}^h = \bar{x}) = \Pi_h(\bar{x}, A) \quad \text{for } \bar{x} \in S_h, A \subset R^d.$$

Proposition 3.1. *Define a continuous-time Markov process $\bar{X}_t^h$ by $\bar{X}_t^h = \bar{Y}_{h[t/h]}^h$, where $[t/h]$ is the largest integer no greater than t/h (i.e., we make $\bar{X}_t^h$ constant on intervals $[mh,(m+1)h]$). Also, define $\bar{a}_{ij}^h(\bar{x}) = \int_{|\bar{y}-\bar{x}|\le 1}(y_i - x_i)(y_j - x_j)\Pi_h(\bar{x}, d\bar{y})$ and $\bar{b}_i^h(\bar{x}) = \int_{|\bar{y}-\bar{x}|\le 1}(y_i - x_i)\Pi_h(\bar{x}, d\bar{y})$. Let $Z_m(\bar{Y}_{mh}^h)$ denote the conditional random variable $(\bar{Y}_{(m+1)h}^h - \bar{Y}_{mh}^h | \bar{Y}_{mh}^h)$ for $m = 0,1,2,\ldots$.*

If for each i, j, and $\varepsilon > 0$,

(i) $\bar{a}^h_{ij}(\bar{x}) = a_{ij}(\bar{x})h + o(h)$
(ii) $\bar{b}^h_i(\bar{x}) = b_i(\bar{x})h + o(h)$
(iii) $|Z_m(\overline{Y}^h_{mh})|$ *is bounded by some deterministic function* $z(h)$ *with probability* 1, $\forall \overline{Y}^h_{mh} \in S_h$, $m = 0,1,2,\ldots$, *moreover,* $\lim_{h\downarrow 0} z(h) = 0$
(iv) $\overline{X}^h_0 = \bar{x}_0$,

then we have $\overline{X}^h_t$ *converging in distribution to* $\overline{X}_t$ *with* $\overline{X}_0 = \bar{x}_0$ *as* $h \to 0$.

PROOF. See Appendix A. ■

The fact that the processes $\{(X_t, Y_t) : t = 0,1,\ldots,N\}$, defined by (6) and (9) in the interior of Λ_t and properly defined on the boundary of Λ_t for all t, converge in distribution to the corresponding mean-reverting processes is then a corollary to Proposition 3.1.

COROLLARY 3.2. *Consider the continuous-time mean-reversion processes* $\{(\widetilde{X}_t, \widetilde{Y}_t) : 0 \le t \le T\}$ *which are the strong solution of the SDEs (1) with constant parameters* $\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2$, *and* ρ *and initial value* $(\widetilde{X}_0, \widetilde{Y}_0) = (x_0, y_0)$. *Suppose* $|\rho| < 1$. *Then, the Markov processes* $\{(X_t, Y_t) : t = 0,1,\ldots,N\}$ *defined by (6) and (9) in the interior of* Λ_t *and properly defined on the boundary of* Λ_t *with* $(X_0, Y_0) = (x_0, y_0)$ *converge in distribution to* $\{(\widetilde{X}_t, \widetilde{Y}_t) : 0 \le t \le T\}$ *as* $\Delta t \to 0$.

PROOF. Since the parameters κ_1, κ_2, θ_1, θ_2, σ_1, σ_2, and ρ are constants, the strong solution to the SDEs (1) exists for any initial value $(\widetilde{X}_0, \widetilde{Y}_0) = (x_0, y_0)$. As long as we can verify that the Markov process $\{(X_t, Y_t) : t = 0,1,\ldots,N\}$ defined by (6) and (9) satisfies the four conditions in Proposition 3.1, the claim of this corollary is true by applying that proposition. Let $h = \Delta t = T/N$. Without loss of generality, consider $\Delta t \ll 1$. By the construction of $\{(X_t, Y_t) : t = 0,1,\ldots,N\}$ through (6) and (9), or (12) and (13), we know that conditions (i), (ii), and (iv) in Proposition 3.1 are satisfied. Moreover, $|(X_{t+1} - X_t|X_t)| \le \alpha\sigma_1\sqrt{\Delta t}$ and $|(Y_{t+1} - Y_t|Y_t)| \le \alpha\sigma_2\sqrt{\Delta t}$ for some constant α for all t, which means that condition (iii) is also satisfied. Therefore, the convergence in distribution is established. ■

Corollary 3.2 holds true when $\kappa_1(t)$, $\kappa_2(t)$, $\theta_1(t)$, $\theta_2(t)$, $\sigma_1(t)$, $\sigma_2(t)$, and $\rho(t)$ in (1) are simple functions of t since a simple function is a piecewise constant function.

3.2. Valuation of a Power Plant with Operational Constraints

Suppose the logarithm of the electricity and the natural gas prices evolve according to the Markov processes $\{(X_t, Y_t) : t = 0,1,\ldots,N\}$ constructed in Section 3.1. Recall from Section 2 that there is a delay (or ramp-up) period of D (called the *ramp-up time*) before a power plant can output electricity after the plant is turned on from the "off" state. Without loss of generality, we assume that $K_N \equiv D/\Delta t$ is an integer. Let $w_t \in W_N \equiv \{0,1,2,\ldots,K_N\}$ denote the operational state of the power plant at time t. Then, w_t takes on $K_N + 1$ possible values:

- $w_t = 0$: This means that the power plant is in the "off" state at time t.
- $w_t = i$: For $i \in \{1,2,\ldots,K_N - 1\}$, it means that the power plant is *on* but in the ith stage of the *ramp-up* period D at time t.
- $w_t = K_N$: This means that the power plant is *on and ready* to generate electricity outputs at time t.

Whereas the value of a power plant certainly depends on (X_t, Y_t) and w_t at each time step t $(t = 0,1,\ldots,N)$, it also depends on the action taken by the power plant operator. Assume that the plant operator can only take the following three possible actions a_i $(i = \text{I,II,III})$ at time t.

- a_{I} = "full": The operator runs the power plant at full-capacity level. The plant generates $\bar{Q}\Delta t$ units of electricity in time Δt with an operating heat rate of $\overline{Hr}$ if it is not in the ramp-up period; otherwise, it generates 0 units of electricity.
- a_{II} = "low": The operator keeps the power plant running at the minimum capacity level. The plant generates $\underline{Q}\Delta t$ units of electricity in time Δt with an operating heat rate of $\underline{Hr}$ if it is not in the ramp-up period; otherwise, it generates 0 units of electricity.
- a_{III} = "off": The operator turns the power plant off from the "on" state.

The admissible control set $A_t \equiv A(X_t, Y_t, w_t)$ is $A \equiv \{a_{\text{I}}, a_{\text{II}}, a_{\text{III}}\}$ for all time t in our formulation. The operator of the power plant seeks to maximize the expected total profit of the power plant with respect to the random price vector (S_t^e, S_t^g) over the operating time horizon by making optimal decisions regarding whether to turn on or shut down the generating unit as well as how to operate the unit. Under the risk-neutral probabilities, the expected total profit of a power plant over its operating time horizon yields the value of the power plant during that time period.

Let $R_t \equiv R(a,x,y,w) : A \times R^2 \times W_N \to R^1$ denote the operating profit of the power plant during time period t in state (x,y,w) if the operator takes action a. The operational characteristics described in Section 2 are reflected in the following definitions of R_t. We assume that the ramp-up cost rate is $c_r(y)$ per unit of time, where $c_r(\cdot) : R^1 \to R^1$ is a positive increasing function.

$$
R(a_t, X_t, Y_t, 0) = \begin{cases} a_t = a_{\text{I}}: & -c_{\text{start}} - c_r(Y_t)\Delta t \\ a_t = a_{\text{II}}: & -c_{\text{start}} - c_r(Y_t)\Delta t \quad \forall (X_t, Y_t), \\ a_t = a_{\text{III}}: & 0 \end{cases}
$$

$$
R(a_t, X_t, Y_t, w) = \begin{cases} a_t = a_{\text{I}}: & -c_r(Y_t)\Delta t \\ a_t = a_{\text{II}}: & -c_r(Y_t)\Delta t \quad \forall (X_t, Y_t),\ w = 1,2,\ldots,K_N - 1, \\ a_t = a_{\text{III}}: & -c_{\text{down}} \end{cases} \tag{14}
$$

$$
R(a_t, X_t, Y_t, w) = \begin{cases} a_t = a_{\text{I}}: & \bar{Q}\Delta t[\exp(X_t) - \overline{Hr}\exp(Y_t)] \\ a_t = a_{\text{II}}: & \underline{Q}\Delta t[\exp(X_t) - \underline{Hr}\exp(Y_t)] \quad \forall (X_t, Y_t),\ w = K_N. \\ a_t = a_{\text{III}}: & -c_{\text{down}} \end{cases}
$$

The plant operator seeks to maximize the expected sum of R_t's over the life span of the power plant by choosing a series of most profitable actions $\{a_t : t = 0, 1, \ldots, N\}$ from the admissible control sets $\{A_t : t = 0, 1, \ldots, N\}$. The value of the power plant at time k $(0 \le k \le N)$, which is a function of the initial states (X_k, Y_k, w_k), is thus given by

$$V_k(x, y, w) = \max_{\{a_t \in A_t : t = k, k+1, \ldots, N\}} E_k\left[\sum_{t=k}^{N} e^{-r(t-k)\Delta t} R(a_t, X_t, Y_t, w_t) | (X_k, Y_k) = (x, y)\right], \tag{15}$$

where $E_k[\cdot | (X_k, Y_k) = (x, y)]$ is the conditional expectation operator conditioning on (X_k, Y_k) and r is the constant discount rate. The value function $V_t(x, y, w)$ can be solved through recursive equations (16), (17), and (18) by a standard backward induction algorithm for SDP problems.

- If the operating state of the power plant is "off" (viz. $w = 0$),

$$V_t(X_t, Y_t, 0) = \max_{a_t} \begin{cases} a_t = a_{\mathrm{I}}: & -c_{\mathrm{start}} - c_r(Y_t)\Delta t + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, 1)] \\ a_t = a_{\mathrm{II}}: & -c_{\mathrm{start}} - c_r(Y_t)\Delta t + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, 1)] \\ a_t = a_{\mathrm{III}}: & e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, 0)]. \end{cases} \tag{16}$$

- If the state of the power plant is in *ramp up* (i.e., $w = 1, \ldots, K_N - 1$),

$$V_t(X_t, Y_t, w) = \max_{a_t} \begin{cases} a_t = a_{\mathrm{I}}: & -c_r(Y_t)\Delta t + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, w+1)] \\ a_t = a_{\mathrm{II}}: & -c_r(Y_t)\Delta t + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, w+1)] \\ a_t = a_{\mathrm{III}}: & -c_{\mathrm{down}} + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, 0)]. \end{cases} \tag{17}$$

- If the state of the power plant is "ready" (viz. $w = K_N$),

$$V_t(X_t, Y_t, K_N) = \max_{a_t} \begin{cases} a_t = a_{\mathrm{I}}: & \bar{Q}\Delta t[\exp(X_t) - \overline{Hr}\exp(Y_t)] + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, K_N)] \\ a_t = a_{\mathrm{II}}: & \underline{Q}\Delta t[\exp(X_t) - \underline{Hr}\exp(Y_t)] + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, K_N)] \\ a_t = a_{\mathrm{III}}: & -c_{\mathrm{down}} + e^{-r\Delta t} E_t[V_{t+1}(X_{t+1}, Y_{t+1}, 0)], \end{cases} \tag{18}$$

where $E_t[\cdot]$ is just an abbreviated notation of $E_t[\cdot | (X_t, Y_t) = (x, y)]$.

The boundary conditions are

$$V_{N+1}(x, y, w) \equiv 0, \qquad \forall (x, y) \in R^2, w = 0, 1, \ldots, K_N. \tag{19}$$

3.3. Structural Property of Value Function and Optimal Policy

We start by proving a useful lemma and using it to show that, at each time step t, the value function $V_t(x, y, w)$ is continuous, and increasing in x and decreasing in y (or equivalently, increasing in $(x, -y)$) for all the states of w. We then demonstrate that if the operating profit function $R(a, x, y, w)$ satisfies certain conditions, then optimal decisions at each time t also have monotonic properties.

We begin with a few notations and definitions.

We define a partial order "$\succcurlyeq$" on R^2 as follows. For two vectors (x_1, y_1) and (x_2, y_2), we say that $(x_1, y_1) \succcurlyeq (x_2, y_2)$ if and only if $x_1 \geq x_2$ and $y_1 \leq y_2$. A Borel measurable set $U \subseteq R^2$ is called an *upper set* (or *increasing set*) if $(x, y) \in U$ whenever $(x, y) \succcurlyeq (\tilde{x}, \tilde{y})$ and $(\tilde{x}, \tilde{y}) \in U$. Let $f(x, y, w)$ be a real function defined on $\Lambda \times W$, where $\Lambda \subset R^2$ and $W \subset R^1$. $f(x, y, w)$ is increasing in (x, y, w) if $f(x, y, w) \geq f(\tilde{x}, \tilde{y}, \widetilde{w})$ whenever $(x, y) \succcurlyeq (\tilde{x}, \tilde{y})$ and $w \geq \widetilde{w}$. $f(x, y, w)$ is said to have *increasing difference* in (x, y) and w if, for any $(x, y), (\tilde{x}, \tilde{y}) \in \Lambda$ and $(x, y) \succcurlyeq (\tilde{x}, \tilde{y})$, $f(x, y, w) - f(\tilde{x}, \tilde{y}, w) \geq f(x, y, \widetilde{w}) - f(\tilde{x}, \tilde{y}, \widetilde{w})$ whenever $w \geq \widetilde{w}$.

Recall that Λ_t denotes the time-t state space of the Markov process $\{(X_t, Y_t) : t = 0, 1, \ldots, N\}$ defined by either (3) and (4), or (6) and (9).

With the help of Lemma B.1 in Appendix B, we obtain the following properties of the value function $V_t(x, y, w)$.

PROPOSITION 3.3. *At each time step t $(t = 0, 1, \ldots, N)$, $\forall w = 0, 1, \ldots, K_N$, we have $V_t(x, y, w)$ continuous in (x, y) and $V_t(x_1, y_1, w) \geq V_t(x_2, y_2, w)$ whenever $(x_1, y_1) \succcurlyeq (x_2, y_2)$.*

PROOF. See Appendix B. ■

We next turn to the discussion of the monotonic optimal decision rules. Let us define a rank order for the three actions to be

$$a_{\mathrm{I}} > a_{\mathrm{II}} > a_{\mathrm{III}}. \tag{20}$$

Let $a_t^*(x, y, w)$ denote the optimal solution of (16)–(18) at time t given the time-t state $(X_t, Y_t, w_t) = (x, y, w)$ (if the optimal solution is not unique, then we set $a_t^*(x, y, w)$ to be the largest one). We say $a_t^*(x, y, w)$ is increasing in (x, y, w) if $a_t^*(x, y, w) > a_t^*(\tilde{x}, \tilde{y}, \widetilde{w})$ whenever $(x, y) \succcurlyeq (\tilde{x}, \tilde{y})$ and $w \geq \widetilde{w}$.

PROPOSITION 3.4. *Assume that operating profit function $R(a, x, y, w)$ satisfies the following conditions:*

1. *$R(a, x, y, w)$ is increasing in (x, y, w) for all $a \in A$,*
2. *$R(a, x, y, w)$ has increasing differences in pairs of $\{a, (x, y, w)\}$, $\{x, (a, y, w)\}$, $\{y, (a, x, w)\}$, $\{w, (a, x, y)\}$, $\{(a, x), (y, w)\}$, $\{(a, y), (x, w)\}$, and $\{(a, w), (x, y)\}$,*

and, $0 \leq \rho < 1$ in (7). Then, at each time step t $(t = 0, 1, \ldots, N)$, given any $(x, y) \succcurlyeq (\tilde{x}, \tilde{y})$ where (x, y) and $(\tilde{x}, \tilde{y}) \in \Lambda_t$ (recall that Λ_t denotes the time-t state space of the Markov process $\{(X_t, Y_t) : t = 0, 1, \ldots, N\}$ generated by (3) and (4), or (6) and (9)), we have the following:

(a) $V_t(x, y, w) - V_t(\tilde{x}, \tilde{y}, w) \geq V_t(x, y, w') - V_t(\tilde{x}, \tilde{y}, w')$ whenever $w \geq w'$, $\forall w, w' \in W_N$.

(b) The optimal action $a_t^*(x, y, w)$ is increasing in (x, y, w).

PROOF. See Appendix C. ■

Remark 3.5. Condition A provides a set of sufficient conditions for function $R(a,x,y,w)$ to satisfy conditions stated in Proposition 3.4:

Condition A
1. $\bar{Q}\overline{Hr} > \underline{Q}\underline{Hr}$ and $c_r(y) \geq \underline{Q}\underline{Hr}e^y$.
2. $c_{\text{down}} = 0$.
3. $\underline{Q}\underline{Hr}(e^y - e^{\tilde{y}}) \geq c_r(y) - c_r(\tilde{y})$.

For discussion purposes, we assume that *the conditions in Proposition 3.4 are satisfied* in the remainder of this section. Proposition 3.4 says that the optimal action $a_t(x,y,w)$ at time t within any operational state w is a threshold type of control on the $X-Y$ plane with both lattice price models introduced in Section 3.1. This implies that there exist optimal action regions with boundary $B_t^a(w)$ on the $X-Y$ plane, where a denotes the action and w denotes the operational state.

When a power plant is in the *off* state (i.e., $w=0$), a turn-on boundary $B_t^{\text{on}}(0) \equiv \{(x_t^*,y_t^*): \text{for every given } x_t^*, y_t^* = \sup_{y \in \{y_t : a_t^*(x_t^*,y_t,0)=a_{\text{I}}\}} y \ (y_t^* = -\infty \text{ if the set } \{y_t : a_t^*(x_t^*,y_t,0) = a_{\text{I}}\} \text{ is empty})\}$ consists of points whose coordinates (x_t^*,y_t^*) are such that for each x_t^*, the corresponding y_t^* is the largest y_t for which the optimal action $a_t^*(x_t^*,y_t,0)$ is a_{I} (viz. to turn a plant from *off* to *on*). (Note that there is no difference between actions a_{I} and a_{II} in states $w=0,1,\ldots,K_N$ because of our assumption about no cost or time delay in switching between a_{I} and a_{II}. We set the optimal action a^* to be a_{I} whenever $a^* = a_{\text{I}} = a_{\text{II}}$ since $a_{\text{I}} > a_{\text{II}}$.) The optimal action $a_t^*(x,y,0)$ at any point (x,y) in the state space Λ_t can be inferred from the relative position of (x,y) with respect to $B_t^{on}(0)$: $a_t^*(x,y,0)$ is to turn on the plant (a_{I}) (or keep the plant in the *off* state (a_{III})) if and only if there exists $(x_t^*,y_t^*) \in B_t^{\text{on}}(0)$ such that $(x,y) \succcurlyeq (x_t^*,y_t^*)$ (or $(x_t^*,y_t^*) \succcurlyeq (x,y)$).

Similarly, the turn-off boundary $B_t^{\text{off}}(w)$ in a *ramp-up* state w ($w = 1,2,\ldots,K_N-1$) is given by the set $\{(x_t^*,y_t^*): \text{for every given } x_t^*, y_t^* = \inf_{y \in \{y_t : a_t^*(x_t^*,y_t,w)=a_{\text{III}}\}} y \ (y_t^* = +\infty \text{ if the set } \{y_t : a_t^*(x_t^*,y_t,w) = a_{\text{III}}\} \text{ is empty})\}$. The optimal action $a_t^*(x,y,w)$ at any point (x,y) in the state space Λ_t is to turn off the plant (a_{III}) (or keep the plant in *ramp up* (a_{I})) if and only if there exists $(x_t^*,y_t^*) \in B_t^{\text{off}}(w)$ such that $(x_t^*,y_t^*) \succcurlyeq (x,y)$ (or $(x,y) \succcurlyeq (x_t^*,y_t^*)$).

In the *on-and-ready* state $w = K_N$, in addition to a turn-off boundary $B_t^{\text{off}}(K_N)$, there may exist a switching boundary $B_t^{\text{switch}}(K_N)$. ("Switching" means that the output level of a power plant is switched between maximum capacity level and minimum capacity level.) By inspecting the value function (18) at $w = K_N$, we know that action a_{I} dominates action a_{II} whenever $\bar{Q}[\exp(x) - \overline{Hr}\exp(y)] > \underline{Q}[\exp(x) - \underline{Hr}\exp(y)]$ and a_{II} dominates a_{I} otherwise. Thus, a switching boundary, if it exists, coincides with the curve Γ, which is independent of time t:

$$\Gamma \equiv \{(x,y): \bar{Q}[\exp(x) - \overline{Hr}\exp(y)] = \underline{Q}[\exp(x) - \underline{Hr}\exp(y)]\}. \tag{21}$$

However, $\Gamma \cap \Lambda_t$ may be empty due to the fact that Λ_t is a discrete set. To avoid confusion, we stick to the generic notations used in previous paragraphs for describ-

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ing $B_t^{\text{switch}}(K_N)$ and $B_t^{\text{off}}(K_N)$. If there exists a $(x,y) \in \Lambda_t$ such that $a_t^*(x,y,K_N)$, then $B_t^{\text{switch}}(K_N) \equiv \{(x_t^*, y_t^*) : \text{for every given } x_t^*, y_t^* = \sup_{y \in \{y_t : a_t^*(x_t^*, y_t, K_N) = a_{\text{I}}\}} y \ (y_t^* = -\infty$ if the set $\{y_t : a_t^*(x_t^*, y_t, K_N) = a_{\text{I}}\}$ is empty$)\}$ and $B_t^{\text{off}}(K_N) \equiv \{(x_t^*, y_t^*) : \text{for every}$ given $x_t^*, y_t^* = \inf_{y \in \{y_t : a_t^*(x_t^*, y_t, K_N) = a_{\text{III}}\}} y \ (y_t^* = +\infty$ if the set $\{y_t : a_t^*(x_t^*, y_t, K_N) = a_{\text{III}}\}$ is empty$)\}$; otherwise, the switching boundary $B_t^{\text{switch}}(K_N)$ does not exist and the turn-off boundary is $\{(x_t^*, y_t^*) : \text{for every given } x_t^*, y_t^* = \inf_{y \in \{y_t : a_t^*(x_t^*, y_t, K_N) = a_{\text{III}}\}} y$ $(y_t^* = +\infty$ if the set $\{y_t : a_t^*(x_t^*, y_t, K_N) = a_{\text{III}}\}$ is empty$)\}$. The optimal action at any point (x,y) in Λ_t can also be inferred based on $B_t^{\text{switch}}(K_N)$ and $B_t^{\text{off}}(K_N)$ accordingly.

Let A be a set in Λ_t. Given any point $(\tilde{x}, \tilde{y})$, we define $\bar{x}_A(\tilde{y}) = \sup_{x \in \{x : (x, \tilde{y}) \in A\}} x$, $\underline{x}_A(\tilde{y}) = \inf_{x \in \{x : (x, \tilde{y}) \in A\}} x$, $\bar{y}_A(\tilde{x}) = \sup_{y \in \{y : (\tilde{x}, y) \in A\}} y$, and $\underline{y}_A(\tilde{x}) = \inf_{y \in \{y : (\tilde{x}, y) \in A\}} y$. Let A and B be two sets in Λ_t; we say that $A \succcurlyeq B$ if $\forall (x,y) \in A$, $\bar{x}_B(y) \le \underline{x}_A(y)$ and $\bar{y}_A(x) \le \underline{y}_B(x)$. Proposition 3.4 also establishes a ranking order in "$\succcurlyeq$" among all the turn-on, turn-off, and switching boundaries (provided it exists) as follows:

$$B_t^{\text{on}}(0) \succcurlyeq B_t^{\text{off}}(1) \succcurlyeq B_t^{\text{off}}(2) \succcurlyeq \cdots \succcurlyeq B_t^{\text{off}}(K_N - 1) \succcurlyeq B_t^{\text{switch}}(K_N) \succcurlyeq B_t^{\text{off}}(K_N). \quad \textbf{(22)}$$

Whenever the log-price vector $(\ln S_t^g, \ln S_t^e)$ lies to the lower right of the boundary $B_t^{\text{on}}(0)$, a power plant operator shall turn the power plant on if the operational state is off ($w = 0$) or maintain current operations if the operational state is $w = 1, 2, \ldots, K_N + 1$. The region between $B_t^{\text{on}}(0)$ and $B_t^{\text{off}}(1)$ is a *"no-action" band* on the $X - Y$ plane in the sense that if the log-price vector falls inside this band, then it is optimal for a power plant operator to maintain the operational state of the power plant as is, regardless of w $(w = 0, 1, \ldots, K_N)$. If the log-price vector lies to the upper left of the boundaries $B_t^{\text{off}}(w)$ $(w = 1, 2, \ldots, K_N)$ or $B_t^{\text{switch}}(K_N)$, then, depending on the state of the power plant, it is optimal for the operator to reduce the output of the power plant to "off" or minimum capacity level depending on the state of the plant.

4. NUMERICAL EXPERIMENTS

We implement this proposed methodology for valuing a fossil-fuel power plant incorporating operational characteristics with a set of sample parameters. This section reports the numerical results on valuing a hypothetical 100-MW natural-gas-fired power plant over a fixed time horizon.

We assume the following specific functional form for the ramp-up cost rate function $c_r(y)$ defined in Section 3.2:

$$c_r(y) = \underline{Q}\underline{H}re^y + M_r, \quad \textbf{(23)}$$

where M_r is a constant. This particular form can be interpreted as follows. The cost rate of ramping up a power plant is a constant markup to the cost rate of operating the power plant at the minimum capacity level. With this $c_r(y)$, the operating profit function $R(a, x, y, w)$ of the power plant satisfies Condition A. It takes one period to ramp up the power plant from the "off" state to a desired output state, but there is no delay in increasing/decreasing output level once the plant is on. Table 1 summarizes the assumed parameter values for the underlying power plant used in our numerical

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Table 1. Parameters for a Hypothetical Natural-Gas-Fired Power Plant

c_{start}	$\underline{Q}:\bar{Q}$	$\underline{Hr}:\overline{Hr}$	M_r	r
\$8000	0.6 : 1	1.38 : 1	1	4.5%

illustration. The maximum and the minimum capacity levels are 100 MW and 60 MW, respectively. The start-up cost is \$8000/start, which is roughly the one-day on-peak operating profit of a 100-MW natural gas power plant. The ratio between the operating heat rates at the minimum and the maximum capacity levels of the power plant is assumed to be 1.38 : 1. The constant risk-free rate r is 4.5%.

4.1. Valuation Under Different Price Models

We first examine the effects of price model specification on power plant valuation for two different price processes: a geometric Brownian motion (GBM) process and a mean-reverting process in which the logarithm of the underlying price is represented by an Ornstein–Uhlenbeck (O-U) process. The assumed parameter values for the two models are given in Tables 2 and 3. A discrete-time trinomial price lattice is constructed according to (3) and (4) with the parameters specified in Table 2 to approximate two correlated GBMs, and a quadrinomial lattice is constructed according to (6) and (9) with the parameters in Table 3 to approximate two correlated O-U processes. The initial prices of electricity and natural gas (NG) are assumed to be \$21.7/MW h and \$3.16/MMBtu, which are sampled from historical market prices. When approximating either the GBM model or the mean-reversion price model for electricity and natural gas, the corresponding lattice model is built with Δt having a granularity of 1 day (our model can handle the hourly granularity but the market information for hourly prices is quite difficult to obtain for a 10-year time horizon). The operator of the power plant makes operational decisions at all nodes of the lattice.

We compute the value for the plant over a 10-year horizon for different possible levels of $\overline{Hr}$, the operating heat rate at the maximum capacity level. Let $\overline{Hr}$ take values of 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, and 13.5 measured in MMBtu/MW h. The start-up cost is \$8000/start as assumed earlier and the initial state of the plant is set

Table 2. Parameters for Correlated Brownian Motions

μ_e	1%	μ_g	1%
σ_e	0.4	σ_g	0.3
ρ	0.3		

TABLE 3. Parameters for Correlated Ornstein–Uhlenbeck Processes

κ_1	3	κ_2	2.25
θ_1	3.2553	θ_2	0.87
σ_1	0.79	σ_2	0.6
ρ	0.3		

to be *off*. The corresponding asset valuation results are reported in Table 4 and illustrated in Figure 2, where the x axis represents the operating heat rate level and the primary y axis on the left represents plant capacity value. The secondary y axis on the right represents the percentage of absolute difference in valuation between the GBM model and the mean-reversion model, denoted by V_{diff}, which is computed as $|V_{\text{GBM}} - V_{\text{mrvt}}|/V_{\text{mrvt}}$. The parameters specified in Tables 2 and 3 were selected so that the asset values V_{GBM} and V_{mrvt} would match if the power plant has a heat rate of $\overline{Hr} = 9.5$ (which is typical for an NG-fired plant) and it is operated at the maximum capacity level. Figure 2 illustrates the sensitivity of plant value to the assumed price process. It demonstrates that the assumption of mean-reverting price processes yields a higher valuation for efficient power plants (e.g., plants with $\overline{Hr}$ smaller than 9.5), whereas the assumption of GBM price models leads to a higher valuation for inefficient plants (e.g., plants with $\overline{Hr}$ greater than 9.5). The dashed curve (without any markers) in Figure 2 indicates that, for inefficient power plants, the plant value resulting from a GBM price model could be more than double (e.g., 116.2% for $\overline{Hr} = 13.5$) the corresponding value obtained under a mean-reverting price model. Moreover, the valuation of a power plant is much more sensitive to operating efficiency under the mean-reversion price models than under the GBM models.

TABLE 4. Value (in Million Dollars) of an NG-Fired Power Plant Under Alternative Price Models: GBM versus Mean Reversion

	Heat Rate ($\overline{Hr}$)						
	7.5	8.5	9.5	10.5	11.5	12.5	13.5
V_{GBM}	31.92	27.99	24.82	22.21	20.03	18.18	16.59
V_{mrvt}	40.80	32.12	24.82	18.88	14.13	10.49	7.68
V_{diff}	21.8%	12.8%	0%	17.7%	41.7%	73.4%	116.2%
$E(\text{Start}_{\text{GBM}})$	0.032	0.028	0.024	0.022	0.019	0.017	0.016
$E(\text{Ramp}_{\text{GBM}})$	0.038	0.036	0.034	0.033	0.031	0.030	0.028
$E(\text{Start}_{\text{mrvt}})$	0.086	0.111	0.133	0.142	0.139	0.131	0.119
$E(\text{Ramp}_{\text{mrvt}})$	0.095	0.132	0.169	0.189	0.197	0.194	0.185

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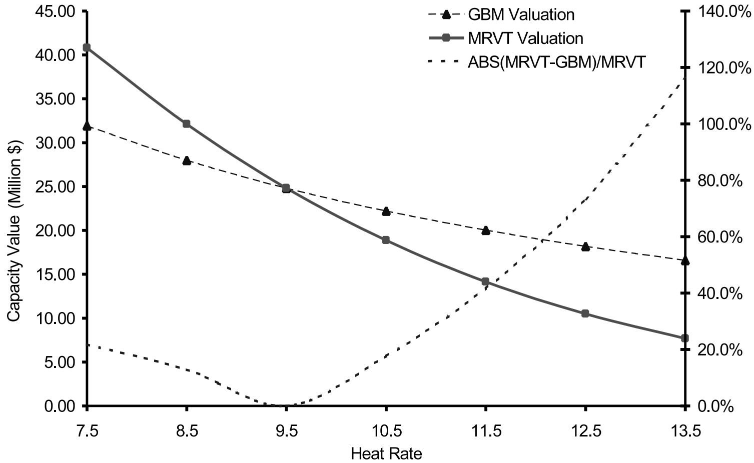

FIGURE 2. Value of an NG-fired power plant under alternative price models: GBM versus mean reversion.

In Table 4, we also present the expected start-up costs, denoted by $E(\text{Start}_*)$, and ramp-up costs, denoted by $E(\text{Ramp}_*)$ incurred by a power plant under the respective power price models. Both the expected start-up and ramp-up costs in the mean-reversion case are significantly higher than those in the GBM case. This phenomenon is demonstrated in Figure 3. The solid and dashed curves illustrate the expected start-up and ramp-up costs for different heat rate levels. Curves with triangular markers correspond to the GBM price assumption and curves with square markers correspond to the mean-reversion assumption. The intuitive explanation of this phenomenon is that the spark spread tends to grow larger (smaller) if the current spread level is large (small) in the GBM price models but not for the mean-reversion price models. Thus, under the GBM price assumptions, a power plant would be turned on and off much less frequently than it would be under the mean-reversion price assumptions. One other observation illustrated by Figure 3 is that the expected start-up and ramp-up costs are peaked at certain intermediate heat rates for the mean-reversion price models, indicating that the operational flexibility option (i.e., the option of turning a power plant on or off) is exercised most frequently for power plants with intermediate operating efficiencies if the underlying commodity prices are mean reverting.

4.2. Impacts of Operational Characteristics on Asset Valuation

We next examine the impacts of operating characteristics on the valuation of a power plant under each of the two price models. The time horizon is set to be 10 years. Again, we use the parameter values given in Tables 2 and 3.

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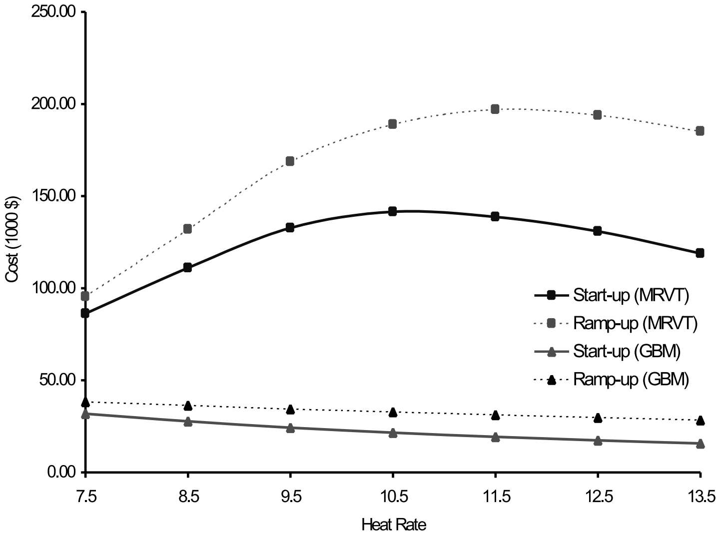

FIGURE 3. Expected start-up and ramp-up costs of an NG-fired power plant under alternative price models: GBM versus mean reversion.

4.2.1. Geometric Brownian motion price model. Based on the trinomial price lattice of electricity and natural gas constructed in Section 4.1, we compute the value of the gas-fired power plant subject to the three operating constraints assuming different operating heat rates ($\overline{Hr}$). We also compute the value of the power plant in the case where none of the three operational characteristics is considered, as well as in the case where only the start-up cost is ignored. The numerical results are reported in Table 5. For instance, if the power plant under consideration has an operating heat rate of 9.5 MMBtu/MW h, then when operated at its best capacity level, its value is $24.82 million. If we ignore all three operating characteristics, then the value of the power plant becomes $24.92 million, which would be 0.41% higher than the value obtained when accounting for all three operating characteristics. The solid curve in Figure 4 illustrates the value of the underlying power plant (incorporating three operating characteristics) across different heat rates against the capacity value axis on the left. The dashed curve with triangular markers in Figure 4 illustrates the percentage of overstated capacity valuation resulting from ignoring all three physical characteristics across different heat rates. We can see that a higher level of operating heat rate corresponds to a higher percentage of overstatement in valuation. Note that the percentage of overstated capacity value is under 1% for operating heat rates up to 13.5 MMBtu/MW h under the GBM price models.

Table 5 also shows the value of the underlying power plant when only the start-up costs are ignored, as well as the corresponding percentage of the overvaluation. These results are represented by the dashed curve with square markers in Figure 4.

TABLE 5. Value of an NG-Fired Power Plant Under GBM Price Models

	Heat Rate ($\overline{Hr}$)				
	7.5	8.5	9.5	11.5	13.5
Power plant value (with 3 operating characteristics)	31.92 million	27.99 million	24.82 million	20.03 million	16.59 million
Power plant value (ignore start-up only)	31.96 million	28.02 million	24.85 million	20.05 million	16.61 million
Percentage of value overstated (ignore start-up only)	0.12%	0.12%	0.12%	0.11%	0.11%
Power plant value (ignore 3 operating characteristics)	32.04 million	28.10 million	24.92 million	20.11 million	16.67 million
Percentage of value overstated (ignore 3 operating characteristics)	0.38%	0.40%	0.41%	0.43%	0.44%

Ignoring the start-up costs alone accounts for 25–31% of the overstated value of a power plant as compared to the overstated valuation when all three operating characteristics are ignored.

4.2.2. Mean-reverting price model. We next examine the impacts of operational characteristics on the valuation of a power plant under the assumption that both the electricity price and the natural gas price are mean reverting, which is a more realistic assumption for energy commodities than the GBM assumption. Based

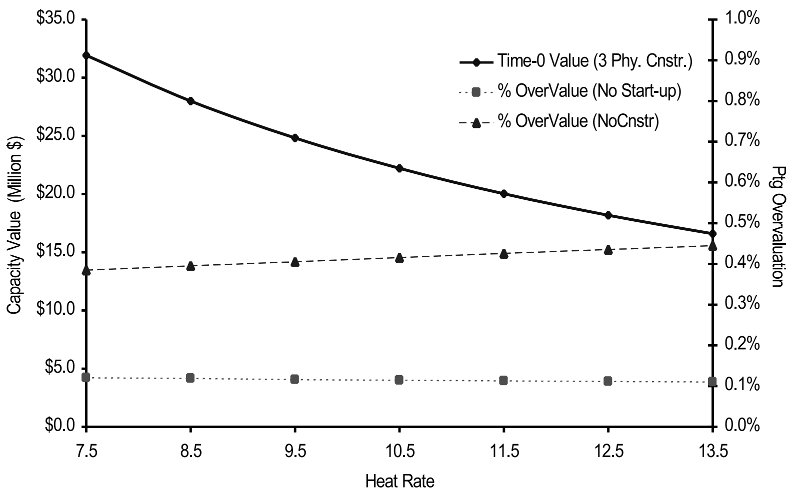

FIGURE 4. Valuation of a power plant with/without operational characteristics under GBM price models.

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TABLE 6. Value of an NG-Fired Power Plant Under Mean-Reversion Price Models

	Heat Rate ($\overline{Hr}$)				
	7.5	8.5	9.5	11.5	13.5
Power plant value (with 3 operating characteristics)	40.80 million	32.12 million	24.82 million	14.13 million	7.68 million
Power plant value (ignore start-up only)	40.89 million	32.24 million	24.96 million	14.28 million	7.80 million
Percentage of value overstated (ignore start-up only)	0.22%	0.37%	0.56%	1.05%	1.64%
Power plant value (ignore 3 operating characteristics)	41.15 million	32.60 million	25.38 million	14.78 million	8.26 million
Percentage of value overstated (ignore 3 operating characteristics)	0.85%	1.50%	2.28%	4.59%	7.60%

on the quadrinomial price lattice of electricity and natural gas constructed in Section 4.1, the value of the underlying power plant is calculated for each of the three cases: considering all three physical operating characteristics, ignoring the three operating characteristics, and ignoring the start-up cost only. The numerical results are presented in Table 6. Similar to Section 4.2.1, we plot the value of the power plant accounting for all three operating characteristics for different heat rates in Figure 5. The x axis represents different heat rates. The solid curve with triangular markers plots the capacity value incorporating all three operational constraints.

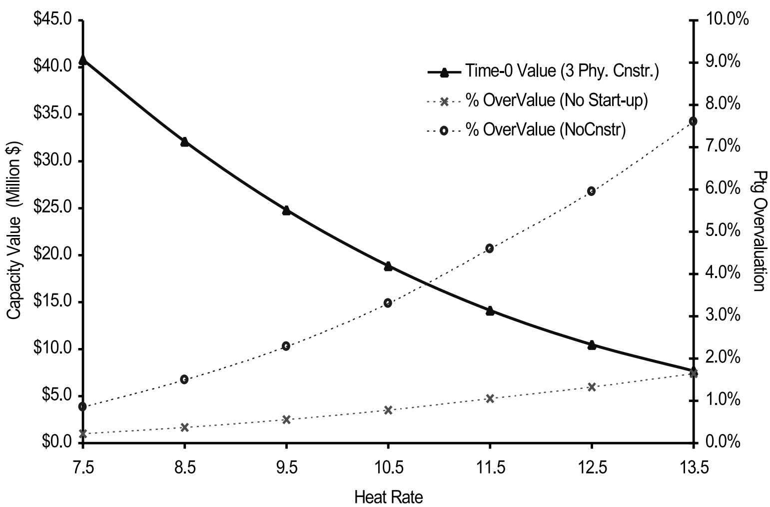

FIGURE 5. Valuation of a power plant with/without operational characteristics under mean-reversion price models.

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The dashed curves with circles illustrate the percentage by which the capacity value is overstated due to ignoring the three operating characteristics. The percentage for which the capacity value is overstated ranges from 0.85% for the most efficient plant to 7.60% for the least efficient plant. The dashed curve with $\times$'s in Figure 5 plots the percentage of overstated value of the power plant due to not accounting for the start-up cost. We can see that the impacts of start-up cost and ramp-up constraint on capacity valuation are significant for a power plant with an operating efficiency below a certain threshold. Furthermore, ignoring the start-up cost while considering the other aspects would result in about 22–31% of the overstated capacity value of the underlying power plant.

5. CONCLUSION

To summarize the above numerical results, we conclude that the operational characteristics affect the valuation of a power plant to different extents, depending on the operating efficiency of the power plant and the assumptions about the electricity and the generating fuel prices. In general, the impacts of physical operating characteristics on the power plant valuation are far more significant under the mean-reversion price models than they are under the geometric Brownian motion price models.

Under each price model, the more efficient a power plant is, the less is its valuation affected by the operational constraints and vice versa. In the examples with mean-reverting commodity price processes, the impacts on capacity valuation range from 0.85% for the most efficient plant to 7.60% for the least efficient plant with a modest level of start-up cost. Among the three operational characteristics of a power plant which we consider here, start-up cost and the ramp-up constraint are the two major factors affecting the capacity valuation. The reason is twofold. The first-order effect of the start-up and the ramp-up costs on capacity valuation is that they directly impose a transaction cost on exercising the embedded spark spread options in a fossil-fuel power plant when the electricity price is greater than the fuel cost. The second-order effect of these costs is that they force the power plant to keep operating at a loss or to forego a profit when the start-up cost cannot be justified by the expected cost-saving or the expected profit that would result from turning the power plant off or on. In other words, the start-up and the ramp-up costs reduce the "option value" of a power plant. Our sensitivity analysis reveals that under the mean-reversion models, ignoring the start-up cost alone can explain a sizable portion of the overstated capacity value of a power plant (as compared to the overstated value when all three operational characteristics are ignored).

The costs associated with the operational characteristics of a power plant have different implications on defining the best operational strategies for a power plant with different efficiency characteristics. In our numerical example under the mean-reverting price assumption, the embedded operational option of a power plant is exercised most frequently when its heat rate is at an intermediate level.

An important conclusion from our analysis is that under the GBM price process, the error introduced by ignoring operational characteristics is rather small. This is important considering that in the absence of these characteristics, pure spark spread valuation models of generating capacity can be evaluated analytically. This does not mean that we would recommend the use of GBM-based valuation since, as indicated, such GBM models do not represent energy commodity prices well. Nevertheless, GBM models are widely used in the energy industry. Thus, it is worth noting that the modeling error will not be noticeably reduced by detailed consideration of operating characteristics. In the case of mean-reverting price models, however, the error is more significant, ranging from 2% for efficient plants to 7.6% for inefficient plants. Considering that plant values are of the order of hundreds of million dollars, these are significant errors with large impacts on profitability. Therefore, in spite of the convenience of analytic solutions available for pure spark spread valuation models, the error at stake calls for the more detailed models and numerical solution approach described in this article. These models can be further refined by including jumps, spikes, and Markovian regime switching phenomena that often characterize energy prices.

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APPENDIX

A. Proof of Proposition 3.1

To prove this proposition, we apply Theorem (7.1) of Durrett [6, p. 297] (referred to below as Theorem (7.1)).

To apply Durrett's Theorem (7.1), we need to verify that the four conditions stated in Proposition 3.1 imply those in the theorem. First, condition (iv) of Proposition 3.1 is the same as condition (iv) in Theorem (7.1). Second, conditions (i) and (ii) in Proposition 3.1 imply $|\bar{a}_{ij}^h(\bar{x})/h - a_{ij}(\bar{x})| = |a_{ij}(\bar{x}) + o(h)/h - a_{ij}(\bar{x})| \to 0$ as $h \to 0$ and $|\bar{b}_i^h(\bar{x})/h - b_i(\bar{x})| = |b_i(\bar{x}) + o(h)/h - b_i(\bar{x})| \to 0$ as $h \to 0$; that is, conditions (i) and (ii) in Theorem (7.1) are satisfied. Finally, condition (iii) in Proposition 3.1 says that the support of the conditional random vector $(X_{t+1}, Y_{t+1})|(X_t, Y_t)$ is uniformly bounded for all possible (X_t, Y_t) across all time t, and the diameter of the support tends to 0 as $h \to 0$. This implies that condition (iii) in Theorem (7.1) is also satisfied. Therefore, Proposition 3.1 is true by applying Theorem (7.1) of Durrett [6]. ■

B. Proof of Proposition 3.3

LEMMA B.1. *Let $f(x,y): R^2 \to R^1$ be an arbitrary finite function increasing in x and decreasing in y. For any $(x_1, y_1) \geqslant (x_2, y_2)$ in Λ_t, we have*

$$E_t[f(X_{t+1}, Y_{t+1})|(X_t, Y_t) = (x_1, y_1)] \geq E_t[f(X_{t+1}, Y_{t+1})|(X_t, Y_t) = (x_2, y_2)] \qquad \textbf{(B.1)}$$

for all t $(t = 0, 1, \ldots, N)$.

PROOF. Let $(X_{t+1}, Y_{t+1})|(x,y)$ denote the random vector of (X_{t+1}, Y_{t+1}) conditioning on $(X_t, Y_t) = (x,y)$. First, it is easy to show (B.1) by definition when $f(x,y)$ is a simple function. Then, for a general real function $f(x,y)$ that increases in x and decreases in y, we can find a sequence of simple functions $\phi_m(x,y)$ so that $\lim_{m\to\infty} \phi_m(x,y) = f(x,y)$ (e.g., Royden [9]) and each $\phi_m(x,y)$ satisfies (B.1). Moreover, $(X_{t+1}, Y_{t+1})|(x,y)$, defined by either (3) and (4), or (6) and (9), has a bounded support. Therefore, we have

$$\lim_{m\to\infty} E_t[\phi_m(X_{t+1}, Y_{t+1})|(x,y)] = E_t[f(X_{t+1}, Y_{t+1})|(x,y)] \qquad \textbf{(B.2)}$$

and (B.1) holds true for $f(x,y)$. ■

PROOF OF PROPOSITION 3.3. We prove by induction. At $t = N$, $\forall w = 0, 1, \ldots, K_N$, we have $V_N(x,y,w) = \max_{a\in A_N} R(a,x,y,w)$ where $R(a,x,y,w)$ is defined in (14). Thus, $V_N(x,y,w)$ is continuous and increasing in $(x,-y)$ since $R(a,x,y,w)$ is continuous and increasing in $(x,-y)$ for each action $a \in A_N$, and $V_N(x,y,w)$ is the upper envelope function of functions $\{R(a,x,y,w): a \in A_N\}$. Suppose we know that, at time $t = n+1 \leq N$, $V_{n+1}(x,y,w)$ is continuous and increasing in $(x,-y)$ for all w. Then, $h(x,y) \equiv E_n[V_{n+1}(X_{n+1}, Y_{n+1}, w)|(X_n, Y_n) = (x,y)]$ is continuous. Moreover, by Lemma B.1, $h(x,y)$ is increasing in $(x,-y)$. Therefore, the recursive equations (16)–(18) imply that $V_n(x,y,w)$ is continuous and increasing in $(x,-y)$. ■

C. Proof of Proposition 3.4

This proposition is slightly more general than Theorem 3.9.2 in Topkis [11, p. 165] (referred to below as Theorem 3.9.2) in the sense that we allow $A \times \Lambda_t \times W_N$, which is the domain of the function $R(a_t, X_t, Y_t, w)$, to be either a lattice or a nonlattice structure, whereas Theorem 3.9.2 covers only the lattice cases.

For the mean-reverting Markov process defined by (6) and (9), $A \times \Lambda_t \times W_N$ is a sublattice of $A \times R^2 \times W_N$. (See Topkis [11] for definitions of lattice/sublattice and related properties.) We can then apply Theorem 3.9.2 to prove the proposition. Condition 1 says that $R(a_t, X_t, Y_t, w)$ is increasing in (x, y, w). Condition 2 implies that $R(a_t, X_t, Y_t, w)$ is supermodular in (a, x, y, w). The conditional random variable $(X_{t+1}, Y_{t+1})|(x, y)$ is independent of a_t and w. The distribution function of $(X_{t+1}, Y_{t+1})|(x, y)$, denoted by $F_{x,y}$, is stochastically increasing in $(x, -y)$ by Lemma B.1. All we need to verify is that $F_{x,y}$ is stochastically supermodular in (x, y); namely $\int_U dF_{x,y}$ is supermodular in (x, y) for any increasing set U. With p_t^i's defined by (9) in the interior of Λ_t and by (7) on the boundary of Λ_t in general, it is true that $\int_U dF_{\tilde{x},\tilde{y}} + \int_U dF_{x,y} \geq \int_U dF_{\tilde{x},y} + \int_U dF_{x,\tilde{y}}$ for all $(\tilde{x}, \tilde{y}) \succcurlyeq (x, y)$ in Λ_t. Therefore, all of the conditions of Theorem 3.9.2 are satisfied. Then, $f_t \equiv V_t$ is supermodular in (x, y, w) (thus having increasing difference in (x, y) and w) and the optimal decision $a_t^*(x, y, w)$ is increasing in (x, y, w).

For the Brownian motion case, we can similarly prove the claims by induction, following the steps of Topkis [11] in proving Theorem 3.9.2, with the property of "supermodularity" replaced by the property of "increasing differences." ∎