Canad. Math. Bull. Vol. **64** (3), 2021, pp. 667–677
http://dx.doi.org/10.4153/S0008439520000739

On a rationality problem for fields of cross-ratios II

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Abstract. Let k be a field, $x_1, \ldots, x_n$ be independent variables and let $L_n = k(x_1, \ldots, x_n)$. The symmetric group Σ_n acts on L_n by permuting the variables, and the projective linear group PGL_2 acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x_i \longmapsto \frac{ax_i + b}{cx_i + d}$$

for each $i = 1, \ldots, n$. The fixed field $L_n^{\mathrm{PGL}_2}$ is called "the field of cross-ratios". Given a subgroup $S \subset \Sigma_n$, H. Tsunogai asked whether L_n^S rational over K_n^S. When $n \geqslant 5$, the second author has shown that L_n^S is rational over K_n^S if and only if S has an orbit of odd order in $\{1, \ldots, n\}$. In this paper, we answer Tsunogai's question for $n \leqslant 4$.

1 Introduction

Let k be a base field, $n \geqslant 1$ be an integer, $x_1, \ldots, x_n$ be independent variables, and $L_n = k(x_1, \ldots, x_n)$. The group PGL_2 acts on L_n via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x_i \longrightarrow \frac{ax_i + b}{cx_i + d}$$

for $i = 1, \ldots, n$. The field of invariants $K_n = L_n^{\mathrm{PGL}_2}$ is generated over k by the cross-ratios

$$\frac{(x_i - x_1)(x_3 - x_2)}{(x_i - x_2)(x_3 - x_1)} \tag{1.1}$$

for $i = 4, \ldots, n$. For this reason K_n is often called the field of cross-ratios. (If $n \leqslant 3$, then $K_n = k$.) The natural action of the symmetric group Σ_n on L_n by permuting the variables descends to a Σ_n-action on K_n. Let S be a subgroup of Σ_n. Motivated by the Noether problem, H. Tsunogai asked the following question [Tsu17, Introduction].

Received by the editors July 10, 2020; revised August 25, 2020.
Published online on Cambridge Core September 4, 2020.
Tran-Trung Nghiem was supported by a Fondation Hadamard Scholarship. Zinovy Reichstein was partially supported by National Sciences and Engineering Research Council of Canada Discovery grant 253424-2017.
AMS subject classification: **14E08, 14M17.**
Keywords: Rationality problem, field extension, cross ratio, conic curve.

Question 1.1 Is L_n^S is rational over K_n^S?

For $n \geqslant 5$, the second author answered Question 1.1 as follows; see [Re20].

Theorem 1.2 *Let S be a subgroup of the symmetric group Σ_n, where $n \geqslant 5$. Then the following conditions are equivalent:*

(i) *L_n^S is rational over K_n^S;*
(ii) *L_n^S is unirational over K_n^S;*
(iii) *S has an orbit of odd order in $\{1, \dots, n\}$.*

The purpose of this paper is to address Question 1.1 in the case where $n \leqslant 4$. For $n \leqslant 3$, there is an easy answer. Here, as we mentioned above, $K_n = k$ and thus $K_n^S = k$ for any $S \subset \Sigma_n$. In other words, for $n \leqslant 3$, Question 1.1 reduces to the following special case of the Noether Problem: Is L_n^S rational over k? The answer is known to be "yes" for every subgroup $S \subset \Sigma_n$ ($n \leqslant 3$); see [KW14, Theorem 3.3].

The case where $n = 4$ is more delicate. Our main result is as follows.

Theorem 1.3 *Let S be a subgroup of Σ_4.*

(i) *Assume S is not isomorphic to a cyclic group of order 4. Then L_4^S is rational over K_4^S for any base field k.*
(ii) *Assume S is cyclic of order 4 and $\mathrm{char}(k) \neq 2$. Then L_4^S is rational over K_4^S if and only if k contains a primitive fourth root of unity.*
(iii) *Assume S is cyclic of order 4 and $\mathrm{char}(k) = 2$. Then L_4^S is rational over K_4^S.*

Note that the symmetric group Σ_4 has exactly 11 subgroups up to conjugacy; see [GPW]. Part (i) covers 10 of them. Note also that for $n \leqslant 3$ and for $n \geqslant 5$, the answer to Question 1.1 is independent of the base field k. For $n \leqslant 3$, it is always "yes", and for $n \geqslant 5$, it depends only on n and the subgroup $S \subset \Sigma_n$, up to conjugacy; see Theorem 1.2. A cyclic group S of order 4 in Σ_4 represents the only instant where the answer to Question 1.1 depends on k.

We will view L_n, K_n, L_n^S and K_n^S as the function fields of $(\mathbb{P}^1)^n$, $(\mathbb{P}^1)^n/\mathrm{PGL}_2$, $(\mathbb{P}^1)^n/S$, and $(\mathbb{P}^1)^n/(\mathrm{PGL}_2 \times S)$, respectively. Here and in the sequel X/G will denote the rational (or Rosenlicht) quotient variety for the action of an algebraic group G on an algebraic variety X defined over k. Recall that the Rosenlicht quotient X/G is only defined up to birational equivalence and that $k(X/G) = k(X)^G$. For details of this construction and further references, see [RS20, Section 2].

The remainder of this paper will be devoted to proving Theorem 1.3. Before proceeding with the proof, we would like to explain a new phenomenon that arises in this case and that motivated our interest in Question 1.1 for $n = 4$. When $n \geqslant 5$, the PGL_2-action on $(\mathbb{P}^1)^n/S$ is generically free. (For the definition of a generically free action, see the beginning of Section 2.) Consequently, the natural projection

$$\pi_S \colon (\mathbb{P}^1)^n/S \longrightarrow (\mathbb{P}^1)^n/(\mathrm{PGL}_2 \times S)$$

is a PGL_2-principal homogeneous space over the generic point $\mathrm{Spec}(K_n^S)$ of $(\mathbb{P}^1)^n/(\mathrm{PGL}_2 \times S)$. It is shown in [Re20] that L_n^S is rational over K_n^S if and only if this principal homogeneous space is split; the proof of Theorem 1.2 in [Re20] is based on this observation. When $n = 4$, π_S is also a PGL_2-homogeneous space over the generic point of $(\mathbb{P}^1)^n/(\mathrm{PGL}_2 \times S)$, but it may not be principal. More precisely, the geometric fibers of π_S in general position are isomorphic to $\mathrm{PGL}_2/S[4]$, where $S[4] = S \cap V[4]$ is the intersection of S with the Klein 4-subgroup

$$V[4] = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}, \tag{1.2}$$

suitably embedded in PGL_2. It is easy to see that $S[4] = \{1\}$ if and only if S has a fixed point in $\{1, 2, 3, 4\}$. In this case, π_S is again a principal homogeneous space over $\mathrm{Spec}(K_4^S)$, and the same argument as in [Re20] shows that L_4^S is rational over K_4^S. If $S[4] \neq 1$, then the arguments from [Re20] no longer apply, and a different approach is required.

2 First Reductions

Recall that the action of an algebraic group G on an irreducible algebraic variety X defined over a field k is called *generically free* if there exists a dense open G-invariant subvariety $X_0 \subset X$ such that the stabilizer G_{x_0} is trivial for every $\overline{k}$-point $x_0 \in X_0$. Here, as usual, $\overline{k}$ denotes the algebraic closure of k.

Let B denote the Borel subgroup of upper triangular matrices in PGL_2. Equivalently, $B \subset \mathrm{PGL}_2$ is the stabilizer of the point $\infty = (1:0) \in \mathbb{P}^1$. The following lemma is undoubtedly well known. For lack of a suitable reference, we include a short proof.

Lemma 2.1 *Let $X_4 = (\mathbb{P}^1)^4/\Sigma_4 \simeq \mathbb{P}^4$ be the space of unordered 4-tuples of points on $\mathbb{P}^1$. Then the B-action on X_4 is generically free.*

Proof Denote the stabilizer of the unordered 4-tuple of points $\{p_1, \dots, p_4\} \in (\mathbb{P}^1)^4/\Sigma_4$ by $H_{p_1,\dots,p_4}$. Assume the contrary: $H_{p_1,\dots,p_4} \neq 1$ for $p_1, \dots, p_4$ in general position.

Now let $q_1, \dots, q_5$ be a 5-tuple of points in $\mathbb{P}^1$. Translating q_5 to ∞ by a suitable element of PGL_2, we obtain a 5-tuple of the form $p_1, \dots, p_4, \infty$. If $q_1, \dots, q_5$ are in general position, then so are $p_1, \dots, p_4$. Hence, by our assumption, $H_{p_1,\dots,p_4} \neq 1$. Since $H_{p_1,\dots,p_4}$ is a subgroup of B, we see that $H_{p_1,\dots,p_4}$ stabilizes ∞. Thus, $H_{p_1,\dots,p_4}$ stabilizes the unordered 5-tuple $\{p_1, \dots, p_4, \infty\}$. Since the unordered 5-tuple $\{q_1, \dots, q_5\}$ lies in the same PGL_2-orbit as $\{p_1, \dots, p_4, \infty\}$, we conclude that the stabilizer of $\{q_1, \dots, q_5\}$ in PGL_2 is non-trivial. On the other hand, it is well known that the stabilizer of an unordered 5-tuple of points of $\mathbb{P}^1$ in general position is trivial, a contradiction. ■

In the sequel, we will denote the field L_4^B by F_4. The various invariant fields we are interested in are pictured in the diagram below.

$$\begin{tikzcd}
(\mathbb{P}^1)^4 \arrow[d, dashed] & L_4 = k(x_1, \dots, x_4) \arrow[d, dash] & 4 \\
(\mathbb{P}^1)^4/S \arrow[d, dashed] & L_4^S \arrow[d, dash, "\text{rational}"] & 4 \\
(\mathbb{P}^1)^4/(B \times S) \arrow[d, dashed] & F_4^S \arrow[d, dash, "\text{transcendence degree1}"] & 2 \\
(\mathbb{P}^1)^4/(S \times \mathrm{PGL}_2) & L_4^{S \times \mathrm{PGL}_2} = K_4^S & 1
\end{tikzcd}$$

Here the fields in the middle column represents the function fields of the varieties on the left. The right column lists the dimension of each variety (or equivalently the transcendence degree of its function field) over k. We now proceed with the main result of this section.

Proposition 2.2 *Let S be a subgroup of Σ_4. Then the following are equivalent:*

(i) *L_4^S is rational over K_4^S;*
(ii) *L_4^S is unirational over K_4^S;*
(iii) *F_4^S is unirational over K_4^S;*
(iv) *F_4^S is rational over K_4^S.*

Proof The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. The field extension F_4^S/K_4^S is of transcendence degree $2 - 1 = 1$. Hence, the implication (iii) $\Rightarrow$ (iv) follows from Lüroth's theorem.[1]

To prove the remaining implication (iv) $\Rightarrow$ (i), it suffices to show that L_4^S is rational over F_4^S. By Lemma 2.1 the action of B on the space $(\mathbb{P}^1)^4/\Sigma_4 \simeq \mathbb{P}^4$ of unordered 4-tuples of points on $\mathbb{P}^1$ is generically free. Hence, so is the action of B on $(\mathbb{P}^1)^4/S$ for any subgroup $S \subset \Sigma_4$. Since B is a special group (see [Se97, Proposition II.1.2.1]), this implies that $(\mathbb{P}^1)^4/S$ is birationally isomorphic to $\big((\mathbb{P}^1)^4/(B \times S)\big) \times B$. Since B is a rational 2-dimensional variety over k, we conclude that L_4^S is rational of transcendence degree 2 over F_4^S, as claimed. ■

3 Proof of Theorem 1.3(i)

Let S be a subgroup of Σ_4 and consider the exact sequence

$$1 \longrightarrow S[4] \longrightarrow S \longrightarrow S/S[4] \longrightarrow 1, \tag{3.1}$$

---

[1] Recall that Lüroth's theorem asserts that a unirational field extension of transcendence degree 1 is rational; a proof can be found, *e.g.*, in [Ja89, Chapter 8] or [vdW91, Chapter 10].

where $V[4]$ is the Klein 4-subgroup as in (1.2), $S[4] := S \cap V[4]$, and $S/S[4]$ is a subgroup of $\Sigma_4/V[4] \simeq \Sigma_3$. Note that $S[4]$ acts trivially on K_4, and $S/S[4]$ acts faithfully.

Lemma 3.1 *Sequence (3.1) does not split if and only if $S \subset \Sigma_4$ is a cyclic subgroup of order 4.*

Proof Observe that sequence (3.1) splits in the following two cases:

$$\text{(i) if } S[4] = 1 \qquad \text{or} \qquad \text{(ii) if } S = \Sigma_4 .$$

Case (i) is obvious, and (ii) follows from the fact that Σ_3, naturally embedded into Σ_4, is a complement to $V[4]$.

Case (ii) implies that sequence (3.1) splits whenever $S[4] = V[4]$. Thus, we can assume without loss of generality that $S[4]$ has order 2. Now $S/S[4]$ is a subgroup of Σ_3, so has order 1, 2, 3 or 6. Let us consider these possibilities in turn.

If $|S/S[4]| = 1$ or 3, then $|S| = 2$ or 6, respectively. Clearly, (3.1) splits in both cases.

If $|S/S[4]| = 6$, then $|S| = 12$, so $S = A_4$ is the alternating group, and $S[4] = V[4]$, contradicting our assumption that $|S[4]| = 2$.

This leaves us with the case where $|S/S[4]| = 2$, *i.e.*, $|S| = 4$. Up to conjugacy, there are only two subgroups of order 4 in Σ_4, namely, $V[4]$ and a cyclic subgroup generated by a 4-cycle. Clearly, $S \neq V[4]$, because we are assuming that $S[4] = S \cap V[4]$ has order 2. Thus, S is the group of order 4 generated by a 4-cycle σ. In this case, $S[4] = \langle \sigma^2 \rangle$, and sequence (3.1) does not split. ■

We are now ready to complete the proof of Theorem 1.3(i).

Case 1: $S[4] = 1$. This is equivalent to the condition that S has a fixed point in $\{1, 2, 3, 4\}$. As we mentioned in the introduction, in this case, the $\mathrm{PGL}_2 \times S$-action on $(\mathbb{P}^1)^4$ is generically free, and the same argument used to prove Theorem 1.2 in [Re20] goes through unchanged. (For a self-contained proof, see Remark 3.2 .) We conclude that L_4^S is rational over K_4^S.

Case 2: $S \subset \Sigma_4$ is not cyclic of order 4. By Lemma 3.1, sequence (3.1) splits. Let $S' \subset S$ be a complement to $S[4]$. By Case 1, $L_4^{S'}$ is rational over K_4^S. The diagram

$$\begin{tikzcd}
L_4^{S'} \arrow[d, dash] \arrow[dd, bend left=60, dash, "\text{rational}"] \\
L_4^{S} \arrow[d, dash] \arrow[d, bend left, dash, "\text{unirational}"] \\
K_4^{S} = K_4^{S'}
\end{tikzcd}$$

now shows that L_4^S is unirational over K_4^S. By Proposition 2.2, we conclude that L_4^S is rational over K_4^S.

Remark 3.2 For the sake of completeness, we will now outline a self-contained geometric proof of Theorem 1.3(i) in Case 1.

As we mentioned above, since $S[4] = 1$, S fixes an element of $\{1, 2, 3, 4\}$. After replacing S by a conjugate in Σ_4 we can assume without loss of generality that S fixes 4, *i.e.*, $S \subset \Sigma_3$. Consider the diagram of PGL_2-equivariant rational maps

$$\begin{tikzcd}
(\mathbb{P}^1)^4 \arrow[d, dashed] \arrow[dr, "p_4"] & & \\
(\mathbb{P}^1)^4/S \arrow[r, dashed, "\overline{p_4}"] \arrow[d, dashed, "\pi"] & \mathbb{P}^1 \arrow[r, leftrightarrow, "\sim"] & \mathrm{PGL}_2/B \\
(\mathbb{P}^1)^4/(S \times \mathrm{PGL}_2). & &
\end{tikzcd}$$

Here, B is the Borel subgroup of upper-triangular matrices in PGL_2, as in Section 2, and $p_4\colon (\mathbb{P}^1)^4 \to \mathbb{P}^1$ is the projection to the fourth factor. Since $S \subset \Sigma_3$, p_4 descends to a PGL_2-equivariant map $(\mathbb{P}^1)^4/S \dashrightarrow \mathbb{P}^1$, which we denote by $\overline{p_4}$. The function fields of $(\mathbb{P}^1)^4$, $(\mathbb{P}^1)^4/S$, and $(\mathbb{P}^1)^4/(S \times \mathrm{PGL}_2)$ are L_4, L_4^S, and K_4^S, respectively; see the diagram in Section 2. The PGL_2-action on $(\mathbb{P}^1)^4/S$ is generically free; hence, π is a PGL_2-torsor over the generic point of $(\mathbb{P}^1)^4/(S \times \mathrm{PGL}_2)$. Our goal is to prove that this torsor is split; this will show that $(\mathbb{P}^1)^4/S$ is birational to $(\mathbb{P}^1)^4/(S \times \mathrm{PGL}_2) \times \mathrm{PGL}_2$, and thus, L_4^S is rational over K_4^S.

Since there exists a PGL_2-equivariant map from the total space of π to PGL_2/B, π admits reduction of structure to the Borel subgroup B. In other words, the class $[\pi]$ of π in $H^1(K_4^S, \mathrm{PGL}_2)$ lies in the image of the natural map $H^1(K_4^S, B) \to H^1(K_4^S, \mathrm{PGL}_2)$. As we mentioned in the proof of Proposition 2.2, B is a special group. In particular, $H^1(K_4^S, B) = 1$, and the desired conclusion follows.

4 A Preliminary Computation

The approach we used in the previous section does not work when S is a cyclic group of order 4. To prove parts (ii) and (iii) of Theorem 1.3, we will resort to explicit calculations in the next section. The following lemma will facilitate these calculations.

Lemma 4.1 *Let k be a field and let a, u be independent variables over k.*

(i) *Let σ be an automorphism of $k(a,u)/k$ of order 2 given by $\sigma(a) = 1 - a$ and $\sigma(u) = -\frac{1}{u}$. Assume* $\mathrm{char}(k) \neq 2$. *Then $k(a,u)^{\langle\sigma\rangle}$ is rational over $k(a)^{\langle\sigma\rangle}$ if and only if k contains a primitive fourth root of unity.*

(ii) *Assume* $\mathrm{char}(k) = 2$. *Let σ be an automorphism of $k(a,u)/k$ of order 2 given by $\sigma(a) = a + 1$ and $\sigma(u) = u + 1$. Then $k(a,u)^{\langle\sigma\rangle}$ is rational over $k(a)^{\langle\sigma\rangle}$.*

Proof (i) Set $b = 1 - 2a$, so that $\sigma(b) = -1 + 2a = -b$. Since $\mathrm{char}(k) \neq 2$, we have $k(a,u) = k(b,u)$ and $k(a) = k(b)$. Thus, part (i) is equivalent to

(a′) *$k(b,u)^{\langle\sigma\rangle}$ is rational over $k(b)^{\langle\sigma\rangle}$ if and only if k contains a fouth root of unity.*

Clearly, $k(b)^{\langle\sigma\rangle} = k(x)$, where $x = b^2$. We claim that $k(b,u)^{\langle\sigma\rangle} = k(x,y,z)$, where

$$y = \frac{b}{2}\Big(u + \frac{1}{u}\Big) \quad \text{and} \quad z = \frac{1}{2}\Big(u - \frac{1}{u}\Big).$$

Indeed, one readily checks that $x, y, z \in k(b,u)^{\langle\sigma\rangle}$. It remains to show that $[k(b,u) : k(x,y,z)] \leqslant 2$; the claim will immediately follow from the diagram below.

$$\begin{tikzcd}
k(b,u) \arrow[d, dash] \arrow[d, dash, bend left, "\text{degree } 2"] \arrow[dd, dash, bend left=60, "\text{degree} \leqslant 2"] \\
k(b,u)^{\langle\sigma\rangle} \arrow[d, dash] \\
k(x,y,z).
\end{tikzcd}$$

To show that $[k(b,u) : k(x,y,z)] \leqslant 2$, note that $k(b,u) = k(x,y,z)(u)$ and u satisfies the quadratic equation $u^2 - 2zu - 1 = 0$ over $k(x,y,z)$. This proves the claim.

We have thus reduced part (i) to the following assertion:

(i$''$) *$k(x,y,z)$ is rational over $k(x)$ if and only if k contains a primitive fourth root of unity.*

Note that $k(x,y,z) = k(b,u)^{\langle\sigma\rangle}$ is of transcendence degree 2 over k and

$$y^2 - xz^2 - x = 0.$$

This equation defines a conic in $\mathbb{A}^2$ over the field $k(x)$. Since this conic is absolutely irreducible, $k(x,y,z)$ is the function field of this conic. This field is rational over $k(x)$ if and only if the projective conic

$$Y^2 - xZ^2 - xW^2 = 0$$

has a $k(x)$-point. Here, Y, Z, and W are homogeneous coordinates in $\mathbb{P}^2$. It thus remains to show that

(i$'''$) *The quadratic form $q(Y,Z,W) = Y^2 - xZ^2 - xW^2$ is isotropic over $k(x)$ if and only if k contains a primitive fourth root of unity.*

Suppose k contains a primitive fourth root of unity. Denote it by i. Then $q(0,i,1) = 0$, so q is isotropic over $k(x)$.

Conversely, assume q is isotropic over $k(x)$. That is, $q(A(x), B(x), C(x)) = 0$ for some $A(x), B(x), C(x) \in k(x)$, not all zero. After clearing denominators, we can assume that $A(x)$, $B(x)$ and $C(x)$ are polynomials with coefficients in k, and $(A(0), B(0), C(0)) \neq (0,0,0)$. Substituting $x = 0$ into the equation $A(x)^2 = x(B(x)^2 + C(x)^2)$, we see that $A(0) = 0$. Thus, the left-hand side is divisible by x^2, and consequently, $B(0)^2 + C(0)^2 = 0$, where $B(0)$ and $C(0)$ are not both 0. This means that neither can be 0, and $B(0)/C(0)$ is a primitive fourth root of unity in k. This completes the proof of (i$'''$) and thus of part (i).

(ii) Set $x = a(1+a), y = u(1+u), z = a + u$. Note that x, y, z are invariant under σ and $[k(a,u) : k(x,y,z)] \leqslant 2$, because $k(a,u)$ is generated by u over $k(x,y,z)$, and u satisfies the quadratic equation $u^2 + u + y = 0$ over $k(x,y,z)$. Using the same argument as in part (i), we see that $k(a)^{\langle\sigma\rangle} = k(x)$, $k(a,u)^{\langle\sigma\rangle} = k(x,y,z)$ and $k(x,y,z)$ is the function field of the affine conic

$$z^2 + z + y + x = 0$$

over $k(x)$. Equivalently, $k(x,y,z)$ is the function field of the projective conic

$$Z^2 + ZW + YW + xW^2 = 0,$$

over $k(x)$, where Y, Z, and W are homogeneous coordinates in $\mathbb{P}^2$. This conic has a $k(x)$-point $(Y : Z : W) = (x : 1 : 1)$. Thus, $k(a,u)^{\langle\sigma\rangle} = k(x,y,z)$ is rational over $k(a)^{\langle\sigma\rangle} = k(x)$. ■

Remark 4.2 In this remark, we will briefly outline an alternative proof of Lemma 4.1, which was suggested to us by the referee.

(i) The argument in [HKK14, p. 371] shows that $k(a,u)^{\langle\sigma\rangle}$ is the function field of the Brauer–Severi variety of the quaternion algebra $(x,-1)$ over the field $k(a)^{\langle\sigma\rangle} = k(x)$, where $x = b^2$, as in our proof. Thus, $k(a,u)^{\langle\sigma\rangle}$ is rational over $k(a)^{\langle\sigma\rangle} = k(x)$ if and only if the quaternion algebra $(x,-1)$ splits over $k(x)$. On the other hand, $(x,-1)$ splits over $k(x)$ if and only if -1 is a square in k; see, *e.g.*, [GS17, Chapter 1, Example 1.3.8].

(ii) follows from [KW14, Theorem 2.2].

5 Conclusion of the Proof of Theorem 1.3

Our goal is to prove parts (ii) and (iii) of Theorem 1.3. We can assume without loss of generality that $S = \langle\sigma\rangle \subset \Sigma_4$, where σ is the 4-cycle $(1\,2\,3\,4)$.

We begin by constructing a convenient birational model for $(\mathbb{P}^1)^4/B$, where the action of Σ_4 is particularly transparent. (Recall that a priori the rational quotient $(\mathbb{P}^1)^4/B$ is only defined up to birational isomorphism.) Let V be the 4-dimensional k-vector space, $x_1, \dots, x_4$ be a basis for the dual space V^*, V_1 is the 1-dimensional subspace of V spanned by the vector $(1,1,1,1)$ and $\overline{V} = V/V_1$. The dual space $\overline{V}^*$ is the 3-dimensional subspace of V^* consisting of linear functions $\lambda_1 x_1 + \cdots + \lambda_1 x_4$ such that $\lambda_1 + \cdots + \lambda_4 = 0$.

The Borel subgroup B of upper triangular matrices in PGL_2 decomposes as a semidirect product $U \rtimes T$, where $T \simeq \mathbb{G}_m$ is the diagonal maximal torus and $U \simeq \mathbb{G}_a$ is the group of matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. One readily checks that the rational quotient $(\mathbb{P}^1)^4/U$ is Σ_4-equivariantly birationally isomorphic to the two-dimensional affine space $\mathbb{A}(\overline{V})$, and the rational quotient $(\mathbb{P}^1)^4/B$ is Σ_4-equivariantly birationally isomorphic to the 2-dimensional projective space

$$\big((\mathbb{P}^1)^4/U\big)/T \simeq \mathbb{A}(\overline{V})/\mathbb{G}_m \simeq \mathbb{P}(\overline{V}).$$

In other words, $F_4 \overset{\text{def}}{=} L_4^B$ is Σ_4-equivariantly isomorphic to the function field of $\mathbb{P}(\overline{V})$ over k. That is,

$$F_4 = k\left(\frac{\lambda_1 x_1 + \cdots + \lambda_4 x_4}{\mu_1 x_1 + \cdots + \mu_4 x_4} \mid \lambda_1 + \ldots \lambda_4 = \mu_1 + \cdots + \mu_4 = 0\right);$$

cf. [Tsu17, formula (4), p. 904].

Proof of Theorem 1.3(ii) Assume $\operatorname{char}(k) \neq 2$. Then the linear functions

$$\begin{aligned} w &= -x_1 - x_2 + x_3 + x_4, \\ y &= -x_1 + x_2 + x_3 - x_4, \\ z &= -x_1 + x_2 - x_3 + x_4 \end{aligned}$$

form a k-basis for $\overline{V}^*$. Note that

$$\begin{aligned} x_4 - x_1 &= \frac{1}{2}(w + z), & x_3 - x_2 &= \frac{1}{2}(w - z), \\ x_4 - x_2 &= \frac{1}{2}(w - y), & x_3 - x_1 &= \frac{1}{2}(w + y). \end{aligned}$$

Now recall that by (1.1),

$$K_4 = k(a), \text{ where } a = \frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_2)(x_3 - x_1)}. \tag{5.1}$$

The generic fiber of the natural projection map $(\mathbb{P}^1)^4/B \to (\mathbb{P}^1)^4/\operatorname{PGL}_2$ is the quadric in $\mathbb{P}^3_{K_4}$ given by

$$(x_4 - x_1)(x_3 - x_2) = a(x_4 - x_2)(x_3 - x_1),$$

or equivalently, $w^2 - z^2 = a(w^2 - y^2)$ or

$$(1 - a)w^2 - z^2 + ay^2 = 0\,.$$

Setting $u = w/y$ and $t = z/y$, we see that $F_4 = k(a, u, t)$, where

$$(1 - a)u^2 - t^2 + a = 0. \tag{5.2}$$

We claim that $F_4^{\langle \sigma^2 \rangle} = k(a, u)$. Indeed, since $\sigma(w) = -y$, $\sigma(y) = w$, $\sigma(z) = -z$, and $\sigma(a) = 1 - a$, we see that

$$\sigma^2(a) = a, \quad \sigma^2(u) = u, \quad \text{and} \quad \sigma^2(t) = -t.$$

Now consider the following diagram of field extensions

(5.3)
$$\begin{tikzcd}
F_4 \arrow[d, dash] \arrow[d, bend left, dash, "\text{degree } 2"] \arrow[dd, bend left=60, dash, "\text{degree} \leqslant 2"] \\
F_4^{\langle\sigma^2\rangle} \arrow[d, dash] \\
k(a,u) \arrow[d, dash] \\
K_4 = k(a).
\end{tikzcd}$$

Here, $[F_4 : F_4^{\langle\sigma^2\rangle}] = 2$, because σ^2 is an automorphism of order 2, and $[F_4 : k(a,u)] \leqslant 2$, because t satisfies a quadratic equation over $k(u,a)$; see (5.2). We conclude that $F_4^{\langle\sigma^2\rangle} = k(a,u)$. This proves the claim.

Note that since $\mathrm{trdeg}_k(F_4^{\langle\sigma^2\rangle}) = \mathrm{trdeg}_k(F_4) = 2$, a and u are algebraically independent over k. The group $S/\langle\sigma^2\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $F_4^{\langle\sigma^2\rangle} = k(a,u)$ by $\sigma\colon a \mapsto 1 - a$ and $\sigma\colon u \mapsto -1/u$. Lemma 4.1(i) now tells us that $F_4^S = (F_4^{\langle\sigma^2\rangle})^{S/\langle\sigma^2\rangle} = k(a,u)^{\langle\sigma\rangle}$ is rational over $K_4^S = k(a)^{\langle\sigma\rangle}$ if and only if k contains a primitive fourth root of unity. ∎

Proof of Theorem 1.3(iii) Now assume that char $k = 2$. The linear functions

$$w = x_1 + x_2 + x_3 + x_4,$$
$$y = x_1 + x_3,$$
$$z = x_1 + x_4$$

form a k-basis for $\overline{V}^*$. Again by (5.1), the fiber of the natural projection map $(\mathbb{P}^1)^4/B \to (\mathbb{P}^1)^4/\mathrm{PGL}_2$ over the generic point $\mathrm{Spec}(K_4) \to (\mathbb{P}^1)^4/\mathrm{PGL}_2$ is the quadric in $\mathbb{P}^3_{K_4}$ given by

$$(x_4 + x_1)(x_3 + x_2) = a(x_4 + x_2)(x_3 + x_1).$$

(Note that in characteristic 2, $x_i - x_j$ is the same as $x_i + x_j$.) In w, y, z-coordinates, this equation can be rewritten as $z(w + z) = a(w + y)y$, or equivalently, as

$$ay^2 + ayw + z^2 + zw = 0.$$

Setting $u = y/w$, $t = z/w$, we see that $F_4 = k(a,u,t)$, where

$$au^2 + au + t^2 + t = 0. \tag{5.4}$$

The action of σ is given by $\sigma(w) = w$, $\sigma(y) = w + y$, $\sigma(z) = w + y + z$. Thus,

$$\sigma^2(w) = w, \quad \sigma^2(y) = y, \quad \sigma^2(z) = w + z.$$

Examining the diagram of field extensions (5.3) and arguing as in the proof of Theorem 1.3(ii), we conclude that $F_4^{\langle\sigma^2\rangle} = k(a,u)$, where a and u are algebraically independent over k. Once again, the group $S/\langle\sigma^2\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts faithfully on $F_4^{\langle\sigma^2\rangle} = k(a,u)$ by $\sigma\colon a \mapsto 1-a = a+1$ and $\sigma\colon u \mapsto u+1$. By Lemma 4.1(ii), $F_4^S = k(a,u)^{\langle\sigma\rangle}$ is rational over $K_4^S = k(a)^{\langle\sigma\rangle}$, as desired. ■

Acknowledgment This paper is based on a summer research project conducted by the first author at the University of British Columbia. Due to the travel restrictions associated with the Covid-19 pandemic, the work was conducted virtually. We are grateful to the anonymous referee for contributing Remark 4.2 and for other constructive comments.

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