ARTICLES

ON THE THREAT
OF COUNTERFEITING

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We study counterfeiting of currency in a search-theoretic model of monetary exchange. In contrast to Nosal and Wallace [*Journal of Monetary Economics* 54, 229–246 (2007)], we establish that counterfeiting does not pose a threat to the existence of a monetary equilibrium; i.e., a monetary equilibrium exists irrespective of the cost of producing counterfeits, or the ease with which genuine money can be authenticated. However, the possibility of counterfeiting fiat money can affect its value, velocity, output, and welfare, even if no counterfeiting occurs in equilibrium. We provide two extensions of the model under which the threat of counterfeiting can materialize: counterfeits can circulate across periods, and sellers set terms of trade in some matches. Policies that make the currency more costly to counterfeit or easier to recognize raise the value of money and society's welfare, but the latter policy does not always decrease counterfeiting.

Keywords: Money, Counterfeiting, Search, Bargaining

1. INTRODUCTION

Recent advances in technology, such as digital printers and higher-quality scanners, have greatly reduced the cost of producing counterfeits, posing serious challenges to currencies all over the world.[1] Despite the extent of counterfeiting being small in the United States—the value of counterfeit currency in 2005 was about one dollar for every \$12,400 in circulation—the U.S. government has responded to this threat by redesigning notes, and by enhancing the public's ability to verify the authenticity of currency through educational programs.[2] To assess the effectiveness of various measures to deter counterfeiting, one needs to understand the exact nature of the threat that counterfeiting poses to economic activity. Does the possibility of counterfeiting threaten the very existence of the fiat money system? Under which conditions does the threat of counterfeiting materialize? And is it necessary

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to continue efforts to deter and suppress counterfeiting even when counterfeiting seems insignificant?

To answer these questions, one needs a model of monetary exchange, where the social benefits of fiat money are explicitly spelled out, and where genuine money is not perfectly recognizable and can be counterfeited at some cost.[3] This is precisely the approach of Nosal and Wallace (2007)—NW hereafter—who study counterfeiting using the off-the-shelf monetary model of Shi (1995) and Trejos and Wright (1995). They derive two main implications: counterfeiting of currency does not occur in equilibrium; the possibility of counterfeiting constitutes a threat to the existence of a monetary equilibrium, but the allocation in a monetary equilibrium is independent of the technology for producing counterfeits and of policies intended to improve the recognizability of genuine money.

The objective of this paper is twofold. First, despite using a similar equilibrium concept, we obtain insights that are in contrast to the main results in NW.[4] We establish that counterfeiting does not pose a threat to the existence of a monetary equilibrium. That is, the cost of producing counterfeits and the recognizability of genuine money do not determine whether fiat money has value. The intuition is simple: because counterfeiting involves a fixed cost, it should not threaten a currency that is almost valueless. However, in contrast with the view that the allocation is independent of the possibility of counterfeiting as long as it is not realized, we find that the cost of producing counterfeits and the recognizability of genuine money can affect the value of fiat money, its velocity, output, and welfare, even though no counterfeiting occurs in equilibrium. Our findings also imply a different objective for anticounterfeiting policies: it is not to prevent the monetary equilibrium from breaking down as suggested in NW; it is to raise output and welfare by mitigating the threat of counterfeiting to the value of money.

Second, the prediction of no counterfeiting in NW is not consistent with the observed counterfeiting of major currencies. In 2005 in the United States, $61 million of counterfeit currency was passed on to the public, 3,717 counterfeiters were arrested, and 611 counterfeiting plants were suppressed.[5] As documented by Mihm (2007), counterfeiting was a widespread phenomenon in the United States during the nineteenth century. Another objective of the paper is thus to reconcile the predictions of the model with the observed counterfeiting, significant or not, in the real world. To do so, we provide two extensions of the model under which the threat of counterfeiting can materialize. The first extension relaxes the assumption of full confiscation in NW. We assume that the government has limited power to withdraw counterfeits, so that individuals who receive them have a chance to pass them on to someone else in the future. We provide a simple condition under which a counterfeiting equilibrium exists, and we show that this equilibrium is separating—in meetings where the quality of money is not recognized, genuine buyers and counterfeiters make different offers.[6] Counterfeiting is more likely to take place if the stock of genuine money is low, if the cost of producing counterfeits is small, and if the ability of the government to confiscate counterfeits is limited.

The second extension consists of assuming that sellers can set terms of trade in some matches. Counterfeiting is shown to prevail even under full confiscation. When the quality of money is not recognized, no trade takes place if buyers make the offer, whereas terms of trade are pooled if sellers make the offer. This implies that sellers accept counterfeits in the instances where they cannot verify the authenticity of currency. The results also demonstrate the distortions that counterfeiting generates in economic activity: on some occasions, mutually beneficial trades cannot take place, while on others, trades occur but terms of trade are distorted by the private information problem. By enhancing the public's ability to authenticate currency, the government can increase the value of money and society's welfare, but it may not always decrease the extent of counterfeiting.

The rest of the paper is organized as follows. Section 2 presents the environment. In Sections 3 and 4 we analyze a simple game where sellers are always uninformed and counterfeits are confiscated at the end of each period. In Section 5 we consider the case of heterogeneously informed sellers. Counterfeits are allowed to circulate across periods in Section 6, and sellers are endowed with some bargaining power in Section 7.

2. THE MODEL

The environment is similar to the one in Shi (1995) and Trejos and Wright (1995). Time is indexed by $t \in \mathbf{N}$. There are a large number of perfectly divisible and perishable goods and a unit measure of agents who are specialized in the goods they produce and consume. Agents do not consume their own output. Each agent's preference at time 0 is

$$\mathbf{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left[u(q_t) - h_t \right] \right],$$

where $q_t \in \mathbf{R}_+$ is consumption at time t, $h_t \in \mathbf{R}_+$ is the effort devoted to production, and $\beta = (1 + r)^{-1} \in (0, 1)$ is the discount factor. We assume that $u(q)$ is twice continuously differentiable, strictly increasing, and strictly concave. Moreover, $u(0) = 0$, and there exists $q^* > 0$ such that $u'(q^*) = 1$. Producing q units of output incurs disutility q.

Agents trade in bilateral matches. The pattern of specialization rules out double–coincidence of wants matches, but it allows single–coincidence matches where one agent wishes to consume the good produced by his partner, but not vice versa. Agents cannot commit to future actions and the trading histories are private information, which eliminates the possibility of credit arrangements. Thus, trade must involve a tangible medium of exchange.

There are two types of money, genuine fiat money and counterfeits. Both objects are indivisible, perfectly durable, and storable. For tractability, we assume money holdings lie in the set $\{0, 1\}$. The quantity of genuine money is $m \in (0, 1)$. Agents who do not hold genuine money at the beginning of a period decide whether or

not to produce counterfeits at a utility cost $k > 0$. As a result of this decision, the measure of agents holding counterfeit notes is $n \in [0, 1 - m]$. At the end of each period, the government is able to take out of circulation a fraction $\delta \in (0, 1]$ of counterfeits.[7] We assume that, though the government confiscates counterfeits, it does not know the identity or trading histories of agents, so the anonymity of agents is preserved. The measure of agents without money is $1 - m - n$. Notice the key differences between genuine and counterfeit monies: whereas the former are in fixed supply, the latter are privately produced and subject to confiscation by the government.

Only meetings between agents holding money (genuine or counterfeit) and agents without money occur. In particular, two money-holders never meet.[8] We assume a simple matching technology where the probability of a buyer being matched with a seller is equal to the measure of sellers in the market, $1 - m - n$.[9] The probability of a single coincidence match for an agent without money is $m + n$. Conditional on being matched, the producer meets a money-holder with genuine money with probability $m/(n + m)$.

In a match, a producer is informed about the quality of the money of his partner with probability η, in which case there is complete information in the match. With the complementary probability $1 - \eta$ the producer is uninformed and there is one-sided incomplete information. After matches are terminated, all agents learn the quality of their money holdings.

Terms of trade in bilateral matches are determined by a bargaining protocol where the buyer is chosen to make a take-it-or-leave-it offer with probability θ, whereas with the complementary probability $1 - \theta$ the seller is the one to make the offer. We assume $\theta = 1$ throughout except in Section 7, where we assume $\theta < 1$. If the other agent in the match accepts the offer, trade is implemented. Agents can offer lotteries to overcome the indivisibility of money, as in Berentsen et al. (2002). The ability to make two-dimensional offers permits buyers with genuine money to potentially separate themselves from counterfeiters.[10]

3. A SIMPLE COUNTERFEITING GAME

Our main insights on how the threat of counterfeiting affects the equilibrium can be presented in a simple environment where sellers are always uninformed ($\eta = 0$), counterfeits cannot circulate across periods ($\delta = 1$), and only buyers make offers ($\theta = 1$). The first assumption allows us to focus on the determination of terms of trade in uninformed matches, which is key for the incentives to produce counterfeits. The second assumption (made in NW) implies that a counterfeit has no value for someone who holds it after matches are dissolved. The third assumption (also used in NW) allows us to focus on a bargaining game that has been used extensively in the literature. All these assumptions will be relaxed in the subsequent sections.

We first analyze the one-period game starting at the beginning of a period, taking as given the discounted continuation values of a money holder, υ_1, and an

agent without money, υ_0. Later we will consider infinitely lived agents that play the game repeatedly in a stationary economy. Formally, the game is defined as follows.

Players. A measure $m \in (0, 1)$ of buyers holding one unit of genuine money and a measure $1 - m$ of agents without money.

Game Tree. First, each agent without money chooses to produce a counterfeit ($\chi = c$) or to become a seller ($\chi = s$). Second, buyers and sellers are matched at random. Third, each matched buyer makes an offer $(q, p) \in \mathbf{R}_+ \times [0, 1]$, where q is the output produced by the seller and p is the probability that the buyer hands over his unit of money. Fourth, matched sellers accept or reject the offers they receive.

Information Structure. The type of money held by an agent is private information. The offers and acceptance decisions in a trade are private information to a match.

Payoffs. The payoff of an agent without money is represented by the following von Neumann–Morgenstern utility function:

$$U_0(\chi, q^b, q^s, d) = -k\mathbf{I}_{\{\chi=c\}} + u(q^b) - q^s + \upsilon_0\mathbf{I}_{\{d=0\}} + \upsilon_1\mathbf{I}_{\{d=1\}},$$

where $\mathbf{I}_A$ is an indicator function equal to one if A holds, $\chi \in \{c, s\}$ is the agent's decision to produce a counterfeit, $q^b \in \mathbf{R}_+$ is his consumption, $q^s \in \mathbf{R}_+$ his production, and $d \in \{0, 1\}$ his money holdings at the end of the game. (Recall that due to the single-coincidence matches, $q^b q^s = 0$.) Similarly, the payoff of an agent holding one unit of genuine money is represented by

$$U_1(q^b, d) = u(q^b) + \upsilon_0\mathbf{I}_{\{d=0\}} + \upsilon_1\mathbf{I}_{\{d=1\}}.$$

A (pure) strategy for an agent without money is composed of the decision to produce a counterfeit ($\chi = c$) or become a seller ($\chi = s$), which offer (q, p) to make if $\chi = c$, and which offer to accept if $\chi = s$. A strategy for a genuine buyer is an offer (q, p). The seller's belief that he faces a genuine buyer following an offer (q, p) is denoted $\lambda(q, p)$. We will assume that all sellers share the same belief system.

An equilibrium of this game is composed of a strategy for each player and a belief system λ that satisfy the following requirements:

Sequential Rationality. Given their belief λ, sellers accept offers that yield a nonnegative surplus, i.e., $-q + p\lambda(q, p)(\upsilon_1 - \upsilon_0) \geq 0$.[11] Offers made by genuine buyers and counterfeiters are best responses to sellers' acceptance rule. [Because the no-trade offer $(0, 0)$ is always accepted, we can, with no loss of generality,

require that buyers only make offers that are accepted.] The offer of a genuine buyer solves

$$\left(q_b^u, p_b^u\right) = \arg\max_{q, p \leq 1} \left[u(q) - p(\upsilon_1 - \upsilon_0)\right] \quad \text{s.t.} \quad -q + p\lambda(q, p)(\upsilon_1 - \upsilon_0) \geq 0, \tag{1}$$

whereas the offer of a counterfeiter solves

$$\left(q_c^u, p_c^u\right) = \arg\max_{q, p \leq 1} u(q) \quad \text{s.t.} \quad -q + p\lambda(q, p)(\upsilon_1 - \upsilon_0) \geq 0. \tag{2}$$

The decision of an agent without money to produce a counterfeit or not is optimal given the measure of other agents who choose to counterfeit and the offers and acceptance rules in the bargaining game, i.e.,

$$\chi \begin{cases} = c & > \\ = s & \text{if } -k + (1 - m - n)u\left(q_c^u\right) < m\left[-q_b^u + p_b^u(\upsilon_1 - \upsilon_0)\right] + n(-q_c^u). \\ \in \{c, s\} & = \end{cases} \tag{3}$$

Consistency of Beliefs with Strategies. For any offer (q, p) made in equilibrium, $\lambda(q, p)$ is derived from Bayes' rule.

Because there is little discipline on the out-of-equilibrium beliefs, this game admits a large number (continuum) of sequential equilibria. In particular, any offer that is acceptable to sellers can be part of an equilibrium by using the threat that all other offers are attributed to counterfeiters. The following lemma shows that some strategies for agents without money are strictly dominated.

LEMMA 1. *The strategy of an agent without money that consists of producing a counterfeit ($\chi = c$) and offering (q, p) such that $-k + (1 - m)u(q) < 0$ is strictly dominated by the strategy that consists of becoming a seller ($\chi = s$) and rejecting all offers.*

A natural requirement is that a seller assigns no probability to his opponent playing a strictly dominated strategy. Together with Lemma 1, this leads to the following refinement of the belief system.[12]

REFINEMENT 1a (Elimination of Strictly Dominated Strategies). $\lambda(q, p) = 1$ *for all (q, p) such that $-k + (1 - m)u(q) < 0$.*

Refinement 1a simply says that, if a seller receives an offer such that the expected utility of consumption adjusted by the matching probability in an economy without counterfeiters is less than the cost of producing a counterfeit, then he should attribute it to a genuine buyer.

LEMMA 2. *The infimum for the surplus of a genuine buyer in a bilateral match for all belief systems λ that satisfy Refinement 1a is*

$$\max_{q,p\leq 1} [u(q) - p(v_1 - v_0)], \tag{4}$$

$$s.t. \quad -q + p(v_1 - v_0) \geq 0, \tag{5}$$

$$-k + (1-m)u(q) \leq 0. \tag{6}$$

Provided that $k > 0$, a genuine buyer can always secure a strictly positive surplus, even though the seller cannot recognize the quality of his money. Intuitively, a genuine buyer can always ask for some low level of output that does not generate enough consumption value to cover the entry cost of a counterfeiter and, hence, the offer would be attributed with probability one to a buyer holding genuine money. Moreover, the higher the production cost of a counterfeiter, the larger the surplus that the genuine buyer can secure.[13] We will see later that this is the insight that explains the robustness of the monetary equilibrium.

In order to refine the beliefs further, we follow NW and adopt a forward-induction argument, in the spirit of the Intuitive Criterion of Cho and Kreps (1987).[14]

REFINEMENT 1b (Forward Induction). *Consider an equilibrium where $\mathcal{U}_1$ and $\mathcal{U}_0$ are the expected payoffs of a genuine buyer and an agent without money, respectively, and n is the measure of counterfeiters. The proposed equilibrium is disqualified if there exists an out-of-equilibrium offer (q', p') such that*

$$-k + (1 - n - m)u(q') + v_0 < \mathcal{U}_0, \tag{7}$$

$$(1 - n - m)[u(q') - p'(v_1 - v_0)] + v_1 > \mathcal{U}_1, \tag{8}$$

$$-q' + p'(v_1 - v_0) \geq 0. \tag{9}$$

From (7), an agent without money would not choose to produce a counterfeit to make an offer (q', p') that reduces his payoff compared to the proposed equilibrium, irrespective of how a seller would interpret this offer (even under the most favorable beliefs of sellers). From (8), the offer (q', p') would benefit a genuine buyer if it were to be accepted; from (9), it is acceptable provided that $\lambda(q', p') = 1$.[15]

PROPOSITION 1. *Under Refinement 1b, there is no equilibrium with $n > 0$. In any equilibrium with $n = 0$ the offer made by a genuine buyer solves (4)–(6).*

There cannot be a pooling offer with active trades under Refinement 1b (same as Proposition 1 in NW). If such a pooling offer were made in equilibrium then a buyer with genuine money could signal the quality of his currency by proposing an (out-of-equilibrium) offer where he transferred money with a smaller probability and consumed less output.

Under the Cho–Kreps refinement the genuine buyer's payoff is exactly the solution to (4)–(6). This suggests that the refinement we use does not confer an additional advantage on fiat money relative to Refinement 1a. Hence, we conjecture that if there is a monetary equilibrium under our refinement, there should also be a monetary equilibrium under an alternative specification for sellers' beliefs, provided that they comply with Refinement 1a.

According to (4)–(6), genuine buyers choose the offer that maximizes their payoff and that deters counterfeiting. The terms of trade in bilateral matches solve

$$
q_b^u = \begin{cases} \min\left[q^*, \upsilon_1 - \upsilon_0\right] & \text{if } u(\min\left[q^*, \upsilon_1 - \upsilon_0\right]) \le \dfrac{k}{1-m} \quad \textbf{(10)} \\[2mm] u^{-1}\left(\dfrac{k}{1-m}\right) & \text{otherwise.} \quad\quad\quad\quad\quad\quad \textbf{(11)} \end{cases}
$$

Moreover,

$$
p_b^u = \frac{q_b^u}{\upsilon_1 - \upsilon_0}. \tag{12}
$$

Provided that k is sufficiently large, buyers make the same offer as the one they would make in an economy where counterfeiting was not even a possibility. However, if k is not too large, so that (6) binds, then buyers lower the output they ask for to deter entry in counterfeiting. A belief system of sellers to sustain the offer (q_b^u, p_b^u) as an equilibrium offer is such that all offers that satisfy $-k + (1 - m)u(q) \le 0$ are attributed to genuine buyers, whereas all other offers are attributed to an agent without money who chose to be a counterfeiter. As a consequence, all offers that violate (6) are rejected by sellers.

In the equilibrium of the one-period game, all agents without money choose to be sellers. Genuine buyers propose the solution (q_b^u, p_b^u) to (10)–(12). Agents without money who would have chosen to produce counterfeits would offer the highest q consistent with $-q + p(\upsilon_1 - \upsilon_0) \ge 0$ and $-k + (1 - m)u(q) \le 0$. Sellers accept all offers such that $-q + p(\upsilon_1 - \upsilon_0) \ge 0$ and $-k + (1 - m)u(q) \le 0$.

4. THE THREAT OF COUNTERFEITING IN EQUILIBRIUM

So far we have studied the game that takes place within a period, taking as given the continuation values υ_1 and υ_0. We now consider our model where the game repeats itself every period and the time horizon is infinite. An agent's strategy is restricted not to depend on his past private histories, because such histories are payoff-irrelevant. Moreover, we focus on stationary equilibria where υ_0 and υ_1 are constant across periods.

From Proposition 1, $n = 0$ and $\upsilon_0 = 0$. The value of a genuine buyer, υ_1, solves the following flow Bellman equation:

$$
r\upsilon_1 = (1 - m)\left[u\left(q_b^u\right) - q_b^u\right], \tag{13}
$$

where $r = \beta^{-1} - 1$ and q_b^u solves (4)–(6).[16] As is standard, (13) states that the flow return of holding genuine money is the expected surplus from a match in case trade occurs.

A monetary equilibrium in this economy consists of $\langle q_b^u, p_b^u, \upsilon_1 \rangle$ such that (q_b^u, p_b^u) solves (4)–(6) and $\upsilon_1 > 0$ solves (13).

PROPOSITION 2. *There exists a unique monetary equilibrium (under Refinement 1b) iff*

$$r < (1 - m)[u'(0) - 1]. \tag{14}$$

According to Proposition 2, the possibility of counterfeiting does not constitute a threat to the existence of a monetary equilibrium, in the sense that the set of parameters that determine whether fiat money is valued does not include the cost to produce counterfeits.[17] In particular, if $u'(0) = +\infty$ then there always exists a monetary equilibrium even if fiat money can be counterfeited and sellers are unable to recognize the quality of the money held by buyers. This result relies on a key characteristic of counterfeiting activities—a fixed entry cost. From Lemma 2, it is this fixed cost that secures a positive surplus for genuine buyers, taking υ_1 as given. Put it differently, nobody would pay the cost to imitate an object which has almost no value, so that the existence of valued fiat money is never threatened.

The threat of counterfeiting, however, matters for allocation and welfare. Let υ_1^* be the unique positive solution to $r\upsilon_1 = (1 - m)\{u[q(\upsilon_1)] - q(\upsilon_1)\}$, where $q(\upsilon_1) = \min(q^*, \upsilon_1)$. Here υ_1^* is the value of genuine money in an economy where there is no possibility of counterfeiting currency.

PROPOSITION 3. *Suppose $r < (1 - m)[u'(0) - 1]$. Then there is $\bar{k} = (1 - m)u[\min(q^*, \upsilon_1^*)]$ such that for all $k < \bar{k}$, $d\upsilon_1/dk > 0$, $dq_b^u/dk > 0$, and $dp_b^u/dk > 0$. As $k \to 0$, $q_b^u \to 0$ and $\upsilon_1 \to 0$.*

Even though no counterfeiting takes place in equilibrium, the mere possibility of counterfeiting affects the equilibrium provided that the cost of producing counterfeits is not too large. This is another example where an out-of-equilibrium threat affects the equilibrium outcome. Intuitively, buyers never trade more than the quantity that would give incentives to agents without money to produce counterfeits; i.e., the existence of a noncounterfeiting equilibrium can be a result of market discipline. Even so, our model legitimizes policies that consist of making a currency harder to counterfeit: by raising the cost of producing counterfeits, the policy makers can increase the velocity of money, output, and welfare. Notice that the monetary equilibrium disappears at the limit when the cost to produce counterfeits is driven to zero. If the cost is above a certain threshold, the possibility of counterfeiting cannot affect the economic activity so that the threat of counterfeiting is inactive.

5. HETEROGENEOUS INFORMATION

We extend the model to allow sellers to receive informative signals about the authenticity of the money held by buyers, i.e., $\eta \in (0, 1)$. The objective is twofold. First, by introducing a proxy for the recognizability of money, we will be able to assess the effects of advertising campaigns that make the public more able to verify the authenticity of currency. Second, we will show that under some conditions the terms of trade can vary across matches depending on whether the quality of money is recognized or not.

The signal received by the seller is common knowledge in the match. The definition of equilibrium must be adapted so that the strategy of an agent without money specifies his acceptance rule depending on the signal he receives. Similarly, the strategy of a buyer must specify an offer as a function of the signal received by the seller he is matched with. The terms of trade in informed matches solve

$$\left(q_b^i, p_b^i\right) = \arg \max_{q,p} \left[u(q) - p\upsilon_1\right] \quad \text{s.t.} \quad -q + p\upsilon_1 \geq 0, \tag{15}$$

if the buyer holds a genuine unit of money. If the buyer in the match holds a counterfeit then $q_c^i = 0$ and $p_c^i \in [0, 1]$.

Regarding the terms of trade in uninformed matches, we adopt a forward-induction argument similar to Refinement 1b. Following the same logic as in Proposition 1, the equilibrium of the counterfeiting game involves no counterfeiting, $n = 0$, and the terms of trade in uninformed matches solve

$$\left(q_b^u, p_b^u\right) = \arg \max_{q,p} \left[u(q) - p\upsilon_1\right] \tag{16}$$

$$\text{s.t.} \quad -q + p\upsilon_1 \geq 0, \tag{17}$$

$$-k + (1 - m)(1 - \eta)u(q) \leq 0. \tag{18}$$

A belief system that sustains (q_b^u, p_b^u) as an equilibrium offer is $\lambda(q, p) = 1$ if (18) holds, and $\lambda(q, p) = 0$ otherwise. The novelty in (18) comes from the fact that an agent who chooses to produce a counterfeit can obtain some output only when the quality of his money is not recognized, with probability $1 - \eta$.

The value of genuine money in a noncounterfeiting equilibrium solves

$$r\upsilon_1 = (1 - m)\eta \left[u\left(q_b^i\right) - q_b^i\right] + (1 - m)(1 - \eta)\left[u\left(q_b^u\right) - q_b^u\right]. \tag{19}$$

A monetary equilibrium is a list $\langle(q_b^i, p_b^i), (q_b^u, p_b^u), \upsilon_1\rangle$ such that (q_b^i, p_b^i) solves (15) and (q_b^u, p_b^u) solves (16)–(18), and $\upsilon_1 > 0$ solves (19). There exists a unique monetary equilibrium under condition (14).

We establish the effects of the recognizability of money on economic activity in the following proposition.

PROPOSITION 4. *Suppose $r < (1 - m)[u'(0) - 1]$ holds. Then there is $\bar{k} = (1 - m)(1 - \eta)u[\min(q^*, \upsilon_1^*)]$ such that*

(1) *For all $k < \bar{k}$, $dq_b^u/d\eta > 0$ and $d\upsilon_1/d\eta > 0$. Moreover, $q_b^u < q_b^i$ and $p_b^u < p_b^i$.*
(2) *For all $k \geq \bar{k}$, $(q_b^u, p_b^u) = (q_b^i, p_b^i)$.*

In equilibrium, sellers know that they meet genuine buyers with probability one. However, provided that the cost of producing counterfeits is sufficiently low, agents trade lower quantities and spend their money with a lower probability in matches where the authenticity of money is not verified. The recognizability of money has real effects—it matters for the distribution of terms of trade, the value and velocity of money, output, and welfare. Once again, even though no counterfeiting occurs in equilibrium, the possibility of counterfeiting (an out-of-equilibrium threat) affects the economic activity. Also, in contrast to standard search equilibrium models, a distribution of terms of trade emerges, even though agents on both sides of the market are homogeneous.

6. THE MATERIALIZING THREAT: CIRCULATING COUNTERFEITS

The model with full confiscation that we have considered so far predicts no counterfeiting in any equilibrium (as in NW). One key assumption that prvents counterfeiting from emerging is that counterfeit money cannot circulate across periods. To account for the extent of counterfeiting, in this section we assume that the government has limited ability to take the counterfeits out of circulation: only a fraction $\delta \in (0, 1)$ of the counterfeits are confiscated at the end of each period. We show that counterfeiting can occur in a monetary equilibrium, and study the effectiveness of the policies against counterfeiting.[18]

As a consequence of the assumption that $\delta \in (0, 1)$, agents at the beginning of each period can be divided into three types: agents holding genuine money, agents holding counterfeits inherited from the previous trades, and agents without money. Let υ_c denote the present value of an agent holding a counterfeit at the beginning of the next period. We denote $\omega_1 \equiv \upsilon_1 - \upsilon_0$ the value of genuine money and $\omega_c \equiv (1 - \delta)(\upsilon_c - \upsilon_0)$ the value of a counterfeit.

As in the preceding sections, an equilibrium consists of a profile of strategies (i.e., offers, acceptance rules, and decisions to produce counterfeits) and a belief system for sellers that satisfy sequential rationality and consistency of beliefs with strategies. The determination of the terms of trade in informed matches is given by (15) with υ_1 replaced by ω_1, if the buyer holds a unit of genuine money, and

$$(q_c^i, p_c^i) = \arg \max_{q, p} [u(q) - p\omega_c] \quad \text{s.t.} \quad -q + p\omega_c \geq 0, \qquad (20)$$

if the buyer holds a counterfeit (irrespective of whether it has been produced or acquired in past trades). In uninformed matches an agent holding a counterfeit

makes an offer that solves

$$\left(q_c^u, p_c^u \right) = \arg \max_{q, p \leq 1} \left[u(q) - p\omega_c \right]$$

$$\text{s.t.} \quad -q + p\{\lambda(q, p)\omega_1 + [1 - \lambda(q, p)]\omega_c\} \geq 0. \tag{21}$$

In this section we look for counterfeiting equilibria with $n > 0$ and $\omega_1 > \omega_c$, i.e., genuine money is more valuable than counterfeits. We adopt the following forward-induction argument to refine sellers' beliefs in uninformed matches.

REFINEMENT 2. *Consider a proposed equilibrium with $n > 0$ where the trade surpluses of a genuine buyer and a counterfeiter in uninformed matches are $\mathcal{S}_1^u$ and $\mathcal{S}_c^u$, respectively. This equilibrium is disqualified if there is an out-of-equilibrium offer (q', p') such that the following is true:*

$$u(q') - p'\omega_1 > \mathcal{S}_1^u, \tag{22}$$

$$u(q') - p'\omega_c < \mathcal{S}_c^u, \tag{23}$$

$$-q' + p'\omega_1 \geq 0. \tag{24}$$

According to (22), the out-of-equilibrium offer (q', p') would make a buyer with genuine money strictly better off if it were accepted. According to (23), the offer (q', p') would make a buyer with a counterfeit strictly worse off. That is, an agent without money would not choose to produce a counterfeit to make such an offer, because in the proposed counterfeiting equilibrium he is just indifferent between being a seller or a counterfeiter. According to (24), the offer is acceptable provided that the seller believes it comes from a buyer with genuine money.

LEMMA 3. *The offer made by a counterfeiter in an uninformed match solves*

$$\left(q_c^u, p_c^u \right) = \arg \max_{q, p \in [0, 1]} \left[u(q) - p\omega_c \right] \quad \text{s.t.} \quad -q + p\omega_c \geq 0. \tag{25}$$

The offer made by a genuine buyer solves

$$\left(q_b^u, p_b^u \right) = \arg \max_{q, p \in [0, 1]} \left[u(q) - p\omega_1 \right] \quad \text{s.t.} \quad -q + p\omega_1 \geq 0, \tag{26}$$

$$u(q) - p\omega_c \leq u\left(q_c^u \right) - p_c^u \omega_c. \tag{27}$$

There cannot be a pooling offer with active trades under our forward-induction refinement. If such a pooling offer were made in equilibrium, then a buyer with genuine money could signal the quality of his currency by proposing an (out-of-equilibrium) offer where he transferred money with a lower probability and he consumed less output. The logic is similar to that used to dismiss pooling equilibria in Proposition 1. A belief system consistent with the offers in (25)–(27) is such that sellers attribute all offers that violate (27) to buyers with counterfeits, and all other out-of-equilibrium offers to buyers with genuine money.

The solution to the problem of the genuine buyer is characterized in the following lemma.

LEMMA 4. *If $\omega_c > 0$ then there is a unique solution to (26) and (27), and it is such that*

$$p_b^u = \frac{q_b^u}{\omega_1}, \tag{28}$$

$$u\left(q_b^u\right) - \frac{\omega_c}{\omega_1} q_b^u = u\left(q_c^u\right) - q_c^u. \tag{29}$$

Moreover, $q_b^u < q_c^u$ and $p_b^u < p_c^u$.

If $\omega_c > 0$ then the offer is separating and the buyer with genuine money obtains less output than the buyer with a counterfeit, but spends his money with a lower probability.

From the characterization of the terms of trade above, it is immediate that $\upsilon_0 = 0$. The flow Bellman equation for the value of an agent holding a counterfeit is[19]

$$r\upsilon_c = (1 - m - n)\left[u\left(q_c^i\right) - q_c^i\right] - \delta\upsilon_c. \tag{30}$$

From Lemma 3, counterfeiters make their complete-information offer in all matches; i.e., $q_c^i = q_c^u = q_c = \min[q^*, \omega_c]$ and $p_c = q_c/\omega_c$. The last term on the right-hand side of (30) captures the fact that counterfeits are confiscated with probability δ. Using the definition $\omega_c \equiv (1 - \delta)\upsilon_c$, it becomes

$$\left(\frac{r + \delta}{1 - \delta}\right)\omega_c = (1 - m - n)\left[u\left(q_c^i\right) - q_c^i\right]. \tag{31}$$

Counterfeiting occurs in equilibrium if $\omega_c = \beta(1 - \delta)k$, which from (31) implies that

$$1 - m - n = \frac{[1 - (1 - \delta)\beta]\,k}{u(q_c^i) - q_c^i}. \tag{32}$$

The value of genuine money, $\upsilon_1 = \omega_1$, obeys

$$r\upsilon_1 = (1 - m - n)\left\{(1 - \eta)\left[u\left(q_b^u\right) - q_b^u\right] + \eta\left[u\left(q_b^i\right) - q_b^i\right]\right\}. \tag{33}$$

Given that $\upsilon_0 = 0$, we define a monetary equilibrium as follows.

DEFINITION 1. *A counterfeiting equilibrium is a list $\langle(q_b^i, p_b^i), (q_b^u, p_b^u), (q_c^i, p_c^i), (q_c^u, p_c^u), \upsilon_1, \upsilon_c, n\rangle$ that satisfies (i) the equations for the determination of the terms of trade, (15), (20), (25)–(27); (ii) the Bellman equations, (30) and (33); (iii) the entry condition in the counterfeiting sector, (32); and (iv) the conditions for a monetary equilibrium, $\upsilon_1 > 0$, and for genuine money to be more valuable than counterfeits, $\upsilon_1 > (1 - \delta)\upsilon_c$.*

The counterfeiting equilibrium has a recursive structure. Equation (32) determines n, whereas, given n, (33) generates υ_1.

PROPOSITION 5. *If*

$$r' < (1 - m)[u'(0) - 1], \tag{34}$$

where $r' = [1 - (1 - \delta)\beta]/(1 - \delta)\beta$, then there is $\bar{k} > 0$ such that for all $k < \bar{k}$ there exists a counterfeiting equilibrium.

If $u'(0) = +\infty$ then there always exists a counterfeiting equilibrium provided that the cost of counterfeiting is small. Counterfeiting can prevail in equilibrium, even though counterfeiters and genuine buyers make different offers; i.e., people accept counterfeits knowingly. Just as with genuine money, there is a chance that counterfeits can be passed on to someone else and traded for some output in the future. From Proposition 5, counterfeiting is more likely to take place if the stock of genuine money is low, if the cost of producing counterfeits is small, and if the ability of the government to confiscate counterfeits is limited.

We now study the effects of various policies on the extent of counterfeiting and the value of money. Let $n(m, k, \eta, \delta)$ and $\omega_1(m, k, \eta, \delta)$ denote the equilibrium measure of counterfeiters and the equilibrium value of genuine money as functions of exogenous variables.

PROPOSITION 6. *Assume (A.3) holds. Then (i) $\partial n/\partial m = -1$ and $\partial \omega_1/\partial m = 0$; (ii) $\partial n/\partial k < 0$ and $\partial \omega_1/\partial k > 0$; (iii) $\partial n/\partial \eta = 0$ and $\partial \omega_1/\partial \eta > 0$; (iv) $\partial n/\partial \delta < 0$ and $\partial \omega_1/\partial \delta > 0$.*

The fraction of agents holding a unit of money (genuine or fake) is uniquely pinned down by the free entry in the counterfeiting sector. Consequently, an increase in the supply of genuine money has a complete crowding-out effect on counterfeits, leaving the value of genuine money unchanged. By endowing a larger number of buyers with genuine money, the monetary authority reduces the available space to counterfeiters.

In terms of anticounterfeiting policies, a higher confiscation rate of counterfeits reduces the extent of counterfeiting and raises the value of genuine money. An increase in the recognizability of genuine money (η) raises the value of genuine money because the surplus enjoyed by genuine buyers is greater in informed matches than in uninformed matches. The improved recognizability, however, does not affect counterfeiting activity, because in a separating equilibrium the offers made by counterfeiters depend on the production cost of counterfeits but not on the value of genuine money.

Society's welfare is measured by the sum of all agents' expected lifetime utility; i.e., $W = \beta^{-1}[(1 - m - n)\upsilon_0 + m\upsilon_1 + n\upsilon_c] - \delta nk$, where the last term comes from the fact that a fraction δ of the counterfeits are confiscated and replaced in every period. Because $\upsilon_0 = 0$ and $\upsilon_c = \beta k$, we have

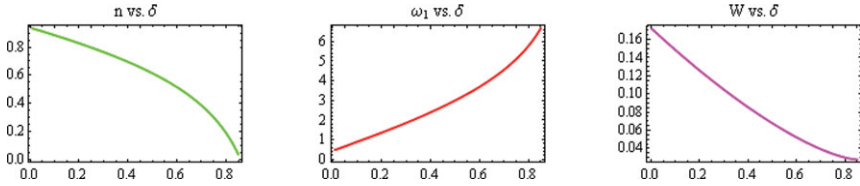$$W = \beta^{-1}(m\omega_1 + n\omega_c). \tag{35}$$

FIGURE 1. Confiscation of counterfeits under a currency shortage.

The society's welfare is equal to the aggregate real balances in the economy, which also include the real value of counterfeits.

PROPOSITION 7. *Suppose (A.3) holds. Then $dW/dm > 0$ and $dW/d\eta > 0$.*

An increase in the stock of genuine money raises welfare by replacing one-for-one counterfeits with more valuable genuine money. An increase in the fraction of informed matches raises welfare by making genuine money more valuable.

Whether a higher confiscation rate δ can improve welfare depends on the stock of genuine money in the economy. In Figure 1, we consider a numerical example where the stock of genuine money is seriously low, $m = 0.01$, a situation that can be interpreted as a currency shortage.[20] An increase in the confiscation rate deters counterfeiting and raises the value of genuine money, but it lowers welfare. Although it is believed that counterfeiting does harm to the economy and should be eliminated, our example suggests that the policy makers should let them circulate as long as the government cannot control the supply of genuine money and there is a severe currency shortage.[21]

To conclude this section, we investigate the limiting case where the government has no enforcement power and cannot confiscate counterfeits; i.e., we consider the limit of the equilibrium as $\delta \to 0$. The following proposition shows the existence of a counterfeiting equilibrium under an extremely lax anticounterfeiting policy.

PROPOSITION 8. *Consider the limit economy as $\delta \to 0$. If*

$$r < (1 - m)[u'(0) - 1],$$

then there is $\bar{k} > 0$ such that for all $k < \bar{k}$ there exists a counterfeiting equilibrium, and it is such that $\omega_1 = \omega_c = \beta k$.

Notice that the condition for the existence of a counterfeiting equilibrium here is identical to the condition for the existence of a monetary equilibrium in the previous sections (e.g., Proposition 2), with an additional requirement that $k < \bar{k}$. When the confiscation rate tends to zero, counterfeits circulate as genuine money does. Furthermore, counterfeits and genuine money trade at the same value, which, given $\omega_c = \beta k$, is determined by the entry cost in the counterfeiting sector.

7. THE MATERIALIZING THREAT: POOLING OFFERS

The counterfeiting equilibria obtained in Section 6 have the feature that genuine buyers are able to separate themselves from counterfeiters by making an offer that the latter would not imitate. As a consequence, sellers accept counterfeits knowingly. In this section we consider the model as in Section 5 but assume $\theta < 1$. We will show that if sellers are informed in some matches ($\eta > 0$) a counterfeiting equilibrium can exist even if counterfeits cannot circulate across periods ($\delta = 1$).[22] In this equilibrium, sellers accept counterfeits in the instances where they cannot recognize the authenticity of currency.

In informed matches, buyers offer a solution (q_b^i, p_b^i) to (15), whereas sellers offer (q_s^i, p_s^i) if the buyer holds a genuine note, with

$$\left(q_s^i, p_s^i\right) = \arg \max_{q,\, p \leq 1} \left[-q + p(v_1 - v_0)\right] \quad \text{s.t.} \quad u(q) - p(v_1 - v_0) \geq 0. \quad \textbf{(36)}$$

The solution to (36) is $q_s^i = \min[q^*, u^{-1}(\omega_1)]$ and $p_s^i = u(q_s^i)/\omega_1$. In uninformed matches, sellers may offer a menu of terms of trade as an attempt to screen buyers holding genuine money. We impose that sellers cannot commit to terms of trade that are not ex post individually rational.

LEMMA 5. *In uninformed matches, sellers offer a single contract solution to*

$$\left(q_s^u, p_s^u\right) = \arg \max \left[\frac{m}{n + m} p(v_1 - v_0) - q\right] \quad \text{s.t.} \quad u(q) - p(v_1 - v_0) \geq 0. \quad \textbf{(37)}$$

Given that counterfeits have no value, sellers cannot credibly offer to accept a counterfeit, and hence they must offer terms of trade that pool genuine buyers and counterfeiters. According to (37), sellers make an offer that maximizes the expected value of the currency they receive minus the production cost, taking into account the acceptance rule of genuine buyers only. The solution to (37) is $q_s^u = \min[u'^{-1}(1 + n/m), u^{-1}(\omega_1)]$ and $p_s^u = u(q_s^u)/\omega_1$.

We focus on a monetary equilibrium with counterfeiting, i.e., $n > 0$. The following lemma shows that no trade takes place in uninformed matches when buyers make the offer.

LEMMA 6. *In any counterfeiting equilibrium ($n > 0$), buyers in uninformed matches offer $(q_b^u, p_b^u) = (0, 0)$.*

Under the Intuitive Criterion (Refinement 1b), a genuine buyer can break out of a pooling equilibrium by offering to consume less output and to transfer money with a lower probability. The no-trade outcome is sustained by the sellers' belief that any offer with positive output comes from a counterfeiter; i.e., $\lambda(q, p) = 0$ whenever $q > 0$. Even though counterfeiters get zero consumption when they make offers, agents may be willing to produce counterfeits because they can obtain some surplus from the pooling offers made by sellers.

The value functions for agents in different states obey the following flow Bellman equations:

$$rv_1 = (1 - m - n)\theta\eta \left[u(q_b^i) - q_b^i \right], \tag{38}$$

$$rv_0 = (1 - \theta)\left\{ \eta m \left[u(q_s^i) - q_s^i \right] \right.$$
$$\left. + (1 - \eta)(n + m) \left[-q_s^u + \frac{m}{n + m} p_s^u (v_1 - v_0) \right] \right\}, \tag{39}$$

$$rv_c = (1 - m - n)(1 - \theta)(1 - \eta)u(q_s^u) - (v_c - v_0). \tag{40}$$

According to (38), a genuine buyer can only obtain a positive surplus in an informed match when he makes the offer. According to (39), a seller enjoys a positive expected surplus whenever he makes the offer. From (40), a counterfeiter enjoys some consumption if the seller makes the offer in an uninformed match.[23] Finally, assuming an interior solution, the measure of counterfeiters solves

$$v_c - v_0 = \beta k. \tag{41}$$

DEFINITION 2. *A counterfeiting equilibrium is a list* $\langle (q_b^i, p_b^i), (q_s^i, p_s^i), (q_s^u, p_s^u), v_0, v_1, v_c, n \rangle$ *that satisfies (i) the equations for the terms of trade, (15), (36), (37); (ii) the flow Bellman equations (38)–(40); (iii) the entry condition in the counterfeiting sector, (41); and (iv) the condition for a monetary equilibrium,* $v_1 > 0$.

Society's welfare, $\mathcal{W}$, is measured as the sum of the surpluses across all matches minus the cost to produce counterfeits; i.e.,

$$\mathcal{W} = m(1 - m - n) \left\{ \theta\eta \left[u(q_b^i) - q_b^i \right] + (1 - \theta)\eta \left[u(q_s^i) - q_s^i \right] \right\}$$
$$+ (m + n)(1 - m - n)(1 - \theta)(1 - \eta) \left[u(q_s^u) - q_s^u \right] - nk. \tag{42}$$

Counterfeiting exerts various adverse effects on society's welfare. No trade takes place in uninformed matches when buyers make offers, even though some buyers hold genuine money. The trade surplus is lower in uninformed matches than in the informed matches because sellers make pooling offers. There is also a direct cost because resources are used to produce counterfeits.

We study the equilibrium through numerical examples. Our benchmark parameterization is $u(q) = 2\sqrt{q}$, $r = 0.1$, $m = 0.3$, $\theta = \eta = 0.5$, and $k = 0.02$. In Figure 2 we report how changes in policy variables (k, η, and m) and market structure (θ) affect the measure of counterfeiters, the value of genuine money, and welfare. In the last column of Figure 2, we express $\mathcal{W}$ as a fraction of the welfare at the first-best allocation.[24]

Policies that raise the cost of producing counterfeits and make genuine money more recognizable have a positive effect on the value of genuine money and social welfare (see the first two rows in Figure 2). Moreover, a higher k reduces n, whereas the relationship between n and η is nonmonotonic. If it is very difficult
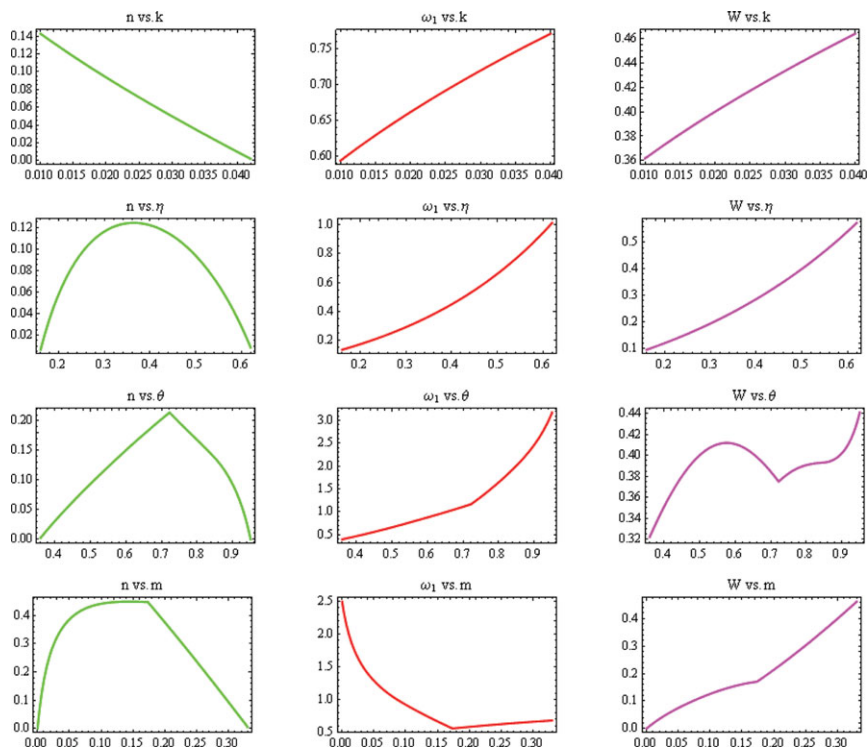
FIGURE 2. Counterfeiting equilibria.

to distinguish a unit of genuine money from a counterfeit, then an improvement in the recognizability of money can in fact promote counterfeiting. The reason is that genuine money becomes more valuable which, in turn makes the return to counterfeiting higher.

The third row in Figure 2 illustrates the effects of a change in buyers' market power. Because buyers holding genuine money get no surplus when sellers make offers, genuine money becomes more valuable as buyers get more opportunities to determine the terms of trade. An increase in θ has two opposite effects on agents' incentives to produce counterfeits. Genuine money becomes more valuable, and hence, agents have higher incentives to enter the counterfeiting sector. But a higher θ also implies that counterfeits can be traded less frequently because no trade takes place in uninformed matches when buyers make offers. The former effect dominates when θ is low.

Finally, the last row in Figure 2 reports the effects of a change in the supply of genuine money. If there is a shortage of genuine money, then agents have an incentive to produce counterfeits, because it is hard to trade as a seller. As the quantity of genuine money becomes sufficiently abundant, the opposite occurs and

an increase in m discourages counterfeiting. Society's welfare increases with the quantity of money (over the range where there is counterfeiting in equilibrium).

8. CONCLUSION

We have studied and extended the model of counterfeiting from Nosal and Wallace (2007). Our results challenge a key proposition in NW, that the threat of counterfeiting makes the existence of a monetary equilibrium less likely. We showed that the existence of a monetary equilibrium is not threatened by the possibility of counterfeiting. However, in contrast with what was previously thought, the value of genuine money and society's welfare can be directly affected by the cost of producing counterfeits and the recognizability of money, even if counterfeiting does not occur in equilibrium. An important policy implication is that, even though counterfeiting seems insignificant, government's efforts, such as introducing a new design of currency or law enforcement that makes counterfeiting more costly, should be necessary to prevent the threat from having adverse effects on output and welfare.[25]

We also showed that the model can account for the counterfeiting activity, even modest, that is observed in the United States and most countries. We have offered two extensions of the model under which the threat of counterfeiting is realized: if counterfeits can circulate across periods, or if sellers set terms of trade in some matches. Under both versions of the model, policies that improve the recognizability of the currency raise the value of genuine money and society's welfare. In some cases, however, those policies are ineffective in terms of reducing the extent of counterfeiting, or may even have adverse effects.

Several other extensions would be worth exploring, such as introducing various punishments for people caught with counterfeits, or considering a costly technology for detecting counterfeits. For tractability, we have assumed that an agent's money holding lies in the set $\{0, 1\}$. This assumption can capture the phenomenon of currency shortage and the role of counterfeiting in promoting trades. Nonetheless, to study the effects of monetary policy such as inflation on counterfeiting, one may like to consider a version of the model with no restrictions on individuals' money holdings. This can be done by using a quasilinear model with divisible money [e.g., Lagos and Wright (2005)], in which some agents can choose to produce any quantity of counterfeits at a fixed cost.[26]

NOTES

1. According to "The use and counterfeiting of United States currency abroad," Part 3, Section 6.6, the cost of producing reasonably deceptive counterfeits could be as low as $300. Digital counterfeited notes as a fraction of all passed notes increased from 0.5% in 1995 to 52% in 2005.

2. In 2005, out of the $760 billion of U.S. banknotes in circulation, $61 million of counterfeit currency was passed on to the public. See "The use and counterfeiting of United States currency abroad," Part 3, p. 47 (United States Treasury Department, September 2006). Also, according to the estimate by Judson and Porter (2003), the counterfeit U.S. currency that has been passed into circulation

is about one note in ten thousand of currency in circulation. Two recent programs to improve the design of the U.S. currency are the New Currency Design introduced in 1996 and the Series 2004 New Color of Money introduced in 2003. Both programs were preceded by educational campaigns.

3. Surprisingly, very little work has been devoted to this topic. Green and Weber (1996) and Kultti (1996) are the first papers to study counterfeiting of currency in a random-matching model with exogenous prices. Williamson (2002) investigates the counterfeiting of banknotes in a random-matching model with indivisible money and divisible output. Nosal and Wallace (2007) introduce lotteries as a proxy for divisible money and show that it allows buyers to signal the quality of their money holdings. Monnet (2005) and Cavalcanti and Nosal (2007) adopt a mechanism design approach and focus on pooling allocations. In addition, Monnet (2005) does not restrict money holdings to lie in {0, 1}. Quercioli and Smith (2007) study counterfeiting in a nonmonetary model, but they introduce multiple denominations and a costly decision to verify currency. Papers that consider competing assets of which one is subject to the counterfeiting problem and study the liquidity differentials or acceptability of assets include, for example, Lester et al. (2007), and Kim and Lee (2008).

4. In short, NW omit some equilibria by implicitly assuming that in any equilibrium with no counterfeiting all offers made by buyers, even out-of-equilibrium ones, should be attributed to genuine buyers. See Appendix B for a more detailed comparison between our approach and the one in NW.

5. These numbers are taken from "The use and counterfeiting of United States currency abroad," Part 3, p. 47.

6. This implies that sellers accept counterfeits knowingly. Mihm (2007, p. 221) provides examples where store keepers in the nineteenth-century United States would accept notes even when they suspected those notes were counterfeits. These episodes seem consistent with the separating equilibrium described here.

7. We assume the absence of punishment for an individual caught with a counterfeit. This is consistent with the situation in the United States in the eighteenth and nineteenth centuries when prosecutions were rare. For a more accurate description of counterfeiting nowadays, one would also need to take into account the usually severe punishments for those producing or attempting to pass counterfeit notes.

8. This rules out the possibility that genuine money is exchanged for a counterfeit note plus some output. See, e.g., Aiygari et al. (1996) and Li (2002).

9. We could adopt different matching technologies without affecting the main results [see, e.g., Li and Rocheteau (2008b)]. We could also add explicitly a probability of single coincidence, as in NW.

10. In search monetary models with indivisible money and complete information, the use of lotteries acts as an imperfect proxy for divisibility of money: it allows agents to extract larger gains from trade, and it eliminates some trade inefficiencies [see, e.g., Berentsen and Rocheteau (2002)].

11. Here we adopt the tie-breaking rule that a seller accepts any offer that makes him indifferent between accepting or rejecting. This enables us to find a solution to the buyer's problem.

12. Our game has a structure similar to that of the one used in Fudenberg and Tirole (1991, p. 438, Figure 11.1) to motivate the notion of strategic stability.

13. This finding is in sharp contrast to the outcome of the same bargaining game under pure adverse selection, i.e., when the distribution of types is exogenous. In that case, the no-trade outcome is obtained under the Intuitive Criterion.

14. The game we are considering is not a signaling game with exogenous types as in Cho and Kreps (1987); i.e., the types that are relevant in the bargaining game are chosen endogenously. The refinement we are using, however, is inspired by the same forward-induction logic as the one underlying the Intuitive Criterion. There are other refinements for signaling games [e.g., the undefeated equilibrium from Mailath et al. (1993)] but they are not directly applicable to games with endogenous types. For a method to analyze a class of signaling games with endogenous types, see In and Wright (2008). We apply this methodology in Li and Rocheteau (2009) with divisible money and obtain similar insights.

15. Notice that Refinement 1b is consistent with Refinement 1a in the following sense. Consider a proposed equilibrium where the surplus of the genuine buyer is less than the lower bound defined

in Lemma 2. This equilibrium would have to be sustained by a belief system that violated Refinement 1a, and it could be dismissed by Refinement 1b.

16. Recall that υ_1 is the present value of an agent holding a unit of genuine money at the beginning of the next period. Let $V_1 = (1 + r)\upsilon_1$. Then V_1 satisfies the following Bellman equation:

$$V_1 = (1 - m)\left[u\left(q_b^u\right) + \left(1 - p_b^u\right)\beta V_1\right] + m\beta V_1.$$

With probability $1 - m$ a genuine buyer meets a seller; then he consumes q_b^u and delivers money with probability p_b^u. Subtract βV_1 on both sides and use from (12) that $p_b^u \beta V_1 = q_b^u$ to get (13).

17. Our proposition is in contrast with Proposition 2 in NW, where a necessary and sufficient condition for the existence of a monetary equilibrium is $k > (1 - m)u(\bar{q})$, where $\bar{q}$ is the equilibrium quantity in an economy where counterfeiting is not possible. The difference between their result and ours can be traced back to their Problem 1 (p. 997), which determines terms of trade. They assume that the seller's belief is independent of the offer made by the buyer and is equal to the fraction of genuine buyers in equilibrium. This belief system is even more restrictive than the Cho–Kreps refinement, which explains why NW does not uncover all the equilibria. We provide a more detailed discussion in Appendix B.

18. In Li and Rocheteau (2008a) we demonstrate that a noncounterfeiting monetary equilibrium can arise provided k or δ is high enough.

19. Let $V_c = (1 + r)\upsilon_c$; then it solves the following Bellman equation:

$$V_c = (1 - m - n)\left[u\left(q_c^i\right) + \left(1 - p_c^i\right)(1 - \delta)\beta V_c\right] + (m + n)(1 - \delta)\beta V_c.$$

Subtract $(1 - \delta)\beta V_c$ from both sides and use, from (20), that $q_c^i = p_c^i(1 - \delta)\beta V_c$ to get (30).

20. The rest of the parameterization is $u(q) = 2\sqrt{q}$, $\eta = 0.5$, $r = 0.1$, and $k = 0.5$. Welfare is measured relative to the first best level, which is $1/4$.

21. Mihm (2007) provides numerous quotes in support of the thesis that counterfeiting might have been a necessary evil to overcome the currency shortage in the United States before the Civil War. For instance, Burroughs, the first famous counterfeiter in the United States, in his memoirs [cited in Mihm (2007, p. 41)] quotes a friend as saying, "An undue scarcity of cash now prevails [and] whoever contributes, really, to increase the quantity of cash, does not only himself, but likewise the community, an essentiel benefit." Daniel Mevis in his *Pioneer Recollections* [cited in Mihm (2007, p. 159)] wrote, "Counterfeiting and issuing worthless 'bank notes' . . . was not looked upon as a felony as it would be today. Of course, it was taken for granted that it was a 'little crooked' but the scarcity of real money, together with the necessity for a medium of exchange, made almost anything that looked like money answer the purpose."

22. In Li and Rocheteau (2008a) we show that counterfeiting cannot occur in any equilibrium if $\eta = 0$.

23. Let $V_c = (1 + r)\upsilon_c$; then it solves the following Bellman equation:

$$V_c = (1 - m - n)(1 - \theta)(1 - \eta)\left[u\left(q_s^u\right) + \beta V_0\right] + [1 - (1 - m - n)(1 - \theta)(1 - \eta)]\beta V_0.$$

Because counterfeits are confiscated at the end of a period, the continuation value is βV_0 no matter whether the counterfeiter trades from a match or not. Subtract βV_c from both sides to get (40).

24. The first-best is achieved when the number of matches is maximum ($m = 1/2$) and agents trade q^* in all matches.

25. "We have to stay ahead of technology, which is developing and progressing at an ever-increasing rate. Items like digital printers and higher quality scanners are becoming more readily available at cheaper prices. So we have to make our currency notes safer, smarter, and more secure in order to stay ahead of the would-be counterfeiters," by Tom Ferguson, Director of the Treasury's Bureau of Engraving and Printing, which produces U.S. currency. The quotation is taken from the *Federal Reserve Bulletin*, Summer 2004.

26. See Li and Rocheteau (2009) and Rocheteau (2008) for such a model.

27. Suppose there is a separating equilibrium where the expected payoff of the buyer with a counterfeit is $\mathcal{S}_c^u$ and the expected payoff of the genuine buyer is $\mathcal{S}_1 \in [0, \mathcal{S}_1^g)$, where $\mathcal{S}_1^g \equiv u(q_b^u) - p_b^u \omega_1$ (this is the payoff of the genuine buyer at the Pareto-efficient separating equilibrium). Replace $u(q_c^u) - p_c^u \omega_c$ in (27) by $\mathcal{S}_c^u - \varepsilon$ with $\varepsilon > 0$, and denote $\mathcal{S}_\varepsilon^1$ the associated payoff for the genuine buyer. The set of acceptable and feasible offers is compact. From the Theorem of the Maximum, $\mathcal{S}_\varepsilon^1$ is continuous in ε and $\lim_{\varepsilon \to 0} \mathcal{S}_\varepsilon^1 = \mathcal{S}_1^g$. Hence, there is an $\varepsilon > 0$ such that $\mathcal{S}_\varepsilon^1 > \mathcal{S}_1$. The associated offer satisfies (22)–(24) so that the proposed equilibrium violates Refinement 2.

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APPENDIX A

A.1. PROOF OF LEMMA 1

The strategy of being a producer yields at least v_0 because a seller always has the option of rejecting an offer that is proposed. The strategy of producing counterfeits and offering (q, p) yields an expected payoff no greater than $-k + (1 - m)u(q) + v_0$ because the matching probability of the buyer is bounded above by $1 - m$ and the buyer enjoys the utility of consuming q if the offer is accepted. Thus, $-k + (1 - m)u(q) + v_0 < v_0$, irrespective of the choices of other agents, such as the decision of agents without money to produce counterfeits or the decision of sellers to accept or reject the offer (q, p). ∎

A.2. PROOF OF LEMMA 2

According to Refinement 1a, any offer (q, p) satisfying $-k + (1-m)u(q) < 0$ is attributed to a genuine buyer ($\lambda(q, p) = 1$), and the expected surplus of the seller is $-q + p(v_1 - v_0)$. Moreover, if (5) holds, then the offer is accepted. To find the greatest lower bound for the buyer's surplus, we consider offers in the closure of the set

$$\{(q, p) \in \mathbf{R}_+ \times [0, 1] : -q + p(v_1 - v_0) \geq 0 \text{ and } -k + (1 - m)u(q) < 0\},$$

which gives (5) and (6). ∎

A.3. PROOF OF PROPOSITION 1

(i) *There is no equilibrium with $n > 0$.* The proof is by contradiction. Suppose $n > 0$. Then, from (3), $\chi = c$ requires that $q_c^u > 0$. For the offer (q_c^u, p_c^u) to be acceptable,

$$-q_c^u + \lambda\left(q_c^u, p_c^u\right) p_c^u (v_1 - v_0) \geq 0.$$

Consider the out-of-equilibrium offer (q', p') such that $p' = p_c^u - \varepsilon$, where $\varepsilon \in (0, p_c^u - \frac{q_c^u}{v_1-v_0})$, and $q' < q_c^u$ satisfies (7) and (8) or, equivalently,

$$0 < \frac{u\left(q_c^u\right) - u\left(q'\right)}{\varepsilon} < v_1 - v_0. \tag{A.1}$$

We first establish that the set of offers (q', p') is not empty. Using the fact that $\lambda(q_c^u, p_c^u) < 1$, the seller's participation constraint at the proposed equilibrium implies $q_c^u < p_c^u(v_1 - v_0)$, and hence $(0, p_c^u - \frac{q_c^u}{v_1-v_0})$ is not empty. Moreover, for any

$\varepsilon \in (0, p_c^u - \frac{q_c^u}{\upsilon_1 - \upsilon_0})$, there is a $q' \geq 0$ that satisfies (A.1). Second, we show that any offer (q', p') disqualifies the proposed equilibrium according to Refinement 1b. By construction, from (A.1), (q', p') satisfies (7) and (8). Moreover, $\varepsilon < p_c^u - q_c^u/(\upsilon_1 - \upsilon_0)$ implies that $q_c^u < (p_c^u - \varepsilon)(\upsilon_1 - \upsilon_0)$. From (A.1), $u(q_c^u) - u(q') > 0$ and therefore $q_c^u - q' > 0$. So (9) is satisfied as well.

(ii) *In any equilibrium with $n = 0$ the offer made by a genuine buyer solves (4)–(6).* Consider an equilibrium with $n = 0$, where the expected surplus of the genuine buyer is greater than that implied by (4)–(6). Thus, (6) must be violated. But then, from (3), $n = 0$ implies that $-q_b^u + p_b^u(\upsilon_1 - \upsilon_0) > 0$. With reasoning similar to that in (i), one can find (q', p') that solves (7)–(9) to disqualify the proposed equilibrium, violating Refinement 1b. To see this, take $q' = q_b^u - \varepsilon$, where $\varepsilon > 0$, and $p' = (q_b^u - \varepsilon)/(\upsilon_1 - \upsilon_0)$. Because $q' < q_b^u$, the offer would not be made by an agent without money who produced a counterfeit. To see this, recall that if $n = 0$ the following inequality holds:

$$-k + (1 - m)u(q') + \upsilon_0 < -k + (1 - m)u(q_b^u) + \upsilon_0 \leq \mathcal{U}_0.$$

As $\varepsilon \to 0$, the buyer's payoff tends to $u(q_b^u) - q_b^u$, which is the total surplus of the match. Because $-q_b^u + p_b^u(\upsilon_1 - \upsilon_0) > 0$, $u(q_b^u) - p_b^u(\upsilon_1 - \upsilon_0) < u(q_b^u) - q_b^u$. Then, provided that ε is small enough, $u(q') - p'(\upsilon_1 - \upsilon_0) > u(q_b^u) - p_b^u(\upsilon_1 - \upsilon_0)$. So the offer is attributed to a genuine buyer and it is accepted.

A belief system for sellers to sustain the offer given by (4)–(6) as an equilibrium offer is such that all offers that satisfy $-k + (1 - m)u(q) \leq 0$ are attributed to genuine buyers, whereas all other offers are attributed to an agent without money who chose to be a counterfeiter. The buyer's offer is optimal given that sellers believe it is made by the genuine buyers and accept it, and all offers that violate (6) are attributed to a counterfeiter and hence are rejected. ∎

A.4. PROOF OF PROPOSITION 2

An equilibrium is a $\upsilon_1 > 0$ solution to $r\upsilon_1 = \Gamma(\upsilon_1)$, where $\Gamma(\upsilon_1) = (1 - m)\{u[q_b^u(\upsilon_1)] - q_b^u(\upsilon_1)\}$. From (10) and (11), $q_b^u(\upsilon_1)$ is a continuous, function of υ_1, and hence $\Gamma(\upsilon_1)$ is continuous, with $\Gamma(0) = 0$. Moreover, $\Gamma(\upsilon_1) = (1 - m)[u(\upsilon_1) - \upsilon_1]$ is strictly concave and increasing if both $\upsilon_1 < q^*$ and $u(\upsilon_1) < k/(1 - m)$. If $u(q^*), \leq k/(1 - m)$ and $\upsilon_1 \geq q^*$, then $\Gamma(\upsilon_1) = (1 - m)[u(q^*) - q^*]$; if $u(q^*) > k/(1 - m)$ and $\upsilon_1 > u^{-1}[k/1 - m]$, then $\Gamma(\upsilon_1) = (1 - m)\{k/(1 - m) - u^{-1}[k/(1 - m)]\}$. Because $\Gamma(\upsilon_1)$ is constant for all υ_1 above a threshold, $r\upsilon_1 > \Gamma(\upsilon_1)$ for υ_1 sufficiently large. Hence, a unique $\upsilon_1 > 0$ solution to $r\upsilon_1 = \Gamma(\upsilon_1)$ exists if and only if $r < \Gamma'(0)$ or, equivalently, $r < \Gamma'(0) = (1 - m)[u'(0) - 1]$. See Figure A.1. ∎

A.5. PROOF OF PROPOSITION 3

From (10) evaluated at υ_1^* and (6) held at equality, one can solve for $\bar{k}$. For all $k < \bar{k}$ the constraint (6) binds. Hence, $q_b^u = u^{-1}[k/(1 - m)]$ and $dq_b^u/dk > 0$. Differentiating (13),

$$\frac{d\upsilon_1}{dk} = r^{-1}\left[1 - \frac{1}{u'(q_b^u)}\right].$$
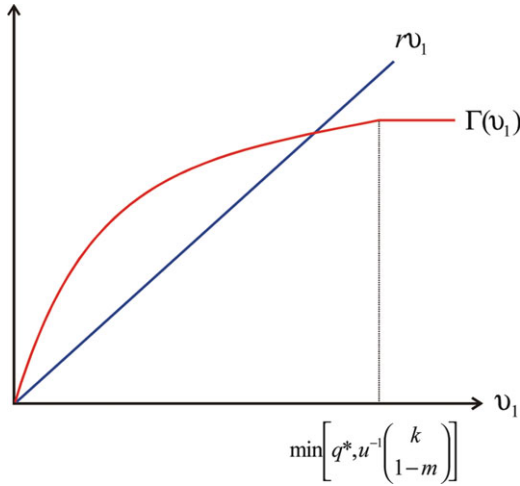
FIGURE A.1. Existence of equilibrium.

Because (6) binds, $q_b^u < \min(v_1, q^*) \leq q^*$. Hence, $u'(q_b^u) > 1$ and $dv_1/dk > 0$. From (12), $p_b^u = q_b^u/v_1 < 1$. Differentiating (12),

$$\frac{dp_b^u}{dk} = \frac{rv_1 - (1-m)\left[q_b^u u'\left(q_b^u\right) - q_b^u\right]}{(1-m)ru'\left(q_b^u\right)(v_1)^2}.$$

Using the strict concavity of $u(q_b^u)$, $rv_1 > (1-m)[q_b^u u'(q_b^u) - q_b^u]$ and hence $dp_b^u/dk > 0$. Because $q_b^u = u^{-1}[k/(1-m)]$, as $k \to 0$ then $q_b^u \to 0$ and $v_1 \to 0$ from (13).   ∎

A.6. PROOF OF PROPOSITION 4

From (16)–(18), the lowest value of k above which (18) does not bind is $\bar{k} = (1-m)(1-\eta)u(q)$ with $q = \min(q^*, v_1^*)$. If $k < \bar{k}$ then $q_b^u = u^{-1}(k/(1-m)(1-\eta))$ and hence $\partial q_b^u/\partial\eta > 0$. Moreover, if (18) binds then $q_b^u < q_b^i$ and hence $p_b^u = q_b^u/v_1 < p_b^i = q_b^i/v_1$. Differentiating (19),

$$\frac{dv_1}{d\eta}$$

$$= \frac{(1-m)\eta\left\{\left[u\left(q_b^i\right) - q_b^i\right] - \left[u\left(q_b^u\right) - q_b^u\right]\right\} + (1-m)(1-\eta)\left[u'\left(q_b^u\right) - 1\right]\dfrac{\partial q_b^u}{\partial\eta}}{r - (1-m)\eta\left[u'\left(q_b^i\right) - 1\right]\dfrac{\partial q_b^i}{\partial v_1}} > 0,$$

because at the monetary equilibrium the right-hand side of (19) intersects the left-hand side of (19) from above so that the denominator of the expression is positive, $u(q_b^i) - q_b^i > u(q_b^u) - q_b^u$ (because $q_b^u < q_b^i \leq q^*$) and $dq_b^u/d\eta > 0$.   ∎

A.7. PROOF OF LEMMA 3

We first show that there cannot be a pooling offer that is accepted such that $q > 0$. Suppose there is an equilibrium where both types of buyers offer $(\bar{q}, \bar{p})$ with $\bar{q} > 0$, i.e., $\lambda(\bar{q}, \bar{p}) \in (0, 1)$. The equilibrium payoffs are $\mathcal{S}_1^u = u(\bar{q}) - \bar{p}\omega_1$ and $\mathcal{S}_c^u = u(\bar{q}) - \bar{p}\omega_c$. Furthermore, the offer $(\bar{q}, \bar{p})$ is accepted if $-\bar{q} + \bar{p}\{\lambda(\bar{q}, \bar{p})\omega_1 + [1 - \lambda(\bar{q}, \bar{p})]\omega_c\} \geq 0$. Consider the out-of-equilibrium offer (q', p') such that $p' = \bar{p} - \varepsilon$, where $\varepsilon \in (0, \bar{p} - \bar{q}/\omega_1)$, and $q' < \bar{q}$ satisfies (22) and (23) or, equivalently,

$$\omega_c < \frac{u(\bar{q}) - u(q')}{\varepsilon} < \omega_1. \tag{A.2}$$

We first establish that the set of offers (q', p') is not empty. Using the fact that $\lambda(\bar{q}, \bar{p}) < 1$, the seller's participation constraint at the proposed equilibrium implies $\bar{q} < \bar{p}\omega_1$ and hence $(0, \bar{p} - \bar{q}/\omega_1)$ is not empty. Moreover, for any $\varepsilon \in (0, \bar{p} - \bar{q}/\omega_1)$ there is a $q' \geq 0$ that satisfies (A.2). To see this, rewrite (A.2) as

$$u(\bar{q}) - \omega_1\varepsilon < u(q') < u(\bar{q}) - \omega_c\varepsilon.$$

Because $\varepsilon < \bar{p} - \bar{q}/\omega_1$, we have $\varepsilon\omega_1 < \bar{p}\omega_1 - \bar{q}$ and hence $u(\bar{q}) - \omega_1\varepsilon > \mathcal{S}_1^u + \bar{q} > 0$. Second, we show that any offer (q', p') disqualifies the proposed equilibrium according to Refinement 2. From (A.2), (q', p') satisfies (22) and (23). Moreover, $\varepsilon < \bar{p} - \bar{q}/\omega_1$ implies $\bar{q} < (\bar{p} - \varepsilon)\omega_1$. From (A.2), $u(\bar{q}) - u(q') > 0$ and therefore $\bar{q} - q' > 0$. So (24) is satisfied as well.

We next ask whether a pooling offer of $q = 0$ exists. If $q = 0$ then $\mathcal{S}_1^u = \mathcal{S}_c^u = 0$ (because the equilibrium payoff of a buyer is always nonnegative). But $\mathcal{S}_c^u \geq \max_{q, p \leq 1}[u(q) - p\omega_c]$ s.t. $-q + p\omega_c \geq 0$, which implies that this case is only relevant when $\omega_c = 0$.

Unless $q = 0$ the equilibrium of the bargaining game is separating. So the buyer with counterfeit money cannot do better than proposing his complete-information offer. Hence, his offer (q_c^u, p_c^u) solves (25). The buyer with genuine money makes the separating offer that maximizes his payoff; any other separating offer violates Refinement 2.[27] Finally, a belief system consistent with the offers in (25)–(27) is such that sellers attribute all offers that violate (27) to buyers with counterfeits, and all other out-of-equilibrium offers to buyers with genuine money. ∎

A.8. PROOF OF LEMMA 4

From Lemma 3, (q_b^u, p_b^u) is the solution to (26) and (27). Suppose first that (27) is not binding. Then, (q_b^u, p_b^u) is the complete-information offer, i.e., $q_b^u = \min[q^*, \omega_1] \geq q_c^u$ and $p_b^u = q_b^u/\omega_1 > 0$. The payoff of a buyer with a counterfeit who offers (q_b^u, p_b^u) is then

$$u\left(q_b^u\right) - p_b^u\omega_c = u\left(q_b^u\right) - q_b^u + p_b^u\left(\omega_1 - \omega_c\right)$$
$$> u(q_c^u) - q_c^u,$$

because $p_b^u > 0$ and $\omega_1 - \omega_c > 0$. But then (27) is violated. A contradiction.

Suppose next that the seller's participation constraint in (26) is not binding. Substitute $u(q_b^u) = p_b^u\omega_c + u(q_c^u) - p_c^u\omega_c$, given by (27) at equality, into the buyer's payoff to get

$$\max_{p \in [0,1]} \left\{ p[\omega_c - \omega_1] + u\left(q_c^u\right) - p_c^u\omega_c \right\},$$

which gives $p_b^u = 0$ and $u(q_b^u) = u(q_c^u) - q_c^u > 0$ (because $\omega_c > 0$). But then the seller's participation constraint, $-q_b^u \geq 0$, is violated. A contradiction.

So (27) and the seller's participation constraint are binding. From the seller's participation constraint we obtain (28). Substitute p_b^u from (28) into (27) at equality to get (29).

In order to establish the existence of a unique solution to (26) and (27), notice that the left-hand side of (29) is first increasing and then decreasing in q_b^u, and it reaches a maximum greater than $u(q^*) - q^* \geq u(q_c^u) - q_c^u$ for some $q_b^u > q^*$ solution to $u'(q_b^u) = \omega_c/\omega_1$. So, there might be multiple solutions (at most two) to (29). However, only the lowest value for q_b^u maximizes the payoff of the buyer with genuine money. To see this, notice that

$$u\left(q_b^u\right) - q_b^u = u\left(q_b^u\right) - \frac{\omega_c}{\omega_1}q_b^u - \left(1 - \frac{\omega_c}{\omega_1}\right)q_b^u.$$

From (29),

$$u\left(q_b^u\right) - q_b^u = u\left(q_c^u\right) - q_c^u - \left(1 - \frac{\omega_c}{\omega_1}\right)q_b^u,$$

The right-hand side of the above equation is decreasing in q_b^u; hence, $u(q_b^u) - q_b^u$ is maximized at the lowest value of q_b^u that solves (29).

Next, we establish that $q_b^u < q_c^u$ and $p_c^u > p_b^u$. The left-hand side of (29) is increasing in q_b^u over $[0, q_c^u]$ from $0 < u(q_c^u) - q_c^u$ to $u(q_c^u) - (\omega_c/\omega_1)q_c^u > u(q_c^u) - q_c^u$. So, there is a unique $q_b^u \in (0, q_c^u)$. From (27), $(p_c^u - p_b^u)\omega_c = u(q_c^u) - u(q_b^u) > 0$, and hence $p_c^u > p_b^u$. ∎

A.9. PROOF OF PROPOSITION 5

The proof proceeds in three parts by following the recursive structure of the equilibrium. First, we show that a unique solution $n \in (0, 1-m)$ to (32) exists provided that

$$1 - m > \frac{[1 - (1-\delta)\beta]k}{u\left(q_c^i\right) - q_c^i} \tag{A.3}$$

holds, where $q_c^i = \min[q^*, \beta(1-\delta)k]$. We then derive condition (34) from (A.3). Second, given condition (A.3), we show that there is a unique $\omega_1 > \omega_c$ that solves (33). Third, we establish that there is no equilibrium with $\omega_1 \leq \omega_c$.

(i) The right-hand side of (32) is independent of n whereas the left-hand side is strictly decreasing for all $n \in [0, 1-m]$. Hence, if a solution exists, it is unique. At $n = 1-m$, the left-hand side is 0. So a solution to (32) exists, and it is such that $n > 0$, if the left-hand side of (32) evaluated at $n = 0$ is greater than the right-hand side, which gives (A.3).

Let $\bar{k}$ denote the value of k that solves $1 - m = [1 - (1-\delta)\beta]k/[u(q_c^i) - q_c^i]$. If $q^* \leq \beta(1-\delta)k$ then $\bar{k} = (1-m)[u(q^*) - q^*]/[1 - (1-\delta)\beta] > 0$. For all $k < \bar{k}$, (A.3) holds, and there exists a counterfeiting equilibrium. Consider next the case $q^* > \beta(1-\delta)k$. Differentiate the following expression with respect to k:

$$\frac{d}{dk}\left[\frac{u\left(q_c^i\right) - q_c^i}{k}\right] = \frac{\left[u'\left(q_c^i\right) - 1\right]\beta(1-\delta)k - \left[u\left(q_c^i\right) - q_c^i\right]}{k^2}$$

$$= k^{-2}\left[u\left(q_c^i\right) - q_c^i\right]\left\{\frac{\left[u'\left(q_c^i\right) - 1\right]q_c^i}{u\left(q_c^i\right) - q_c^i} - 1\right\},$$

where we have used that $q_c^i = \min[q^*, \beta(1 - \delta)k] = \beta(1 - \delta)k$. Using the strict concavity of $u(q_c^i) - q_c^i$ over $(0, q^*)$ and the fact that $u(0) = 0$, we have $[u'(q_c^i) - 1]q_c^i/[u(q_c^i) - q_c^i] < 1$, and so $d/dk\{[u(q_c^i) - q_c^i]/k\} < 0$. Hence, the right-hand side of (A.3) is increasing in k. As $k \to 0$, $[1 - (1 - \delta)\beta]k/[u(q_c^i) - q_c^i] \to [1 - (1 - \delta)\beta]/[u'(0) - 1](1 - \delta)\beta$ (from L'Hôpital's rule). Then, (A.3) becomes $(1 - m)[u'(0) - 1] > [1 - (1 - \delta)\beta]/(1 - \delta)\beta = r'$, which gives (34).

(ii) Take as given the solution $n \in (0, 1 - m)$ to (32). We start by establishing that a solution $\omega_1 > \omega_c$ to (33) exists. From the comparison of (15) and (26) and (27),

$$u\left(q_b^u\right) - q_b^u \leq u\left(q_b^i\right) - q_b^i,$$

with a strict inequality when $\omega_1 > \omega_c$. (To see this, recall from Lemma 4 that the incentive-compatibility condition (27) is binding.) At $\omega_1 = \omega_c$, $q_b^u = q_c^u = q_c^i$ (from (29)) and the right-hand side of (33) is

$$(1 - m - n)\left[u\left(q_c^i\right) - q_c^i\right] = [1 - (1 - \delta)\beta]k > (1 - \beta)(1 - \delta)k = r\omega_c,$$

where we have used (32) to obtain the first equality, the assumption $\delta > 0$ to get the inequality, and $\omega_c = \beta(1 - \delta)k$ to obtain the last equality. Consider next $\omega_1 = \tilde{\omega}_1 > \omega_c$, where $\tilde{\omega}_1$ is the unique solution to $r\tilde{\omega}_1 = (1 - m - n)[u(q_b^i) - q_b^i]$ with $q_b^i = \min[q^*, \tilde{\omega}_1]$. In order to establish that $\tilde{\omega}_1 > \omega_c$, notice from (31) that ω_c solves $[(r + \delta)/(1 - \delta)]\omega_c = (1 - m - n)[u(q_c) - q_c]$, where $q_c = \min[q^*, \omega_c]$. Because $\delta > 0$, the result is immediate. Thus,

$$(1 - m - n)\left\{(1 - \eta)\left[u\left(q_b^u\right) - q_b^u\right] + \eta\left[u\left(q_b^i\right) - q_b^i\right]\right\}$$
$$< (1 - m - n)\left[u\left(q_b^i\right) - q_b^i\right] = r\tilde{\omega}_1.$$

By the Intermediate Value Theorem, there is a $\omega_1 \in (\omega_c, \tilde{\omega}_1)$ that solves (33).

In order to establish uniqueness, rewrite (33) as

$$r\omega_1 - (1 - m - n)\eta\left[u\left(q_b^i\right) - q_b^i\right] = (1 - m - n)(1 - \eta)\left[u\left(q_b^u\right) - q_b^u\right].$$

The left-hand side (LHS) is convex in ω_1 (because $u(q_b^i) - q_b^i$ is concave), it is equal to 0 at $\omega_1 = 0$, it reaches a negative minimum at $\omega_1 = \bar{q}$, where $\bar{q}$ solves

$$u'(\bar{q}) = 1 + \frac{r}{(1 - m - n)\eta},$$

and it is equal to $r\omega_1 - (1 - m - n)[u(q^*) - q^*]$ for all $\omega_1 \geq q^*$. Hence, the LHS is increasing for all $\omega_1 > \bar{q}$ and it becomes positive for sufficiently large ω_1. The right-hand side is decreasing in ω_1. To see this, differentiate (29) to get

$$\frac{\partial q_b^u}{\partial \omega_1} = \frac{-\omega_c q_b^u}{(\omega_1)^2\left[u'\left(q_b^u\right) - \omega_c/\omega_1\right]} < 0$$

for all $\omega_1 > \omega_c$. (Recall from Lemma 4 that $q_b^u < q_c^u \leq q^*$ so that the denominator of the previous expression is positive.) Consequently, the solution $\omega_1 \in (\omega_c, \tilde{\omega}_1)$ to (33) is unique.

(iii) Suppose there is an equilibrium with $\omega_1 \leq \omega_c$. Following a similar reasoning as in the text, ω_1 and ω_c solve

$$r\omega_1 = (1 - m - n)\left[u\left(q_b^i\right) - q_b^i\right], \tag{A.4}$$

$$\left(\frac{r + \delta}{1 - \delta}\right)\omega_c = (1 - m - n)\left\{(1 - \eta)\left[u\left(q_c^u\right) - q_c^u\right] + \eta\left[u\left(q_c^i\right) - q_c^i\right]\right\}, \tag{A.5}$$

where, from Lemma 4, q_c^u is the smallest solution to

$$u\left(q_c^u\right) - \frac{\omega_1}{\omega_c}q_c^u = u\left(q_b^i\right) - q_b^i, \tag{A.6}$$

and $q_c^u \leq q_b^i \leq q_c^i$. Consequently, $u(q_c^u) - q_c^u \leq u(q_c^i) - q_c^i$, and hence, from (A.5), $[(r + \delta)/(1 - \delta)]\omega_c \leq (1 - m - n)[u(q_c^i) - q_c^i]$. From (A.4), $\omega_1 \leq \omega_c$ implies $r\omega_c \geq (1 - m - n)[u(q_c^i) - q_c^i]$. Then,

$$\left(\frac{r + \delta}{1 - \delta}\right)\omega_c \leq (1 - m - n)\left[u\left(q_c^i\right) - q_c^i\right] \leq r\omega_c.$$

A contradiction whenever $\delta > 0$. ∎

A.10. PROOF OF PROPOSITION 6

(i) *Change in m*. The right-hand side of (32) is independent of n and m. Hence, (32) determines a unique fraction of buyers, $n + m$. Consequently, $\partial(n + m)/\partial m = 0$ and $\partial n/\partial m = -1$. Because $\partial(n + m)/\partial m = 0$, from (33), $\partial\omega_1/\partial m = 0$.

(ii) *Change in k*. If $q^* \leq \beta(1 - \delta)k$ then $q_c^i = q^*$ is independent of k and hence the right-hand side of (32) is increasing in k. Consider next the case $q^* > \beta(1 - \delta)k$. Using the proof of Proposition 5, we know that $\{[u(q_c^i) - q_c^i]k\}' < 0$. The right-hand side of (32) is thus increasing in k. Hence, $\partial n/\partial k < 0$.

Let RHS denote the right-hand side of (33). Then, differentiating (33),

$$\frac{\partial\omega_1}{\partial k} = \frac{-r\omega_1}{(r - \partial\text{RHS}/\partial\omega_1)(1 - m - n)}\frac{\partial n}{\partial k},$$

where $\partial\text{RHS}/\partial\omega_1$ is evaluated at the equilibrium. From the proof of Proposition 5, and the fact that there exists a unique $\omega_1 > 0$ solution to (33), $r - \partial\text{RHS}/\partial\omega_1 > 0$ and $\partial\omega_1/\partial k > 0$.

(iii) *Change in η*. From (32), $\partial n/\partial\eta = 0$. From (33),

$$\frac{\partial\omega_1}{\partial\eta} = \frac{(1 - m - n)\left\{\left[u\left(q_b^i\right) - q_b^i\right] - \left[u\left(q_b^u\right) - q_b^u\right]\right\}}{r - \partial\text{RHS}/\partial\omega_1} > 0,$$

where $\partial\text{RHS}/\partial\omega_1$ is evaluated at the equilibrium.

(iv) *Change in δ*. The numerator on the right-hand side of (32) is increasing in δ whereas the denominator is decreasing in δ, because $q_c^i = \beta(1 - \delta)k$. Hence, $\partial n/\partial\delta < 0$. Following reasoning similar to that in (ii), $\partial\omega_1/\partial\delta > 0$. ∎

A.11. PROOF OF PROPOSITION 7

(i) From Proposition 6 (i) and (35),

$$\frac{dW}{dm} = \beta^{-1}(\omega_1 - \omega_c) > 0.$$

(ii) From Proposition 6 (iii), $\partial n/\partial\eta = 0$ and $\partial\omega_1/\partial\eta > 0$. Hence, W increases with η. ∎

A.12. PROOF OF PROPOSITION 8

The conditions are derived directly from (A.3) and the proof of Proposition 5 by taking $\delta \to 0$. Next we show that $\omega_1 \to \omega_c$ as $\delta \to 0$. We have shown that $\omega_1 \geq \omega_c$ when $\delta > 0$ in the proof of Proposition 5. Hence, it suffices to show that the inequality is not strict when $\delta \to 0$. Suppose not; $\omega_1 > \omega_c$ as $\delta \to 0$. Then ω_c and ω_1 solve

$$r\omega_c = (1 - m - n)\left[u\left(q_c^i\right) - q_c^i\right], \tag{A.7}$$

$$r\omega_1 = (1 - m - n)\left\{(1 - \eta)\left[u\left(q_b^u\right) - q_b^u\right] + \eta\left[u\left(q_b^i\right) - q_b^i\right]\right\}, \tag{A.8}$$

respectively, as $\delta \to 0$. Thus, $q_c^i = \min[q^*, \omega_c]$. From Lemma 4, $\omega_1 > \omega_c$ implies that $q_b^u < q_c^i < q_b^i$. Consequently, $u(q_b^u) - q_b^u < u(q_b^i) - q_b^i$, and hence, from (A.8), $r\omega_1 < (1 - m - n)[u(q_b^i) - q_b^i]$. From (A.7) $\omega_1 > \omega_c$ implies that $r\omega_1 > (1 - m - n)[u(q_b^i) - q_b^i]$. But then,

$$r\omega_1 < (1 - m - n)\left[u\left(q_b^i\right) - q_b^i\right] < r\omega_1,$$

a contradiction. Hence, $\omega_1 \to \omega_c$ as $\delta \to 0$. ∎

A.13. PROOF OF LEMMA 5

Consider the problem of the seller if we do not impose the contracts offered by the seller being ex post individually rational. Suppose that the seller chooses a menu of contracts $\{(q_g, p_g), (q_c, p_c)\}$, where (q_g, p_g) is the contract intended for genuine buyers and (q_c, p_c) is intended for counterfeiters, to maximize his expected surplus. The problem of the seller is thus

$$\max_{(q_g, p_g), (q_c, p_c)} \left[\frac{n}{n+m}(-q_c) + \frac{m}{n+m}(p_g\omega_1 - q_g)\right] \tag{A.9}$$

$$\text{s.t. } u(q_g) - p_g\omega_1 \geq 0, \tag{A.10}$$

$$u(q_g) - p_g\omega_1 \geq u(q_c) - p_c\omega_1, \tag{A.11}$$

$$q_c \geq q_g. \tag{A.12}$$

According to (A.9), the expected payoff of the seller is his production cost $-q_c$ if he trades with a counterfeiter, and the transfer of genuine money minus his production cost, $p_g\omega_1 - q_g$, if he trades with a genuine buyer. The constraint (A.10) indicates that the genuine buyer must obtain a positive surplus, whereas (A.11) requires that the genuine buyer prefer the contract that is intended for him. An incentive-compatibility constraint similar to (A.11) also holds for the counterfeiters, which reduces to (A.12), because counterfeits are not valued across dates. According to (A.12), a counterfeiter chooses the contract with the highest output. First, the constraint $q_c \geq q_g$ is binding. The proof is by contradiction. Suppose $q_c > q_g$. Then the seller could reduce q_c, which would increase the payoff (see (A.9)) and would relax the incentive-compatibility condition (A.11) for genuine buyers. Second, given $q_c = q_g$, the seller can choose $p_c = p_g$, as p_c does not affect the seller's objective function (A.9), to guarantee that the incentive-compatibility condition is satisfied.

The seller's problem becomes

$$\max_{(q_g, p_g)} \left[\frac{m}{n+m} p_g \omega_1 - q_g \right] \tag{A.13}$$

$$\text{s.t. } u(q_g) - p_g \omega_1 \geq 0. \tag{A.14}$$

This pooling contract is ex post individually rational because it does not reveal the type of the buyer. The menus $\{(q_g, p_g), (q_c, p_c)\}$ such that (q_g, p_g) solves (A.13) and (A.14), $q_c = q_g$, and $p_c > p_g$ would be payoff-equivalent but would not be ex post individually rational because a seller would not want to trade with a buyer choosing the contract (q_c, p_c). ∎

A.14. PROOF OF LEMMA 6

The proof is by contradiction. Suppose there is an equilibrium such that $(q_b^u, p_b^u) \neq (0, 0)$. By incentive-compatibility, a genuine buyer cannot make an active offer (with positive consumption) that is different from the one of a counterfeiter, i.e., $(q_b^u, p_b^u) = (q_c^u, p_c^u)$. By a reasoning similar to that in the proof of Proposition 1, Part (i), or in Lemma 3, genuine buyers could make an alternative offer (q', p') such that $q' < q_c^u$, $p' < p_c^u$, $u(q') - p'\omega_1 > u(q_b^u) - p_b^u \omega_1$, and $p'\omega_1 - q' \geq 0$. Such an offer disqualifies the proposed pooling outcome according to our forward-induction argument. A belief system that sustains $(q_b^u, p_b^u) = (0, 0)$ as a solution to (1) is $\lambda(q, p) = 0$ for all (q, p) such that $q > 0$. ∎

APPENDIX B. ACCOUNTING FOR THE DIFFERENCES WITH NW

According to Problem 1 in NW, sellers' beliefs are such that $\lambda(q, p) = m/(m + n)$ for all (q, p) in uninformed matches; i.e., an offer is attributed to a genuine buyer with a probability equal to the fraction of genuine buyers among all buyers. This belief is more restrictive than is warranted by the Intuitive Criterion, and this is why NW cannot uncover all the monetary equilibria. In a noncounterfeiting equilibrium ($n = 0$), this belief system becomes $\lambda(q, p) = 1$ for all (q, p). Under this assumption, the genuine buyer offers

$$(q_b, p_b) = \max_{q, p \in [0,1]} [u(q) - p\upsilon_1] \text{ s.t. } -q + p\upsilon_1 = 0$$

in all matches.

NW assume that genuine buyers and counterfeiters make the same offer in uninformed matches [see definition of C in Eq. (3)]. This assumption is not consistent with the assumed belief system. Instead, under the belief system $\lambda \equiv 1$, an agent who deviates and produces a counterfeit offers

$$(q_c, p_c) = \max_{q, p \in [0,1]} u(q) \text{ s.t. } -q + p\upsilon_1 = 0$$

in an uninformed match. Thus, $(q_c, p_c) \neq (q_b, p_b)$ whenever $\upsilon_1 > q^*$ because in that case $(q_b, p_b) = (q^*, q^*/\upsilon_1)$ and $(q_c, p_c) = (\upsilon_1, 1)$. The value of genuine money solves

$$r\upsilon_1 = (1 - m) [u(q_b) - q_b],$$

where $q_b = \min[q^*, \upsilon_1]$. There is a $\upsilon_1 > 0$ solution to the above equation, and it is unique, if and only if $r < (1 - m)[u'(0) - 1]$. The condition for $n = 0$ requires that

$$-k + (1 - m)u(q_c) \leq 0,$$

where $q_c = \upsilon_1$. This condition differs from the one in NW because $q_c \neq q_b$ when $\upsilon_1 > q^*$. So the condition for the existence of a monetary equilibrium derived in NW under the belief system $\lambda \equiv 1$ is incorrect, and the restriction on beliefs is not implied by the Intuitive Criterion.