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# A global attracting set for nonlocal Kuramoto–Sivashinsky equations arising in interfacial electrohydrodynamics

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We study a generalized class of nonlocal evolution equations which includes those arising in the modelling of electrified film flow down an inclined plane, with applications in enhanced heat or mass transfer through interfacial turbulence. Global existence and uniqueness results are proved and refined estimates of the radius of the absorbing ball in $L^2$ are obtained in terms of the parameters of the equations (the length of the system and the dimensionless electric field-measuring parameter multiplying the nonlocal term). The established estimates are compared with numerical solutions of the equations which in turn suggest an optimal upper bound for the radius of the absorbing ball. A scaling argument is used to explain this and a general conjecture is made based on extensive computations.

## 1 Introduction

In this paper we study equations which arise in the problem of a perfectly conducting thin film flow down an inclined plane in the presence of an electric field which is uniform in its undisturbed state, and normal to the plate at infinity (see Gonzales & Castellanos [9] and Tseluiko & Papageorgiou [34]). In the absence of an electric field, the flow is linearly unstable/stable depending on whether the Reynolds number $R$ is above/below the critical value $R_c = (5\cot\beta)/4$, where $\beta$ is the angle of inclination of the plate with the horizontal. The presence of the electric field acts to destabilize the flow even when this is viscously dominated and stable – this phenomenon opens the way for possible control of wave formation and physical consequences such as enhanced heat and mass transfer (see Tseluiko & Papageorgiou [34] for numerous references). A weakly nonlinear analysis of the Navier–Stokes equations, the electrostatics equations and associated free surface conditions, leads to a modified Kuramoto–Sivashinsky (MKS), or a modified damped Kuramoto–Sivashinsky (DMKS) equation which have an additional nonlocal term due to the effect of the electric field. This equation was first derived by Gonzales & Castellanos [9] and recently by Tseluiko & Papageorgiou [34] using formal asymptotics. When rescaled to $2\pi$-periodic domains, the canonical equations take the form

$$u_t + uu_x \pm u_{xx} + \nu u_{xxxx} + \mu \mathscr{H}[u]_{xxx} = 0, \tag{1.1}$$

where $\mathscr{H}$ is the Hilbert transform operator defined by

$$\mathscr{H}[f](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi} d\xi, \tag{1.2}$$

and the integral is understood in the sense of a Cauchy principal value. (For the properties of the Hilbert transform see for example Abdelouhab *et al.* [1], and Appendix A.) A plus sign in front of the $u_{xx}$ term corresponds to the linearly unstable hydrodynamic regime (the MKS equation) and a minus sign to the stable one (the DMKS equation). In addition, $\nu = (\pi/L)^2$ and $\mu = (\pi/L)\gamma$, where $2L$ is the length (period) of the system and $\gamma = 2\overline{W}_e\overline{C}/\sqrt{|\frac{4}{5}R - \cot\beta|}$, with $\overline{W}_e$ and $\overline{C}$ being the rescaled electric Weber and Capillary numbers respectively (see Tseluiko & Papageorgiou [34]). When the electric field is absent, i.e. $\mu = 0$, and $R > R_c$, we obtain the usual KS equation on $2\pi$-periodic domains:

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0. \tag{1.3}$$

This equation arises in a variety of applications and describes the asymptotic behavior of many physical systems. It occurs in free surface film flows (Benney [2], Hooper & Grimshaw [13], Shlang & Sivashinsky [26], Sivashinsky & Michelson [29]), in two-phase flows in cylindrical and plane geometries (Coward *et al.* [6], Papageorgiou *et al.* [23], Tilley *et al.* [33]) flame-front instabilities and reaction diffusion combustion dynamics (Sivashinsky [27], Sivashinsky [28]), chemical physics for propagation of concentration waves (Kuramoto [19], Kuramoto & Tsuzuki [20], Kuramoto & Tsuzuki [21]), and plasma physics (Cohen *et al.* [3]). Due to its practical applications, there have been many computational (Frisch *et al.* [8], Greene & Kim [11], Hyman & Nicolaenko [14], Hyman *et al.* [15], Kevrekidis *et al.* [18], Papageorgiou & Smyrlis [24], Smyrlis & Papageorgiou [30], Sivashinsky & Michelson [29], Smyrlis & Papageorgiou [31]) as well as analytical studies of this equation (Collet *et al.* [5], Collet *et al.* [4], Goodman [10], Il'yashenko [16], Jolly *et al.* [17], Nicolaenko *et al.* [22]). The results show that the KS equation is one of the simplest one-dimensional evolution equations exhibiting chaotic behavior. This complex behavior emerges from a balance between active and dissipative linear terms (a negative diffusion and a fourth derivative damping term) and the Burgers-type nonlinearity.

It was also observed numerically, and established analytically, that the solutions of the KS equation do not grow exponentially as linear theory would predict, but remain bounded as time goes to infinity due to a nonlinear transfer of energy from low active modes to high dissipative ones. The existence and uniqueness of the solutions and the first analytical estimates for the $L^2$-norm for odd-parity solutions were obtained by Nicolaenko et al. [22]. The approach of Nicolaenko *et al.* [22] was extended by several authors for the case of general periodic solutions (Collet *et al.* [5], Goodman [10], Il'yashenko [16], Jolly *et al.* [17]). The best known upper bound for the $L^2$-norm of the solutions (i.e. the radius of the absorbing ball) was obtained by Jolly *et al.* [17], who reworked the analysis by Collet *et al.* [5]. (The approach is based on the careful selection of an appropriate gauge function, which is constructed in Fourier space.) The analyticity properties of the solutions of the KS equation were studied by Collet *et al.* [4]. They show that the solutions are analytic in a strip around the real axis, and give a bound for the width of this strip. They also provide several stronger conjectures regarding the analyticity of the solutions based on a series of numerical experiments.

The electric field induces a linear growth which is worse than the negative diffusion but is still dominated by the fourth order damping. The numerical study of the nonlocal KS equation was performed by Tseluiko & Papageorgiou [34], where it is observed numerically that the solutions remain bounded and exhibit a complex behavior including chaotic oscillations as in the case of the usual KS equation. Analytical results of global existence, uniqueness and uniform boundedness of solutions of the MKS equation were obtained by Duan & Ervin [7], who also obtain a bound for the radius of the absorbing ball in $L^2$.

In the present paper we consider the following generalization of equations (1.1):

$$u_t + uu_x \pm u_{xx} + \nu u_{xxxx} - \mu(\mathscr{H} \circ \partial_x)^p[u] = 0, \tag{1.4}$$

on a $2\pi$-periodic interval with $\nu > 0$, $\mu \geqslant 0$. Here $p \in [3, 4)$ (for $p = 3$ equations (1.1) and (1.4) are identical), and the operator $(\mathscr{H} \circ \partial_x)^p$ is defined by its symbol in Fourier space. In particular for our purposes the Fourier transform of the nonlocal term in equation (1.4), gives $\mathscr{F}\left[(\mathscr{H} \circ \partial_x)^p[u]\right](k) = |k|^p\hat{u}(k)$, with $\hat{u}(k)$ denoting the Fourier transform of $u$ – see property (A 7) in Appendix A. Linear stability of (1.4) follows by writing $u = \epsilon \exp[ikx + \omega t]$ $(k \in \mathbb{Z})$, linearizing with respect to $\epsilon$ and working in Fourier space using the properties in Appendix A, to obtain

$$\omega_+(k) = k^2 + \mu|k|^p - \nu k^4, \tag{1.5}$$

$$\omega_-(k) = -k^2 + \mu|k|^p - \nu k^4, \tag{1.6}$$

where $\omega_+$ and $\omega_-$ correspond to the MKS and DMKS equations, respectively. The nonlocal term is always destabilizing and enhances the hydrodynamic instability (for MKS) and can make a hydrodynamically stable flow unstable (for DMKS) if $\mu$ is sufficiently large (for fixed $\nu$). The extension of the nonlocal operator as defined above, to the interval $p \in [3, 4)$, enables a parametric study of the increasing instability and global features such as the radius of the attracting set. We expressly exclude the case $p = 4$ because equation (1.4) becomes ill-posed when $\mu > \nu$ (it becomes a fourth derivative negative diffusion equation). In all the results presented in the sequel the dependence on $p \in [3, 4)$ is explicit and the physical problem results follow readily. In fact we derive estimates for the $L^2$-norm of the solution as a function of $\mu$, $\nu$ and $p$.

Throughout this paper we denote by $L^2_{\text{per}}$, $H^k_{\text{per}}$, $k = 1, 2, \ldots,$ the subspaces of the Sobolev spaces $L^2(-\pi, \pi)$, $H^k(-\pi, \pi)$ consisting of periodic functions with period $2\pi$. We also use $\dot{L}^2_{\text{per}}$, $\dot{H}^k_{\text{per}}$ to denote the subspaces of $L^2_{\text{per}}$, $H^k_{\text{per}}$ consisting of functions with zero mean, and use $L^2_{\text{odd}}$, $H^k_{\text{odd}}$ to denote the subspaces of $\dot{L}^2_{\text{per}}$, $\dot{H}^k_{\text{per}}$ consisting of odd functions.

The paper is organized as follows. In § 2 we compile some relevant general results regarding existence and uniqueness of the solutions for Cauchy problems for nonlinear evolutionary equations on Banach spaces. In § 3 we use these results to prove global existence and uniqueness of the solutions of the nonlocal Kuramoto–Sivashinsky equations (1.4) in $\dot{H}^1_{\text{per}}$. First, we prove local results following the approach of Duan & Ervin [7] (see also Henry [12]) and then establish global results by proving uniform boundedness of the solutions in $\dot{H}^1_{per}$ on each time interval. To establish uniform boundedness of the solutions in $\dot{H}^1_{per}$ we first prove uniform boundedness in $\dot{L}^2_{per}$, which is done by a modification of the method of Collet *et al.* [5]. After proving global existence this also provides the existence of the absorbing ball in $\dot{L}^2_{per}$ as well as estimates for its radius. As $p$ increases from the value 3, we can determine how the nonlocal term affects the radius $\mathscr{R}$ of the absorbing ball. In particular, when we consider our bound for large values of $\gamma = \mu\nu^{-1/2}$ we obtain the estimate $\mathscr{R} = \mathcal{O}\big(\gamma^{\frac{23p-28}{10(4-p)}}\nu^{-21/20}\big)$. The corresponding estimate for $\gamma = 0$ is $\mathscr{R} = \mathcal{O}\big(\nu^{-21/20}\big)$, in agreement with Collet *et al.* [5]. Both these estimates represent improvements over those found by Duan & Ervin [7], who considered $p = 3$ and whose values are $\mathcal{O}\big(\gamma^6\nu^{-3/2}\big)$ and $\mathcal{O}\big(\nu^{-3/2}\big)$ for $\gamma$ large and zero, respectively. In § 4 we compare the analytical estimate with values obtained from numerical computations for different values of $p$ and guided by the computations we give a heuristic argument for the best bound of $\mathscr{R}$ for large values of $\mu$ (or $\gamma$). Conclusions are given in § 5.

## 2 Existence and uniqueness theory for nonlinear Cauchy problems

In this section we review some basic results regarding existence and uniqueness theory of the solutions of nonlinear partial differential equations which are relevant to our problem. For more information see for example Henry [12] or Sell & You [25].

Consider the following Initial Value Problem for an abstract nonlinear evolutionary equation:

$$\partial_t u + Au = F(u, t), \quad \text{for } u(t_0) = u_0 \in W \text{ and } t \geqslant t_0 \geqslant 0, \tag{2.1}$$

on a Banach space $W$, where $A$ is a sectorial operator. We also assume that $F$ maps some open subset $U \subset W^\alpha \times \mathbb{R}^+$ into $W$, for some $\alpha \in [0, 1)$, and that $F \in C_{\mathrm{Lip};\theta}(U, W)$, where $C_{\mathrm{Lip};\theta}(U, W)$ are those functions which are locally Lipschitz continuous in $u$ and locally Hölder continuous in $t$ on $U$, i.e. for each $(u_1, t_1) \in U$ there exists a neighborhood $V \subset U$ of $(u_1, t_1)$ such that for any $(v_1, s_1) \in V$, $(v_2, s_2) \in V$,

$$\|F(v_1, s_1) - F(v_2, s_2)\| \leqslant L(\|v_1 - v_2\|_\alpha + |s_1 - s_2|^\theta), \tag{2.2}$$

for some constants $L > 0$, $\theta \in (0, 1]$. It is also assumed that for every bounded set $B \subset U$, the image $F(B)$ is bounded in $W$.

Let $\tau > 0$ and $I = [t_0, t_0 + \tau)$ be an interval in $\mathbb{R}^+$.

**Definition 2.1** *A pair $(u, I)$ is said to be a solution of (2.1) in the space $W^\alpha$ on $I$ if $u : I \to W$ is (strongly) continuous, $u(t_0) = u_0$, and on $(t_0, t_0 + \tau)$ we have $(u(t), t) \in U$, $u(t) \in D(A)$ (where $D(A)$ is the domain of the operator $A$), the mapping $t \to F(u(t), t)$ is locally Hölder continuous, $u$ is (strongly) differentiable, and $u$ satisfies the equation*

$$\partial_t u(t) + Au(t) = F(u(t), t) \tag{2.3}$$

*in $W$, at each $t \in (t_0, t_0 + \tau)$.*

The following local existence and uniqueness result holds:

**Theorem 2.2** *Let $A$ be a sectorial operator, $\alpha \in [0, 1)$, $F \in C_{\mathrm{Lip};\theta}(U, W)$, where $U$ is an open subset of $W^\alpha \times \mathbb{R}^+$. Then for every $(u_0, t_0) \in U$ the initial value problem* (2.1) *has a unique solution in $W^\alpha$ on some interval $I = [t_0, t_0 + \tau)$, for some $\tau > 0$.*

Let $(u_1, I_1)$ and $(u_2, I_2)$ be two solutions of (2.1), where $I_i = [t_0, t_0 + \tau_i)$, $i = 1, 2$, and $\tau_1 \leqslant \tau_2$. The uniqueness of solutions implies $u_1(t) = u_2(t)$ for $t \in I_1$. Hence $(u_2, I_2)$ is an extension of $(u_1, I_1)$. If $\tau_1 < \tau_2$ then $(u_2, I_2)$ is said to be a proper extension of $(u_1, I_1)$. A solution $(u, I)$ is said to be a maximally defined solution if it has no proper extensions.

**Theorem 2.3** *Let $A$ and $F$ be as in Theorem 2.2 above. Then for every $(u_0, t_0) \in U$ the initial value problem* (2.1) *has a unique maximally defined solution $(u, I)$ of* (2.1) *in $W^\alpha$, where $I = [t_0, T)$. Furthermore, either $T = \infty$, or there exists a sequence $t_n \to T^-$, $n = 1, 2, \ldots$, such that $(u(t_n), t_n) \to \partial U$ as $n \to \infty$. (If $U$ is unbounded, the point at infinity is included in $\partial U$, e.g. if $\partial U$ has only the point at infinity, then $\lim_{t\to T^-} \|u(t)\|_\alpha = \infty$.)*

## 3 Results for the MKS and DMKS equations

In this section we study the behavior of the solutions of equations (1.4) with periodic boundary conditions. Note that the operator $\mathscr{H} \circ \partial_x$ is self adjoint, densely defined and bounded below in $L^2_{\mathrm{per}}$. Hence it is sectorial, and the powers $(\mathscr{H} \circ \partial_x)^p$ can be considered which are linear operators.

We will consider the solutions having a vanishing spatial integral. This assumption is correct due to the conservation of spatial integrals, which can be seen by integrating equation (1.4):

$$\frac{d}{dt}\int_{-\pi}^{\pi} u(x, t)dx = 0. \tag{3.1}$$

So, if initially the spatial integral is zero, it remains zero for all time.

First, we will show local existence and uniqueness of the solutions in $\dot{H}^1_{\mathrm{per}}$ on some time interval $[0, T(u_0))$ using the results above (Theorem 2.2) and then we will show that if $T(u_0)$ is finite then the solutions are uniformly bounded for all time in $\dot{H}^1_{\mathrm{per}}$, which by Theorem 2.3 also implies global existence.

An estimate for the upper bound of the $L^2$-norm of the solutions in terms of the parameters of the equation will also be obtained. This will be done using the method of Collet *et al.* [5] by considering first the problem for antisymmetric (odd) solutions, and then expanding the results for general (not necessarily odd) solutions.

### 3.1 Local existence and uniqueness

For equations (1.4) we fix the basic space to be the real Hilbert space $\dot{L}^2_{\mathrm{per}}$. We define the operator $A_1 : \mathscr{D}(A_1) \to \dot{L}^2_{\mathrm{per}}$ by

$$A_1\varphi = \nu\partial_x^4\varphi \pm \partial_x^2\varphi + a\varphi, \quad \text{for } \varphi \in \mathscr{D}(A_1), \tag{3.2}$$

where $\mathscr{D}(A_1) = \dot{H}^4_{\mathrm{per}}$, and we also define the operator $A_2 : \mathscr{D}(A_2) \to \dot{L}^2_{\mathrm{per}}$ by

$$A_2\varphi = -\mu(\mathscr{H} \circ \partial_x)^p[\varphi], \quad \text{for } \varphi \in \mathscr{D}(A_2), \tag{3.3}$$

where $\mathscr{D}(A_2) = \{\varphi \in \dot{L}^2_{\mathrm{per}} : \sum_{k=-\infty}^{\infty} |k|^{2p}\varphi_k^2 < \infty\}$. (Here $\varphi_k$, $k = 0, \pm1, \pm2, \ldots$, are the Fourier coefficients of $\varphi$.) (Note that $\mathscr{D}(A_1) \subset \mathscr{D}(A_2)$.)

Let $a$ be chosen such that the eigenvalues of $A_1$ are all positive, i.e.

$$\nu k^4 \mp k^2 + a > 0, \quad \text{for all } k = 0, \pm1, \pm2, \ldots. \tag{3.4}$$

Then $A_1$ is a positive sectorial operator. By Theorem 1.4.2 in Henry [12] the operator $A_1^{-p/4}$ is a bounded linear operator. It is easy to see that the operator $A_2 \circ A_1^{-p/4}$ is also bounded. Hence, Corollary 1.45 of Henry [12] implies that $A = A_1 + A_2$ is a sectorial operator.

Equations (1.4) take the form

$$\partial_t u + Au = F(u), \quad \text{for } t > 0, \tag{3.5}$$

where the nonlinear operator $F : \dot{H}^1_{\mathrm{per}} \to \dot{L}^2_{\mathrm{per}}$ is defined by

$$F(\varphi) = -\varphi\varphi_x + a\varphi, \quad \text{for } \varphi \in \dot{H}^1_{\mathrm{per}}. \tag{3.6}$$

It can be verified that $F \in C_{\mathrm{Lip}}(\dot{H}^1_{\mathrm{per}}, \dot{L}^2_{\mathrm{per}})$. Therefore, by Theorem 2.2, for every $u_0 \in \dot{H}^1_{\mathrm{per}}$ there exists a unique maximally defined solution in $\dot{H}^1_{\mathrm{per}}$ on the interval $[0, T)$, where $0 < T = T(u_0)$.

It remains to prove that the solution is uniformly bounded in $\dot{H}^1_{\mathrm{per}}$ on every finite time interval. Then, from the theory in the previous section (Theorem 2.3) it follows that $T(u_0) = \infty$. In order to accomplish the proof, we first need to establish uniform boundedness of the solutions in $\dot{L}^2_{\mathrm{per}}$. This is done next, for both odd-parity and non-parity solutions.

### 3.2 Uniform boundedness of the solutions in $\dot{L}^2_{\mathrm{per}}$

In what follows we analyze the MKS equation (the plus sign is taken in (1.4)) and prove uniform boundedness of the solutions in the $\dot{L}^2_{\mathrm{per}}$ and $\dot{H}^1_{\mathrm{per}}$ norms, along with estimates for the radius of the absorbing ball. When the DMKS equation is considered, the results are similar and are briefly discussed in the Conclusions section.

#### 3.2.1 *The antisymmetric case*

First we consider the antisymmetric case, i.e., we consider the solutions in $L^2_{\mathrm{odd}}$. It is noticed that if a solution of 1.4) is initially in $L^2_{\mathrm{odd}}$ then it remains in $L^2_{\mathrm{odd}}$ for all time.

Define the linear operator $\mathscr{L} = \mathscr{L}_{\mu,\nu}$:

$$\mathscr{L} : f \mapsto -f_{xx} - \nu f_{xxxx} + \mu(\mathscr{H} \circ \partial_x)^p[f], \tag{3.7}$$

then (1.4) can be written as

$$u_t = \mathscr{L}u - uu_x. \tag{3.8}$$

If we express $u$ as $u(x, t) = v(x, t) + \varphi(x)$, where $\varphi \in L^2_{\mathrm{odd}}$ is an appropriately chosen gauge function found in the sequel, then the equation becomes:

$$v_t = \mathscr{L}v + \mathscr{L}\varphi - vv_x - v\varphi' - \varphi v_x - \varphi\varphi'. \tag{3.9}$$

Multiplying the last equation by $v$ and integrating over the interval $[-\pi, \pi]$ gives

$$\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} v^2 dx = \int_{-\pi}^{\pi} v\mathscr{L}v dx + \int_{-\pi}^{\pi} v\mathscr{L}\varphi dx - \int_{-\pi}^{\pi} v^2 v_x dx - \int_{-\pi}^{\pi} v^2\varphi' dx - \int_{-\pi}^{\pi} vv_x\varphi dx - \int_{-\pi}^{\pi} v\varphi\varphi' dx. \tag{3.10}$$

Integrating by parts and using periodicity, yields,

$$\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} v^2 dx = \int_{-\pi}^{\pi} v\mathscr{L}v dx + \int_{-\pi}^{\pi} v\mathscr{L}\varphi dx - \frac{1}{2}\int_{-\pi}^{\pi} v^2\varphi' dx - \int_{-\pi}^{\pi} v\varphi\varphi' dx. \tag{3.11}$$

Next, define a bilinear form

$$(f, g)_{\alpha\varphi} = v\int_{-\pi}^{\pi} f_{xx}g_{xx} dx - \int_{-\pi}^{\pi} f_x g_x dx + \mu\int_{-\pi}^{\pi} f_{xx}(\mathscr{H}\circ\partial_x)^{p-2}[g] dx + \alpha\int_{-\pi}^{\pi} fg\varphi' dx, \tag{3.12}$$

which also can be written as

$$(f, g)_{\alpha\varphi} = -\int_{-\pi}^{\pi} f(\mathscr{L} - \alpha\varphi')g dx. \tag{3.13}$$

Then (3.11) takes the form

$$\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} v^2 dx = -(v, v)_{\frac{1}{2}\varphi} - (v, \varphi)_{\varphi}. \tag{3.14}$$

The main idea now is to find an odd function $\varphi$ such that the bilinear form (3.13) becomes positive definite and for large enough $L^2$-norm of $v$ the right-hand side of (3.14) becomes negative.

We define the following auxiliary quadratic forms:

$$R_{\alpha\varphi}(u) = (u, u)_{\alpha\varphi}, \tag{3.15}$$

$$Q(u) = \frac{v}{4}\int_{-\pi}^{\pi} u_{xx}^2 dx + \frac{\eta(\gamma)}{4v}\int_{-\pi}^{\pi} u^2 dx, \tag{3.16}$$

where $\gamma = v^{1-p/2}\mu$, and $\eta$ is a function of $\gamma$, which is defined as follows:

$$\eta(\gamma) = \begin{cases} 1, & \text{if } \gamma \leqslant 1, \\ \gamma^4, & \text{if } \gamma > 1. \end{cases} \tag{3.17}$$

Using (3.12) we can write

$$R_{\alpha\varphi}(u) = v\int_{-\pi}^{\pi} u_{xx}^2 dx - \int_{-\pi}^{\pi} u_x^2 dx + \mu\int_{-\pi}^{\pi} u_{xx}(\mathscr{H}\circ\partial_x)^{p-2}[u] dx + \alpha\int_{-\pi}^{\pi} u^2\varphi' dx. \tag{3.18}$$

**Proposition 3.1** *There exists a function $\varphi \in H^2_{\text{odd}}$ such that for $\mu \geqslant 0$ and $v \in (0, v_0(\mu))$, and for all $v \in H^2_{\text{odd}}$ and all $\alpha \in [\alpha_0, 1]$*

$$R_{\alpha\varphi}(v) \geqslant Q(v), \tag{3.19}$$

$$R_{\alpha\varphi}(\varphi) \leqslant C(\gamma, v). \tag{3.20}$$

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*Here* $\alpha_0$ *is some fixed number in* $(0, 0.5)$ *and* $C(\gamma, \nu)$ *is a function of* $\gamma$ *and* $\nu$ *only, which will be determined later.*

**Remark**. The upper bound $\nu_0(\mu)$ is the unique positive solution of $k_0(\mu, \nu) = 1$ where $k_0$ is the unique positive solution of $\omega_+(k_0) = 0$. This follows from the dispersion relation (1.5) and the fact that we are considering $2\pi$-periodic solutions, meaning that $k$ is a positive integer. In general, (1.5) gives a band of unstable waves $0 < k < k_0(\mu, \nu)$, which may or may not include $k = 1$. Hence instability first enters when $k_0(\mu, \nu) = 1$, which is in turn solved to obtain the unique critical value of $\nu = \nu_0$ (for fixed $\mu$). If $\nu > \nu_0(\mu)$, it can be proved, using Poincaré inequalities, that $\|u\|_2$ decays to zero uniformly as $t$ tends to infinity. Note that for $p = 3$ we can solve for $\nu_0$ analytically to obtain $\nu_0(\mu) = 1 + \mu$.

**Proof** We work with the Fourier series of $v$ and $\varphi'$. Since $v$ and $\varphi$ are odd functions with respect to $x$, we get

$$v(x) = i \sum_{n\in\mathbb{Z}} v_n e^{inx}, \tag{3.21}$$

where $v_n$ are all real, and $v_0 = 0$, $v_n = -v_{-n}$ for $n = 1, 2, \dots$, and

$$\varphi'(x) = -\sum_{n\in\mathbb{Z}} \psi_n e^{inx}, \tag{3.22}$$

where $\psi_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$, and $\psi_0 = 0$, $\psi_n = \psi_{-n}$ for $n = 1, 2, \dots$.

Next we will find the expressions for $R_{\alpha\varphi}(v)$ and $Q(v)$ in terms of the coefficients $v_n$ and $\psi_n$. First, note that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} v^2\varphi' dx &= \frac{1}{2\pi}\sum_{k,l,m}\int_{-\pi}^{\pi} v_k v_l \psi_m e^{i(k+l+m)x} dx \\ &= \sum_{k+l+m=0} v_k v_l \psi_m = \sum_{k,l} v_k v_l \psi_{-k-l} = \sum_{k,l} v_k v_l \psi_{|k+l|}. \end{aligned} \tag{3.23}$$

Using (3.18) we get

$$\begin{aligned} \frac{1}{2\pi} R_{\alpha\varphi}(v) &= \nu \sum_{n\in\mathbb{Z}} n^4 v_n^2 - \sum_{n\in\mathbb{Z}} n^2 v_n^2 - \mu \sum_{n\in\mathbb{Z}} |n|^p v_n^2 + \alpha \sum_{k,l} v_k v_l \psi_{|k+l|} \\ &= -2\sum_{n=1}^{\infty} \omega_+(n) v_n^2 + \alpha \sum_{k,l>0} v_k v_l (\psi_{|k+l|} + \psi_{|-k-l|} - \psi_{|k-l|} - \psi_{|-k+l|}) \\ &= -2\sum_{n=1}^{\infty} \omega_+(n) v_n^2 + 2\alpha \sum_{k,l>0} v_k v_l (\psi_{|k+l|} - \psi_{|k-l|}) \\ &= 2\sum_{n=1}^{\infty} (-\omega_+(n) + \alpha\psi_{2n}) v_n^2 + 2\alpha \sum_{k,l>0\, k\neq l} v_k v_l (\psi_{|k+l|} - \psi_{|k-l|}) \\ &= 2\left[\sum_{n=1}^{\infty} (-\omega_+(n) + \alpha\psi_{2n}) v_n^2 + 2\alpha \sum_{k>l>0} v_k v_l (\psi_{|k+l|} - \psi_{|k-l|})\right], \end{aligned} \tag{3.24}$$

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where $\omega_+(n) = -\nu n^4 + \mu|n|^p + n^2$, as was defined above. Also,

$$\frac{1}{2\pi}Q(v) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\nu n^4 + \frac{\eta(\gamma)}{\nu}\right)v_n^2. \tag{3.25}$$

To prove the first part of the proposition, we need

$$2\left[\sum_{n=1}^{\infty}(-\omega_+(n) + \alpha\psi_{2n})v_n^2 + 2\alpha\sum_{k>l>0}v_k v_l(\psi_{|k+l|} - \psi_{|k-l|})\right] \geqslant \frac{1}{2}\sum_{n=1}^{\infty}\left(\nu n^4 + \frac{\eta(\gamma)}{\nu}\right)v_n^2. \tag{3.26}$$

First, let $N(\gamma, \nu)$ be an integer, such that

$$-\omega_+(n) \geqslant \frac{1}{2}\left(\nu n^4 + \frac{\eta(\gamma)}{\nu}\right) \quad \text{for } n > N. \tag{3.27}$$

The inequality we need to solve takes the form

$$\frac{\nu}{2}n^4 - \mu n^p - n^2 - \frac{\eta(\gamma)}{2\nu} \geqslant 0, \tag{3.28}$$

or

$$\frac{1}{2}x^4 - \gamma x^p - x^2 - \frac{\eta(\gamma)}{2} \geqslant 0, \tag{3.29}$$

where $\gamma = \nu^{1-p/2}\mu$, as was defined before, and also $x = \nu^{1/2}n$.

In the general case $3 \leqslant p < 4$, consideration of the large $\gamma$ behavior of equation (3.29), yields $N = \mathcal{O}\left(\nu^{-1/2}\gamma^{1/(4-p)}\right)$. Note that as $p \to 4^-$, $N \to \infty$ and the analysis breaks down as expected. When $p = 3$ it can be easily verified that inequality (3.29) holds for $x > 2.2(\gamma + 1)$, which implies $n > 2.2\nu^{-1/2}(\gamma + 1)$. Therefore, when $p = 3$, we can take $N = \lceil 2.2\nu^{-1/2}(\gamma + 1)\rceil$.

Next, let $B = B(\gamma, \nu)$ be determined as follows:

$$\begin{aligned} B &= \min_{0\leqslant n\leqslant N}\left(\frac{\nu}{2}n^4 - \mu n^p - n^2 - \frac{\eta(\gamma)}{2\nu}\right) \\ &= \frac{1}{\nu}\min_{0\leqslant x\leqslant \nu^{1/2}N}\left(\frac{1}{2}x^4 - \gamma x^p - x^2 - \frac{\eta(\gamma)}{2}\right). \end{aligned} \tag{3.30}$$

For simplicity, we denote $\delta = \delta(\gamma) \equiv -\min_{0\leqslant x\leqslant \nu^{1/2}N}\left(\frac{1}{2}x^4 - \gamma x^p - x^2 - \frac{\eta(\gamma)}{2}\right)$.

Even though $\delta$ cannot be obtained in closed form for general $p$, its large $\gamma$ behavior can be calculated asymptotically, yielding

$$\delta = \mathcal{O}\left(\gamma^{4/(4-p)}\right) \quad \text{as } \gamma \to \infty. \tag{3.31}$$

When $p = 3$, the exact expression $\delta = \frac{1+\eta(\gamma)}{2} + (72\gamma^2 + 27\gamma^4 + \gamma(9\gamma^2 + 16)^{3/2})/64$ follows, which in turn implies that

$$\delta = \mathcal{O}\left(\gamma^4\right) \quad \text{as } \gamma \to \infty. \tag{3.32}$$

We choose

$$\psi_{2n} = -\frac{1}{\alpha_0}B = \frac{\delta}{\alpha_0\nu} \quad \text{for } n \leqslant N. \tag{3.33}$$

The remaining coefficients $\psi_k$ are chosen to be nonnegative and will be specified later. Now

$$-\omega_+(n) + \alpha\psi_{2n} \geqslant \frac{1}{2}\left(vn^4 + \frac{\eta(\gamma)}{v}\right) \tag{3.34}$$

for all $n = 1, 2, \ldots$, and $\alpha \in [\alpha_0, 1]$. We define

$$\tau_n = \sqrt{\frac{1}{2}\left(vn^4 + \frac{\eta(\gamma)}{v}\right)} \tag{3.35}$$

and set $w_n = \tau_n v_n$ for $n = 1, 2, \ldots$. Then,

$$\frac{1}{2\pi} R_{\alpha\varphi}(v) \geqslant 2\left[\sum_{n=1}^{\infty} w_n^2 + 2\alpha \sum_{k>l>0} w_k \frac{\psi_{|k+l|} - \psi_{|k-l|}}{\tau_k \tau_l} w_l\right] \equiv 2\big(w, (\mathbf{Id} + 2\alpha\boldsymbol{\Gamma})w\big). \tag{3.36}$$

Hence the first part of the proposition will be proved if we find appropriate coefficients $\psi_k$, such that

$$\big(w, (\mathbf{Id} + 2\alpha\boldsymbol{\Gamma})w\big) \geqslant \frac{1}{2}(w, w), \quad \text{for } \alpha \in [\alpha_0, 1]. \tag{3.37}$$

So, we need to get $(w, w) \geqslant -4\alpha(w, \boldsymbol{\Gamma} w)$. The sufficient condition for this is that the Hilbert–Schmidt norm of $4\alpha\boldsymbol{\Gamma}$ is less than 1, i.e.,

$$(4\alpha)^2 \|\boldsymbol{\Gamma}\|_{\mathrm{HS}}^2 \equiv 16\alpha^2 \sum_{k>l>0} \left|\frac{\psi_{|k+l|} - \psi_{|k-l|}}{\tau_k \tau_l}\right|^2 < 1 \tag{3.38}$$

for all $\alpha \in [\alpha_0, 1]$. So, it is enough to find coefficients $\psi_k$ such that

$$\|\boldsymbol{\Gamma}\|_{\mathrm{HS}}^2 \equiv \sum_{k>l>0} \left|\frac{\psi_{|k+l|} - \psi_{|k-l|}}{\tau_k \tau_l}\right|^2 < \frac{1}{16}. \tag{3.39}$$

But on the other hand we need to find the coefficients $\psi_k$ such that $(\varphi, \varphi)_{\alpha\varphi}$ is minimized (this is needed in the estimates that come later). We can satisfy (3.39) by choosing $\psi_k$ to be constant. But then the corresponding Fourier series does not converge and the norm of $\varphi$ becomes infinite. Therefore, we choose $\psi_k$ to be a nonnegative and non-increasing function of $k$, vanishing sufficiently fast as $k$ goes to infinity. This can be done by taking

$$\psi_{2m} = \begin{cases} \frac{\delta}{\alpha_0 v}, & \text{if } 1 \leqslant |m| \leqslant M, \\ \frac{\delta}{\alpha_0 v} f\left(\frac{|m|}{M} - 1\right), & \text{if } |m| > M, \end{cases} \tag{3.40}$$

where $M$ is an integer which will be chosen later (of course we should take $M \geqslant N$ to be consistent with (3.33)), and $f \in C^1[0, \infty]$ is a nonnegative non-increasing function satisfying the conditions $f(0) = 1$, $f'(0) = 0$, $\sup|f'| < 1$, and

$$\int_0^{\infty} (1 + k^2) f^2(k) dk < \infty. \tag{3.41}$$

A plot of the function $\varphi(x)$ is provided in Figure 1.

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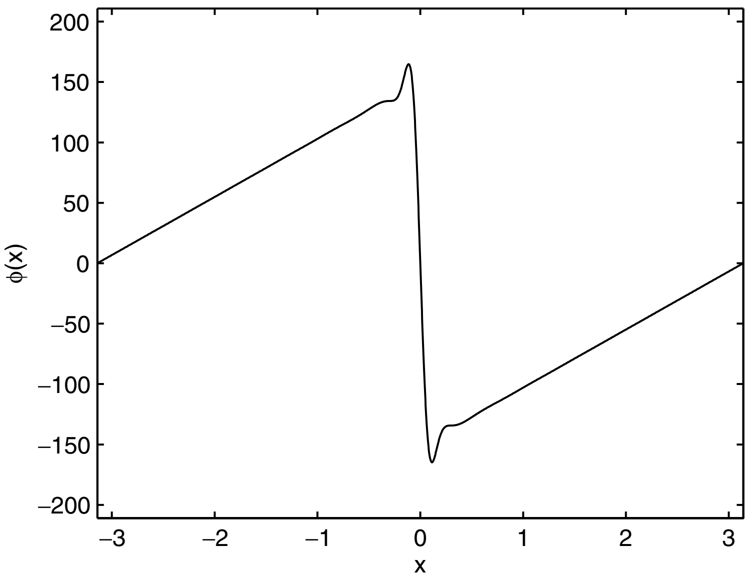


FIGURE 1. The graph of the gauge function $\varphi(x)$ for $v = 0.5$, $\mu = 1$, in physical space, when the function $f(k)$ in equation (3.40) is chosen as $f(k) = e^{-k^2}$.

For $k > l$ we get $|\psi_{k+l} - \psi_{k-l}| = 0$ if $k + l \leqslant 2M$ and also if $k + l$ and $k - l$ are both odd numbers (i.e., if $k$ and $l$ are of a different parity).

Next, if $k$ and $l$ are of the same parity and $k + l > 2M$, we get:

- If $k - l \leqslant 2M$

$$|\psi_{k+l} - \psi_{k-l}| = \frac{\delta}{\alpha_0 v}\left|f\left(\frac{k+l}{2M} - 1\right) - 1\right| = \frac{\delta}{\alpha_0 v}\left|f\left(\frac{k+l}{2M} - 1\right) - f(0)\right|. \tag{3.42}$$

Since $\sup|f'| < 1$ then by the mean value theorem we get

$$\begin{aligned} |\psi_{k+l} - \psi_{k-l}| &\leqslant \frac{\delta}{\alpha_0 v}\left(\frac{k+l}{2M} - 1\right) = \frac{\delta}{\alpha_0 v}\left(\frac{k+l-2M}{2M}\right) \\ &\leqslant \frac{\delta}{\alpha_0 v}\left(\frac{k+l-(k-l)}{2M}\right) = \frac{\delta}{\alpha_0 v}\frac{l}{M}. \end{aligned} \tag{3.43}$$

- Similarly, if $k - l > 2M$

$$\begin{aligned} |\psi_{k+l} - \psi_{k-l}| &= \frac{\delta}{\alpha_0 v}\left|f\left(\frac{k+l}{2M} - 1\right) - f\left(\frac{k-l}{2M} - 1\right)\right| \\ &\leqslant \frac{\delta}{\alpha_0 v}\left[\left(\frac{k+l}{2M} - 1\right) - \left(\frac{k-l}{2M} - 1\right)\right] = \frac{\delta}{\alpha_0 v}\frac{l}{M}. \end{aligned} \tag{3.44}$$

Hence, for all $k > l > 0$, $|\psi_{k+l} - \psi_{k-l}| = 0$ if $k + l \leqslant 2M$ or if $k$ and $l$ are of a different parity, and $|\psi_{k+l} - \psi_{k-l}| \leqslant \frac{\delta}{\alpha_0 v}\frac{l}{M}$ if $k + l > 2M$.

So, we get

$$\|\boldsymbol{\Gamma}\|_{\mathrm{HS}}^2 = \sum_{l=1}^{M}\sum_{k=l+1}^{\infty}\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2 + \sum_{l=M+1}^{\infty}\sum_{k=l+1}^{\infty}\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2. \tag{3.45}$$

For the first term we get

$$\begin{aligned}\sum_{l=1}^{M}\sum_{k=l+1}^{\infty}\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2 &= \left(\sum_{l=1}^{M}\sum_{k=l+1}^{2M-l}+\sum_{l=1}^{M}\sum_{k=2M-l+1}^{\infty}\right)\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2\\ &= 0+\sum_{l=1}^{M}\sum_{k=2M-l+1}^{\infty}\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2 = \sum_{l=1}^{M}\sum_{k=2M-l+1}^{\infty}\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2. \end{aligned}\tag{3.46}$$

Hence

$$\begin{aligned}\|\boldsymbol{\Gamma}\|_{\mathrm{HS}}^2 &= \sum_{l=1}^{M}\sum_{k=2M-l+1}^{\infty}\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2 + \sum_{l=M+1}^{\infty}\sum_{k=l+1}^{\infty}\left|\frac{\psi_{|k+l|}-\psi_{|k-l|}}{\tau_k\tau_l}\right|^2\\ &\leqslant \sum_{l=1}^{M}\sum_{k=2M-l+1}^{\infty}\frac{\delta^2}{\alpha_0^2\nu^2}\frac{l^2}{M^2}\tau_k^{-2}\tau_l^{-2} + \sum_{l=M+1}^{\infty}\sum_{k=l+1}^{\infty}\frac{\delta^2}{\alpha_0^2\nu^2}\frac{l^2}{M^2}\tau_k^{-2}\tau_l^{-2}\\ &= \frac{\delta^2}{\alpha_0^2\nu^2M^2}\sum_{l=1}^{M}l^2\tau_l^{-2}\sum_{k=2M-l+1}^{\infty}\tau_k^{-2} + \frac{\delta^2}{\alpha_0^2\nu^2M^2}\sum_{l=M+1}^{\infty}l^2\tau_l^{-2}\sum_{k=l+1}^{\infty}\tau_k^{-2}\\ &\leqslant \frac{\delta^2}{\alpha_0^2\nu^2M^2}\sum_{l=1}^{M}l^2\tau_l^{-2}\int_{2M-l}^{\infty}\tau_k^{-2}dk + \frac{\delta^2}{\alpha_0^2\nu^2M^2}\sum_{l=M+1}^{\infty}l^2\tau_l^{-2}\int_{l}^{\infty}\tau_k^{-2}dk. \end{aligned}\tag{3.47}$$

Now

$$\int_a^\infty \tau_k^{-2}dk = \int_a^\infty \frac{2}{\nu k^4+\eta(\gamma)/\nu}dk < \frac{2}{\nu}\int_a^\infty k^{-4}dk = \frac{2}{3}\nu^{-1}a^{-3}. \tag{3.48}$$

Thus,

$$\int_{2M-l}^{\infty}\tau_k^{-2}dk < \frac{2}{3}\nu^{-1}(2M-l)^{-3} \leqslant \frac{2}{3}\nu^{-1}M^{-3}, \tag{3.49}$$

(the last step is true for $1 \leqslant l \leqslant M$) and

$$\int_l^\infty \tau_k^{-2}dk < \frac{2}{3}\nu^{-1}l^{-3}. \tag{3.50}$$

Therefore,

$$\begin{aligned}\|\boldsymbol{\Gamma}\|_{\mathrm{HS}}^2 &< \frac{2\delta^2}{3\alpha_0^2\nu^3}M^{-5}\sum_{l=1}^{M}l^2\tau_l^{-2} + \frac{2\delta^2}{3\alpha_0^2\nu^3}M^{-2}\sum_{l=M+1}^{\infty}l^{-1}\tau_l^{-2}\\ &< \frac{2\delta^2}{3\alpha_0^2\nu^3}M^{-5}\int_0^M l^2\tau_l^{-2}dl + \frac{2\delta^2}{3\alpha_0^2\nu^3}M^{-2}\int_M^\infty l^{-1}\tau_l^{-2}dl\\ &= \frac{4\delta^2}{3\alpha_0^2\nu^3}M^{-5}\int_0^M\frac{l^2}{\nu l^4+\eta(\gamma)/\nu}dl\\ &\quad + \frac{4\delta^2}{3\alpha_0^2\nu^3}M^{-2}\int_M^\infty\frac{1}{\nu l^5+(\eta(\gamma)/\nu)l}dl. \end{aligned}\tag{3.51}$$

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Next we estimate the integrals in the last expression:

$$\int_0^M \frac{l^2}{\nu l^4+\eta(\gamma)/\nu}dl < \int_0^\infty \frac{\nu l^2}{\nu^2 l^4+\eta(\gamma)}dl < \int_0^\infty \frac{1}{\sqrt{\nu^2 l^4+\eta(\gamma)}}dl$$
$$= \int_0^a \frac{1}{\sqrt{\nu^2 l^4+\eta(\gamma)}}dl + \int_a^\infty \frac{1}{\sqrt{\nu^2 l^4+\eta(\gamma)}}dl$$
$$< \int_0^a \big(\eta(\gamma)\big)^{-1/2}dl + \int_a^\infty \frac{1}{\nu l^2}dl = a\big(\eta(\gamma)\big)^{-1/2} + (a\nu)^{-1}. \tag{3.52}$$

Taking $a = \nu^{-1/2}\big(\eta(\gamma)\big)^{1/4}$ gives

$$\int_0^M \frac{l^2}{\nu l^4+\eta(\gamma)/\nu}dl < 2\nu^{-1/2}\big(\eta(\gamma)\big)^{-1/4}. \tag{3.53}$$

Estimating the other integral, we get

$$\int_M^\infty \frac{1}{\nu l^5+(\eta(\gamma)/\nu)l}dl < \int_M^\infty \frac{1}{\nu l^5}dl = \frac{1}{4}\nu^{-1}M^{-4}. \tag{3.54}$$

So,

$$\|\boldsymbol{\Gamma}\|_{\mathrm{HS}}^2 < \frac{8\delta^2}{3\alpha_0^2}\nu^{-7/2}\big(\eta(\gamma)\big)^{-1/4}M^{-5} + \frac{\delta^2}{3\alpha_0^2}\nu^{-4}M^{-6}. \tag{3.55}$$

We choose $M \geqslant N$ such that the right hand side of the last inequality becomes less than $1/16$. This can be done since the right hand side of (3.55) is a positive function of $M$ that decays to zero as $M \to \infty$. For "small" $\nu$ (that is $\nu \in (0, 1]$) if we take $M \geqslant \frac{5}{2}\alpha_0^{-2/5}\delta^{2/5}\eta(\gamma)^{-1/20}\nu^{-7/10}$, the condition $M \geqslant N$ is always satisfied. Hence, $M = \big\lceil \frac{5}{2}\alpha_0^{-2/5}\delta^{2/5}\eta(\gamma)^{-1/20}\nu^{-7/10}\big\rceil$; for large $\gamma$, therefore, it follows using (3.17) and (3.31), that

$$M = \mathcal{O}\big(\gamma^{\frac{4+p}{5(4-p)}}\nu^{-7/10}\big). \tag{3.56}$$

When $p = 3$ we obtain $M = \mathcal{O}\big(\gamma^{7/5}\nu^{-7/10}\big)$ which is used later.

Next we estimate the value $R_{\alpha\varphi}(\varphi)$. Using (3.18) we first note that $R_{\alpha\varphi}(\varphi) = R_0(\varphi)$, since the part which is dependent on $\alpha$ is equal to zero due to periodicity of $\varphi$. Also, if

$$\varphi'(x) = -\sum_{n\in\mathbb{Z}} \psi_n e^{inx}, \tag{3.57}$$

then

$$\varphi(x) = i\sum_{n\in\mathbb{Z}} \frac{\psi_n}{n}e^{inx}. \tag{3.58}$$

Therefore,

$$R_{\alpha\varphi}(\varphi) = -4\pi\sum_{n=1}^\infty \omega_+(n)\left(\frac{\psi_n}{n}\right)^2 = -4\pi\sum_{k=1}^\infty \omega_+(2k)\left(\frac{\psi_{2k}}{2k}\right)^2$$
$$= 4\pi\sum_{k=1}^\infty \big(\nu(2k)^2 - \mu(2k)^{p-2} - 1\big)\psi_{2k}^2 < 4\pi\sum_{k=1}^\infty 4\nu k^2\psi_{2k}^2$$

$$
\begin{aligned}
&= 4\pi \sum_{k=1}^{M} 4\nu k^2 \psi_{2k}^2 + 4\pi \sum_{k=M+1}^{\infty} 4\nu k^2 \psi_{2k}^2 \\
&= 16\pi \frac{\delta^2}{\alpha_0^2 \nu} \sum_{k=1}^{M} k^2 + 16\pi \frac{\delta^2}{\alpha_0^2 \nu} \sum_{k=M+1}^{\infty} k^2 f^2\left(\frac{k}{M} - 1\right) \\
&< 16\pi \frac{\delta^2}{\alpha_0^2 \nu} M^3 + 16\pi \frac{\delta^2}{\alpha_0^2 \nu} \int_M^{\infty} k^2 f^2\left(\frac{k}{M} - 1\right) dk. \qquad (3.59)
\end{aligned}
$$

Now, using the substitution $t = \frac{k}{M} - 1$ in the integral above, we get

$$
\begin{aligned}
R_{\alpha\varphi}(\varphi) &< 16\pi \frac{\delta^2}{\alpha_0^2 \nu} M^3 + 16\pi \frac{\delta^2}{\alpha_0^2 \nu} M^3 \int_0^{\infty} (t+1)^2 f^2(t) dt \\
&= 16\pi \frac{\delta^2}{\alpha_0^2 \nu} M^3 \left(1 + \int_0^{\infty} (t+1)^2 f^2(t) dt\right). \qquad (3.60)
\end{aligned}
$$

So, $R_{\alpha\varphi}(\varphi) < C(\gamma, \nu)$, where

$$
C(\gamma, \nu) = 16\pi \frac{\delta^2}{\alpha_0^2 \nu} M^3 \left(1 + \int_0^{\infty} (t+1)^2 f^2(t) dt\right). \qquad (3.61)
$$

For large $\gamma$ and small $\nu$ the value of $C(\gamma, \nu)$ is estimated using (3.31) and (3.56) to be

$$
C(\gamma, \nu) = \mathcal{O}\left(\gamma^{\frac{52+3p}{5(4-p)}} \nu^{-31/10}\right). \qquad (3.62)
$$

Note that when $p = 3$, we get $C(\gamma, \nu) = \mathcal{O}\left(\gamma^{61/5} \nu^{-31/10}\right)$. □

Now we are ready to prove uniform boundedness of the $L^2$-norm of the solutions of the MKS equation.

**Theorem 3.2** *Let $\mu$ be any positive number and $\nu \in (0, \nu_0(\mu))$. If $u(x, t)$ is a solution of equation (1.4) such that $u(x, 0) = u_0(x) \in H^1_{\text{odd}}$, then there is a constant $K > 0$ (independent of $\mu$, $\nu$, $u_0$), and a constant $D > 0$ (independent of $\mu$, $u_0$) such that*

$$
\|u\|_2 \leqslant (\|u_0\|_2 + \|\varphi\|_2) \exp(-Dt) + K\sqrt{\frac{\nu}{\eta(\gamma)} C(\gamma, \nu)} + \|\varphi\|_2, \qquad (3.63)
$$

*where $\varphi$ is the function constructed in Proposition 3.1, and $C(\gamma, \nu)$ is given by* (3.61).

**Proof** Let $\varphi$ be the function defined in Proposition 3.1. Then the bilinear form (3.13) is positive definite. Indeed, for any nonzero function $w$ we get $(w, w)_{\alpha\varphi} = R_{\alpha\varphi}(w) \geqslant Q(w) > 0$. Then, applying first the Cauchy–Shwarz inequality and then Young's inequality for the second term in (3.14) we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_{-\pi}^{\pi} v^2 dx &= -(v, v)_{\frac{1}{2}\varphi} - (v, \varphi)_{\varphi} \\
&\leqslant -(v, v)_{\frac{1}{2}\varphi} + \frac{\varepsilon}{2}(v, v)_{\varphi} + \frac{1}{2\varepsilon}(\varphi, \varphi)_{\varphi}, \qquad (3.64)
\end{aligned}
$$

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where $\varepsilon$ is a positive number which will be chosen later. So,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} v^2 dx &\leqslant \int_{-\pi}^{\pi} v\left(\mathscr{L}-\frac{1}{2}\varphi'\right)v dx - \frac{\varepsilon}{2}\int_{-\pi}^{\pi} v(\mathscr{L}-\varphi')v dx + \frac{1}{2\varepsilon}R_\varphi(\varphi)\\ &= \left(1-\frac{\varepsilon}{2}\right)\int_{-\pi}^{\pi} v\left(\mathscr{L}-\frac{1-\varepsilon}{2-\varepsilon}\varphi'\right)v dx + \frac{1}{2\varepsilon}R_\varphi(\varphi)\\ &= -\left(1-\frac{\varepsilon}{2}\right)R_{\frac{1-\varepsilon}{2-\varepsilon}\varphi}(v) + \frac{1}{2\varepsilon}R_\varphi(\varphi)\\ &\leqslant -\left(1-\frac{\varepsilon}{2}\right)Q(v) + \frac{1}{2\varepsilon}R_\varphi(\varphi). \end{aligned} \tag{3.65}$$

The last step is true when $\frac{1-\varepsilon}{2-\varepsilon} \in [\alpha_0, 1]$, or $\varepsilon \leqslant \frac{1-2\alpha_0}{1-\alpha_0}$. Inequality (3.65) implies

$$\frac{d}{dt}\|v\|_2^2 \leqslant -\frac{\eta(\gamma)}{2\nu}\left(1-\frac{\varepsilon}{2}\right)\|v\|_2^2 + \frac{1}{\varepsilon}C(\gamma, \nu). \tag{3.66}$$

The Gronwall inequality (see Temam [32]) then implies:

$$\|v\|_2^2 \leqslant \|v_{t=0}\|_2^2 \exp\left[-\frac{\eta(\gamma)}{2\nu}\left(1-\frac{\varepsilon}{2}\right)t\right] + \frac{4\nu C(\gamma, \nu)}{\varepsilon(2-\varepsilon)\eta(\gamma)}. \tag{3.67}$$

Therefore

$$\|v\|_2 \leqslant \|v_{t=0}\|_2 \exp\left[-\frac{\eta(\gamma)}{4\nu}\left(1-\frac{\varepsilon}{2}\right)t\right] + \frac{2}{\sqrt{\varepsilon(2-\varepsilon)}}\sqrt{\frac{\nu}{\eta(\gamma)}C(\gamma, \nu)}. \tag{3.68}$$

Since $\|v\|_2 = \|u-\varphi\|_2 \geqslant \big|\|u\|_2 - \|\varphi\|_2\big|$ and $\|v_{t=0}\|_2 = \|u_0-\varphi\|_2 \leqslant \|u_0\|_2 + \|\varphi\|_2$, we get

$$\begin{aligned}\|u\|_2 \leqslant (\|u_0\|_2 + \|\varphi\|_2)\exp\left[-\frac{\eta(\gamma)}{4\nu}\left(1-\frac{\varepsilon}{2}\right)t\right]\\ + \frac{2}{\sqrt{\varepsilon(2-\varepsilon)}}\sqrt{\frac{\nu}{\eta(\gamma)}C(\gamma, \nu)} + \|\varphi\|_2. \end{aligned} \tag{3.69}$$

Thus,

$$\|u\|_2 \leqslant (\|u_0\|_2 + \|\varphi\|_2)\exp(-Dt) + K\sqrt{\frac{\nu}{\eta(\gamma)}C(\gamma, \nu)} + \|\varphi\|_2, \tag{3.70}$$

where $D = \frac{\eta(\gamma)}{4\nu}\left(1-\frac{\varepsilon}{2}\right)$ and $K = \frac{2}{\sqrt{\varepsilon(2-\varepsilon)}}$.

We can chose $\varepsilon$ to minimize the constant $K$. Since

$$\min_{0<\varepsilon\leqslant\frac{1-2\alpha_0}{1-\alpha_0}} \frac{1}{\sqrt{\varepsilon(2-\varepsilon)}} = \left.\frac{1}{\sqrt{\varepsilon(2-\varepsilon)}}\right|_{\varepsilon=\frac{1-2\alpha_0}{1-\alpha_0}} = \frac{1-\alpha_0}{\sqrt{1-2\alpha_0}}, \tag{3.71}$$

we get a better estimate for $K$ if $\varepsilon = \frac{1-2\alpha_0}{1-\alpha_0}$:

$$K = \frac{2(1-\alpha_0)}{\sqrt{1-2\alpha_0}}. \tag{3.72}$$

The constant $\alpha_0$ can be chosen to minimize the righthand side of inequality (3.70) to get a better estimate. □

**Remark**. We also have the following estimate for $\|\varphi\|_2$:

$$\begin{aligned}
\|\varphi\|_2^2 &= 4\pi \sum_{n=1}^{\infty} \left(\frac{\psi_n}{n}\right)^2 = 4\pi \sum_{k=1}^{\infty} \frac{\psi_{2k}^2}{4k^2} \\
&= \pi \sum_{k=1}^{M} \frac{\psi_{2k}^2}{k^2} + \pi \sum_{k=M+1}^{\infty} \frac{\psi_{2k}^2}{k^2} \\
&= \frac{\pi\delta^2}{\alpha_0^2 \nu^2} \sum_{k=1}^{M} \frac{1}{k^2} + \frac{\pi\delta^2}{\alpha_0^2 \nu^2} \sum_{k=M+1}^{\infty} \frac{1}{k^2} f^2\left(\frac{k}{M} - 1\right) \\
&< \frac{2\pi\delta^2}{\alpha_0^2 \nu^2} + \frac{\pi\delta^2}{\alpha_0^2 \nu^2} \int_M^{\infty} \frac{1}{k^2} f^2\left(\frac{k}{M} - 1\right) dk \\
&= \frac{2\pi\delta^2}{\alpha_0^2 \nu^2} + \frac{\pi\delta^2}{\alpha_0^2 \nu^2 M} \int_0^{\infty} \frac{f^2(t)}{(t+1)^2} dt \\
&\leqslant \frac{\pi\delta^2}{\alpha_0^2 \nu^2} \left(2 + \int_0^{\infty} \frac{f^2(t)}{(t+1)^2} dt\right) = \mathcal{O}\big(M^{-3} C(\gamma, \nu)\big).
\end{aligned} \tag{3.73}$$

Using the large $\gamma$ and $0 < \nu \leqslant 1$ result (3.56), we show that $\|\varphi\|_2 \ll \sqrt{\frac{\nu}{\eta(\gamma)} C(\gamma, \nu)}$. Thus for the radius $\mathcal{R}$ of the absorbing ball in $L^2_{\text{odd}}$ we get the following estimate:

$$\mathcal{R} = \mathcal{O}\left(\sqrt{\frac{\nu}{\eta(\gamma)} C(\gamma, \nu)}\right) = \mathcal{O}\big(\gamma^{\frac{23p-28}{10(4-p)}} \nu^{-21/20}\big), \tag{3.74}$$

which gives, for $p = 3$,

$$\mathcal{R} = \mathcal{O}\left(\sqrt{\frac{\nu}{\eta(\gamma)} C(\gamma, \nu)}\right) = \mathcal{O}\big(\gamma^{41/10} \nu^{-21/20}\big). \tag{3.75}$$

Note that in this case the equation we study is

$$u_t + u u_x + u_{xx} + \nu u_{xxxx} + \mu \mathcal{H}[u]_{xxx} = 0, \tag{3.76}$$

which is obtained from the following equation given on $2L$-periodic interval

$$u_t + u u_x + u_{xx} + u_{xxxx} + \gamma \mathcal{H}[u]_{xxx} = 0, \tag{3.77}$$

by the following rescaling (dropping the bars):

$$\bar{t} = \nu t, \quad \bar{x} = \nu^{1/2} x, \quad \bar{u} = \nu^{-1/2} u, \tag{3.78}$$

where $\nu = (\pi/L)^2$ and $\mu = (\pi/L)\gamma$ (see Introduction and Tseluiko & Papageorgiou [34]).

In unscaled variables the estimate for the radius of the absorbing ball takes the following form:

$$\widetilde{\mathcal{R}} = \nu^{1/4} \mathcal{R} = \mathcal{O}\big(\gamma^{41/10} \nu^{-4/5}\big) = \mathcal{O}\big(\gamma^{41/10} L^{8/5}\big), \tag{3.79}$$

which significantly improves the estimate $\mathcal{O}(\gamma^6 L^{5/2})$ obtained by Duan & Ervin [7].

**Remark** If $\mu = 0$ then $\gamma = 0$ and $\delta = 1$ which implies $\mathcal{R} = \mathcal{O}(\nu^{-21/10})$. This value corresponds to the case of the usual Kuramoto–Sivashinsky equation:

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0 \tag{3.80}$$

In unscaled variables the estimate for the radius of the absorbing ball in $L^2_{\text{odd}}$ is $\mathcal{O}(\nu^{1/4}\nu^{-21/20}) = \mathcal{O}(\nu^{-4/5}) = \mathcal{O}(L^{8/5})$. This is the estimate which was obtained by Collet et al. [5] for the case of the usual Kuramoto–Sivashinsky equation.

**Remark** It can be seen from the estimate (3.74) that in the general $p$ case (recall that $3 \leqslant p < 4$), the estimated radius of the absorbing ball is an increasing function of $p$ which blows up as $p \to 4^-$. This is expected due to the ill-posedness of the equation when $p = 4$.

### 3.2.2 *The general case*

For the general case, when solutions of (1.4) are not necessarily odd functions, the idea is to consider a generalization of the gauge function $\varphi$. We start by introducing the following Liapunov function:

$$F[u] = \text{dist}^2(u, S) = \inf_{\psi \in S} \|u - \psi\|_2^2, \tag{3.81}$$

where $S$ is the following translation-invariant set of functions:

$$S = \{\psi : \exists a, \text{ s.t. } \psi(x) \equiv \varphi(x + a)\}. \tag{3.82}$$

This is equivalent to saying that

$$F[u] = \|u(x, t) - \varphi_a(x)\|_2^2, \tag{3.83}$$

where $\varphi_a(x) = \varphi(x + a)$, and $a = a(t)$ is a suitably chosen translation function:

$$\|u(x, t) - \varphi(x + a(t))\|_2^2 = \inf_{\psi \in S} \|u(x, t) - \psi(x)\|_2^2, \tag{3.84}$$

for all $t > 0$. So, $a = a(t)$ must satisfy the optimality condition $dF/da|_{a=a(t)} = 0$, which can also be written as

$$\left. \int_{-\pi}^{\pi} u\varphi_a' dx \right|_{a=a(t)} = 0. \tag{3.85}$$

The function $u$ is expressed as $u(x, t) = v(x, t) + \varphi_a(x)$. Equation (1.4) becomes

$$v_t + \varphi_a' a' = \mathcal{L}v + \mathcal{L}\varphi_a - vv_x - v\varphi_a' - \varphi_a v_x - \varphi_a\varphi_a'. \tag{3.86}$$

Multiplying the last equation by $v$ and integrating over the interval $[-\pi, \pi]$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} v^2 dx + a'\int_{-\pi}^{\pi} v\varphi_a' dx &= \int_{-\pi}^{\pi} v\mathcal{L}v dx + \int_{-\pi}^{\pi} v\mathcal{L}\varphi_a dx - \int_{-\pi}^{\pi} v^2 v_x dx \\ &\quad - \int_{-\pi}^{\pi} v^2\varphi_a' dx - \int_{-\pi}^{\pi} vv_x\varphi_a dx - \int_{-\pi}^{\pi} v\varphi_a\varphi_a' dx. \end{aligned}\tag{3.87}$$

This implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}F[u] &= \frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} v^2 dx \\ &= \int_{-\pi}^{\pi} v\mathscr{L}v dx + \int_{-\pi}^{\pi} v\mathscr{L}\varphi_a dx - \frac{1}{2}\int_{-\pi}^{\pi} v^2\varphi_a' dx \\ &\quad - \int_{-\pi}^{\pi} v\varphi_a\varphi_a' dx - a'\int_{-\pi}^{\pi} v\varphi_a' dx. \end{aligned} \tag{3.88}$$

Noting that the last term is zero due to our choice of $a$ and using bilinear form (3.13) we can write this as

$$\frac{1}{2}\frac{d}{dt}F[u] = -(v, v)_{\frac{1}{2}\varphi_a} - (v, \varphi_a)_{\varphi_a}. \tag{3.89}$$

For a given $t$ we can assume for simplicity that $a(t) = 0$ in (3.89) (since the right hand side of this equality is invariant under translations $a \to a + \text{constant}$).

Next we want to get a result similar to Proposition 3.1 for the general case. So, let $w$ be any function in $\dot{H}^2_{\text{per}}$. This function can be decomposed as follows:

$$w(x) = w(0) + \frac{1}{2}[w(x) + w(-x) - 2w(0)] + \frac{1}{2}[w(x) - w(-x)]. \tag{3.90}$$

So,

$$w(x) = w(0) + w_s(x) + w_a(x), \tag{3.91}$$

where $w_s(x) = \frac{1}{2}[w(x) + w(-x) - 2w(0)]$ is an even $2\pi$-periodic function of $x$, and $w_a(x) = \frac{1}{2}[w(x) - w(-x)]$ is an odd $2\pi$-periodic function of $x$. Besides, $w_s(0) = 0$.

Now, let us introduce the following operator:

$$\mathscr{T}[f](x) = \begin{cases} f(x), & \text{if } x \in [0, \pi], \\ -f(x), & \text{if } x \in [-\pi, 0). \end{cases} \tag{3.92}$$

For simplicity, let us reduce our consideration to half the interval, i.e. let us assume that all the functions are $\pi$-periodic. (We can do this reduction without loss of generality, since it is always possible to transfer from any $L$-periodic intervals to $2\pi$-periodic domains and vice versa, without changing the form of the equation, see for example transformation (3.78). The only impact of such transforms on the equation is that the coefficients $\mu$ and $\nu$ are rescaled.) Then $\mathscr{T}[w_s]$ is an odd $2\pi$-periodic function. Also, since $R_{\alpha\varphi_a}(w_s) = R_{\alpha\varphi_a}(\mathscr{T}[w_s])$ and $Q(w_s) = Q(\mathscr{T}[w_s])$ we get that Proposition 3.1 holds not only for $w_a$ but for $w_s$ too:

$$R_{\alpha\varphi_a}(w_a) \geqslant Q(w_a), \tag{3.93}$$

$$R_{\alpha\varphi_a}(w_s) \geqslant Q(w_s), \tag{3.94}$$

for all $\alpha \in [\alpha_0, 1]$. Next, it can be easily checked that

$$R_{\alpha\varphi_a}(w) = R_{\alpha\varphi_a}(w_a) + R_{\alpha\varphi_a}(w_s), \tag{3.95}$$

$$Q(w) = Q(w_a) + Q(w_s) - \frac{\pi}{2\nu}w^2(0), \tag{3.96}$$

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for all $\alpha \in [\alpha_0, 1]$. Hence, (3.93) and (3.94) imply

$$R_{\alpha\varphi_a}(w) \geqslant Q(w) + \frac{\pi}{2v}w^2(0) \geqslant Q(w) \quad \text{for } \alpha \in [\alpha_0, 1]. \tag{3.97}$$

This means that with our choice of $a = a(t)$ Proposition 3.1 holds for the general case too.

After this point the proof of the nonlinear stability becomes the same as for the antisymmetric case. Thus the following result holds:

**Theorem 3.3** *Let $\mu$ be any positive number and $v \in (0, v_0(\mu))$. If $u(x, t)$ is a solution of equation (1.4) such that $u(x, 0) = u_0(x) \in \dot{H}^1_{\text{per}}$, then there is a constant $K > 0$ (independent of $\mu$, $v$, $u_0$), and a constant $D > 0$ (independent of $\mu$, $u_0$) such that*

$$\|u\|_2 \leqslant (\|u_0\|_2 + \|\varphi\|_2)\exp(-Dt) + K\sqrt{\frac{v}{\eta(\gamma)}C(\gamma, v)} + \|\varphi\|_2, \tag{3.98}$$

*where $\varphi$ is the function constructed in Proposition 3.1, and $C(\gamma, v)$ is given by* (3.61).

(The constants $D$ and $K$ are the same as in Theorem 3.2.)

### 3.3 Uniform boundedness of the solutions in $\dot{H}^1_{\text{per}}$

To prove global existence of the solutions in $\dot{H}^1_{\text{per}}$ it is enough (according to Theorem 2.3) to prove uniform boundedness of the solutions in $\dot{H}^1_{\text{per}}$. This will be established by showing uniform boundedness of the $L^2$-norm of $u_{xx}$, which by Poincaré inequality also implies boundedness of the $L^2$-norm of $u_x$.

Multiplying (1.4) by $u_{xxxx}$ and integrating over $[-\pi, \pi]$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|u_{xx}\|_2^2 + v\|u_{xxxx}\|_2^2 \\ &= -\int_{-\pi}^{\pi} u_{xx}u_{xxxx}dx - \int_{-\pi}^{\pi} uu_x u_{xxxx}dx + \mu\int_{-\pi}^{\pi}(\mathscr{H}\circ\partial_x)^p[u]u_{xxxx}dx.\end{aligned} \tag{3.99}$$

Using Young's inequality, Agmon' inequality, the Nirenberg–Gagliardo inequalities and the interpolation inequalities (see Henry [12], Sell & You [25], Temam [32]), the integrals on the right hand side of this expression can be estimated as follows:

$$-\int_{-\pi}^{\pi} u_{xx}u_{xxxx}dx \leqslant \|u_{xx}\|_2\|u_{xxxx}\|_2 \leqslant \varepsilon_1\|u_{xxxx}\|_2^2 + \frac{\varepsilon_1^{-1}}{4}\|u_{xx}\|_2^2, \tag{3.100}$$

$$\begin{aligned}-\int_{-\pi}^{\pi} uu_x u_{xxxx}dx &\leqslant \|u\|_\infty\|u_x\|_2\|u_{xxxx}\|_2 \leqslant \sqrt{2}\|u\|_2^{1/2}\|u_x\|_2^{3/2}\|u_{xxxx}\|_2 \\ &\leqslant \sqrt{2\left[\|u_0\|_2 + 2\|\varphi\|_2 + K\sqrt{\frac{v}{\eta(\gamma)}C(\gamma, v)}\right]}\|u_x\|_2^{3/2}\|u_{xxxx}\|_2.\end{aligned} \tag{3.101}$$

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Denoting for simplicity $\|u_0\|_2 + 2\|\varphi\|_2 + K\sqrt{\frac{\nu}{\eta(\gamma)}C(\gamma, \nu)}$ by $C_1$ gives

$$\begin{aligned}
-\int_{-\pi}^{\pi} uu_x u_{xxxx} dx &\leqslant \sqrt{2C_1}\|u_x\|_2^{3/2}\|u_{xxxx}\|_2 \\
&\leqslant \sqrt{2C_1}\left(C_2\|u_{xx}\|_2^{1/2}\|u\|_2^{1/2}\right)^{3/2}\|u_{xxxx}\|_2 \\
&\leqslant \sqrt{2}C_1^{5/4}C_2^{3/2}\|u_{xx}\|_2^{3/4}\|u_{xxxx}\|_2 \\
&\leqslant \sqrt{2}C_1^{5/4}C_2^{3/2}(\varepsilon_2\|u_{xxxx}\|_2^2 + \frac{\varepsilon_2^{-1}}{4}\|u_{xx}\|_2^{3/2}) \\
&\leqslant \sqrt{2}C_1^{5/4}C_2^{3/2}\left(\varepsilon_2\|u_{xxxx}\|_2^2 + \frac{\varepsilon_2^{-1}}{4}\|u_{xx}\|_2^{3/2}\right) \\
&\leqslant \sqrt{2}C_1^{5/4}C_2^{3/2}\left(\varepsilon_2\|u_{xxxx}\|_2^2 + \frac{\varepsilon_2^{-1}}{4}\left[\frac{3}{4}\|u_{xx}\|_2^2 + \frac{1}{4}\right]\right).
\end{aligned} \tag{3.102}$$

Denoting $A_1 = \sqrt{2}C_1^{5/4}C_2^{3/2}$, $A_2 = \frac{\sqrt{2}}{16}\varepsilon_2^{-1}C_1^{5/4}C_2^{3/2}$ gives

$$-\int_{-\pi}^{\pi} uu_x u_{xxxx} dx \leqslant \varepsilon_2 A_1\|u_{xxxx}\|_2^2 + 3A_2\|u_{xx}\|_2^2 + A_2. \tag{3.103}$$

Next,

$$\int_{-\pi}^{\pi} (\mathscr{H}\circ\partial_x)^p[u]u_{xxxx} dx \leqslant \left\|(\mathscr{H}\circ\partial_x)^p[u]\right\|_2\|u_{xxxx}\|_2. \tag{3.104}$$

Also,

$$\begin{aligned}
\left\|(\mathscr{H}\circ\partial_x)^p[u]\right\|_2 &= \left\|(\mathscr{H}\circ\partial_x)^{p-3}\left[(\mathscr{H}\circ\partial_x)^3[u]\right]\right\|_2 \\
&\leqslant \left\|(\mathscr{H}\circ\partial_x)\left[(\mathscr{H}\circ\partial_x)^3[u]\right]\right\|_2^{(p-3)}\left\|(\mathscr{H}\circ\partial_x)^3[u]\right\|_2^{(4-p)} \\
&= \left\|(\mathscr{H}\circ\partial_x)^4[u]\right\|_2^{(p-3)}\left\|(\mathscr{H}\circ\partial_x)^3[u]\right\|_2^{(4-p)} \\
&= \left\|u_{xxxx}\right\|_2^{(p-3)}\left\|u_{xxx}\right\|_2^{(4-p)} \\
&\leqslant (p-3)\varepsilon_3\|u_{xxxx}\|_2 + (4-p)\varepsilon_3^{-\frac{p-3}{4-p}}\|u_{xxx}\|_2 \\
&\leqslant (p-3)\varepsilon_3\|u_{xxxx}\|_2 + (4-p)\varepsilon_3^{-\frac{p-3}{4-p}}\left(C_4\|u_{xx}\|_2^{1/2}\|u_{xxxx}\|_2^{1/2}\right) \\
&\leqslant (p-3)\varepsilon_3\|u_{xxxx}\|_2 + \varepsilon_4\|u_{xxxx}\|_2 \\
&\quad + (4-p)^2C_4^2\varepsilon_3^{-\frac{2(p-3)}{4-p}}\frac{\varepsilon_4^{-1}}{4}\|u_{xx}\|_2 \\
&\leqslant [(p-3)\varepsilon_3 + \varepsilon_4]\|u_{xxxx}\|_2 \\
&\quad + (4-p)^2C_4^2\varepsilon_3^{-\frac{2(p-3)}{4-p}}\frac{\varepsilon_4^{-1}}{4}\|u_{xx}\|_2.
\end{aligned} \tag{3.105}$$

Denoting $A_3 = (4-p)^2C_4^2\varepsilon_3^{-\frac{2(p-3)}{4-p}}$ gives

$$\begin{aligned}
\int_{-\pi}^{\pi} (\mathscr{H}\circ\partial_x)^p[u]u_{xxxx} dx &\leqslant [(p-3)\varepsilon_3 + \varepsilon_4]\|u_{xxxx}\|_2^2 + A_3\|u_{xx}\|_2\|u_{xxxx}\|_2 \\
&\leqslant [(p-3)\varepsilon_3 + \varepsilon_4]\|u_{xxxx}\|_2^2 + A_3\varepsilon_5\|u_{xxxx}\|_2^2 + A_3\frac{\varepsilon_5^{-1}}{4}\|u_{xx}\|_2^2 \\
&= [(p-3)\varepsilon_3 + \varepsilon_4 + A_3\varepsilon_5]\|u_{xxxx}\|_2^2 + A_3\frac{\varepsilon_5^{-1}}{4}\|u_{xx}\|_2^2.
\end{aligned} \tag{3.106}$$

We finally obtain:

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_2^2 + \nu\|u_{xxxx}\|_2^2 \leqslant [\varepsilon_1 + \varepsilon_2 A_1 + \mu(\varepsilon_3(p-3) + \varepsilon_4 + \varepsilon_5 A_3)]\,\|u_{xxxx}\|_2^2 + \left[\frac{\varepsilon_1^{-1}}{4} + 3A_2 + \mu A_3\frac{\varepsilon_5^{-1}}{4}\right]\|u_{xx}\|_2^2 + A_2. \quad (3.107)$$

Choosing $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\varepsilon_4$, $\varepsilon_5$ to be sufficiently small and denoting for simplicity $A_4 = \frac{\varepsilon_1^{-1}}{4} + 3A_2 + \mu A_3\frac{\varepsilon_5^{-1}}{4}$ gives:

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_2^2 \leqslant A_4\|u_{xx}\|_2^2 + A_2. \quad (3.108)$$

Applying the uniform Gronwall inequality (see Temam [32]) implies that $\|u_{xx}\|$ is bounded on the time interval $[0, T(u_0))$ if $T(u_0)$ is finite. This also implies boundedness of $\|u_x\|_2$. Theorem 2.3 then gives $T(u_0) = \infty$. Therefore the following result holds:

**Theorem 3.4** *For every $u_0 \in \dot{H}^1_{\text{per}}$ there exists a unique globally defined solution of equation* (1.4).

Now, having global existence of the solutions of (1.4), Theorem 3.3 implies existence of an absorbing ball in $\dot{L}^2_{\text{per}}$ and the following estimate for the radius of this absorbing ball:

**Corollary 3.5** *Let $\mu$ be any positive number and $\nu \in (0, \nu_0(\mu))$. If $u(x, t)$ is a solution of* (1.4) *such that $u(x, 0) = u_0(x) \in \dot{H}^1_{\text{per}}$, then there is a constant $K$ (independent of $\mu$, $\nu$, $u_0$), such that*

$$\limsup_{t\to\infty} \|u\|_2 \leqslant K\sqrt{\frac{\nu}{\eta(\gamma)}C(\gamma, \nu)} + \|\varphi\|_2. \quad (3.109)$$

(Here $K$, $C(\gamma, \nu)$, $\varphi$ are the same as in Theorem 3.2.)

## 4 Numerical evaluation of the analytical results

The MKS equation (1.4) was solved numerically with periodic boundary conditions using a modification (when $p \neq 3$) of the methods used in Tseluiko & Papageorgiou [34]. Our main objective is to compare the analytical bound (3.109) for the radius of the absorbing ball in the space $\dot{L}^2_{\text{per}}$, with the "exact" numerically computed value $\|u\|_2$. A comparison at large values of $\gamma$ (equivalently large $\mu = \nu^{1/2}\gamma$) for fixed $\nu$ , is particularly amenable due to the simple algebraic nature of the estimate in this limit (see (3.74)) – the large $\gamma$ and large $\mu$ behaviors are identical due to the fact that $\nu$ is fixed. The computations are carried out to values of time beyond which transient behavior dies out and we can be confident that the computed trajectories lie close to the attractor. Given values of $p$, $\nu$, and $\mu$, the quantity $\max(\|u\|_2)(p, \nu; \mu)$ is found over a time interval beyond transients. This is repeated for a fixed $\nu = 0.5$ and a range of increasing values of $\mu = 2^2, 2^3, \ldots, 2^8$. The results are presented on logarithmic scales in Figures 2 and 3 for $p = 3$ and $p = 3.2$, respectively. As can be seen from the Figures, the behavior is linear and an estimate for the slope, providing a numerical best bound for the $\dot{L}^2_{\text{per}}$ norm as a power of $\mu$ (for large

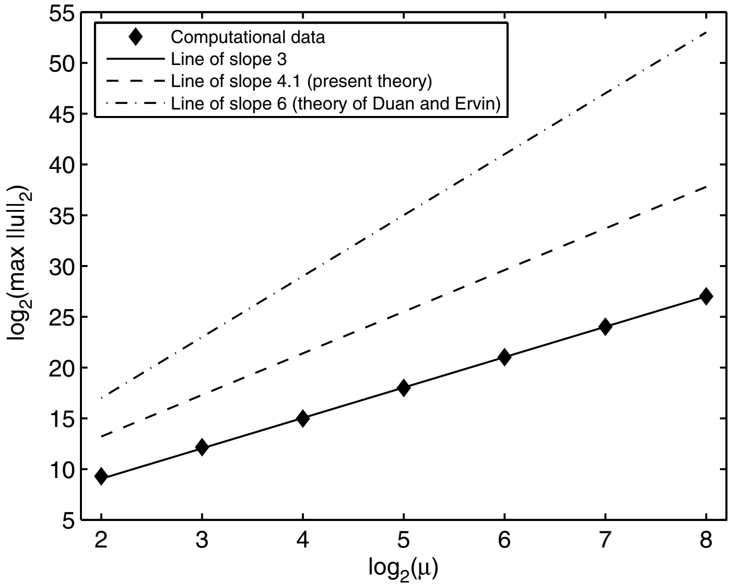


FIGURE 2. Variation of max $\|u\|_2$ with increasing $\mu$ for fixed $v = 0.5$, $p = 3$. Diamonds – numerical computation (solid line is of slope 3); dashed line – current theoretical estimate as given by equation (3.74); dashdot line – estimate of Duan & Ervin [7].

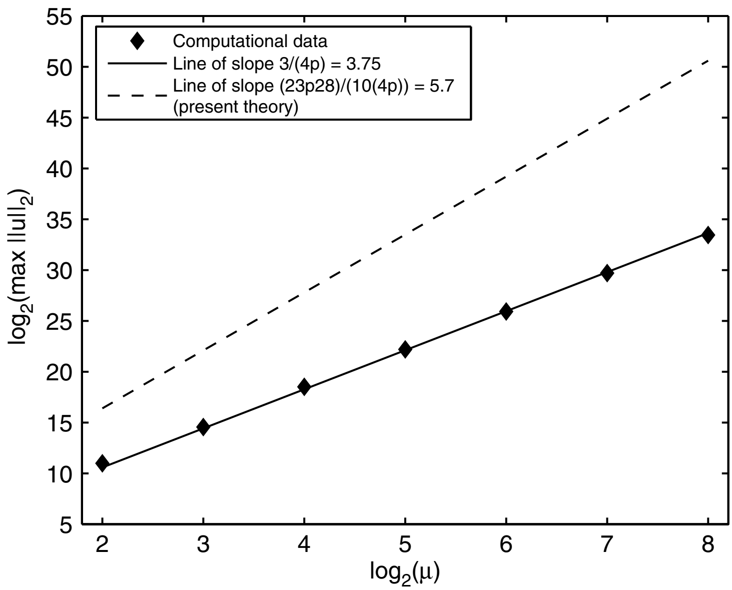


FIGURE 3. Variation of max $\|u\|_2$ with increasing $\mu$ for fixed $v = 0.5$, $p = 3.2$. Diamonds – numerical computation (solid line is of slope $3/(4-p) = 3.75$); dashed line – current theoretical estimate as given by equation (3.74).

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$\mu$), is determined. The numerical results give the behavior

$$\max(\|u\|_2)(p = 3,\ \nu = 0.5;\ \mu) = \mathcal{O}\big(\mu^3\big), \tag{4.1}$$

$$\max(\|u\|_2)(p = 3.2,\ \nu = 0.5;\ \mu) = \mathcal{O}\big(\mu^{3.75}\big) = \mathcal{O}\big(\mu^{\frac{3}{(4-p)}}\big). \tag{4.2}$$

Our corresponding analytical estimates (the values are 4.1 and 5.7, for $p = 3, 3.2$, respectively) are also given in the Figures along with the estimate of Duan & Ervin [7] when $p = 3$. The analytical results overestimate the numerically constructed bound, but the present estimates are better than those found in the literature.

As indicated in (4.1)–(4.2), the numerically computed large $\mu$ estimates are $\mathcal{O}\big(\mu^{\frac{3}{(4-p)}}\big)$. This can be understood by the following order-of-magnitude argument valid for large $\mu$. Considering the equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} - \mu(\mathscr{H} \circ \partial_x)^p[u] = 0, \tag{4.3}$$

a balance must take place between the nonlinearity, the fourth order diffusion and the nonlocal term (the unsteady term provides a time scale *a posteriori*). We have, then,

$$\frac{[u]}{[t]} \sim \frac{[u]^2}{[x]} \sim \frac{[u]}{[x]^4} \sim \mu\frac{[u]}{[x]^p}, \tag{4.4}$$

which in turn provides the scalings

$$[t] \sim \mu^{-\frac{4}{4-p}}, \quad [x] \sim \mu^{-\frac{1}{4-p}}, \quad [u] \sim \mu^{\frac{3}{4-p}}. \tag{4.5}$$

These scalings suggest that as $\mu$ increases typical amplitudes increase and at the same time the spatial scale over which the solution varies, decreases; in addition, the solution varies over typical time scales which are also decreasing asymptotically. This behavior places severe restrictions on the numerical parameters. Taking the case $p = 3$, for example, we observe that doubling the value of $\mu$ requires a decrease of the time-step by a factor of $2^4$ and a doubling of the number of modes. In accordance with the scalings (4.5), in order to resolve the solution for the largest value $\mu = 256$, we used a time-step of $1.16 \times 10^{-11}$ and $2^{14}$ Fourier modes. As $p$ increases the situation worsens as evidenced by (4.5).

As shown in Figures 2 and 3, the values of $\|u\|_2$ follow the scaling for $u$ shown in equation (4.5). This is consistent with the numerical solutions which exhibit pulses of width $\mathcal{O}\big(\mu^{-\frac{1}{4-p}}\big)$ and height $\mathcal{O}\big(\mu^{\frac{3}{4-p}}\big)$ whose net contribution over the interval $[-\pi, \pi]$ gives $\|u\|_2 = \mathcal{O}\big(\mu^{\frac{3}{4-p}}\big)$. This pulse behavior is indicated in Figures 4 and 5 for $p = 3$ and $\mu = 16$ and 32 respectively. There are approximately 28 pulses for $\mu = 16$ and 52 pulses for $\mu = 32$, while the pulse heights are approximately equal to $2 \times 10^4$ and $1.5 \times 10^5$; these numerical observations are in agreement with the scaling laws (4.5) and have been confirmed for all the computations presented here.

These findings enable us to formulate the following conjecture.

**Conjecture 4.1** *Let* $\nu \in (0, \nu_0(\mu))$ *and* $p \in [3, 4)$ *be fixed. If* $u(x, t)$ *is a solution of* (1.4) *such that* $u(x, 0) = u_0(x) \in \dot{H}^1_{\text{per}}$, *then for* $\mu$ *sufficiently large*

$$\limsup_{t\to\infty} \|u\|_2 = \mathcal{O}(\mu^{\frac{3}{4-p}}). \tag{4.6}$$

Note that the values of $\mu$ need not be too large for (4.6) to hold. The numerical results give values of approximately 2.

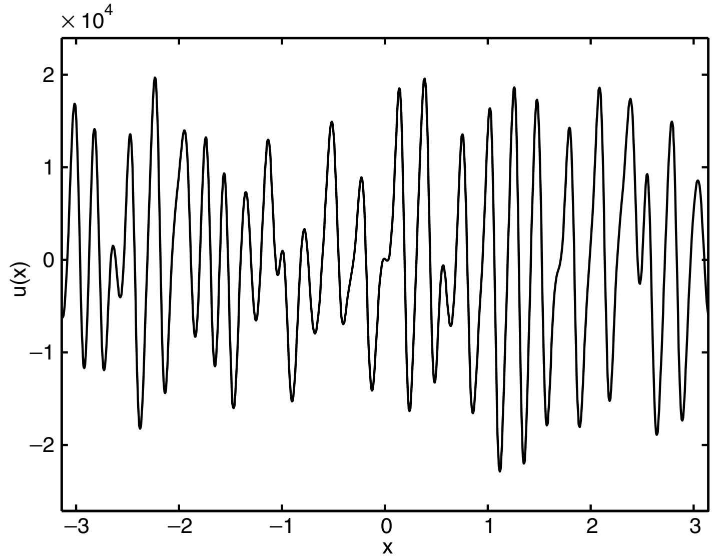


FIGURE 4. The solution $u(x,t)$ after 20000 time-steps; $\mu = 2^4$, $v = 0.5$, $p = 3$.

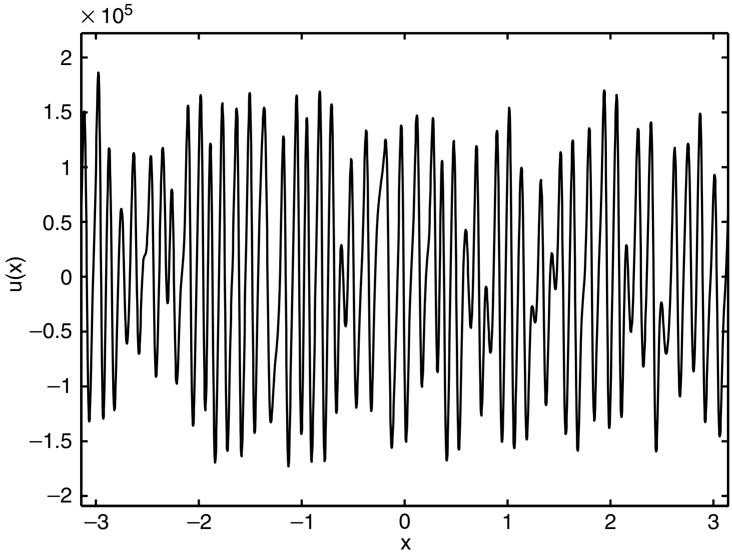


FIGURE 5. The solution $u(x,t)$ after 20000 time-steps; $\mu = 2^5$, $v = 0.5$, $p = 3$.

## 5 Conclusions

We have studied a class of nonlocal Kuramoto–Sivashinsky (KS) equations arising in interfacial electrohydrodynamics. Two modifications of the well-known KS equation are afforded by the model: (i) the addition of a nonlocal Hilbert transform term that enhances

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the usual second derivative negative diffusion (equation (1.4) with the plus sign), and, (ii) the case when the nonlocal term is the only term providing instability (equation (1.4) with the minus sign), all other linear terms being diffusive. We have presented in detail rigorous results for case (i). In particular we proved global existence and uniqueness of the solutions in $\dot{H}^1_{\text{per}}$ by first proving local results and then establishing global results by proving uniform boundedness of the solutions in $\dot{H}^1_{\text{per}}$ on each time interval. We also established uniform boundedness of the solutions in $\dot{H}^1_{\text{per}}$ after proving uniform boundedness in $\dot{L}^2_{\text{per}}$, using a modification of the method of Collet et al. [5]. Along with global existence, this proves the existence of an absorbing ball in $\dot{L}^2_{\text{per}}$ and provides estimates for its radius. Our estimates improve those of Duan & Ervin [7], for $p = 3$, who used a different gauge function. An evaluation of the rigorous estimates valid at large values of the electrical parameter $\mu$, as compared to numerical solutions of the equations is also carried out (see Figures 2, 3). The numerical work indicates that an optimal $\dot{L}^2_{\text{per}}$ solution bound arises; this is explained by a simple scaling argument. A conjecture valid for all $p$ (and verified by extensive numerical simulations) is made regarding these findings.

In case (ii) results which parallel those above have been obtained when $\mu$ is larger than the threshold value $\mu_0(\nu; p)$ above which linearly unstable modes enter – for example when $p = 3$, we can take $\mu_0 = 2\sqrt{\nu}$. (When $\mu < \mu_0$ the value of $\|u\|_2$ decays to zero as $t \to \infty$.) The main differences are technical and result in other expressions for the quantities $N$, $M$, $\delta$ and $C(\gamma, \nu)$ and in turn result in different estimates for the radius of the absorbing ball. The large $\gamma$ (note that $\gamma = \nu^{1-p/2}\mu$) behavior of this radius is identical to that given for case (i) – see estimate (3.74).


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## Appendix A: Properties of the Hilbert Transform

Here we list some important properties of the Hilbert transform. The Hilbert transform operator $\mathscr{H}$ is defined by

$$\mathscr{H}[f](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(\xi)}{x - \xi} d\xi, \tag{A 1}$$

where the integral is understood in the sense of Cauchy principal value.

The Hilbert transform $\mathscr{H} : L^2(I) \to L^2(I)$ (or $\mathscr{H} : H^k(I) \to H^k(I)$) is a linear, invertible, bounded operator from $L^2$ to $L^2$ (and from the Sobolev space $H^k$ to $H^k$).We note:

$$\partial_x \circ \mathscr{H} = \mathscr{H} \circ \partial_x, \tag{A 2}$$

$$\mathscr{H}^{-1} = -\mathscr{H}, \tag{A 3}$$

$$\int_I u(x)\mathscr{H}[v](x)dx = -\int_I v(x)\mathscr{H}[u](x)dx, \tag{A 4}$$

$$\int_I u(x)\mathscr{H}[u](x)dx = 0, \tag{A 5}$$

$$\|\mathscr{H}[u]\| = \|u\|, \tag{A 6}$$

$$\mathscr{F}\big[\mathscr{H}[u]\big](k) = -i\,\mathrm{sign}\,(\mathrm{Re}\,k)\hat{u}(k). \tag{A 7}$$

Here $\mathscr{F}$ is the Fourier transform operator and $I$ is either $\mathbb{R}$ or a periodic interval.

For periodic functions on $[-\pi, \pi]$ we have

$$\mathscr{H}[f](x) = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} f(\xi)\cot\left(\frac{x-\xi}{2}\right)d\xi. \tag{A 8}$$

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