STOCHASTIC COMPARISONS OF PARALLEL SYSTEMS WHEN COMPONENTS HAVE PROPORTIONAL HAZARD RATES

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Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$, and let $Y_1, \ldots, Y_n$ be a random sample with common population survival distribution $\bar{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. Let $X_{n:n}$ and $Y_{n:n}$ denote the lifetimes of the parallel systems consisting of these components, respectively. It is shown that $X_{n:n}$ is greater than $Y_{n:n}$ in terms of likelihood ratio order. It is also proved that the sample range $X_{n:n} - X_{1:n}$ is larger than $Y_{n:n} - Y_{1:n}$ according to reverse hazard rate ordering. These two results strengthen and generalize the results in Dykstra, Kochar, and Rojo [6] and Kochar and Rojo [11], respectively.

1. INTRODUCTION

Order statistics have received a great amount of attention from many researchers since they play an important role in reliability, data analysis, goodness-of-fit tests, statistical inference, and other applied probability areas. Please refer to David and Nagaraja [5] and Balakrishnan and Rao [1, 2] for more details. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of random variables $X_1, X_2, \ldots, X_n$. In the reliability context, the lifetimes of parallel and series systems correspond to order statistics, $X_{n:n}$ and $X_{1:n}$, respectively, and they have been extensively studied when the components are independent and identically distributed (i.i.d.). However, in practice, usually, the observations are not i.i.d. Due to the complicated nature of the problem, not much work has been done for the non-i.i.d. case.

For ease of reference, let us first recall some stochastic orders that will be used in the sequel. Let X and Y be two nonnegative random variables with distribution functions F and G; survival functions $\bar{F}$ and $\bar{G}$; and density functions f and g, respectively.

DEFINITION 1.1 (Shaked and Shanthikumar [17] and Müller and Stoyan [14]): *If the ratios below are well defined, X is said to be smaller than Y in the following:*

1. *likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in x*
2. *hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x*
3. *reversed hazard rate order (denoted by $X \leq_{\mathrm{rh}} Y$) if $G(x)/F(x)$ is increasing in x*
4. *stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x.*

It is well known that

$$X \leq_{\mathrm{lr}} Y \Longrightarrow X \leq_{\mathrm{hr(rh)}} Y \Longrightarrow X \leq_{\mathrm{st}} Y.$$

Let $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

DEFINITION 1.2: *The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \ldots, n - 1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

For extensive and comprehensive details on the theory of the majorization order and its applications, please refer to Marshall and Olkin [12]. Another interesting order related to the majorization order introduced by Bon and Paltanea [4] is the p-larger order.

DEFINITION 1.3: *A vector $\mathbf{x}$ in $\mathbb{R}^{+^n}$ is said to be p-larger than another vector $\mathbf{y}$ in $\mathbb{R}^{+^n}$ (denoted by $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if*

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, \qquad j = 1, \ldots, n.$$

Khaledi and Kochar [9] proved that, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{+^n}$,

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}.$$

However, the converse is not true.

Random variables $X_1, X_2, \ldots, X_n$ are said to follow the proportional hazard rates (PHR) model if for $i = 1, 2, \ldots, n$, the survival function of X_i can be expressed as

$$\bar{F}_i(x) = [\bar{F}(x)]^{\lambda_i}, \tag{1.1}$$

where $\bar{F}(x)$ is the survival function of some random variable X. If $r(t)$ is the hazard rate corresponding to the base line distribution F, then the hazard rate of X_i is $\lambda_i r(t)$, $i = 1, 2, \ldots, n$. We can express (1.1) as

$$\bar{F}_i(x) = e^{-\lambda_i R(x)}, \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

where $R(x) = \int_0^x r(t)\,dt$, is the cumulative hazard rate of X. Exponential random variables with hazard rates $\lambda_1, \lambda_2, \ldots, \lambda_n$ is a special case of the PHR model with $R(x) = x$. Many interesting results have been obtained in the literature for the PHR model. Pledger and Proschan [15] proved that if $(X_1, \ldots, X_n)$ and $(X_1^*, \ldots, X_n^*)$ have proportional hazard rate vectors $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1^*, \ldots, \lambda_n^*)$, respectively, then

$$(\lambda_1, \ldots, \lambda_n) \overset{m}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*)$$

implies that, for $i = 1, \ldots, n$,

$$X_{i:n} \geq_{\text{st}} X_{i:n}^*. \tag{1.3}$$

Subsequently, Proschan and Sethuraman [16] generalized this result from component-wise stochastic ordering to multivariate stochastic ordering. Boland, El-Neweihi, and Proschear [3] showed by a counterexample that (1.3) cannot be strengthened from stochastic ordering to hazard rate ordering. This topic is followed up by Dykstra, Kochar, and Rojo [6], where they showed that if $X_1, \ldots, X_n$ are independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, and if $Y_1, \ldots, Y_n$ is a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^n \lambda_i / n$, then

$$Y_{n:n} \leq_{\text{hr}} X_{n:n}. \tag{1.4}$$

Under a weeker condition that if $Z_1, \ldots, Z_n$ are a random sample with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$, the geometric mean of the λ's, Khaledi and Kochar [7] proved that

$$Z_{n:n} \leq_{\text{hr}} X_{n:n}. \tag{1.5}$$

They also showed there that

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\text{st}} X_{n:n}^*, \tag{1.6}$$

which improved the bound given by (1.3). Recently, Khaledi and Kochar [10] extended the results (1.5) and (1.6) from the exponential case to the PHR model.

Another interesting topic that has attracted much attention is the sample range, one of the criteria for comparing variabilities among distributions. Kochar and Rojo [11] pointed out that in the case of heterogeneous exponentials,

$$Y_{n:n} - Y_{1:n} \leq_{\text{st}} X_{n:n} - X_{1:n}. \tag{1.7}$$

Later, Khaledi and Kochar [8] improved upon this result. They proved that

$$Z_{n:n} - Z_{1:n} \leq_{\text{st}} X_{n:n} - X_{1:n},$$

where $Z_{n:n}$ is the maximum of a random sample from exponential distribution with common parameter as the geometric mean of the λ_i's.

In this article, the above topics are further studied. We prove that if $X_1, \ldots, X_n$ are independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ are a random sample with common population survival distribution $\bar{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$, then

$$Y_{n:n} \leq_{\text{lr}} X_{n:n}$$

and

$$Y_{n:n} - Y_{1:n} \leq_{\text{rh}} X_{n:n} - X_{1:n}.$$

These two results strengthen and generalize (1.4) and (1.7), respectively.

For the sake of convenience, throughout this article, the term *increasing* is used for *monotone nondecreasing* and *decreasing* is used for *monotone nonincreasing*.

2. STOCHASTIC COMPARISONS OF PARALLEL SYSTEMS

The following two lemmas will be used to prove our main result.

LEMMA 2.1 (Khaledi and Kochar [7]): *For $x \geq 0$, the functions*

$$\frac{1 - e^{-x}}{x} \quad \text{and} \quad \frac{x^2 e^{-x}}{(1 - e^{-x})^2}$$

are both decreasing.

LEMMA 2.2: *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. Then*

$$Y_{n:n} \leq_{\text{lr}} X_{n:n}. \tag{2.1}$$

PROOF: For $x \geq 0$, the distribution function of $X_{n:n}$ is

$$F_{n:n}(x) = \prod_{i=1}^{n} (1 - e^{-\lambda_i x}),$$

with density function as

$$f_{n:n}(x) = F_{n:n}(x) \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}.$$

Similarly, the distribution function of $Y_{n:n}$ for $x \geq 0$ is

$$G_{n:n}(x) = (1 - e^{-\bar{\lambda} x})^n,$$

with density function

$$g_{n:n}(x) = G_{n:n}(x) \frac{n \bar{\lambda} e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}}.$$

Note that, for $x \geq 0$,

$$\frac{f_{n:n}(x)}{g_{n:n}(x)} = \frac{\sum_{i=1}^{n} \left[\dfrac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} \right]}{\dfrac{n \bar{\lambda} e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}}} \cdot \frac{F_{n:n}(x)}{G_{n:n}(x)}$$

$$= \frac{h_1(x)}{n \bar{\lambda}} \frac{F_{n:n}(x)}{G_{n:n}(x)},$$

Where

$$h_1(x) = \frac{\sum_{i=1}^{n} \left[\dfrac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} \right]}{\dfrac{e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}}}$$

$$= \sum_{i=1}^{n} \lambda_i \frac{e^{\bar{\lambda} x} - 1}{e^{\lambda_i x} - 1}.$$

Since

$$(\bar{\lambda}, \ldots, \bar{\lambda}) \preceq_m (\lambda_1, \ldots, \lambda_n),$$

it follows from Theorem 3.2 of Dykstra et al. [6] that

$$\frac{F_{n:n}(x)}{G_{n:n}(x)}$$

is increasing in $x \geq 0$. Thus, it is sufficient to prove that $h_1(x)$ is increasing in $x \geq 0$.

The derivative of $h_1(x)$ is, for $x \geq 0$,

$$h_1'(x) = \bar{\lambda} e^{\bar{\lambda} x} \sum_{i=1}^{n} \frac{\lambda_i}{e^{\lambda_i x} - 1} - (e^{\bar{\lambda} x} - 1) \sum_{i=1}^{n} \frac{\lambda_i^2 e^{\lambda_i x}}{(e^{\lambda_i x} - 1)^2}.$$

By Lemma 2.1 and Čebyšev's sum inequality (Mitrinović [5, Thm. 1, p. 36]), it holds that, for $x \geq 0$,

$$\bar{\lambda}e^{\bar{\lambda}x}\sum_{i=1}^{n}\frac{\lambda_i}{e^{\lambda_i x}-1} \geq \frac{\bar{\lambda}e^{\bar{\lambda}x}}{n}\sum_{i=1}^{n}\frac{\lambda_i^2 e^{-\lambda_i x}}{(1-e^{-\lambda_i x})^2}\sum_{i=1}^{n}\frac{1-e^{-\lambda_i x}}{\lambda_i}. \tag{2.2}$$

Thus, $h_1'(x)$ will be nonnegative if, for $x \geq 0$,

$$\frac{\bar{\lambda}e^{\bar{\lambda}x}}{n}\sum_{i=1}^{n}\frac{1-e^{-\lambda_i x}}{\lambda_i} \geq e^{\bar{\lambda}x}-1$$

holds.

Denote, for $x \geq 0$,

$$h_2(x) = \frac{\bar{\lambda}}{n}\sum_{i=1}^{n}\frac{1-e^{-\lambda_i x}}{\lambda_i} - (1-e^{-\bar{\lambda}x}).$$

Since the derivative of $h_2(x)$ is

$$h_2'(x) = \frac{\bar{\lambda}}{n}\sum_{i=1}^{n}e^{-\lambda_i x} - \bar{\lambda}e^{-\bar{\lambda}x}$$

and by the arithmetic–geometric mean inequality, for $x \geq 0$,

$$\frac{\sum_{i=1}^{n}e^{-\lambda_i x}}{n} \geq \sqrt[n]{\prod_{i=1}^{n}e^{-\lambda_i x}} = e^{-\bar{\lambda}x},$$

it follows that $h_2'(x) \geq 0$; for $x \geq 0$; that is, $h_2(x)$ is increasing in $x \geq 0$. Observing that $h_2(0) = 0$, we have $h_2(x) \geq 0$ for $x \geq 0$. Hence, $h_1(x)$ is increasing in $x \geq 0$. The required result follows immediately. ∎

Now, we are ready to extend the above result to the PHR family.

THEOREM 2.3: *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample with common population survival distribution $\bar{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{i=1}^{n}\lambda_i/n$. Then*

$$Y_{n:n} \leq_{\text{lr}} X_{n:n}.$$

PROOF: Note that the cumulative hazard of F is

$$H(x) = -\log \bar{F}(x).$$

Now, for $x \geq 0$, $i = 1, \ldots, n$,

$$P(H(X_i) > x) = P(X_i > H^{-1}(x)) = \bar{F}^{\lambda_i}(\bar{F}^{-1}(e^{-x})) = e^{-\lambda_i x},$$

Where H^{-1} is the right inverse of H. Denoting $X_i' = H(X_i)$, we notice that X_i' is exponential with hazard rate λ_i for $i = 1, \ldots, n$. Similarly, let $Y_i' = H(Y_i)$ be exponential with hazard rate $\bar{\lambda}$ for $i = 1, \ldots, n$. It follows from Lemma 2.2 that

$$Y_{n:n}' \leq_{\text{lr}} X_{n:n}';$$

that is,

$$H(Y_{n:n}) \leq_{\text{lr}} H(X_{n:n}).$$

Since H^{-1} is an increasing function, it follows from Theorem 1.C.4 in Shaked and Shanthikumar [17] that

$$Y_{n:n} \leq_{\text{lr}} X_{n:n}. \qquad \blacksquare$$

One might wonder whether (1.5) of Khaledi and Kochar [7] can be strengthened from the hazard rate order to the likelihood ratio order. The following example serves as a counterexample.

Example 2.4: Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, and $Z_1, \ldots, Z_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda} = \left(\prod_{i=1}^{n} \lambda_i\right)^{1/n}$. Then the reversed hazard rate of $X_{n:n}$ is

$$\frac{f_{n:n}(x)}{F_{n:n}(x)} = \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}.$$

Similarly, the reversed hazard rate of $Z_{n:n}$ is

$$\frac{g_{n:n}(x)}{G_{n:n}(x)} = n\tilde{\lambda} \frac{e^{-\tilde{\lambda}x}}{1 - e^{-\tilde{\lambda}x}}.$$

Let $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 3$, and $n = 3$; then

$$\frac{f_{n:n}(1)}{F_{n:n}(1)} \approx 1.321 \leq 1.339 \approx \frac{g_{n:n}(1)}{G_{n:n}(1)}.$$

Thus,

$$X_{n:n} \not\geq_{\text{rh}} Z_{n:n},$$

which implies that

$$X_{n:n} \not\geq_{\text{lr}} Z_{n:n}.$$

Remark: Remark 2.2 of Khaledi and Kochar [7] asserted that the stochastic order in (1.6) cannot be extended to the hazard rate order. Example 2.4 also shows that

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \not\Rightarrow X_{n:n}^* \leq_{\text{rh}} X_{n:n}.$$

3. REVERSE HAZARD RATE ORDERING BETWEEN THE SAMPLE RANGES

Theorem 3.2 will strengthen (1.7) from the stochastic order to the reversed hazard rate order and also generalize it to the PHR family. First, let us prove the following lemma.

LEMMA 3.1: *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^n \lambda_i / n$. Then*

$$Y_{n:n} - Y_{1:n} \leq_{\text{rh}} X_{n:n} - X_{1:n}. \tag{3.1}$$

PROOF: Denote by $R_X = X_{n:n} - X_{1:n}$ and $R_Y = Y_{n:n} - Y_{1:n}$ the sample ranges of X_i's and Y_i's, respectively. From David and Nagaraja [5, p. 26], the distribution function of R_X is, for $x \geq 0$,

$$F_{R_X}(x) = \frac{1}{\sum_{i=1}^n \lambda_i} \sum_{i=1}^n \lambda_i \prod_{j=1, j \neq i}^n (1 - e^{-\lambda_j x}).$$

Thus, we have the density function of R_X as, for $x \geq 0$,

$$f_{R_X}(x) = \frac{1}{\sum_{i=1}^n \lambda_i} \left(\sum_{i=1}^n \lambda_i \prod_{j=1, j \neq i}^n (1 - e^{-\lambda_j x}) \right)'$$

$$= \frac{1}{\sum_{i=1}^n \lambda_i} \left[\sum_{i=1}^n \lambda_i \sum_{j=1, j \neq i}^n \lambda_j e^{-\lambda_j x} \prod_{k=1, k \neq i, j}^n (1 - e^{-\lambda_k x}) \right]$$

$$= \frac{1}{\sum_{i=1}^n \lambda_i} \left[\sum_{i=1}^n \lambda_i \sum_{j=1, j \neq i}^n \frac{\lambda_j e^{-\lambda_j x}}{(1 - e^{-\lambda_i x})(1 - e^{-\lambda_j x})} \right] \prod_{i=1}^n (1 - e^{-\lambda_i x})$$

$$= \frac{\prod_{i=1}^n (1 - e^{-\lambda_i x})}{\sum_{i=1}^n \lambda_i} \sum_{i=1}^n \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{j=1, j \neq i}^n \frac{\lambda_j e^{-\lambda_j x}}{1 - e^{-\lambda_j x}}.$$

Hence, the reversed hazard rate of R_X is, for $x \geq 0$,

$$\tilde{r}_{R_X}(x) = \left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{j=1, j \neq i}^{n} \frac{\lambda_j e^{-\lambda_j x}}{1 - e^{-\lambda_j x}} \right) \left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \right)^{-1}.$$

The reversed hazard rate of R_Y is, for $x \geq 0$,

$$\tilde{r}_{R_Y}(x) = (n-1) \frac{\bar{\lambda} e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}}.$$

Since, for $x \geq 0$,

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \geq n \sqrt[n]{\prod_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}}}$$

and

$$\sum_{i=1}^{n} \frac{1 - e^{-\lambda_i x}}{\lambda_i} \geq n \sqrt[n]{\prod_{i=1}^{n} \frac{1 - e^{-\lambda_i x}}{\lambda_i}},$$

it holds that

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{i=1}^{n} \frac{1 - e^{-\lambda_i x}}{\lambda_i} \geq n^2. \tag{3.2}$$

Note that, from inequality (2.2),

$$\sum_{i=1}^{n} \frac{\lambda_i^2 e^{-\lambda_i x}}{(1 - e^{-\lambda_i x})^2} \sum_{i=1}^{n} \frac{1 - e^{-\lambda_i x}}{\lambda_i} \leq n \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}.$$

Combining this with inequality (3.2), we get, for $x \geq 0$,

$$\sum_{i=1}^{n} \frac{\lambda_i^2 e^{-\lambda_i x}}{(1 - e^{-\lambda_i x})^2} \leq \frac{1}{n} \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}};$$

that is, for $x \geq 0$,

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} - \sum_{i=1}^{n} \frac{\lambda_i^2 e^{-\lambda_i x}}{(1 - e^{-\lambda_i x})^2}$$

$$\geq \frac{n-1}{n} \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}. \tag{3.3}$$

Observe that

$$f(x) = \frac{x e^{-x}}{1 - e^{-x}}$$

is convex in $x \geq 0$. It follows from Jensen's inequality that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\lambda_i x e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} \geq \frac{\bar{\lambda} x e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}};$$

that is,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} \geq \frac{\bar{\lambda} e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}}. \tag{3.4}$$

Using inequalities (3.3) and (3.4), it holds that, for $x \geq 0$,

$$\sum_{i=1}^{n}\frac{\lambda_i}{1 - e^{-\lambda_i x}}\sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} - \sum_{i=1}^{n}\frac{\lambda_i^2 e^{-\lambda_i x}}{(1 - e^{-\lambda_i x})^2} \geq (n-1)\frac{\bar{\lambda} e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}}\sum_{i=1}^{n}\frac{\lambda_i}{1 - e^{-\lambda_i x}}.$$

Hence, for $x \geq 0$,

$$\left(\sum_{i=1}^{n}\frac{\lambda_i}{1 - e^{-\lambda_i x}}\sum_{j=1,j\neq i}^{n}\frac{\lambda_j e^{-\lambda_j x}}{1 - e^{-\lambda_j x}}\right)\left(\sum_{i=1}^{n}\frac{\lambda_i}{1 - e^{-\lambda_i x}}\right)^{-1} \geq (n-1)\frac{\bar{\lambda} e^{-\bar{\lambda} x}}{1 - e^{-\bar{\lambda} x}}; \tag{3.5}$$

that is,

$$\tilde{r}_{R_X}(x) \geq \tilde{r}_{R_Y}(x).$$

The required result follows immediately. ∎

Now, we extend the above result to the PHR family.

THEOREM 3.2: *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample with common population survival distribution $\bar{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{i=1}^{n}\lambda_i/n$. Then*

$$Y_{n:n} - Y_{1:n} \leq_{\text{rh}} X_{n:n} - X_{1:n}.$$

PROOF: From David and Nagaraja [5, p. 26], the distribution function of R_X is, for $x \geq 0$,

$$F_{R_X}(x) = \sum_{i=1}^{n}\int_0^{\infty}\lambda_i\bar{F}^{\lambda_i-1}(u)f(u)\prod_{j=1,j\neq i}^{n}[\bar{F}^{\lambda_j}(u) - \bar{F}^{\lambda_j}(u+x)]du.$$

Hence, the density function is, for $x \geq 0$,

$$f_{R_X}(x) = \sum_{i=1}^{n}\int_0^{\infty}\lambda_i\bar{F}^{\lambda_i-1}(u)f(u)\sum_{j=1,j\neq i}^{n}\lambda_j\bar{F}^{\lambda_j-1}(x+u)f(x+u)$$

$$\times \prod_{k=1,k\neq i,j}^{n}[\bar{F}^{\lambda_k}(u) - \bar{F}^{\lambda_k}(u+x)]\,du.$$

Similarly, the distribution function of R_Y is, for $x \geq 0$,

$$F_{R_Y}(x) = n \int_0^\infty \bar{\lambda} \bar{F}^{\bar{\lambda}-1}(u) f(u) \left[\bar{F}^{\bar{\lambda}}(u) - \bar{F}^{\bar{\lambda}}(u+x) \right]^{n-1} du.$$

Hence, the density function is, for $x \geq 0$,

$$f_{R_Y}(x) = n(n-1) \int_0^\infty \bar{\lambda}^2 \bar{F}^{\bar{\lambda}-1}(u) f(u) \bar{F}^{\bar{\lambda}-1}(x+u) f(x+u) \left[\bar{F}^{\bar{\lambda}}(u) - \bar{F}^{\bar{\lambda}}(u+x) \right]^{n-2} du.$$

From the definition, we need to prove that, for $x \geq 0$,

$$\frac{f_{R_X}(x)}{F_{R_X}(x)} \geq \frac{f_{R_Y}(x)}{F_{R_Y}(x)}.$$

Thus, it is sufficient for us to prove that the following inequality holds:

$$\left\{ \sum_{i=1}^n \lambda_i \bar{F}^{\lambda_i-1}(u) f(u) \sum_{j=1, j \neq i}^n \lambda_j \bar{F}^{\lambda_j-1}(x+u) f(x+u) \prod_{k=1, k \neq i,j}^n \left[\bar{F}^{\lambda_k}(u) - \bar{F}^{\lambda_k}(u+x) \right] \right\}$$

$$\times \left\{ n\bar{\lambda} \bar{F}^{\bar{\lambda}-1}(u) f(u) \left[\bar{F}^{\bar{\lambda}}(u) - \bar{F}^{\bar{\lambda}}(u+x) \right]^{n-1} \right\}$$

$$\geq \left\{ n(n-1) \bar{\lambda}^2 \bar{F}^{\bar{\lambda}-1}(u) f(u) \bar{F}^{\bar{\lambda}-1}(x+u) f(x+u) \left[\bar{F}^{\bar{\lambda}}(u) - \bar{F}^{\bar{\lambda}}(u+x) \right]^{n-2} \right\}$$

$$\times \left\{ \sum_{i=1}^n \lambda_i \bar{F}^{\lambda_i-1}(u) f(u) \prod_{j=1, j \neq i}^n \left[\bar{F}^{\lambda_j}(u) - \bar{F}^{\lambda_j}(u+x) \right] \right\}.$$

After some simplifications, the above inequality is reduced to, for $x, u \geq 0$,

$$\sum_{i=1}^n \frac{\lambda_i \bar{F}^{\lambda_i}(u)}{\bar{F}^{\lambda_i}(u) - \bar{F}^{\lambda_i}(x+u)} \sum_{j=1, j \neq i}^n \frac{\lambda_j \bar{F}^{\lambda_j}(x+u)}{\bar{F}^{\lambda_j}(u) - \bar{F}^{\lambda_j}(x+u)} \left[\bar{F}^{\bar{\lambda}}(u) - \bar{F}^{\bar{\lambda}}(u+x) \right]$$

$$\geq \sum_{i=1}^n \frac{\lambda_i \bar{F}^{\lambda_i}(u)}{\bar{F}^{\lambda_i}(u) - \bar{F}^{\lambda_i}(x+u)} (n-1) \bar{\lambda} \bar{F}^{\bar{\lambda}}(u+x);$$

that is, for $x, u \geq 0$,

$$\sum_{i=1}^n \frac{\lambda_i}{1 - \bar{F}_u^{\lambda_i}(x)} \sum_{j=1, j \neq i}^n \frac{\lambda_j}{\bar{F}_u^{-\lambda_j}(x) - 1} \left[\bar{F}_u^{-\bar{\lambda}}(x) - 1 \right] \geq \sum_{i=1}^n \frac{\lambda_i}{1 - \bar{F}_u^{\lambda_i}(x)} (n-1) \bar{\lambda}, \quad (3.6)$$

where

$$\bar{F}_u(x) = \frac{\bar{F}(u+x)}{\bar{F}(u)},$$

which is the survival function of $X_u = X - u | X > u$, the residual life of X at time $u \geq 0$. Now, using the transform

$$H(x) = -\log \bar{F}_u(x), \qquad u \geq 0,$$

(3.6) is equivalent to

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i H(x)}} \sum_{j=1, j \neq i}^{n} \frac{\lambda_j}{e^{\lambda_j H(x)} - 1} \left[e^{\bar{\lambda} H(x)} - 1 \right] \geq \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i H(x)}} (n-1)\bar{\lambda};$$

that is, for $x \geq 0$,

$$\left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i H(x)}} \sum_{j=1, j \neq i}^{n} \frac{\lambda_j e^{-\lambda_j H(x)}}{1 - e^{-\lambda_j H(x)}} \right) \left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i H(x)}} \right)^{-1}$$

$$\geq (n-1)\bar{\lambda} \frac{e^{-\bar{\lambda} H(x)}}{1 - e^{-\bar{\lambda} H(x)}}.$$

Thus, the required result follows from inequality (3.5) ∎

Acknowledgment

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