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TAIL ASYMPTOTICS FOR THE SUPREMUM OF AN INFINITELY DIVISIBLE FIELD WITH CONVOLUTION EQUIVALENT LÉVY MEASURE

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Abstract

We consider a continuous, infinitely divisible random field in $\mathbb{R}^d$ given as an integral of a kernel function with respect to a Lévy basis with convolution equivalent Lévy measure. For a large class of such random fields we compute the asymptotic probability that the supremum of the field exceeds the level x as $x \to \infty$. Our main result is that the asymptotic probability is equivalent to the right tail of the underlying Lévy measure.

Keywords: Asymptotic supremum; convolution equivalence; infinite divisibility; Lévy-based modelling

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Secondary 60E07; 60D05

1. Introduction

In this paper we investigate the extremal behaviour of a field $(X_t)_{t\in B}$ defined by

$$X_t = \int_{\mathbb{R}^d} f(|t-s|)M(\mathrm{d}s), \tag{1.1}$$

where M is an infinitely divisible, independently scattered random measure on $\mathbb{R}^d$, f is some kernel function, and B is a compact index set. We will assume that the Lévy measure of the random measure M has a convolution equivalent right tail [11], [12], [21].

In this paper we derive for a random field (1.1) the very useful result that the asymptotic behaviour of the supremum of X_t, $t \in B$, has a tail that is equivalent to the tail of the underlying Lévy measure. More precisely under the assumption that the underlying Lévy measure ρ of M has a tail that is convolution equivalent, we show that

$$\mathbb{P}\Big(\sup_{t\in B} X_t > x\Big) \sim C\rho((x,\infty))\mathbb{E}\exp(\beta X_{t_0})m_d(B) \quad \text{as } x \to \infty,$$

where C is a known constant and $m_d(B)$ is the Lebesgue measure of B. The proof of this result uses an important lemma from a paper by Braverman and Samorodnitsky; see [10, Lemma 2.1]. Measures with a convolution equivalent tail cover the important cases of an inverse Gaussian (IG) and a normal inverse Gaussian (NIG) basis, respectively; see Section 2 below.

Lévy models as defined in (1.1) provide a flexible and tractable modelling framework that has recently been used for a variety of modelling purposes, including modelling of turbulent

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flows [9], growth processes [17], Cox point processes [16], and brain imaging data [18]. In [18], a model (1.1) with M following a NIG distribution was suitable for modelling the neuroscience data under consideration. For such data it is typically of interest to detect for which $t \in B$ a given field obtains values that are significantly large.

To the best of the authors' knowledge, the extremal behaviour of a NIG field or more generally a field (1.1) with convolution equivalent Lévy measure has not yet been studied in detail. For Gaussian random fields it is known that the distribution of the supremum of the field can be approximated by the expected Euler characteristic of an excursion set (see [2] and the references therein). In [15] an exact asymptotic result was obtained for Gaussian random fields under the assumption of $\alpha(t)$-local stationarity. However, in [18] it was shown by simulations that using a model based on the NIG distribution gives results that are substantially different from those obtained by Gaussian models.

The supremum of a non-Gaussian field given by integrals with respect to an infinitely divisible random measure has already been studied, when the random measure has regularly varying tails. Results for the asymptotic distribution of the supremum are found in [27], and these results were refined in [3] and [4], where results were obtained on the asymptotic joint distribution of the number of critical points of the excursion sets. The arguments are—as in this paper—based on finding the Lévy measure of a dense countable subset of the field. However, the remaining proofs rely heavily on the assumption of regularly varying tails and can therefore not be translated into the convolution equivalent framework.

Note that convolution equivalent distributions have heavier tails than Gaussian distributions and lighter tails than those of regularly varying distributions. The latter statement follows from the fact that convolution equivalent distributions have exponential tails while regularly varying distributions have power function tails.

For real-valued one-dimensional infinitely divisible distributions it was shown in [11], [12], and [21] that if the Lévy measure has a convolution equivalent right tail, then the distribution has a right tail that is asymptotically equivalent. The proofs are based on a decomposition of the distribution into a compound Poisson part that is dominating in the tail and a part with a lighter tail. The arguments in this paper apply a similar decomposition to the distribution of a dense countable subset of the field.

In [13], results for a moving average process on $\mathbb{R}$, obtained as an integral with respect to a Lévy process with convolution equivalent tail, were derived. But here the kernel function f satisfies $f(t) = 0$ for $t < 0$ such that

$$X_t = \int_{-\infty}^{t} f(t-s) M(\mathrm{d}s).$$

This paper is organised as follows. In Section 2 we give a short introduction to random fields defined as an integral of a kernel function with respect to a Lévy basis. Such a field X can be decomposed into a sum $X^1 + X^2$ of two independent fields, where X^1 is a compound Poisson sum. In Section 3 the tail asymptotics for X^1 are studied, while it is shown in Section 4 that the supremum of the field X^2 has lighter tails than the supremum of X^1. This makes it possible to derive the overall extremal behaviour of the supremum of X, which is also done in Section 4. The asymptotic behaviour of excursion sets is briefly discussed in Section 5. Proofs concerning the existence of continuous versions of the random fields considered are deferred to Appendix A.

2. Preliminaries

Consider an independently scattered random measure M on $\mathbb{R}^d$. Then for a sequence of disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ in $\mathcal{B}(\mathbb{R}^d)$ the random variables $(M(A_n))_{n\in\mathbb{N}}$ are independent and satisfy $M(\cup A_n) = \sum M(A_n)$. Assume furthermore that $M(A)$ is infinitely divisible for all $A \in \mathcal{B}(\mathbb{R}^d)$. Then M is called a Lévy basis; see [9] and the references therein.

For a random variable X let $C(\lambda \star X)$ denote its cumulant function $\log \mathbb{E}(\mathrm{e}^{\mathrm{i}\lambda X})$. We shall assume that the Lévy basis is stationary and isotropic such that for $A \in \mathcal{B}(\mathbb{R}^d)$ the variable $M(A)$ has a Lévy–Khintchine representation given by

$$C(\lambda \star M(A)) = \mathrm{i}\lambda a m_d(A) + \frac{1}{2}\lambda^2\theta m_d(A) + \int_{A\times\mathbb{R}} (\mathrm{e}^{\mathrm{i}\lambda u} - 1 - \mathrm{i}\lambda u \mathbf{1}_{[-1,1]}(u)) F(\mathrm{d}s, \mathrm{d}u), \quad (2.1)$$

where m_d is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $a \in \mathbb{R}$, $\theta \geq 0$, and F is a measure on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ of the form

$$F(A \times B) = m_d(A)\rho(B). \quad (2.2)$$

We assume that ρ has an exponential tail with index $\beta > 0$, i.e. for all $y \in \mathbb{R}$,

$$\frac{\rho((x-y,\infty))}{\rho((x,\infty))} \to \mathrm{e}^{\beta y} \quad \text{as } x \to \infty. \quad (2.3)$$

Furthermore, letting ρ_1 be a normalization of the restriction of ρ to $(1,\infty)$, we assume that ρ_1 has a convolution equivalent right tail, i.e.

$$\frac{(\rho_1 * \rho_1)((x,\infty))}{\rho_1((x,\infty))} \to 2M \quad \text{as } x \to \infty, \quad (2.4)$$

where $M < \infty$. Here, $\rho_1 * \rho_1$ denotes the convolution. In fact, $M = \int \mathrm{e}^{\beta y}\rho_1(\mathrm{d}y)$, cf. [21, Corollary 2.1(ii)]. Writing $\rho((x,\infty)) = L(x)\mathrm{e}^{-\beta x}$, it is seen from (2.3) that, for all $y \in \mathbb{R}$,

$$\frac{L(x-y)}{L(x)} \to 1 \quad \text{as } x \to \infty. \quad (2.5)$$

For each $a, b \in \mathbb{R}$, the limit (2.5) holds uniformly in $y \in [a,b]$, cf. [21, p. 408].

We furthermore assume that

$$\int z^2 \rho(\mathrm{d}z) < \infty. \quad (2.6)$$

Note that integrability along the right tail is already assumed and that

$$\int_{[-1,1]} z^2 \rho(\mathrm{d}z) < \infty$$

is needed for ρ to be a Lévy measure.

It follows from [21, Lemma 2.4] that if ρ satisfies, for $x > 0$,

$$\rho((x,\infty)) \sim a x^{-\delta}\mathrm{e}^{-\beta x}, \quad (2.7)$$

where $a > 0$, $\delta > 1$, and $\beta > 0$, then (2.3) and (2.4) are fulfilled. Here we use the convention that $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ as $x \to \infty$.

Example 2.1. (*IG basis.*) Suppose that M is inverse Gaussian,

$$M(A) \sim \mathrm{IG}(\eta m_d(A), \gamma), \qquad \eta, \gamma > 0.$$

Then $C(\lambda \star M(A))$ has the representation (2.1) from above with

$$a = \frac{\eta}{\gamma} \int_0^1 x^{-1/2} \mathrm{e}^{(-1/2)\gamma^2 x} \, \mathrm{d}x, \quad \theta = 0, \qquad \rho(\mathrm{d}x) = \frac{\eta}{\sqrt{2\pi}} \mathbf{1}_{\mathbb{R}_+}(x) x^{-3/2} \mathrm{e}^{(-1/2)\gamma^2 x} \, \mathrm{d}x;$$

see, e.g. [7], [8], [17]. Thereby,

$$\rho((x, \infty)) = \frac{\eta}{\sqrt{2\pi}} \int_x^\infty y^{-3/2} \mathrm{e}^{(-1/2)\gamma^2 y} \, \mathrm{d}y \sim \frac{\eta}{\gamma^2} \sqrt{\frac{2}{\pi}} x^{-3/2} \mathrm{e}^{(-1/2)\gamma^2 x} \quad \text{as } x \to \infty.$$

Thus, (2.7) is fulfilled with $\delta = \frac{3}{2}$.

Example 2.2. (*NIG basis.*) Suppose that M is normal inverse Gaussian,

$$M(A) \sim \mathrm{NIG}(\alpha, \beta, \mu m_d(A), \delta m_d(A)),$$

$0 \le |\beta| < \alpha$, $\mu \in \mathbb{R}$, and $0 < \delta$. Then $C(\lambda \star M(A))$ has the representation (2.1) from above with

$$a = \mu + \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x) \, \mathrm{d}x, \quad \theta = 0, \qquad \rho(\mathrm{d}x) = \frac{\delta\alpha}{\pi} \frac{1}{|x|} K_1(\alpha|x|) \mathrm{e}^{\beta x} \, \mathrm{d}x,$$

where K_1 is the modified Bessel function of the second kind and index 1. For further details concerning the Lévy measure of the NIG distribution; see [5] and [6]. Using an asymptotic formula for K_1 it can be shown that

$$\rho((x, \infty)) \sim \frac{\delta\sqrt{\alpha}}{\sqrt{2\pi}} \int_x^\infty |y|^{-3/2} \mathrm{e}^{-\alpha|y| + \beta y} \, \mathrm{d}y \sim \frac{\delta}{\alpha - \beta} \sqrt{\frac{\alpha}{2\pi}} x^{-3/2} \mathrm{e}^{-(\alpha-\beta)x} \quad \text{as } x \to \infty.$$

For details, see [26, Example 2.2]. Again, (2.7) is fulfilled with $\delta = \frac{3}{2}$.

Now assume that $f : [0, \infty) \to [0, \infty)$ is an integration kernel satisfying

$$f(0) = 1, \qquad f(x) < 1 \quad \text{for } x > 0, \qquad \int_{\mathbb{R}^d} f(|s|) \, \mathrm{d}s < \infty, \tag{2.8}$$

and

$$f(x) \le \frac{K_1}{(x+1)^d} \quad \text{for all } x \ge 0 \tag{2.9}$$

for a finite, positive constant K_1. Assume furthermore that f is differentiable with f' satisfying

$$|f'(x)| \le \frac{K_2}{(x+1)^d} \quad \text{for all } x \ge 0 \tag{2.10}$$

for a finite, positive constant K_2. Let B be a compact subset of $\mathbb{R}^d$ with $m_d(B) > 0$ and consider the family of random variables $(X_t)_{t \in B}$ defined by

$$X_t = \int_{\mathbb{R}^d} f(|t - s|) M(\mathrm{d}s).$$

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The integrals defining each X_t exist according to [24, Theorem 2.7], where the conditions (i)–(iii) can be easily verified under the given assumptions on M and f. As explained in Appendix A, Theorem A.1, there furthermore exists a version of $(X_t)_{t\in B}$ with continuous sample paths. In the following, it will be useful to note that

$$\int_{\mathbb{R}^d} \sup_{t\in B} f(|t-s|)\,\mathrm{d}s < \infty. \tag{2.11}$$

Example 2.3. (*Exponential kernel function.*) Suppose that $f(x) = \mathrm{e}^{-\sigma x}$, $\sigma > 0$, then the assumptions (2.8)–(2.10) are satisfied.

Example 2.4. (*Gaussian kernel function.*) Suppose that $f(x) = \mathrm{e}^{-\sigma x^2}$, $\sigma > 0$, then the assumptions (2.8)–(2.10) are satisfied.

Example 2.5. (*Matérn kernel function.*) Suppose that

$$f(x) = \frac{1}{2^{\eta-1}\Gamma(\eta)}|\lambda x|^{\eta} K_{\eta}(\lambda|x|),$$

where K_η is the modified Bessel function of the second kind, index η, and $\lambda > 0$. The use of this kernel function in Lévy-based modelling and its relation to the so-called Matérn correlation structure of the field $(X_t)_{t\in B}$ were discussed in [18]. For a further discussion of modelling, using a Matérn correlation structure, see [14]. Since for $\eta = \frac{1}{2}$,

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2}}x^{-1/2}\mathrm{e}^{-x},$$

the Matérn kernel reduces to the exponential kernel for $\eta = \frac{1}{2}$. It can be shown that for $\eta \geq \frac{1}{2}$ the Matérn kernel satisfies the assumptions (2.8)–(2.10). In the arguments it is essential that, for all $\eta > 0$,

$$K_\eta(x) \sim \sqrt{\frac{\pi}{2}}x^{-1/2}\mathrm{e}^{-x} \quad \text{as } x \to \infty.$$

See [26, Example 2.5] for details.

For the study of the extremal behaviour of $(X_t)_{t\in B}$, we will use the fact that the cumulant function of $X_t = \int_{\mathbb{R}^d} f(|t-s|)M(\mathrm{d}s)$ takes the following form:

$$\begin{aligned} C(\lambda \star X_t) = \mathrm{i}\lambda a \int_{\mathbb{R}^d} f(|t-s|)\,\mathrm{d}s + \frac{1}{2}\lambda^2\theta \int_{\mathbb{R}^d} f(|t-s|)^2\,\mathrm{d}s \\ + \int_{\mathbb{R}^d}\int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}f(|t-s|)\lambda u} - 1 - \mathrm{i}f(|t-s|)\lambda u \mathbf{1}_{[-1,1]}(u))\rho(\mathrm{d}u)\,\mathrm{d}s, \end{aligned}$$

cf. e.g. [24, Theorem 2.7]. A similar formula holds for finite linear combinations of the X_ts. Here, $f(|t-s|)$ is substituted by $\sum_t \beta_t f(|t-s|)$. It follows that all finite-dimensional distributions of $(X_t)_{t\in B}$ are infinitely divisible. As a consequence, any countable field $(X_t)_{t\in T}$ is itself infinitely divisible; see [20] for existence and uniqueness of the infinite divisibility of the entire field. It follows from direct manipulations and it is also noted in, e.g. [27] that the Lévy measure of $(X_t)_{t\in T}$ is the measure ν on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ defined by $\nu = F \circ V^{-1}$, where $V\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^T$ is given by

$$V(s,z) = (zf(|t-s|))_{t\in T}.$$

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We will from now on assume that $T = B \cap \mathbb{Q}^d$, where $\mathbb{Q}^d$ are the rational numbers in $\mathbb{R}^d$. For $\beta \in \mathbb{R}^T$ with $\beta_t = 0$ for all but finitely many $t \in T$, we find that

$$\log \mathbb{E}\bigg(\exp\bigg(\mathrm{i}\sum_t \beta_t X_t\bigg)\bigg) = \mathrm{i}\sum_t \beta_t a_t + \frac{1}{2}\theta \int_{\mathbb{R}^d}\bigg(\sum_t \beta_t f(|t-s|)\bigg)^2 \,\mathrm{d}s + \int_{\mathbb{R}^T}\bigg(\exp\bigg(\mathrm{i}\sum_t \beta_t x_t\bigg) - 1 - \mathrm{i}\sum_t \beta_t x_t \mathbf{1}_{[-1,1]^T}(x)\bigg)\nu(\mathrm{d}x) \tag{2.12}$$

for an appropriate choice of $(a_t)_{t\in T} \in \mathbb{R}^T$. It is furthermore seen that $(a_t)_{t\in T}$ is bounded. Because of the infinite divisibility of $(X_t)_{t\in T}$, we have the following decomposition; see, e.g. [27]:

$$X_t = X_t^1 + X_t^2,$$

where the fields $(X_t^1)_{t\in T}$ and $(X_t^2)_{t\in T}$ are independent. The first field $(X_t^1)_{t\in T}$ is a compound Poisson sum

$$X_t^1 = \sum_{n=1}^{N} U_t^n,$$

where N is Poisson distributed with parameter $\nu(A)$ and

$$A = \Big\{x \in \mathbb{R}^T : \sup_{t\in T} x_t > 1\Big\}.$$

In Appendix A, it is shown that $\nu(A) < \infty$; see Lemma A.1. The fields $(U_t^n)_{t\in T}$ are independent and identically distributed with common distribution $\nu_1 = \nu_A/\nu(A)$, where ν_A is the measure on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ obtained by restricting ν to A. Furthermore, $(X_t^2)_{t\in T}$ is infinitely divisible with a Lévy measure ν_{A^c}, the restriction of ν to A^c, and a cumulant function that is similar to (2.12) but with ν replaced by ν_{A^c}.

It will be crucial for the arguments in the following sections that the fields X^1 and X^2 can be extended to continuous fields indexed by B. Note that each of the fields $(U_t^n)_{t\in T}$ almost surely has the form $zf(|t-s|)$. Hence, there almost surely exists a continuous extension $(U_t^n)_{t\in B}$. Since X^1 is a finite sum of such fields it has a continuous extension to B as well. As already stated, the field $(X_t)_{t\in B}$ has continuous sample paths, see also Theorem A.1. Thereby also X^2 has continuous sample paths.

3. Tail asymptotics for compound Poisson sum of Lévy fields

In this section we will determine the extremal behaviour of $\mathbb{P}(X_t^1 > x - y_t \text{ for some } t)$ for increasing values of x and $(y_t)_{t\in B}$ a continuous field. The main result, formulated in Theorem 3.3 below, will be used in the next section to study the extremal behaviour of $\mathbb{P}(\sup_{t\in B} X_t > x)$, using the fact that $X = X^1 + X^2$ and conditioning on X^2.

It is convenient to introduce a notation that can be seen as a refinement of the event $\{\sup_{t\in T} X_t > x\}$. If $(x_t)_{t\in T}$ is a field in $\mathbb{R}^T$, we define $\Gamma((x_t)_{t\in T})$ to be the following subset in $\mathcal{B}(\mathbb{R}^T)$:

$$\Gamma((x_t)_{t\in T}) = \{(y_t)_{t\in T} : y_t > x_t \text{ for some } t \in T\}.$$

If $x_t = x$ for all $t \in T$ we shall use the notation $\Gamma(x)$. Note that $\{\sup_{t\in T} X_t > x\} = \{X \in \Gamma(x)\}$.

The first step will be determining the behaviour of $\mathbb{P}(U \in \Gamma((x - y_t)_{t\in T}))$, when U is a field with distribution ν_1.

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Theorem 3.1. *Let $(y_t)_{t\in B}$ be continuous and bounded on B. Then*

$$\frac{\nu_1(\Gamma((x-y_t)_{t\in T}))}{L(x)\exp(-\beta x)} \to \frac{1}{\nu(A)}\int_B \exp(\beta y_s)\,\mathrm{d}s \quad \textit{as } x\to\infty. \tag{3.1}$$

Furthermore,

$$\frac{\nu_1(\Gamma(x))}{L(x)\exp(-\beta x)} \to \frac{1}{\nu(A)} m_d(B) \quad \textit{as } x\to\infty, \tag{3.2}$$

and

$$\frac{\nu_1(\Gamma((x-y_t)_{t\in T}))}{\nu_1(\Gamma(x))} \to \frac{\int_B \exp(\beta y_s)\,\mathrm{d}s}{m_d(B)} \quad \textit{as } x\to\infty. \tag{3.3}$$

Proof. The results (3.2) and (3.3) are direct consequences of (3.1), so we focus on the proof of (3.1). We can assume that $(y_t)_{t\in B}$ is nonnegative. Simply write $x = x' - x_0$ for a suitable x_0 such that $(x_0 + y_t)_{t\in B}$ is nonnegative, and find the limit of

$$\frac{\nu_1(\Gamma((x'-(x_0+y_t))_{t\in T}))}{L(x')\exp(-\beta x')} \quad \text{as } x'\to\infty.$$

We find that

$$\begin{aligned}
&\nu_1(\Gamma((x-y_t)_{t\in T}))\\
&\quad = \frac{1}{\nu(A)} F\circ V^{-1}(\Gamma((x-y_t)_{t\in T}))\\
&\quad = \frac{1}{\nu(A)} F(\{(s,z)\in\mathbb{R}^d\times\mathbb{R}\colon \text{there exists } t\in T\colon zf(|t-s|) > x-y_t\})\\
&\quad = \frac{1}{\nu(A)} F\biggl(\biggl\{(s,z)\in\mathbb{R}^d\times\mathbb{R}\colon z > \inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr\}\biggr)\\
&\quad = \frac{1}{\nu(A)}\int_{\mathbb{R}^d} L\biggl(\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr)\exp\biggl(-\beta\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr)\,\mathrm{d}s\\
&\quad = \frac{1}{\nu(A)}\int_{B} L\biggl(\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr)\exp\biggl(-\beta\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr)\,\mathrm{d}s\\
&\qquad + \frac{1}{\nu(A)}\int_{\mathbb{R}^d\setminus B} L\biggl(\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr)\exp\biggl(-\beta\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr)\,\mathrm{d}s.
\end{aligned}\tag{3.4}$$

First, we show that the second term in (3.4) is $o(L(x)\exp(-\beta x))$. Let $y^* = \sup_{s\in T} y_s$. Utilising the fact that $L(x)\exp(-\beta x)$ is decreasing the second term is, for $x > y^*$,

$$\le \frac{1}{\nu(A)}\int_{\mathbb{R}^d\setminus B} L\biggl(\frac{x-y^*}{\sup_{t\in T} f(|t-s|)}\biggr)\exp\biggl(-\beta\frac{x-y^*}{\sup_{t\in T} f(|t-s|)}\biggr)\,\mathrm{d}s. \tag{3.5}$$

Now we use the fact that (2.5) implies that $L(\log(x))$ is slowly varying. Then from the representation theorem for slowly varying functions, we obtain

$$L(x) = a(x)\exp\biggl(-\int_0^x \varepsilon(y)\,\mathrm{d}y\biggr),$$

where $a(x)\to a>0$ and $\varepsilon(x)\to 0$. It follows that for all $\gamma>0$ there exists $x_0>0$ and $C>0$ such that

$$\frac{L(\alpha x)}{L(x)} \le C\mathrm{e}^{(\alpha-1)\gamma x} \quad \text{for all } x\ge x_0,\ \alpha\ge 1. \tag{3.6}$$

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Using (2.5), (3.6), and the facts that $\sup_{t\in T} f(|t-s|) < 1$ for all $s \in \mathbb{R}^d \setminus B$ and

$$L(x)\exp(-\gamma x) \to 0 \quad \text{for all } \gamma > 0,$$

it is seen that the integrand in (3.5) is $o(L(x)\exp(-\beta x))$. If we denote the integrand of (3.5) by $h(s; x)$, it follows by the dominated convergence theorem that (3.5) is $o(L(x)\exp(-\beta x))$ if we can find an integrable function g such that

$$\frac{h(s;x)}{L(x)\exp(-\beta x)} \le g(s), \qquad s \in \mathbb{R}^d.$$

Let $0 < \gamma < \beta$ and $f_0(s) = \sup_{t\in T} f(|t-s|)$. Then, using (3.6) and the boundedness of $L(x-y^*)/L(x)$, we can find a constant $\tilde{C}$ and $x_0 > y^*$ such that, for $x \ge x_0$,

$$\frac{h(s;x)}{L(x)\exp(-\beta x)} \le \tilde{C}\exp(\beta y^*)\exp\biggl(-(\beta-\gamma)\biggl(\frac{1}{f_0(s)}-1\biggr)(x_0-y^*)\biggr). \tag{3.7}$$

Now, choose $r > 0$ such that $B \subseteq C_r(0)$, where $C_r(0)$ is the ball with radius r and centre $0 \in \mathbb{R}^d$. Then, using (2.9), we obtain, for $s \notin C_r(0)$,

$$f_0(s) \le \sup_{t\in C_r(0)} f(|t-s|) \le \sup_{t\in C_r(0)} \frac{1}{(|t-s|+1)^d} = \frac{1}{(|s|-r+1)^d}.$$

It follows that (3.7) is integrable.

The theorem now follows from applying dominated convergence to the first term of (3.4). Since, for $s \in B$,

$$\inf_{t\in T}\frac{x-y_t}{f(|t-s|)} - (x-y_s) \to 0 \quad \text{as } x \to \infty,$$

we have

$$L\biggl(\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr)\exp\biggl(-\beta\inf_{t\in T}\frac{x-y_t}{f(|t-s|)}\biggr) \sim L(x-y_s)\exp(-\beta(x-y_s)),$$

so

$$\frac{L(\inf_{t\in T}((x-y_t)/f(|t-s|)))\exp(-\beta\inf_{t\in T}((x-y_t)/f(|t-s|)))}{L(x)\exp(-\beta x)} \to \mathrm{e}^{\beta y_s}.$$

Using again the fact that $L(x)\exp(-\beta x)$ is decreasing, we have, for large x,

$$\begin{aligned}
&\biggl|\frac{L(\inf_{t\in T}((x-y_t)/f(|t-s|)))\exp(-\beta\inf_{t\in T}((x-y_t)/(f(|t-s|))))}{L(x)\exp(-\beta x)} - \mathrm{e}^{\beta y_s}\biggr| \\
&\qquad \le \frac{L(x-y^*)\exp(-\beta(x-y^*))}{L(x)\exp(-\beta x)} + \mathrm{e}^{\beta y_s} \\
&\qquad \le (C+1)\mathrm{e}^{\beta y^*},
\end{aligned}$$

where C is chosen such that $L(x-y^*)/L(x) \le C$. The result is integrable over B. □

Below, we extend the result of Theorem 3.1 to the $\mathbb{P}(U_1+\cdots+U_n \in \Gamma((x-y_t)_{t\in T}))$ case, where U_i, $i = 1,\dots,n$, are independent with common distribution ν_1. For this purpose, we need the corollary below.

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Corollary 3.1. *Let $(U_t)_{t\in T}$ be distributed according to ν_1. Then, the distribution of* $\sup_{t\in T} U_t$ *is convolution equivalent. In particular, we have*

$$\int e^{\beta \sup_{t\in T} z_t} \nu_1(dz) < \infty.$$

Proof. From Theorem 3.1 and [21, Lemma 2.4(i)], the distribution of $\sup_{t\in T} U_t$ has a convolution equivalent right tail and then the result follows from [21, Corollary 2.1(ii)]. □

If $(U_t)_{t\in T}$ and $(V_t)_{t\in T}$ are independent random fields with distributions ν and μ on

$$(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T)),$$

then we will use the notation $\nu * \mu$ for the distribution of $(U_t + V_t)_{t\in T}$. Similarly, we write ν^{*n} for the n-fold convolution of ν. We have the following theorem.

Theorem 3.2. *For all $n \in \mathbb{N}$ and $(y_t)_{t\in B}$ bounded and continuous, it holds that*

$$\frac{\nu_1^{*n}(\Gamma(x - y_t)_{t\in T})}{\nu_1(\Gamma(x))} \to \frac{n}{m_d(B)}\bigg(\int_B e^{\beta y_s} \int e^{\beta z_s} \nu_1^{*(n-1)}(dz)\, ds\bigg) \quad \text{as } x \to \infty.$$

Proof. As in the proof of Theorem 3.1, we can assume that $(y_t)_{t\in B}$ is nonnegative. The result is shown by induction over n. For $n = 1$, the result is shown in Theorem 3.1. Assume now that the theorem is correct for some $n \in \mathbb{N}$. Let $(U_t)_{t\in T}$ and $(V_t)_{t\in T}$ be independent and with distribution ν_1 and ν_1^{*n}, respectively. Then, we have

$$\begin{aligned}
&(\nu_1^{*n} * \nu_1)(\Gamma((x - y_t)_{t\in T})) \\
&\quad = \mathbb{P}(\text{there exists } t\colon U_t + V_t > x - y_t) \\
&\quad = \mathbb{P}\bigg(\text{there exists } t\colon U_t > \frac{x - y_t}{2}, \text{ there exists } t\colon V_t > \frac{x - y_t}{2}, \\
&\qquad\qquad \text{there exists } t\colon U_t + V_t > x - y_t\bigg) \\
&\qquad + \mathbb{P}\bigg(\text{for all } t\colon U_t < \frac{x - y_t}{2}, \text{ there exists } t\colon U_t + V_t > x - y_t\bigg) \\
&\qquad + \mathbb{P}\bigg(\text{for all } t\colon V_t < \frac{x - y_t}{2}, \text{ there exists } t\colon U_t + V_t > x - y_t\bigg). \qquad (3.8)
\end{aligned}$$

The first term is bounded from above by

$$\begin{aligned}
&\mathbb{P}\bigg(\text{there exists } t\colon U_t > \frac{x - y_t}{2}, \text{ there exists } t\colon V_t > \frac{x - y_t}{2}\bigg) \\
&\qquad \le \nu_1\bigg(\Gamma\bigg(\frac{x - y^*}{2}\bigg)\bigg)\nu_1^{*n}\bigg(\Gamma\bigg(\frac{x - y^*}{2}\bigg)\bigg),
\end{aligned}$$

where $y^* = \sup_{t\in T} y_t$. Since both factors are equivalent to $\rho_1((x/2, \infty))$ it follows from the proof of [11, Lemma 2] that the product is $o((\rho_1 * \rho_1)((x, \infty)))$. In particular, the product above is $o(\rho_1((x, \infty)))$ due to the convolution equivalence.

For the evaluation of the remaining terms in (3.8), we can assume that all the fields $z = (z_t)_{t\in T}$ have continuous extensions to B, since the distribution ν_1 is concentrated on a set of

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fields with this property. The two remaining terms in (3.8) divided by $\nu_1(\Gamma(x))$ can be written as

$$\int_{C_{x,y}} \frac{\nu_1^{*n}(\Gamma((x-y_t-z_t)_{t\in T}))}{\nu_1(\Gamma(x))}\nu_1(\mathrm{d}z) + \int_{\tilde{C}_{x,y}} \frac{\nu_1(\Gamma((x-y_t-\sum_{k=1}^n z_t^k)_{t\in T}))}{\nu_1(\Gamma(x))}\nu_1^{*\otimes n}(\mathrm{d}(z^1,\ldots,z^n)), \tag{3.9}$$

where $\nu_1^{*\otimes n}$ is the n-fold product measure of ν_1, $C_{x,y} = \{z\colon z_t < (x-y_t)/2 \text{ for all } t\}$, and $\tilde{C}_{x,y} = \{(z^1,\ldots,z^n)\colon \sum_{k=1}^n z_t < (x-y_t)/2 \text{ for all } t\}$. Using the induction assumption and Theorem 3.1, the two integrands of (3.9) converge to, as $x\to\infty$,

$$f_1(z) = \frac{n}{m_d(B)}\int_B \mathrm{e}^{\beta(y_s+z_s)}\int \mathrm{e}^{\beta u_s}\nu_1^{*(n-1)}(\mathrm{d}u)\,\mathrm{d}s$$

and

$$f_2(z^1,\ldots,z^n) = \frac{1}{m_d(B)}\int_B \mathrm{e}^{\beta(y_s+\sum_{k=1}^n z_s^k)}\,\mathrm{d}s,$$

respectively. We want to show that (3.9) converges to

$$\int f_1(z)\nu_1(\mathrm{d}z) + \int f_2(z^1,\ldots,z^n)\nu_1^{*\otimes n}(\mathrm{d}(z^1,\ldots,z^n)) = \frac{n+1}{m_d(B)}\bigg(\int_B \mathrm{e}^{\beta(y_s+z_s)}\int \mathrm{e}^{\beta u_s}\nu_1^{*n}(\mathrm{d}u)\,\mathrm{d}s\bigg).$$

Using Fatou's lemma, it is enough to find integrable functions $g_1(z;x)$ and $g_2(z^1,\ldots,z^n;x)$ that are upper bounds of the two integrands of (3.9) such that $g_1(z)=\lim_{x\to\infty} g_1(z;x)$ and $g_2(z^1,\ldots,z^n)=\lim_{x\to\infty} g_2(z^1,\ldots,z^n;x)$ exist with

$$\int g_1(z;x)\nu_1(\mathrm{d}z) + \int g_2(z^1,\ldots,z^n;x)\nu_1^{*\otimes n}(\mathrm{d}(z^1,\ldots,z^n)) \tag{3.10}$$

converging to the similar integrals with $g_1(z)$ and $g_2(z^1,\ldots,z^n)$. Let $\tilde{\nu}_1^{*n}$ be the n-fold convolution of the distribution $\tilde{\nu}_1$ of $\sup_{t\in T} U_t$. Then as functions $g_1(z;x)$ and $g_2(z^1,\ldots,z^n;x)$, we can use

$$g_1(z;x) = \mathbf{1}_{C_x}(z)\frac{\tilde{\nu}_1^{*n}((x-\sup_{t\in T} y_t - \sup_{t\in T} z_t,\infty))}{\nu_1(\Gamma(x))},$$

where $C_x = \{z\colon \sup_{t\in T} z_t < x/2\}$, and

$$g_2(z^1,\ldots,z^n;x) = \mathbf{1}_{\tilde{C}_x}(z^1,\ldots,z^n)\frac{\tilde{\nu}_1((x-\sup_{t\in T} y_t - \sum_{k=1}^n \sup_{t\in T} z_t^k,\infty))}{\nu_1(\Gamma(x))},$$

where $\tilde{C}_x = \{(z^1,\ldots,z^n)\colon \sum_{k=1}^n \sup_{t\in T} z_t^k < x/2\}$. Noting that $\nu_1(\Gamma(x)) = \tilde{\nu}_1((x,\infty))$ and using that $\tilde{\nu}_1$ is convolution equivalent, from [12, Corollary 2.11], we obtain

$$g_1(z;x)\to g_1(z) = n\mathrm{e}^{\beta(\sup_{t\in T} y_t+\sup_{t\in T} z_t)}(\mathbb{E}(\mathrm{e}^{\beta\sup_{t\in T} U_t}))^{n-1}.$$

According to Theorem 3.1, we have

$$g_2(z^1,\ldots,z^n;x)\to g_2(z^1,\ldots,z^n) = \exp\bigg(\beta\bigg(\sup_{t\in T} y_t + \sum_{k=1}^n \sup_{t\in T} z_t^k\bigg)\bigg).$$

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We observe that

$$\int g_1(z)\nu_1(\mathrm{d}z) + \int g_2(z^1, \dots, z^n)\nu_1^{*\otimes n}(\mathrm{d}(z^1, \dots, z^n)) = (n+1)\exp\Big(\beta \sup_{t\in T} y_t\Big)\Big(\mathbb{E}\Big(\exp\Big(\beta \sup_{t\in T} U_t\Big)\Big)\Big)^n. \tag{3.11}$$

Furthermore, both $\tilde{\nu}_1^{*n}$ and $\tilde{\nu}_1$ have exponential tails, so according to [11, Lemma 2], (3.10) is asymptotically equal to

$$\exp\Big(\beta \sup_{t\in T} y_t\Big)\frac{\tilde{\nu}_1^{*(n+1)}((x,\infty))}{\nu_1(\Gamma(x))}$$

which, by another reference to [12, Corollary 2.11], is seen to converge to (3.11). □

We are now ready to prove the main result of this section concerning the extremal behaviour of $\mathbb{P}(X \in \Gamma((x - y_t)_{t\in T}))$ for large x. For a dominated convergence argument, we need the lemma below. Recall that $(U_t^1)_{t\in T}, (U_t^2)_{t\in T}, \dots$ are independent and identically distributed fields with common distribution ν_1.

Lemma 3.1. *There exists a constant K such that for all $n \in \mathbb{N}$ and all $x \ge 0$,*

$$\nu_1^{*n}(\Gamma(x)) \le K^n \nu_1(\Gamma(x)).$$

Proof. Since $\sup_{t\in T} U_t^1$ has a convolution equivalent tail according to Corollary 3.1, it follows from [12, Lemma 2.8] that there exists K such that

$$\mathbb{P}\Big(\sum_{k=1}^n \sup_{t\in T} U_t^k > x\Big) \le K^n \mathbb{P}\Big(\sup_{t\in T} U_t^1 > x\Big).$$

The result is seen directly by noting that $\mathbb{P}(\sup_{t\in T} U_t^1 > x) = \nu_1(\Gamma(x))$ and

$$\nu_1^{*n}(\Gamma(x)) \le \mathbb{P}\Big(\sum_{k=1}^n \sup_{t\in T} U_t^k > x\Big).$$

This completes the proof. □

We have defined the field X^1 from the fields $(U_t^1)_{t\in T}, (U_t^2)_{t\in T}, \dots$ and an independent Poisson distributed variable N with parameter $\nu(A)$ by

$$X_t^1 = \sum_{n=1}^N U_t^n.$$

Theorem 3.3. *We have $\mathbb{E}\exp(\beta \sup_{t\in T} X_t^1) < \infty$ and for a continuous field, $(y_t)_{t\in B}$,*

$$\lim_{x\to\infty}\frac{\mathbb{P}(X^1 \in \Gamma((x - y_t)_{t\in T}))}{\nu(\Gamma(x))} = \frac{\int_B \mathrm{e}^{\beta y_s}\mathbb{E}(\mathrm{e}^{\beta X_s^1})\,\mathrm{d}s}{m_d(B)}.$$

Proof. The first result follows, since $\sup_{t\in T} X_t^1 \le \sum_{n=0}^N \sup_{t\in T} U_t^n$ and $\mathbb{E}\exp(\beta \sup_{t\in T} U_t^1)$ is finite. For the proof of the limit result, we use the fact that

$$\mathbb{P}(X^1 \in \Gamma((x - y_t)_{t\in T})) = \mathrm{e}^{-\nu(A)}\sum_{n=1}^\infty \frac{\nu(A)^n}{n!}\nu_1^{*n}(\Gamma((x - y_t)_{t\in T})).$$

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Utilising Lemma 3.1, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\nu(A)^n}{n!}\frac{\nu_1^{*n}(\Gamma((x-y_t)_{t\in T}))}{\nu_1(\Gamma(x-\sup_{t\in T}y_t))} &\le \sum_{n=1}^{\infty}\frac{\nu(A)^n}{n!}\frac{\nu_1^{*n}(\Gamma(x-\sup_{t\in T}y_t))}{\nu_1(\Gamma(x-\sup_{t\in T}y_t))}\\ &\le \sum_{n=1}^{\infty}\frac{K^n\nu(A)^n}{n!}\frac{\nu_1(\Gamma(x-\sup_{t\in T}y_t))}{\nu_1(\Gamma(x-\sup_{t\in T}y_t))}\\ &= \sum_{n=1}^{\infty}\frac{K^n\nu(A)^n}{n!}\\ &< \infty,\end{aligned}$$

and, furthermore, from Theorem 3.2, we obtain

$$\lim_{x\to\infty}\frac{\nu_1^{*n}(\Gamma((x-y_t)_{t\in T}))}{\nu_1(\Gamma(x-\sup_t y_t))} = \frac{n}{\mathrm{e}^{\beta\sup_{t\in T}y_t}m_d(B)}\bigg(\int_B \mathrm{e}^{\beta y_s}\mathbb{E}(\mathrm{e}^{\beta U_s})^{n-1}\,\mathrm{d}s\bigg).$$

Then, dominated convergence gives

$$\begin{aligned}&\lim_{x\to\infty}\frac{\mathbb{P}(X^1\in\Gamma((x-y_t)_{t\in T}))}{\nu_1(\Gamma(x-\sup_{t\in T}y_t))}\\ &\quad= \mathrm{e}^{-\nu(A)}\frac{1}{\mathrm{e}^{\beta\sup_{t\in T}y_t}m_d(B)}\sum_{n=1}^{\infty}\frac{\nu(A)^n}{n!}n\bigg(\int_B \mathrm{e}^{\beta y_s}\mathbb{E}(\mathrm{e}^{\beta U_s})^{n-1}\,\mathrm{d}s\bigg)\\ &\quad= \nu(A)\frac{1}{\mathrm{e}^{\beta\sup_{t\in T}y_t}m_d(B)}\int_B \mathrm{e}^{\beta y_s}\exp[\nu(A)(\mathbb{E}(\mathrm{e}^{\beta U_s})-1)]\,\mathrm{d}s\\ &\quad= \nu(A)\frac{\int_B \mathrm{e}^{\beta y_s}\mathbb{E}(\mathrm{e}^{\beta X_s^1})\,\mathrm{d}s}{\mathrm{e}^{\beta\sup_{t\in T}y_t}m_d(B)},\end{aligned}$$

which with a final reference to Theorem 3.1 and the definition of ν_1 concludes the proof. □

4. The main theorem

Recall that we can write the field $(X_t)_{t\in T}$ as $X_t = X_t^1 + X_t^2$, where the fields $(X_t^1)_{t\in T}$ and $(X_t^2)_{t\in T}$ are independent, and $(X_t^1)_{t\in T}$ is a compound Poisson sum of fields with distribution ν_1. Each of the fields in the decomposition has a continuous extension to B. In Theorem 3.3 of Section 3 the extremal behaviour of X^1 was determined. In this section we shall investigate the extremal behaviour of X^2 in order to obtain the main result on X, presented in Theorem 4.2 below.

Lemma 4.1. *We have for all* $\gamma > 0$ *that* $\mathbb{E}(\exp(\gamma\sup_{t\in T}X_t^2)) < \infty$ *and*

$$\lim_{x\to\infty}\mathrm{e}^{\gamma x}\mathbb{P}(X^2\in\Gamma(x)) = 0.$$

Proof. Recall the disjoint decomposition $\nu = \nu_A + \nu_{A^c}$, where ν_A is the restriction of ν to the set A. For this proof we shall decompose ν_{A^c} such that $\nu_{A^c} = \nu_{(-\infty,-1]} + \nu_{[-1,1]}$, where $\nu_{(-\infty,-1)}$ is the restriction of ν_{A^c} to the set $\{x\in\mathbb{R}^T\colon \inf_{t\in T}x_t < -1\}$. We can write X^2 as the independent sum of two fields $X_t^2 = Z_t^1 + Z_t^2$, where (Z_t) is an infinitely divisible field with cumulant function identical to the cumulant function of (X_t) in (2.12), but with ν replaced

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by $\nu_{[-1,1]}$. Since $\nu_{(-\infty,-1)}$ is finite due to Lemma A.1, we can write the field $(Z_t^1)_{t\in T}$ as a compound Poisson sum

$$Z_t^1 = \sum_{k=1}^{M} V_t^k,$$

where M is Poisson distributed with parameter $\nu_{(-\infty,-1)}(\mathbb{R}^T)$, and the fields $(V_t^k)_{t\in T}$ are independent and having identical distributions given by a normalized version of $\nu_{(-\infty,-1)}$. Note that for ν almost all $x \in \mathbb{R}^T$ it holds that either $x_t > 0$ for all $t \in T$ or $x_t < 0$ for all $t \in T$. Thus, for each k we have $\sup_{t\in T} V_t^k \le 0$, and we see in particular that $\mathbb{E}(\exp(\gamma \sup_t Z_t^1))$ is finite for all $\gamma > 0$.

We note that both Z^1 and Z^2 have continuous extensions to B, since X^2 has a continuous extension, and Z^1 is finite sum of fields with continuous extensions. Thus, $\mathbb{P}(\sup_{t\in T}|Z_t^2| < \infty) = 1$. Since clearly $\nu_{[-1,1]}(\{x \in \mathbb{R}^T : \sup_{t\in T}|x_t| > 1\}) = 0$, it follows from [10, Lemma 2.1] that $\mathbb{E}(\exp(\gamma \sup_{t\in T}|Z_t^2|))$ is finite for all $\gamma > 0$. □

Theorem 4.1. *It holds that*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\sup_{t\in T} X_t > x)}{\nu(\Gamma(x))} = \mathbb{E}\exp(\beta X_{t_0}) \quad \text{as } x \to \infty$$

with $t_0 \in B$ arbitrarily chosen.

Proof. Let π_1 be the distribution of $(X_t^1)_{t\in T}$ and π_2 be the distribution of $(X_t^2)_{t\in T}$. First note that

$$\frac{\mathbb{P}(\sup_{t\in T} X_t > x)}{\nu(\Gamma(x))} = \frac{\mathbb{P}(\sup_{t\in T} X_t > x)}{\pi_1(\Gamma(x))}\frac{\pi_1(\Gamma(x))}{\nu(\Gamma(x))},$$

where, according to Theorem 3.3,

$$\frac{\pi_1(\Gamma(x))}{\nu(\Gamma(x))} \to \frac{\int_B \mathbb{E}(\mathrm{e}^{\beta X_s^1})\,\mathrm{d}s}{m_d(B)} \quad \text{as } x \to \infty.$$

It therefore suffices to show that

$$\frac{\mathbb{P}(\sup_{t\in T} X_t > x)}{\pi_1(\Gamma(x))} \to \frac{m_d(B)\mathbb{E}(\mathrm{e}^{\beta X_{t_0}})}{\int_B \mathbb{E}(\mathrm{e}^{\beta X_s^1})\,\mathrm{d}s} \quad \text{as } x \to \infty.$$

We have

$$\frac{\mathbb{P}(\sup_{t\in T} X_t > x)}{\pi_1(\Gamma(x))} = \int \frac{\pi_1(\Gamma((x - y_t)_{t\in T}))}{\pi_1(\Gamma(x))}\pi_2(\mathrm{d}y) = \int f(y; x)\pi_2(\mathrm{d}y),$$

say. Letting

$$f(y) = \frac{\int_B \mathrm{e}^{\beta y_s}\mathbb{E}(\mathrm{e}^{\beta X_s^1})\,\mathrm{d}s}{\int_B \mathbb{E}(\mathrm{e}^{\beta X_s^1})\,\mathrm{d}s},$$

Theorem 3.3 implies that $f(y; x) \to f(y)$ as $x \to \infty$. It remains to prove that

$$\int f(y; x)\pi_2(\mathrm{d}y) \to \int f(y)\pi_2(\mathrm{d}y) \quad \text{as } x \to \infty. \tag{4.1}$$

According to Fatou's lemma, (4.1) follows if we can find integrable nonnegative functions $g(y; x)$ and $g(y)$ such that

$$f(y; x) \le g(y; x), \tag{4.2}$$

$$g(y; x) \to g(y), \tag{4.3}$$

$$\int g(y; x)\pi_2(\mathrm{d}y) \to \int g(y)\pi_2(\mathrm{d}y). \tag{4.4}$$

For this purpose, let

$$g(y; x) = \frac{\pi_1(\Gamma(x - \sup_{t\in T} y_t))}{\pi_1(\Gamma(x))}.$$

Then, (4.2) is satisfied. Furthermore, using Theorem 3.3, we find that (4.3) is fulfilled with $g(y) = \mathrm{e}^{\beta \sup_{t\in T} y_t}$. To prove (4.4), note that

$$\int g(y; x)\pi_2(\mathrm{d}y) = \frac{\mathbb{P}(\sup_{t\in T} X_t^1 + \sup_{t\in T} X_t^2 > x)}{\pi_1(\Gamma(x))}.$$

Note that $\sup_{t\in T} X_t^1$ has a convolution equivalent tail according to Theorem 3.3 and [21, Lemma 2.4(i)]. Combining this with Lemma 4.1, [21, Lemma 2.1], and [21, Lemma 2.4(ii)] we obtain

$$\begin{aligned}\lim_{x\to\infty} \frac{\mathbb{P}(\sup_{t\in T} X_t^1 + \sup_{t\in T} X_t^2 > x)}{\pi_1(\Gamma(x))} &= \mathbb{E}\Big(\exp\Big(\beta \sup_{t\in T} X_t^2\Big)\Big) \\ &= \int \lim_{x\to\infty} \frac{\pi_1(\Gamma(x - \sup_{t\in T} y_t))}{\pi_1(\Gamma(x))} \pi_2(\mathrm{d}y).\end{aligned}$$

It follows that (4.4) is fulfilled. □

The theorem below is the main result of our paper. In the formulation of the theorem, we explicitly state the assumptions under which the limit holds.

Theorem 4.2. *Under the assumptions (2.1)–(2.6) on M and (2.8)–(2.10) on f,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\sup_{t\in B} X_t > x)}{L(x)\exp(-\beta x)} = \mathbb{E}\exp(\beta X_{t_0}) m_d(B) \quad \textit{as } x \to \infty$$

with $t_0 \in B$ arbitrarily chosen.

Proof. This follows from Theorem 4.1 and Theorem 3.1. Note that due to the continuity of X $\sup_{t\in T}$ can be replaced by $\sup_{t\in B}$. □

Example 4.1. We consider a model with a NIG basis with parameters $\alpha = 0.8$, $\beta = 0.6$, $\mu = 0.1$, $\delta = 0.1$, and an exponential kernel function with parameter $\sigma = 0.1$; see Examples 2.2 and 2.3. Furthermore, we let $B = [0, 20]^2$. The level of these parameters is—after a reparameterisation—similar to the level of the parameters estimated in [18]. In Figure 1, simulations of the probabilities $\mathbb{P}(\sup_{t\in B} X_t > x)$ based on 2500 replications of the field are plotted together with the function

$$\mathbb{E}\exp(\beta X_{t_0}) m_d(B) x^{-3/2} \exp(-(\alpha - \beta)x).$$

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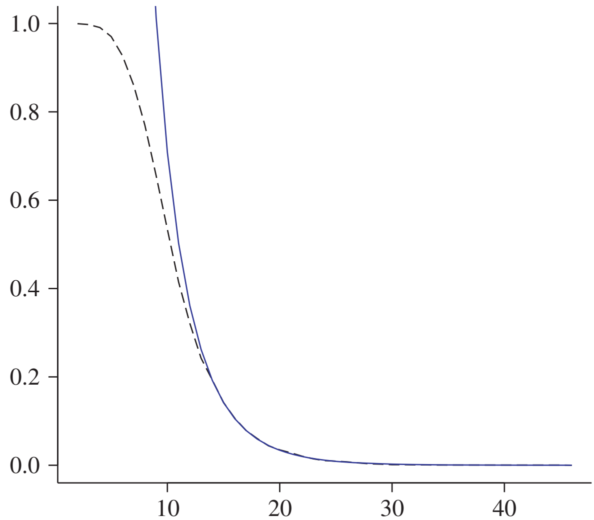

FIGURE 1: Simulated values of $\mathbb{P}(\sup_{t\in B} X_t > x)$ are plotted as a function of x (*dashed*) together with the asymptotic theoretical curve (*solid*) in the case of a NIG basis and an exponential kernel function.

5. Excursion sets

In this paper we have been focusing on the asymptotic probability that the supremum of the random field $(X_t)_{t\in B}$ exceeds a level x as $x \to \infty$. Under the assumptions of our paper, it is also possible to obtain asymptotic results for excursion sets

$$A_x = \{t \in B : X_t \geq x\}, \qquad x \in \mathbb{R}.$$

One example is the asymptotic behaviour of the probability that an excursion set contains a ball of a given size, i.e. the probability of the event

$$\Big\{\text{there exists } t_0 \in B : \inf_{s\in C_r(t_0)} X_s \geq x\Big\},$$

where $C_r(t_0)$ is the ball in $\mathbb{R}^d$ with radius r and centre t_0. Also, this probability is asymptotically described by the right tail of the Lévy measure. The proof is based on the same type of reasoning as in Sections 3 and 4 and is part of a forthcoming paper [25].

Appendix A. Continuous versions of the relevant random fields

In this appendix we make the assumptions (2.1)–(2.6) on M and (2.8)–(2.10) on f.

Theorem A.1. *There exists a continuous version of* $(X_t)_{t\in B}$.

Proof. We can write X as the independent sum $X = Y^1 + Y^2$, where Y^2 is the Gaussian part of X, such that

$$Y_t^2 = \int_{\mathbb{R}^d} f(|t-s|)M^2(\mathrm{d}s),$$

where M^2 is a Gaussian Lévy measure satisfying $C(\lambda \star M^2(A)) = \mathrm{i}\lambda^2\theta m_d(A)$. We will find continuous versions of Y^1 and Y^2 separately.

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For Y^1 we shall refer to [19, Theorem 2.1] (see also [19, Theorem 3.1] that corresponds to the case where B is one-dimensional). Note that Theorem 2.1 requires a separable field, but a separable version can be chosen for all random fields; see, e.g. [23]. Let $\tilde{B} \supseteq B$ be a box in $\mathbb{R}^d$ containing B. With a change of measure (see [22] for the existence of a Lévy–Ito decomposition of M), we can write $Y_t = Y_t^1 - \mathbb{E}Y_t^1$ in the form

$$Y_t = \int_{\mathbb{R}^d \times \mathbb{R}} x f(|t-s|)[N(\mathrm{d}s, \mathrm{d}x) - F(\mathrm{d}s, \mathrm{d}x)],$$

where N is a Poisson random measure on $\mathbb{R}^d \times \mathbb{R}$ with intensity measure F. The integral is well defined since (as is easily verified)

$$\int_{\mathbb{R}^d \times \mathbb{R}} ((xf(|t-s|))^2 \wedge |x| f(|t-s|)) F(\mathrm{d}s, \mathrm{d}x) < \infty;$$

see, e.g. [19, Section 2] and the references therein. Let D be the diameter of $\tilde{B}$. Since $\tilde{B}$ is a box in $\mathbb{R}^d$ it follows that there exists $a > 0$ such that $am_d(C_r(t) \cap \tilde{B}) \geq (r/D)^d$ for all $t \in \tilde{B}$ and $r \in (0, D)$. Using the notation from [19], we have

$$I_q(am_d, |\cdot|, \delta) = \sup_{t \in \tilde{B}} \int_0^D \left(\log \frac{1}{am_d(C_r(t) \cap \tilde{B})} \right)^{1/q} \mathrm{d}r \leq \int_0^D d(\log D - \log r)^{1/q} \, \mathrm{d}r$$

which is finite for all $q \geq 1$, $\delta \in (0, D]$, and in particular for $q = 2$ and $\delta = D$. Furthermore, we see that $\lim_{\delta \downarrow 0} I_q(am_d, |\cdot|, \delta) = 0$. From (2.10) and the mean value theorem, we find constants $C_1, C_2 > 0$ such that

$$\sup_{0 < h \leq D} \frac{|f(x+h) - f(x)|}{h} \leq \frac{C_1}{(x + C_2)^d} \quad \text{for all } x > 0.$$

Thus, with $g(t, (s, x)) = xf(|t-s|)$ (recalling (2.9)), we can find

$$\|g\|(s, x) = |x| \left(D^{-1} f(|s|) + \sup_{t, u \in \tilde{B}, t \neq u} \frac{|f(|t-s|) - f(|u-s|)|}{|t-u|} \right) \leq |x| \frac{K_3}{(|s| + K_4)^d}$$

for some constants $K_3, K_4 \geq 1$ such that, for all $c \in (0, 1]$,

$$\begin{aligned}
c^2 F(\{(s, x) \colon \|g\| \wedge 1 > c\}) &\leq c^2 F\left(\left\{ (s, x) \colon |x| \frac{K_3}{(|s| + K_4)^d} \wedge 1 > c \right\} \right) \\
&\leq c^2 F\left(\left\{ (s, x) \colon \frac{|x| K_3}{c} > |s|^d, |x| > \frac{c}{K_3} \right\} \right) \\
&= c^2 \int_{[-c/K_3, c/K_3]^c} m_d(C_{(K_3|x|/c)^{1/d}}(0)) \rho(\mathrm{d}x) \\
&\propto \int_{[-c/K_3, c/K_3]^c} |x| \left(\frac{c}{K_3} \right) \rho(\mathrm{d}x) \\
&\leq \int_0^\infty x^2 \rho(\mathrm{d}x) \\
&< \infty.
\end{aligned}$$

Furthermore, f is bounded and continuous so it follows from [19, Theorem 2.1] that $(Y_t)_{t \in \tilde{B}}$ has a version with continuous sample paths. In particular, $(Y_t^1)_{t \in B}$ has a continuous version.

The field $(Y_t^2)_{t\in B}$ has covariance function $\mathrm{cov}(Y_{t_1}^2, Y_{t_2}^2) = R(|t_2 - t_1|)$, where for some constant V,

$$R(t) = V \int_{\mathbb{R}^d} f(|s|) f(|t+s|)\,\mathrm{d}s;$$

see, e.g. [18, Section 2]. The sum $s + t$ is interpreted as adding a vector of length t and fixed direction to s. We note that R is continuous and, furthermore,

$$R(0) - R(t) \le V \int_{\mathbb{R}^d} f(|s|) | f(|t+s|) - f(|s|)|\,\mathrm{d}s \le VC'|t| \int_{\mathbb{R}^d} f(|s|)\,\mathrm{d}s,$$

where we have used the mean value theorem to obtain

$$|f(|t+s|) - f(|s|)| = |f'(\xi)|\,||t+s| - |s|| \le C'|t|$$

with $\xi \in (|s| \wedge |t+s|, |s| \vee |t+s|)$ and C' an upper bound for f' chosen according to (2.10). In particular, for given $\varepsilon > 0$ there exists $C > 0$ such that $R(0) - R(t) \le C/|\log(t)|^{1+\varepsilon}$ for all $t > 0$ smaller than the diameter of B. The existence of a continuous version of $(Y_t^2)_{t\in B}$ now follows from a corollary to [1, Theorem 3.4.1]. □

Define for all $\varepsilon > 0$ the subsets of $\mathbb{R}^T$,

$$A_\varepsilon = \{x \in \mathbb{R}^T : \sup x_t > \varepsilon\} \quad \text{and} \quad B_\varepsilon = \{x \in \mathbb{R}^T : \inf x_t < -\varepsilon\}$$

Lemma A.1. *For all $\varepsilon > 0$, we have $\nu(A_\varepsilon) < \infty$ and $\nu(B_\varepsilon) < \infty$.*

Proof. For A_ε, we obtain

$$\begin{aligned}
\nu(A_\varepsilon) &= F(\{(s,z) \in \mathbb{R}^d \times \mathbb{R} : \sup_{t\in T} z f(|t-s| > \varepsilon\}) \\
&= \int_{\mathbb{R}^d} \rho\bigg(\bigg(\frac{\varepsilon}{\sup_{t\in T} f(|t-s|)}, \infty\bigg)\bigg)\,\mathrm{d}s \\
&= \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \sup_{t\in T} f(|t-s|) \int x \mathbf{1}_{\{\varepsilon/\sup_{t\in T} f(|t-s|), \infty\}} \rho(\mathrm{d}x)\,\mathrm{d}s \\
&\le \frac{1}{\varepsilon} \bigg(\int_{\mathbb{R}^d} \sup_{t\in T} f(|t-s|)\,\mathrm{d}s\bigg)\bigg(\int_\varepsilon^\infty x\rho(\mathrm{d}x)\bigg)
\end{aligned}$$

which is finite due to (2.6) and (2.11). The proof for $\nu(B_\varepsilon) < \infty$ is identical. □

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