J. Inst. Math. Jussieu (2022), **21**(2), 675–711
doi:10.1017/S1474748020000249
Published by Cambridge University Press

GLOBAL SUBELLIPTIC ESTIMATES FOR KRAMERS–FOKKER–PLANCK OPERATORS WITH SOME CLASS OF POLYNOMIALS

MONA BEN SAID

Laboratoire Analyse, Géométrie et Applications, Université Paris 13, 99 Avenue, Jean Baptiste Clément, 93430 Villetaneuse, France (bensaid@univ-paris13.fr)

(Received 17 January 2019; revised 8 May 2020; accepted 15 May 2020; first published online 22 June 2020)

Abstract In this article, we study some Kramers–Fokker–Planck operators with a polynomial potential $V(q)$ of degree greater than two having quadratic limiting behaviour. This work provides an accurate global subelliptic estimate for Kramers–Fokker–Planck operators under some conditions imposed on the potential.

Keywords: subelliptic estimates; compact resolvent; Kramers–Fokker–Planck operator

2010 *Mathematics subject classification*: Primary 35Q84; 35H20; 47A10
Secondary 35P05; 14P10

1. Introduction and main results

The Kramers–Fokker–Planck operator reads as

$$K_V = p\partial_q - \partial_q V(q)\partial_p + \tfrac{1}{2}(-\Delta_p + p^2), \quad (q, p) \in \mathbb{R}^{2d}, \tag{1.1}$$

where q denotes the space variable, p denotes the velocity variable, $x.y = \sum_{j=1}^{d} x_j y_j$, $x^2 = \sum_{j=1}^{d} x_j^2$ and the potential $V(q) = \sum_{|\alpha|\leqslant r} V_\alpha q^\alpha$ is a real-valued polynomial function on $\mathbb{R}^d$ with $d^\circ V = r$.

There have been several works concerned with the operator K_V with diversified approaches. In this article, we impose some kinds of assumptions on the polynomial potential $V(q)$ so that the Kramers–Fokker–Planck operator K_V admits a global subelliptic estimate and has a compact resolvent. This problem is closely related to the return to equilibrium for the Kramers–Fokker–Planck operator (see [4, 13, 14]). As mentioned in [6] and [13], the analysis of K_V is also strongly linked to that of the Witten Laplacian $\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q)$. This relation yielded the Helffer–Nier conjecture stated by Helffer and Nier:

$$(1 + K_V)^{-1} \text{ compact} \Leftrightarrow (1 + \Delta_V^{(0)})^{-1} \text{ compact}. \tag{1.2}$$

This conjecture has been partially solved in basic cases (see for example [4, 6] and [11]), whereas for the operator $\Delta_V^{(0)}$ very general criteria of compactness work for polynomial potential $V(q)$ of arbitrary degree. These last criteria require an analysis of the degeneracies at infinity of the potential and rely on extremely sophisticated tools of hypoellipticity developed by Helffer and Nourrigat in the 1980s (see [5, 13]). Among the particularities of these last analyses, we mention that the compactness results obtained for degenerate potentials at infinity were not the same for $\Delta_{+V}^{(0)}$ as $\Delta_{-V}^{(0)}$. The typical example that was considered is the case $V(q_1, q_2) = q_1^2 q_2^2$ in dimension $d = 2$: The operator $\Delta_{-V}^{(0)}$ has a compact resolvent, while $\Delta_{+V}^{(0)}$ has not.

In the case of the Kramers–Fokker–Planck operator, there have been extensive works concerned with the case $d^\circ V \leqslant 2$ (see [1, 2, 7, 8, 18, 19]). Nevertheless, as far as general potential is concerned, different kinds of sufficient conditions on $V(q)$ had been examined by Hérau–Nier [6], Helffer–Nier [4], Villani [17] and Wei-Xi Li [11]. These first results considered only variants of the elliptic situation at infinity (for nondegenerate potential), which did not distinguish the sign $\pm V(q)$. Lately, a significant improvement of those works has been done by Li [12] based on some multiplier methods. In [12], Li showed that for potentials similar to $V(q_1, q_2) = q_1^2 q_2^2$, the results for $K_{\pm V}$ were the same as for $\Delta_{\pm V}^{(0)}$, thus confirming the idea that conjecture (1.2) is true.

The ultimate goal would be to develop a complete recurrence with respect to $d^\circ V$ for the Kramers–Fokker–Planck operator like it is possible to do for the Witten Laplacian as recalled in [4] (cf. Theorem 10.16, p. 106) and [13] by following the general approach of Helffer–Nourrigat in [5] and [14]. Although we are not able to write a complete induction, we establish here subelliptic estimates for K_V for a rather general class of polynomial potentials with criteria that distinguish clearly the sign of $V(q)$. The asymptotic behaviour of those polynomials is governed by at most quadratic parameter dependent potentials, and the global subelliptic estimates in which some logarithmic weights arise are known to be essentially optimal in the quadratic case (see [2]).

Denoting

$$O_p = \tfrac{1}{2}(D_p^2 + p^2),$$

and

$$X_V = p\partial_q - \partial_q V(q)\partial_p,$$

we can rewrite the Kramers–Fokker–Planck operator K_V defined in (1.1) as $K_V = X_V + O_p$.

Notations: Throughout the paper, we use the notation

$$\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}.$$

For an arbitrary polynomial $V(q)$ of degree r, we denote for all $q \in \mathbb{R}^d$

$$\mathrm{Tr}_{+,V}(q) = \sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\ V(q)) \\ \nu > 0}} \nu(q),$$

$$\mathrm{Tr}_{-,V}(q) = -\sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\ V(q)) \\ \nu \leqslant 0}} \nu(q).$$

Furthermore, for a polynomial $P \in E_r := \{P \in \mathbb{R}[X_1, \dots, X_d],\ d^\circ P \leqslant r\}$ and all natural numbers $n \in \{1, \dots, r\}$, we define the functions $R_P^{\geqslant n} : \mathbb{R}^d \to \mathbb{R}$ and $R_P^{=n} : \mathbb{R}^d \to \mathbb{R}$ by

$$R_P^{\geqslant n}(q) = \sum_{n \leqslant |\alpha| \leqslant r} \left| \partial_q^\alpha P(q) \right|^{\frac{1}{|\alpha|}}, \tag{1.3}$$

$$R_P^{=n}(q) = \sum_{|\alpha| = n} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}}. \tag{1.4}$$

For arbitrary real functions A and B, we also make use of the following notation:

$$A \asymp B \iff \exists c \geqslant 1 : c^{-1} |B| \leqslant |A| \leqslant c\, |B|.$$

This work is essentially based on the recent publication by Ben Said, Nier and Viola [2], which deals with the analysis of Kramers–Fokker–Planck operators with polynomials of degree less than 3. In this case, we define the constants A_V and B_V by

$$A_V = \max\{(1 + \mathrm{Tr}_{+,V})^{2/3}, 1 + \mathrm{Tr}_{-,V}\},$$
$$B_V = \max\left\{\min_{q \in \mathbb{R}^d} |\nabla\ V(q)|^{4/3}, \frac{1 + \mathrm{Tr}_{-,V}}{(\log(2 + \mathrm{Tr}_{-,V}))^2}\right\}.$$

As proved in [2], there is a constant $c > 0$ such that the following global subelliptic estimate with remainder,

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + A_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \geqslant c\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})}\Big), \tag{1.5}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Moreover, there exists a constant $c > 0$ such that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geqslant c\, B_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \tag{1.6}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Hence combining (1.5) and (1.6), there is a constant $c > 0$ so that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geqslant \frac{c}{1 + \frac{A_V}{B_V}}\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})}\Big) \tag{1.7}$$

is valid for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. The constants appearing in (1.5)–(1.7) are independent of the potential V and depend only on the dimension d and the degree of the polynomial V. We recall here that for a smooth potential $V \in \mathcal{C}^\infty(\mathbb{R}^d)$, our operator K_V is essentially maximal accretive when endowed with the domain $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ [4] (cf. Proposition 5.5, p. 44). As a result, the domain of its closure is given by

$$D(K_V) = \Big\{u \in L^2(\mathbb{R}^{2d}),\ K_V u \in L^2(\mathbb{R}^{2d})\Big\}.$$

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Consequently, by the density of $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ in $D(K_V)$, all estimates stated in this paper, which are checked with $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ functions, can be extended to the domain of K_V.

Given a polynomial $V(q)$ with degree r greater than two, our result will require the following assumption after setting for $\kappa > 0$,

$$\Sigma(\kappa) = \left\{ q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geqslant \kappa \Big(|\text{Hess } V(q)| + R_V^{\geqslant 3}(q)^4 + 1 \Big) \right\},$$

where $|\text{Hess } V(q)|$ is the norm of the matrix $(\partial^2_{q_i,q_j} V(q))_{1 \leqslant i,j \leqslant d}$.

Assumption 1. *There exist large constants $\kappa_0, C_1 > 1$ such that for all $\kappa \geqslant \kappa_0$, the polynomial $V(q)$ satisfies the properties*

$$\text{Tr}_{-,V}(q) > \frac{1}{C_1} \text{Tr}_{+,V}(q)\ , \quad \textit{for all } q \in \mathbb{R}^d \setminus \Sigma(\kappa) \textit{ with } |q| \geqslant C_1. \tag{1.8}$$

Moreover, if $\mathbb{R}^d \setminus \Sigma(\kappa)$ is not bounded,

$$\lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^d \setminus \Sigma(\kappa)}} \frac{R_V^{\geqslant 3}(q)^4}{|\text{Hess } V(q)|} = 0. \tag{1.9}$$

Those assumptions and in particular the partition $\mathbb{R}^d = \Sigma(\kappa) \sqcup (\mathbb{R}^d \setminus \Sigma(\kappa))$ have a simple interpretation. The region $\Sigma(\kappa)$ is the one where the gradient dominates the Hessian and the higher order derivatives so that the analysis in this region is essentially the same as in the various elliptic cases discussed in [4, 6] and [11]. On the contrary, the Hessian dominates the gradient and the derivatives of higher degree in the region $\mathbb{R}^d \setminus \Sigma(\kappa)$ and the accurate estimates of the quadratic model given by the second order Taylor expansion have to be used. Finally, the parameter κ will be adjusted at the end of the proof so that the main subelliptic estimates control the error terms due to partitions of unity and Taylor expansions. Distinguishing the sign of the potential arises in particular when the region $\mathbb{R}^d \setminus \Sigma(\kappa)$ is considered. Actually, $\text{Tr}_{+,V}$ and $\text{Tr}_{-,V}$ play different roles in the accurate subelliptic estimate without remainder (1.7) for polynomials of degree less than 3.

The Tarski–Seidenberg theorem and some of its consequences reviewed in Appendix B transform Assumption (1.9) into $R_V^{\geqslant 3}(q)^4 = O(|\text{Hess } V(q)|^{1-\nu})$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$, for some $\nu > 0$ (with $|\text{Hess } V(q)| \to +\infty$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$). Alternatively, one could simply assume from the beginning the existence of such a $\nu > 0$. We mention here that one knows that for a potential V satisfying assumption 1, the resolvent of the Witten Laplacian $\Delta_V^{(0)}$ is compact (since the asymptotic models at infinity are of degree less than 3 without a local minimum; cf. Theorem 10.16 [4]).

In Section 4, we provide some explicit families of polynomial potentials for which conditions (1.8) and (1.9) both hold.

Our main result is the following.

Theorem 1.1. *Let $V(q)$ be a polynomial of degree r greater than two verifying Assumption 1. Then there exists a strictly positive constant $C_V > 1$ (depending on V)*

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such that

$$\|K_V u\|_{L^2}^2 + C_V \|u\|_{L^2}^2 \geqslant \frac{1}{C_V}\Big(\|L(O_p)u\|_{L^2}^2 + \|L(\langle \nabla V(q)\rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle \text{Hess } V(q)\rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big) \tag{1.10}$$

holds for all $u \in D(K_V)$, *where* $L(s) = \frac{s+1}{\log(s+1)}$ *for any* $s \geqslant 1$.

Corollary 1.2. *If* $V(q)$ *is a polynomial of degree greater than two that satisfies Assumption 1, then the Kramers–Fokker–Planck operator* K_V *has a compact resolvent.*

Proof of Corollary 1.2. Assume $0 < \delta < 1$. Define the functions $f_\delta : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\delta(q) = |\nabla V(q)|^{\frac{4}{3}(1-\delta)} + |\text{Hess } V(q)|^{1-\delta}.$$

From (1.10) in Theorem 1.1, there is a constant $C_V > 1$ such that

$$\|K_V u\|_{L^2}^2 + C_V \|u\|_{L^2}^2 \geqslant \frac{1}{C_V}\Big(\langle u, f_\delta u\rangle + \|L(O_p)u\|_{L^2}^2 + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big)$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and all $\delta \in (0,1)$. In order to prove that the operator K_V has a compact resolvent, it is sufficient to show that $\lim_{|q|\to+\infty} f_\delta(q) = +\infty$.

To do so, assume $A > 0$ and denote $\kappa = A^{\frac{1}{1-\delta}}$. If $q \in \Sigma(\kappa)$, one has

$$|\nabla V(q)|^{\frac{4}{3}(1-\delta)} \geqslant \kappa^{1-\delta} = A.$$

If $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ by (1.9) in Assumption 1, $\lim_{\substack{|q|\to+\infty \\ q\in\mathbb{R}^d\setminus\Sigma(\kappa)}} |\text{Hess } V(q)| = +\infty$. Hence there exists a constant $\eta > 0$ such that $|\text{Hess } V(q)|^{1-\delta} \geqslant A$ for all $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ with $|q| \geqslant \eta$. □

Remark 1.3. *The results of Theorem 1.1 and Corollary 1.2 can be extended in the case* $V = V_1 + V_2$, *where* V_1 *is a polynomial satisfying Assumption 1 and* V_2 *is a function in* $\mathcal{S}(\mathbb{R}^d)$.

2. Preliminary results

This work is essentially based on two main strategies. The first one consists in the use of a partition of unity, which is the most important tool that allows one to pass from local to global estimates.

In this paper, given a polynomial $V(q)$, we make use of a locally finite partition of unity with respect to the position variable $q \in \mathbb{R}^d$

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \tilde{\chi}_j^2\Big(R_V^{\geqslant 3}(q_j)(q - q_j)\Big) = 1, \tag{2.1}$$

where

$$\text{supp } \tilde{\chi}_j \subset B(0,a) \text{ and } \tilde{\chi}_j \equiv 1 \text{ in } \quad B(0,b)$$

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for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of $j \in \mathbb{N}$. In our work, we need to choose the constant a less than or equal to $\min(C^{-1}, C'^{-1})$, where the constants C and C' are those in Lemma A.5. Such a partition is described more precisely in Lemma A.8 after taking $n = 3$. In our study, introducing this partition yields errors that are under control.

The second approach lies in the decomposition of the operator K_V onto two parts so that the first one is a Kramers–Fokker–Planck operator with polynomial potential of degree less than three. In this way, based on [2], we derive the result of Theorem 1.1.

In order to prove Theorem 1.1, we need the following basic lemmas.

Lemma 2.1. *Assume $V \in \mathbb{R}[q_1, \dots, q_d]$ with degree $r \in \mathbb{N}$. Consider the Kramers-Fokker–Planck operator K_V defined as in (1.1). For a locally finite partition of unity, namely $\sum_{j\in\mathbb{N}} \chi_j^2(q) = 1$, one has*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} = \sum_{j\in\mathbb{N}} \Big(\|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - \|(p\partial_q \chi_j)u\|^2_{L^2(\mathbb{R}^{2d})} \Big) \tag{2.2}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

In particular, when the degree of V is larger than two and the cutoff functions χ_j have the form (2.1), there exists a constant $c_{d,r} > 0$ (depending only on the dimension d and the degree of V) so that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geqslant \sum_{j\in\mathbb{N}} \Big(\|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - c_{d,r} R_V^{\geqslant 3}(q_j)^2 \|p\chi_j u\|^2_{L^2(\mathbb{R}^{2d})} \Big) \tag{2.3}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Proof. First, let V be a real-valued polynomial on $\mathbb{R}^d$ of degree $r \in \mathbb{N}$. Assume that $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. On the one hand,

$$\|K_V u\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle K_V u, \chi_j^2 K_V u\rangle = \sum_{j\in\mathbb{N}} \langle u, K_V^* \chi_j^2 K_V u\rangle.$$

On the other hand,

$$\sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle u, \chi_j K_V^* K_V \chi_j u\rangle.$$

Putting the above equalities together,

$$\|K_V u\|^2_{L^2} - \sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle u, (K_V^* \chi_j^2 K_V - \chi_j K_V^* K_V \chi_j)u\rangle.$$

Using commutators, we compute

$$\begin{aligned} K_V^* \chi_j^2 K_V &= K_V^* \chi_j [\chi_j, K_V] + K_V^* \chi_j K_V \chi_j \\ &= K_V^* \chi_j [\chi_j, K_V] + [K_V^*, \chi_j] K_V \chi_j + \chi_j K_V^* K_V \chi_j \\ &= K_V^* \chi_j [\chi_j, K_V] + [K_V^*, \chi_j]\Big([K_V, \chi_j] + \chi_j K_V\Big) + \chi_j K_V^* K_V \chi_j. \end{aligned}$$

Thus

$$K_V^*\chi_j^2K_V-\chi_jK_V^*K_V\chi_j=K_V^*\chi_j[\,\chi_j,K_V\,]+[\,K_V^*,\chi_j\,]\chi_jK_V+[\,K_V^*,\chi_j\,]\circ[\,K_V,\chi_j\,].$$

Now it is easy to check the following commutation relations:

$$\begin{cases}[\,\chi_j,K_V\,]=-[\,K_V,\chi_j\,]=-[\,p\partial_q,\chi_j(q)\,]=-p\partial_q\chi_j\\ [\,K_V^*,\chi_j\,]=[\,-p\partial_q,\chi_j(q)\,]=-p\partial_q\chi_j\\ [\,K_V^*,\chi_j\,]\circ[\,K_V,\chi_j\,]=-(p\partial_q\chi_j)^2.\end{cases}$$

Collecting the terms, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{N}}(K_V^*\chi_j^2K_V-\chi_jK_V^*K_V\chi_j)&=\sum_{j\in\mathbb{N}}\Big(K_V^*\chi_j(-p\partial_q\chi_j)+(-p\partial_q\chi_j)\chi_jK_V-(p\partial_q\chi_j)^2\Big)\\&=\sum_{j\in\mathbb{N}}\Big(K_V^*\Big(-p\partial_q\Big(\frac{\chi_j^2}{2}\Big)\Big)-p\partial_q\Big(\frac{\chi_j^2}{2}\Big)K_V-(p\partial_q\chi_j)^2\Big)\\&=-\sum_{j\in\mathbb{N}}(p\partial_q\chi_j)^2,\end{aligned}$$

where in the last line, we make use of $\sum_{j\in\mathbb{N}}\chi_j^2(q)=1$.

From this, it follows immediately that

$$\|K_Vu\|_{L^2}^2=\sum_{j\in\mathbb{N}}\Big(\|K_V(\chi_ju)\|_{L^2}^2-\|(p\partial_q\chi_j)u\|_{L^2}^2\Big)$$

for all $u\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Next, suppose that the degree of V is greater than two and $\chi_j(q)=\widetilde{\chi}_j(R_V^{\geqslant 3}(q_j)(q-q_j))$ for all indices j and any $q\in\mathbb{R}^d$ with

$$\operatorname{supp}\widetilde{\chi}_j\subset B(0,a)\quad\text{and}\quad\widetilde{\chi}_j\equiv 1\quad\text{in }B(0,b).$$

Then we can write

$$\begin{aligned}\sum_{j\in\mathbb{N}}\|(p\partial_q\chi_j)u\|^2&=\sum_{j\in\mathbb{N}}\sum_{j'\in\mathbb{N}}\|(p\partial_q\chi_j)\chi_{j'}u\|^2\\&\leqslant c_{d,r}\sum_{j\in\mathbb{N}}R_V^{\geqslant 3}(q_j)^2\|p\chi_ju\|^2,\end{aligned}$$

where $c_{d,r}$ is a constant that depends only on the dimension d and the degree of V. Here the last inequality is due to the fact that for each index j, there are finitely many j' such that $(\partial_q\chi_j)\chi_{j'}$ is nonzero. □

Before stating the following lemma, we fix and recall some notations.

Notations 2.2. *Let V be a polynomial of degree r larger than two. Consider a locally finite partition of unity $\sum_{j\in\mathbb{N}}\chi_j^2(q)=1$ described as in (2.1).*

Set for all $\kappa>0$

$$J(\kappa)=\Big\{j\in\mathbb{N},\ \textit{such that }\operatorname{supp}\chi_j\subset\Sigma(\kappa)\Big\},$$

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where we recall that

$$\Sigma(\kappa) = \left\{ q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geqslant \kappa \Big(|\mathrm{Hess}\ V(q)| + R_V^{\geqslant 3}(q)^4 + 1 \Big) \right\}.$$

For a given $\kappa > 0$ *and all indices* $j \in \mathbb{N}$, *let* $V_j^{(2)}$ *be the polynomial of degree less than three given by*

$$V_j^{(2)}(q) = \sum_{0 \leqslant |\alpha| \leqslant 2} \frac{\partial_q^\alpha V(q_j')}{\alpha!} (q - q_j')^\alpha, \tag{2.4}$$

where

$$\begin{cases} q_j' = q_j & \text{if } j \in J(\kappa) \\ q_j' \in (\mathrm{supp}\ \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa) \Big) & \text{otherwise.} \end{cases}$$

Lemma 2.3. *Assume that* V *is a polynomial of degree* r *larger than two. Consider a locally finite partition of unity described as in (2.1). For a multi-index* $\alpha \in \mathbb{N}^d$ *of length* $|\alpha| \in \{1, 2\}$ *and all* $j \in \mathbb{N}$, *one has*

$$\left| \partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q) \right| \leqslant c_{\alpha,d,r} \Big(R_V^{\geqslant 3}(q_j') \Big)^{|\alpha|} \tag{2.5}$$

for any $q \in \mathrm{supp}\ \chi_j = B(q_j, a R_V^{\geqslant 3}(q_j)^{-1})$, *where* $c_{\alpha,d,r} = \sum_{3 \leqslant |\beta| \leqslant r} (a\, C)^{|\beta| - |\alpha|}$.

As a consequence, if V *satisfies Assumption 1, there exists a large constant* $\kappa_1 \geqslant \kappa_0$ *so that for all* $\kappa \geqslant \kappa_1$ *and every* $j \in \mathbb{N}$,

$$2^{-1} \left| \partial_q V_j^{(2)}(q) \right| \leqslant \left| \partial_q V(q) \right| \leqslant 2 \left| \partial_q V_j^{(2)}(q) \right| , \tag{2.6}$$

for every $q \in (\mathrm{supp}\ \chi_j) \cap \Sigma(\kappa)$ *and*

$$2^{-1} \left| \mathrm{Hess}\ V_j^{(2)}(q) \right| \leqslant |\mathrm{Hess}\ V(q)| \leqslant 2 \left| \mathrm{Hess}\ V_j^{(2)}(q) \right| , \tag{2.7}$$

for any $q \in (\mathrm{supp}\ \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa) \Big)$ *with* $|q| \geqslant C_2(\kappa)$, *where* $C_2(\kappa) > 0$ *is a large constant that depends on* κ.

Proof. Let V be a polynomial of degree r greater than two. In this proof, we are going to need the following equivalence,

$$R_V^{\geqslant 3}(q) \asymp R_V^{\geqslant 3}(q') , \tag{2.8}$$

satisfied for all $q, q' \in \mathrm{supp}\ \chi_j$ and proved in Lemma A.5. That is, there is a constant $C > 1$ such that for every $q, q' \in \mathrm{supp}\ \chi_j$,

$$\left(\frac{R_V^{\geqslant 3}(q)}{R_V^{\geqslant 3}(q')} \right)^{\pm 1} \leqslant C. \tag{2.9}$$

Assume $\alpha \in \mathbb{N}^d$ of length $|\alpha| \in \{1, 2\}$. For every $j \in \mathbb{N}$, observe that

$$\left|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)\right| = \left|\sum_{\substack{3\leqslant|\beta|\leqslant r\\ \beta\geqslant\alpha}} \frac{\partial_q^\beta V(q_j')}{(\beta-\alpha)!}(q-q_j')^{\beta-\alpha}\right|$$
$$\leqslant \sum_{\substack{3\leqslant|\beta|\leqslant r\\ \beta\geqslant\alpha}} \frac{\left|\partial_q^\beta V(q_j')\right|}{(\beta-\alpha)!}\,|q-q_j'|^{|\beta|-|\alpha|}$$

for any $q \in \mathbb{R}^d$. Hence regarding equivalence (2.9), there exists a constant $c_{\alpha,d,r} > 0$ (depending as well on the multi-index α, the dimension d and the degree r of V) so that

$$\left|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)\right| \leqslant \sum_{\substack{3\leqslant|\beta|\leqslant r\\ \beta\geqslant\alpha}} \frac{1}{(\beta-\alpha)!}\left(R_V^{\geqslant 3}(q_j')\right)^{|\beta|}\left(a^{-1}R_V^{\geqslant 3}(q_j)\right)^{-|\beta|+|\alpha|}$$
$$\leqslant \sum_{\substack{3\leqslant|\beta|\leqslant r\\ \beta\geqslant\alpha}} \frac{1}{(\beta-\alpha)!}(a\,C)^{|\beta|-|\alpha|}\left(R_V^{\geqslant 3}(q_j')\right)^{|\alpha|}$$
$$\leqslant c_{\alpha,d,r}\left(R_V^{\geqslant 3}(q_j')\right)^{|\alpha|} \tag{2.10}$$

holds for all q in the support of χ_j, where the constant $C > 1$ is the one in (2.9) and $c_{\alpha,d,r} = \sum_{3\leqslant|\beta|\leqslant r}(a\,C)^{|\beta|-|\alpha|}$.

In the rest of the proof, let the polynomial $V(q)$ satisfy Assumption 1. In view of (2.10), when $|\alpha| = 1$, we get

$$|\nabla V(q) - \nabla V_j^{(2)}(q)| \leqslant c_{1,d,r}\ R_V^{\geqslant 3}(q_j') \tag{2.11}$$

for all $j \in \mathbb{N}$ and any $q \in \operatorname{supp}\chi_j$, where $c_{1,d,r} = \sum_{3\leqslant|\beta|\leqslant r}(a\,C)^{|\beta|-1}$. By virtue of the equivalence (2.9), it results from (2.11)

$$\left|\nabla V(q) - \nabla V_j^{(2)}(q)\right| \leqslant c_{1,d,r}C\ R_V^{\geqslant 3}(q) \tag{2.12}$$

for every $q \in \operatorname{supp}\chi_j$. Given $\kappa \geqslant \kappa_0$, it follows from (2.12) and the definition of $\Sigma(\kappa)$ that

$$\left|\nabla V(q) - \nabla V_j^{(2)}(q)\right| \leqslant \frac{c_{1,d,r}C}{\kappa^{\frac{1}{4}}}\,|\nabla V(q)|^{\frac{1}{3}}$$
$$\leqslant \frac{c_{1,d,r}C}{\kappa^{\frac{1}{4}}}\,|\nabla V(q)| \tag{2.13}$$

for all $q \in (\operatorname{supp}\chi_j)\cap\Sigma(\kappa)$. For the above second inequality, we know that $|\nabla V(q)| \geqslant 1$ for every $q \in (\operatorname{supp}\chi_j)\cap\Sigma(\kappa)$. Indeed, since $q \in (\operatorname{supp}\chi_j)\cap\Sigma(\kappa)$,

$$|\nabla V(q)| \geqslant \kappa^{\frac{3}{4}} \geqslant \kappa_0^{\frac{3}{4}} \geqslant 1.$$

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Taking the constant $\kappa_1 \geqslant \kappa_0$ such that $\frac{c_{1,d,r}C}{\kappa_1^{\frac{1}{4}}} \leqslant \frac{1}{2}$, we get for every $\kappa \geqslant \kappa_1$

$$\Big| |\nabla V(q)| - |\nabla V_j^{(2)}(q)| \Big| \leqslant |\nabla V(q) - \nabla V_j^{(2)}(q)| \leqslant \tfrac{1}{2}|\nabla V(q)|$$

for any $q \in (\text{supp } \chi_j) \cap \Sigma(\kappa)$. Therefore, for every $\kappa \geqslant \kappa_1$,

$$\tfrac{1}{2}|\nabla V_j^{(2)}(q)| \leqslant |\nabla V(q)| \leqslant \tfrac{3}{2}|\nabla V_j^{(2)}(q)|$$

holds for all $q \in (\text{supp } \chi_j) \cap \Sigma(\kappa)$.

On the other hand when $|\alpha| = 2$, by (2.10), there is a constant $c_{2,d,r} > 0$ so that for all $j \in \mathbb{N}$,

$$|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)| \leqslant c_{2,d,r} R_V^{\geqslant 3}(q_j')^2 \tag{2.14}$$

holds for every $q \in \text{supp } \chi_j$, where $c_{2,d,r} = \sum_{3 \leqslant |\beta| \leqslant r} (a\, C)^{|\beta|-2}$.

Using the fact that $R_V^{\geqslant 3}(q) \geqslant R_V^{=r}(0)$ for every $q \in \mathbb{R}^d$, we derive from (2.14) that

$$|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)| \leqslant c_{2,d,r} \frac{R_V^{\geqslant 3}(q_j')^4}{R_V^{=r}(0)^2}$$

for all $q \in \text{supp } \chi_j$.

Assuming $\kappa \geqslant \kappa_0$, we obtain using (1.9) in Assumption 1, if $|q_j'|$ is large enough

$$\Big| \sum_{|\beta|=2} |\partial_q^\beta V(q)| - \sum_{|\beta|=2} |\partial_q^\beta V_j^{(2)}(q)| \Big| \leqslant \sum_{|\beta|=2} |\partial_q^\beta V(q) - \partial_q^\beta V_j^{(2)}(q)| \leqslant \frac{1}{2}|\text{Hess } V(q_j')|$$

for any $q \in (\text{supp } \chi_j) \cap (\mathbb{R}^d \setminus \Sigma(\kappa))$. In other words,

$$\tfrac{1}{2}|\text{Hess } V_j^{(2)}(q)| \leqslant |\text{Hess } V(q)| \leqslant \tfrac{3}{2}|\text{Hess } V_j^{(2)}(q)|$$

holds for all $q \in (\text{supp } \chi_j) \cap (\mathbb{R}^d \setminus \Sigma(\kappa))$ with $|q| \geqslant C_2(\kappa)$, where $C_2(\kappa)$ is a strictly positive large constant depending on κ. □

Lemma 2.4. *Consider two positive operators A and B on a Hilbert space $\mathcal{H}$ such that*

$$\nu\|u\|^2 < \langle u, Au\rangle \leqslant \langle u, Bu\rangle$$

for all $u \in \mathcal{D}$, where $\mathcal{D}$ is dense in $D(B^{1/2})$ with $\nu > 1$. For all $\alpha_0 \in [0, 1]$ and $k \in \mathbb{N}$, there exists $C_{k,\alpha_0,\nu} > 1$ such that

$$\left\langle u, \frac{A^{\alpha_0}}{(\log(A^{\alpha_0/2}))^k} u \right\rangle \leqslant C_{k,\alpha_0,\nu} \left\langle u, \frac{B^{\alpha_0}}{(\log(B^{\alpha_0/2}))^k} u \right\rangle \tag{2.15}$$

for any $u \in \mathcal{D}$.

Proof. Assume that A, B are two positive operators so that

$$\nu\|u\|^2 < \langle u, Au\rangle \leqslant \langle u, Bu\rangle \tag{2.16}$$

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holds for all $u \in \mathcal{D}$ with $\nu > 1$. For $\alpha \in [0,1]$, the application $T \mapsto T^\alpha$ is operator monotone according to [16, Example 6.8]. This provides the inequality

$$\nu^\alpha \|u\|^2 < \langle u, A^\alpha u\rangle \leqslant \langle u, B^\alpha u\rangle \tag{2.17}$$

for any $u \in \mathcal{D}$ and every $\alpha \in [0,1]$, which is of course related with interpolation in Hilbert spaces.

Furthermore, for any positive operator $C \geqslant c\,\mathrm{Id}_{\mathcal{H}}$, $c > 0$, with domain $D(C)$, the logarithm of C (defined by the functional calculus) satisfies for all $u \in D(C)$ and $v \in \mathcal{H}$:

$$\langle v, \log(C)u\rangle = \lim_{\alpha\to 0^+} \left\langle v, \frac{C^\alpha - 1}{\alpha} u\right\rangle. \tag{2.18}$$

Using (2.16),

$$\underbrace{\log(\nu)}_{>0} \|u\|^2 < \langle u, \log(A)u\rangle \leqslant \langle u, \log(B)u\rangle \tag{2.19}$$

holds for all $u \in \mathcal{D}$. Integrating (2.17) with respect to α over $[0, \alpha_0]$, where $\alpha_0 \in [0,1]$, we get

$$\left\langle u, \frac{1}{\log(A)}(A^{\alpha_0} - I)u\right\rangle \leqslant \left\langle u, \frac{1}{\log(B)}(B^{\alpha_0} - I)u\right\rangle. \tag{2.20}$$

Furthermore, by (2.19),

$$\left\langle u, \frac{1}{\log(B)}u\right\rangle \leqslant \left\langle u, \frac{1}{\log(A)}u\right\rangle < \frac{1}{\log(\nu)}\|u\|^2. \tag{2.21}$$

Therefore from (2.20) and (2.21), for any $\alpha_0 \in [0,1]$, there exist $C_\nu, C_{1,\alpha_0,\nu} > 1$ such that

$$\left\langle u, \frac{A^{\alpha_0}}{\log(A)}u\right\rangle \leqslant \left\langle u, \frac{B^{\alpha_0}}{\log(B)}u\right\rangle + C_\nu\|u\|^2 \leqslant C_{1,\alpha_0,\nu}\left\langle u, \frac{B^{\alpha_0}}{\log(B)}u\right\rangle.$$

Once the constant $C_{k,\alpha_0,\nu} \geqslant 1$ is known for $k \geqslant 1$, the same integration with respect to $\alpha \in [0, \alpha_0]$ provides the constant $C_{k+1,\alpha_0,\nu} \geqslant 1$. We proved by induction on $k \in \mathbb{N}$, the existence of a constant $C_{k,\alpha_0,\nu} > 1$ such that

$$\left\langle u, \frac{A^{\alpha_0}}{(\log(A))^k}u\right\rangle \leqslant C_{k,\alpha_0,\nu}\left\langle u, \frac{B^{\alpha_0}}{(\log(B))^k}u\right\rangle,$$

or equivalently

$$\left\langle u, \frac{A^{\alpha_0}}{(\log(A^{\alpha_0/2}))^k}u\right\rangle \leqslant C_{k,\alpha_0,\nu}\left\langle u, \frac{B^{\alpha_0}}{(\log(B^{\alpha_0/2}))^k}u\right\rangle. \qquad \square$$

Lemma 2.5. *Assume that $V(q)$ is a polynomial of degree r greater than two. Let $\sum_{j\in\mathbb{N}} \chi_j^2(q)$ be a locally finite partition of unity defined as in (2.1). For each $j \in \mathbb{N}$, choose any $q'_j \in \operatorname{supp} \chi_j$.*

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There is a constant $c = c(d, r) > 1$ such that

$$\langle u, (2 - \Delta_q + R_V^{\geqslant 3}(q)^4)^\alpha u\rangle \leqslant c \sum_{j\in\mathbb{N}} \langle u, \chi_j (2 - \Delta_q + R_V^{\geqslant 3}(q'_j)^4)^\alpha \chi_j u\rangle \tag{2.22}$$

is valid for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0, 1]$.

As a consequence, there exists a constant $\widetilde{c} = \widetilde{c}(d, r) > 1$ so that

$$\sum_{j\in\mathbb{N}} \left\| L\Big((1 - \Delta_q + R_V^{\geqslant 3}(q'_j)^4)^{\frac{1}{3}}\Big)\chi_j u\right\|^2 \geqslant \frac{1}{c} \left\| L\Big((1 - \Delta_q + R_V^{\geqslant 3}(q)^4)^{\frac{1}{3}}\Big)u\right\|^2 \tag{2.23}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, where $L(s) = \frac{s+1}{\log(s+1)}$ for all $s \geqslant 1$.

Proof. We first set $E_0 = L^2(\mathbb{R}^{2d})$ and $E_1 = \{u \in L^2(\mathbb{R}^{2d}),\ \langle u, (2 - \Delta_q + R_V^{\geqslant 3}(q)^4)u\rangle < +\infty\}$ endowed respectively with the norms $\|\cdot\|_{E_0} = \|\cdot\|_{L^2(\mathbb{R}^{2d})}$ and $\|\cdot\|_{E_1}$ defined as follows for all $u \in L^2(\mathbb{R}^{2d})$:

$$\begin{aligned}\|u\|_{E_1}^2 &= 2\|u\|_{L^2(\mathbb{R}^{2d})}^2 + \|D_q u\|_{L^2(\mathbb{R}^{2d})}^2 + \|R_V^{\geqslant 3}(q)^2 u\|_{L^2(\mathbb{R}^{2d})}^2\\ &= \|(2 - \Delta_q + R_V^{\geqslant 3}(q)^4)^{1/2} u\|_{L^2(\mathbb{R}^{2d})}^2.\end{aligned}$$

By [15, Theorem X.29], the operator $2 - \Delta_q + R_V^{\geqslant 3}(q)^4$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and hence E_1 corresponds to the spectrally defined subspace of $L^2(\mathbb{R}^{2d})$.

Given a partition of unity as in (2.1), define the linear map

$$T : E_0 \to (L^2(\mathbb{R}^{2d}))^{\mathbb{N}},\ u \mapsto (u_j)_{j\in\mathbb{N}} = (\chi_j u)_{j\in\mathbb{N}},$$

and denote $F_0 := \operatorname{Im} T$. Note that $T : E_0 \to F_0$ is unitary. Indeed, for all $u \in E_0$,

$$\|Tu\|_{F_0}^2 = \sum_{j\in\mathbb{N}} \|\chi_j u\|_{L^2}^2 = \|u\|_{L^2}^2 = \|u\|_{E_0}^2. \tag{2.24}$$

Further, the inverse map of T is well defined by

$$T^{-1} : F_0 \to E_0,\ (u_j)_{j\in\mathbb{N}} \mapsto u = \sum_{j\in\mathbb{N}} \chi_j u_j.$$

Now introduce the set

$$F_1 = \left\{(u_j)_{j\in\mathbb{N}} \in F_0,\ \sum_{j\in\mathbb{N}} \langle u_j, (2 - \Delta_q + R_V^{\geqslant 3}(q'_j)^4)u_j\rangle < +\infty\right\},$$

with its associated norm defined for all $(u_j)_{j\in\mathbb{N}} \in F_1$ by

$$\begin{aligned}\|(u_j)_{j\in\mathbb{N}}\|_{F_1}^2 &= \sum_{j\in\mathbb{N}} \Big(2\|u_j\|_{L^2(\mathbb{R}^{2d})}^2 + \|D_q u_j\|_{L^2(\mathbb{R}^{2d})}^2 + \|R_V^{\geqslant 3}(q'_j)^2 u_j\|_{L^2(\mathbb{R}^{2d})}^2\Big)\\ &= \sum_{j\in\mathbb{N}} \|(2 - \Delta_q + R_V^{\geqslant 3}(q'_j)^4)^{1/2} u_j\|_{L^2(\mathbb{R}^{2d})}^2.\end{aligned}$$

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Assume $u \in E_0$. For all $j \in \mathbb{N}$, let $q'_j \in \operatorname{supp} \chi_j$. Observe that

$$\begin{aligned} |\, \|Tu\|^2_{F_1} - \|u\|^2_{E_1} | &= \Big| \sum_{j\in\mathbb{N}} \langle u_j, (2-\Delta_q + R_V^{\geqslant 3}(q'_j)^4) u_j\rangle - \langle u, (2-\Delta_q + R_V^{\geqslant 3}(q)^4)u\rangle \Big| \\ &= \Big| \sum_{j\in\mathbb{N}} \langle u_j, -\Delta_q u_j\rangle - \langle u, -\Delta_q u\rangle + \sum_{j\in\mathbb{N}} \langle u_j, (R_V^{\geqslant 3}(q'_j)^4 - R_V^{\geqslant 3}(q)^4) u_j\rangle \Big| \\ &\leqslant \Big| \sum_{j\in\mathbb{N}} \langle u_j, -\Delta_q u_j\rangle - \langle u, -\Delta_q u\rangle \Big| + \sum_{j\in\mathbb{N}} \langle u_j, |R_V^{\geqslant 3}(q'_j)^4 - R_V^{\geqslant 3}(q)^4| u_j\rangle. \end{aligned} \tag{2.25}$$

Since we are dealing with cutoff functions satisfying $\sum_{j\in\mathbb{N}} |\nabla \chi_j|^2 \leqslant c\, R_V^{\geqslant 3}(q)^2 \leqslant c\, \frac{R_V^{\geqslant 3}(q)^4}{R_V^{=r}(0)^2}$ and owing to the equivalence $R_V^{\geqslant 3}(q) \asymp R_V^{\geqslant 3}(q'_j)$ for all $q \in \operatorname{supp} \chi_j$, it follows from (2.25)

$$|\, \|Tu\|^2_{F_1} - \|u\|^2_{E_1} | \leqslant c_1 \sum_{j\in\mathbb{N}} \langle u_j, R_V^{\geqslant 3}(q'_j)^4 u_j\rangle \leqslant c_1 \|Tu\|^2_{F_1}$$

and

$$|\, \|Tu\|^2_{F_1} - \|u\|^2_{E_1} | \leqslant c'_1 \langle u, R_V^{\geqslant 3}(q)^4 u\rangle \leqslant c'_1 \|u\|^2_{E_1},$$

where c_1, c'_1 are two strictly positive constants. As a result,

$$\frac{1}{\sqrt{(c_1+1)}} \|u\|_{E_1} \leqslant \|Tu\|_{F_1} \leqslant \sqrt{(c'_1+1)} \|u\|_{E_1}. \tag{2.26}$$

In view of (2.24) and (2.26), we conclude by interpolation that for all $\alpha \in [0, 1]$,

$$T : E_\alpha \to F_\alpha$$

verifies $\|T\|_{\mathcal{L}(E_\alpha, F_\alpha)} \leqslant (c'_1+1)^{\frac{\alpha}{2}}$ and $\|T^{-1}\|_{\mathcal{L}(F_\alpha, E_\alpha)} \leqslant (c_1+1)^{\frac{\alpha}{2}}$, where E_α and F_α are two interpolated spaces endowed respectively with the norms

$$\|u\|_{E_\alpha} = \|(2-\Delta_q + R_V^{\geqslant 3}(q)^4)^{\alpha/2} u\|_{L^2(\mathbb{R}^{2d})}$$

and

$$\|(v_j)_{j\in\mathbb{N}}\|_{F_\alpha} = \sum_{j\in\mathbb{N}} \|(2-\Delta_q + R_V^{\geqslant 3}(q'_j)^4)^{\alpha/2} u_j\|_{L^2(\mathbb{R}^{2d})}.$$

Hence there is a constant $c > 0$ so that

$$\langle u, (2-\Delta_q + R_V^{\geqslant 3}(q)^4)^\alpha u\rangle \leqslant c \sum_{j\in\mathbb{N}} \langle u_j, (2-\Delta_q + R_V^{\geqslant 3}(q'_j)^4)^\alpha u_j\rangle \tag{2.27}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0, 1]$. In order to prove (2.23), repeat the same process as in Lemma 2.4. Starting with

$$2^\alpha \|u\|^2 \leqslant \langle u, (2-\Delta_q + R_V^{\geqslant 3}(q)^4)^\alpha u\rangle \leqslant c \sum_{j\in\mathbb{N}} \langle u_j, (2-\Delta_q + R_V^{\geqslant 3}(q'_j)^4)^\alpha u_j\rangle, \tag{2.28}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0,1]$, use the functional calculus on the left-hand side and the Fourier transform on the right-hand side. When integrating with respect to $\alpha \in [0, \frac{2}{3}]$, we can interchange for any fixed $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ the sum and the integral on the right-hand side of (2.28) since the partition of unity is locally finite. This leads to

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}, \quad \langle u, \phi(1-\Delta_q + R_V^{\geqslant 3}(q)^4)u\rangle \leqslant c' \sum_{j\in\mathbb{N}} \langle u, \phi(1-\Delta_q + R_V^{\geqslant 3}(q'_j)^4)u\rangle$$

with $\phi(t) = \frac{(1+t)^{1/3}}{\log((1+t)^{1/3}}$. By referring again to the functional calculus for the left-hand side and the Fourier transform for the right-hand side, the proof is finished after noting the uniform equivalence

$$\sup_{t\in[1,+\infty)} \left(\frac{\phi(t)}{\psi(t)}\right)^{\pm 1} \leqslant \mu$$

when $\psi(t) = \frac{1+t^{1/3}}{\log(1+t^{1/3}}$. □

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. In the sequel, for a given polynomial $V(q)$ with degree r greater than two, we always use a locally finite partition of unity

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \widetilde{\chi}_j^2\Big(R_V^{\geqslant 3}(q_j)(q-q_j)\Big) = 1,$$

where

$$\operatorname{supp} \widetilde{\chi}_j \subset B(0,a) \quad \text{and} \quad \widetilde{\chi}_j \equiv 1 \text{ in } B(0,b)$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of the natural numbers j, defined more specifically as in Lemma A.8 with $n = 3$. As mentioned before, we choose the constant a less than or equal to $\min(C^{-1}, C'^{-1})$, where the constants C and C' are those in Lemma A.5.

Proof. Let $V(q)$ be a polynomial with degree larger than two that satisfies Assumption 1. Assume $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. In the whole proof, we denote $u_j = \chi_j u$ for all natural numbers j.

From Lemma 2.1, we get

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geqslant \sum_{j\in\mathbb{N}} \Big(\|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_{d,r} R_V^{\geqslant 3}(q_j)^2 \|p u_j\|^2_{L^2(\mathbb{R}^{2d})}\Big). \tag{3.1}$$

Given $\kappa \geqslant \kappa_0$, set

$$J(\kappa) = \{j \in \mathbb{N}, \text{ such that } \operatorname{supp} \chi_j \subset \Sigma(\kappa)\}.$$

For all indices $j \in \mathbb{N}$, let $V_j^{(2)}$ be the polynomial of degree less than three given by

$$V_j^{(2)}(q) = \sum_{0\leqslant|\alpha|\leqslant 2} \frac{\partial_q^\alpha V(q'_j)}{\alpha!}(q-q'_j)^\alpha,$$

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where

$$\begin{cases} q'_j = q_j & \text{if } j \in J(\kappa) \\ q'_j \in (\operatorname{supp} \chi_j) \cap \left(\mathbb{R}^d \setminus \Sigma(\kappa)\right) & \text{otherwise.} \end{cases}$$

We associate with each polynomial $V_j^{(2)}$ the Kramers–Fokker–Planck operator $K_{V_j^{(2)}}$. Observe that using the parallelogram law $2(\|x\|^2 + \|y\|^2) - \|x+y\|^2 = \|x-y\|^2 \geqslant 0$,

$$\begin{aligned} \sum_{j\in\mathbb{N}} \|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})} &= \sum_{j\in\mathbb{N}} \|K_{V_j^{(2)}} u_j + (K_V - K_{V_j^{(2)}}) u_j\|^2_{L^2(\mathbb{R}^{2d})} \\ &\geqslant \sum_{j\in\mathbb{N}} \left(\frac{1}{2} \left\| K_{V_j^{(2)}} u_j \right\|^2_{L^2(\mathbb{R}^{2d})} - \left\| (\nabla V(q) - \nabla V_j^{(2)}(q)) \partial_p u_j \right\|^2_{L^2(\mathbb{R}^{2d})} \right). \end{aligned} \tag{3.2}$$

On the other hand, by (2.5) in Lemma 2.3,

$$\sum_{j\in\mathbb{N}} \|(\nabla V(q) - \nabla V_j^{(2)}(q)) \partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} \leqslant c_{1,d,r} \sum_{j\in\mathbb{N}} R_V^{\geqslant 3}(q'_j)^2 \|\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})}. \tag{3.3}$$

Combining (3.1)–(3.3), we get immediately

$$\begin{aligned} \|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geqslant \sum_{j\in\mathbb{N}} \Big(&\frac{1}{2} \|K_{V_j^{(2)}} u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_{1,d,r} R_V^{\geqslant 3}(q'_j)^2 \|\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} \\ &- c_{d,r} R_V^{\geqslant 3}(q_j)^2 \|p u_j\|^2_{L^2(\mathbb{R}^{2d})} \Big). \end{aligned}$$

Therefore, making use of equivalence (*A* 5), it follows

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geqslant \sum_{j\in\mathbb{N}} \left(\frac{1}{2} \|K_{V_j^{(2)}} u_j\|^2_{L^2(\mathbb{R}^{2d})} - c'_{d,r} R_V^{\geqslant 3}(q'_j)^2 \langle u_j, O_p u_j \rangle_{L^2(\mathbb{R}^{2d})} \right), \tag{3.4}$$

where $c'_{d,r} = 2(c^2_{1,d,r} + c_{d,r} C^2)$.

Using the Cauchy–Schwarz inequality and then the Cauchy inequality with epsilon (for any real numbers a, b and all $\epsilon > 0$, $ab \leqslant \epsilon a^2 + \frac{1}{4\epsilon} b^2$),

$$\begin{aligned} c'_{d,r} R_V^{\geqslant 3}(q'_j)^2 \langle u_j, O_p u_j \rangle &= c'_{d,r} R_V^{\geqslant 3}(q'_j)^2 \operatorname{Re}\langle u_j, K_{V_j^{(2)}} u_j \rangle \\ &\leqslant c'_{d,r} R_V^{\geqslant 3}(q'_j)^2 \|u_j\| \cdot \|K_{V_j^{(2)}} u_j\| \\ &\leqslant \left(c'_{d,r} R_V^{\geqslant 3}(q'_j)^2 \right)^2 \|u_j\|^2 + \frac{1}{4} \|K_{V_j^{(2)}} u_j\|^2. \end{aligned}$$

Putting the above estimate and (3.4) together, we obtain

$$\|K_V u\|^2 \geqslant \sum_{j\in\mathbb{N}} \left(\frac{1}{4} \|K_{V_j^{(2)}} u_j\|^2 - (c'_{d,r})^2 R_V^{\geqslant 3}(q'_j)^4 \|u_j\|^2 \right). \tag{3.5}$$

From now on, assume $\kappa \geqslant \kappa_1$, where $\kappa_1 \geqslant \kappa_0$ is introduced in Lemma 2.3. Remember as well that the constants $C_1, C_2(\kappa)$ are given respectively in Assumption 1 (see (1.8)) and

Lemma 2.3 (see (2.7)). By introducing $C(\kappa) \geqslant \max(C_1, C_2(\kappa))$, which will be fixed later, we set for each κ,

$$I(\kappa) = \{j \in \mathbb{N}, \text{ such that } \operatorname{supp} \chi_j \subset \{q \in \mathbb{R}^d, \ \ |q| \geqslant C(\kappa)\}\}.$$

The rest of the proof is divided into three steps. The first one is devoted to the control of the terms on the left-hand side of (3.5) for which $j \in I(\kappa)$ for some large $\kappa \geqslant \kappa_0$ to be chosen. At the end of the first step, the constants $\kappa > \kappa_1$ and $C(\kappa) \geqslant \max(C_1, C_2(\kappa))$ will be fixed. The second step is concerned with the remaining terms for which the support of the cutoff functions χ_j is included in some closed ball $B(0, C'(\kappa))$. We finally sum up all the terms in Step 3 and refer to Lemma 2.5 after some elementary optimization trick in the last step.

Step 1, $\mathbf{j} \in \mathbf{I}(\kappa)$, $\kappa \geqslant \kappa_1$ to be fixed: As proved in [2], there is a constant $c > 0$ such that for all $j \in I(\kappa)$,

$$\|K_{V_j^{(2)}} u_j\|^2 + A_{V_j^{(2)}} \|u_j\|^2 \geqslant c\Big(\|O_p u_j\|^2 + \|\langle \partial_q V_j^{(2)}(q)\rangle^{2/3} u_j\|^2 + \|\langle D_q\rangle^{2/3} u_j\|^2\Big), \tag{3.6}$$

where

$$\begin{aligned} A_{V_j^{(2)}} &= \max\{(1 + \mathrm{Tr}_{+,V_j^{(2)}})^{2/3}, 1 + \mathrm{Tr}_{-,V_j^{(2)}}\} \\ &= \max\{(1 + \mathrm{Tr}_{+,V}(q_j'))^{2/3}, 1 + \mathrm{Tr}_{-,V}(q_j')\}. \end{aligned}$$

Hence there is a constant $C_0 > 0$ so that

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 + (1 + 10C_0) t_j^4 \|u_j\|^2 \geqslant C_0\Big(&\|O_p u_j\|^2 + \|\langle \partial_q V_j^{(2)}(q)\rangle^{2/3} u_j\|^2 \\ &+ \|\langle D_q\rangle^{2/3} u_j\|^2 + 10 t_j^4 \|u_j\|^2\Big), \end{aligned} \tag{3.7}$$

where we use the notation $t_j = 2\langle \mathrm{Hess}\, V(q_j')\rangle^{1/4}$ throughout the proof.

Recall that as mentioned in [2], the constant c in (3.6) does not depend on the polynomial $V_j^{(2)}$ and then so is the constant C_0 in (3.7).

Now for all indices $j \in I(\kappa)$, we distinguish two cases: either $j \in J(\kappa)$ or $j \notin J(\kappa)$.

Case 1. Assume $j \in J(\kappa)$. Then taking into account inequality (2.6) in Lemma 2.3 and using estimate (3.7), we obtain

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 + (1 + 10C) t_j^4 \|u_j\|^2 \geqslant C\Big(&\|O_p u_j\|^2 + \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 \\ &+ \|\langle D_q\rangle^{2/3} u_j\|^2 + 10 t_j^4 \|u_j\|^2\Big). \end{aligned} \tag{3.8}$$

Furthermore, since for all indices $j \in \mathbb{N}$ the quantity $R_V^{\geqslant 2}(q_j')^2$ is always greater than $|\mathrm{Hess}\, V(q_j')|$, there exists a constant $c_d > 0$ so that for every $j \in J(\kappa)$,

$$t_j^4 = 16\langle \mathrm{Hess}\, V(q_j')\rangle \leqslant c_d \langle R_V^{\geqslant 2}(q_j')^2\rangle.$$

Using the fact that the metric $R_V^{\geqslant 2}(q)\, dq^2$ is $R_V^{\geqslant 3}(q)\, dq^2$-slow (see Definition (A 2) and Lemma A.5), it follows

$$t_j^4 \leqslant c_d \langle R_V^{\geqslant 2}(q)^2\rangle$$

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for every $q \in \operatorname{supp} \chi_j$. Hence there is a constant $c_d' > 0$ (depending on the dimension d) such that

$$\begin{aligned} t_j^4 &\leqslant c_d \Big\langle \Big(\sum_{|\alpha|=2} |\partial_q^\alpha V(q)|^{\frac{1}{|\alpha|}} + R_V^{\geqslant 3}(q) \Big)^2 \Big\rangle \\ &\leqslant 3c_d \Big\langle \Big(\sum_{|\alpha|=2} |\partial_q^\alpha V(q)|^{\frac{1}{|\alpha|}} \Big)^2 + R_V^{\geqslant 3}(q)^2 \Big\rangle \\ &\leqslant c_d' \langle |\mathrm{Hess}\ V(q)| + R_V^{\geqslant 3}(q)^2 \rangle \end{aligned}$$

holds for any $q \in \operatorname{supp} \chi_j$. Now, since for every $q \in \mathbb{R}^d$, one has $R_V^{\geqslant 3}(q) \geqslant R_V^{=r}(0)$, we derive from the previous estimate that for any $q \in \operatorname{supp} \chi_j$,

$$\begin{aligned} t_j^4 &\leqslant c_d' \left\langle |\mathrm{Hess}\ V(q)| + \frac{R_V^{\geqslant 3}(q)^4}{R_V^{=r}(0)^2} \right\rangle \\ &\leqslant \frac{c_d''}{\kappa} \max(1, R_V^{=r}(0)^{-2}) \langle \partial_q V(q) \rangle^{\frac{4}{3}}. \end{aligned} \tag{3.9}$$

Collecting estimates (3.8) and (3.9), we get for $\kappa \geqslant \kappa_1$

$$\begin{aligned} &\|K_{V_j^{(2)}} u_j\|^2 + (1 + 10C) \frac{c_d''}{\kappa} \max(1, R_V^{=r}(0)^{-2}) \| \langle \partial_q V(q) \rangle^{\frac{2}{3}} u_j \|^2 \\ &\quad \geqslant C \Big(\|O_p u_j\|^2 + \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 + \|\langle D_q \rangle^{2/3} u_j\|^2 + 10 t_j^4 \|u_j\|^2 \Big). \end{aligned}$$

Choosing $\kappa_2 \geqslant \kappa_1$ so that

$$\frac{C}{2} \geqslant (1 + 10C) \frac{c_d''}{\kappa_2} \max(1, R_V^{=r}(0)^{-2}),$$

the following inequality

$$\|K_{V_j^{(2)}} u_j\|^2 \geqslant C \big(\|O_p u_j\|^2 + \tfrac{1}{2} \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 + \|\langle D_q \rangle^{2/3} u_j\|^2 + 10 t_j^4 \|u_j\|^2 \big) \tag{3.10}$$

holds for all $j \in J(\kappa)$ with $\kappa \geqslant \kappa_2$.

Since $j \in J(\kappa)$, there is a constant $c_1 > 0$ so that

$$\tfrac{1}{8} \langle \partial_q V(q) \rangle^{\frac{4}{3}} \geqslant c_1 \langle \mathrm{Hess}\ V(q) \rangle \tag{3.11}$$

holds for all $q \in \operatorname{supp} \chi_j$. In addition, using equivalence (A 5), it follows

$$\tfrac{1}{8} \langle \partial_q V(q) \rangle^{\frac{4}{3}} \geqslant c_2 |\partial_q V(q)|^{\frac{4}{3}} \geqslant c_2 \kappa\, R_V^{\geqslant 3}(q)^4 \geqslant c_2' \kappa\, R_V^{\geqslant 3}(q_j')^4 \tag{3.12}$$

for any $q \in \operatorname{supp} \chi_j$.

Putting (3.10)–(3.12) together,

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 \geqslant C \Big(&\|O_p u_j\|^2 + \tfrac{1}{4} \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 + \|\langle D_q \rangle^{2/3} u_j\|^2 \\ &+ c_1 \|\langle \mathrm{Hess}\ V(q)\rangle^{1/2} u_j\|^2 + c_2' \kappa\, R_V^{\geqslant 3}(q_j')^4 \|u_j\|^2 + 10 \|t_j^2 u_j\|^2 \Big) \end{aligned} \tag{3.13}$$

holds for all $\kappa \geqslant \kappa_2$.

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Case 2. Assume $j \notin J(\kappa)$, with $\kappa \geqslant \kappa_2 \geqslant \kappa_1 \geqslant \kappa_0$. Hence by Assumption 1 (see (1.8)), one has

$$\mathrm{Tr}_{-,V}(q) \neq 0 \quad \text{for all } \; q \in (\operatorname{supp} \chi_j) \cap (\mathbb{R}^d \setminus \Sigma(\kappa)) \; \text{ such that } \; |q| \geqslant C_1.$$

In particular, since $|q'_j| \geqslant C(\kappa) \geqslant C_1$,

$$\mathrm{Tr}_{-,V_j^{(2)}} = \mathrm{Tr}_{-,V}(q'_j) \neq 0.$$

Referring again to [2],

$$\|K_{V_j^{(2)}} u_j\|^2 \geqslant c\, B_{V_j^{(2)}} \|u_j\|^2,$$

where

$$\begin{aligned} B_{V_j^{(2)}} &= \max\Big(\min_{q \in \mathbb{R}^d} |\nabla V_j^{(2)}(q)|^{4/3}, \frac{1 + \mathrm{Tr}_{-,V_j^{(2)}}}{(\log(2 + \mathrm{Tr}_{-,V_j^2}))^2} \Big) \\ &= \max\Big(\min_{q \in \mathbb{R}^d} |\nabla V_j^{(2)}(q)|^{4/3}, \frac{1 + \mathrm{Tr}_{-,V}(q'_j)}{(\log(2 + \mathrm{Tr}_{-,V}(q'_j)))^2} \Big) \neq 0. \end{aligned}$$

Hence we get in particular,

$$\|K_{V_j^{(2)}} u_j\|^2 \geqslant c \frac{1 + \mathrm{Tr}_{-,V}(q'_j)}{(\log(2 + \mathrm{Tr}_{-,V}(q'_j)))^2} \|u_j\|^2. \tag{3.14}$$

Using again condition (1.8) in Assumption 1, there is a constant $C_1 \geqslant 1$ so that

$$\frac{1}{2} \mathrm{Tr}_{-,V}(q'_j) > \frac{1}{2C_1} \mathrm{Tr}_{+,V}(q'_j)$$

holds, which in turn implies

$$\mathrm{Tr}_{-,V}(q'_j) \geqslant \frac{1}{2} \mathrm{Tr}_{-,V}(q'_j) + \frac{1}{2C_1} \mathrm{Tr}_{+,V}(q'_j) \geqslant \frac{1}{2C_1} (\mathrm{Tr}_{-,V}(q'_j) + \mathrm{Tr}_{+,V}(q'_j)). \tag{3.15}$$

Then it follows from (3.14) and (3.15)

$$\|K_{V_j^{(2)}} u_j\|^2 \geqslant c' \left\| \frac{\sqrt{1 + |\mathrm{Hess}\; V(q'_j)|}}{\log(2 + |\mathrm{Hess}\; V(q'_j)|)} u_j \right\|^2. \tag{3.16}$$

By Assumption 1 (see condition (1.9)) and (3.16), applying Lemma B.6, there are $\delta \in (0, 1)$ and a positive nondecreasing function $\Lambda_{\Sigma(\kappa)}$ on $(0, +\infty)$ such that $\Lambda_{\Sigma(\kappa)}(\varrho) \to +\infty$ as $\varrho \to +\infty$, and such that

$$\begin{aligned} \frac{1 + |\mathrm{Hess}\; V(q'_j)|}{(\log(2 + |\mathrm{Hess}\; V(q'_j)|))^2} &\geqslant \frac{1}{2^\delta} (1 + |\mathrm{Hess}\; V(q'_j)|)^{1-\delta} \\ &\geqslant \frac{1}{2} |\mathrm{Hess}\; V(q'_j)|^{1-\delta} \\ &\geqslant \frac{\Lambda_{\Sigma(\kappa)}(|q'_j|)}{2} R_V^{\geqslant 3}(q'_j)^4 \geqslant \frac{\Lambda_{\Sigma(\kappa)}(C(\kappa))}{2} R_V^{\geqslant 3}(q'_j)^4. \end{aligned}$$

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Here, Lemma B.6 relying on the Tarski–Seidenberg theorem is crucial as shown by the following argument:

$$R^4(q) \overset{|q|\to+\infty}{\sim} \frac{H(q)}{\log(H(q))} \quad \text{and} \quad \lim_{|q|\to+\infty} H(q) = +\infty,$$

where $R(q)$ is a function that plays the same role as $R_V^{\geqslant 3}(q)$ and still satisfies $\lim_{|q|\to+\infty} \frac{R^4(q)}{H(q)} = 0$. For a nonpolynomial function V, we may think of a function $R(q)$ that satisfies

$$\frac{1}{C} R(q) \leqslant \max_{q' \in B(q, \frac{b}{R(q)}),\ |\alpha|=3} |\partial_q^\alpha V(q)|^{\frac{1}{3}} \leqslant C R(q)\ ,$$

with $C > 1$ and $b > 0$ independent of q for $|q|$ large enough.[1]

Alternatively, the asymptotic behaviour (1.9) of Assumption 1 should be replaced by something like $R(q)^4 = O(H(q)^{1-\nu})$ with $\nu > 0$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ or $R(q)^4 = O\left(\left(\frac{H(q)}{(\log H(q))^2}\right)\right)$ (with $|\text{Hess } V(q)| \to +\infty$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$).
Therefore we get from the above inequality and (3.16)

$$\|K_{V_j^{(2)}} u_j\|^2 \geqslant c' \frac{\Lambda_{\Sigma(\kappa)}(C(\kappa))}{2} R_V^{\geqslant 3}(q_j')^4 \|u_j\|^2. \tag{3.17}$$

Next, recall that $t_j = 2\langle \text{Hess } V(q_j')\rangle^{1/4}$. By (2.7) in Lemma 2.3, the equivalence

$$t_j \asymp 2\langle \text{Hess } V(q)\rangle^{1/4} \tag{3.18}$$

holds for any $q \in \text{supp } \chi_j$ with $|q| \geqslant C(\kappa) \geqslant C_2(\kappa)$. From (3.7) and (3.18), we see that

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 + (1+10C)t_j^4 \|u_j\|^2 \geqslant C\Big(&\|O_p u_j\|^2 + \|\langle \partial_q V_j^{(2)}(q)\rangle^{2/3} u_j\|^2 \\ &+ \|\langle \text{Hess } V(q)\rangle^{1/2} u_j\|^2 + \|\langle D_q\rangle^{2/3} u_j\|^2 + 9t_j^4 \|u_j\|^2\Big). \end{aligned} \tag{3.19}$$

One has by (2.6) in Lemma 2.3,

$$\langle \partial_q V_j^{(2)}(q)\rangle \geqslant \tfrac{1}{2} \langle \partial_q V(q)\rangle \tag{3.20}$$

for all $q \in (\text{supp } \chi_j) \cap \Sigma(\kappa)$. On the other hand, for every $q \in (\text{supp } \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$,

$$|\text{Hess } V(q)| + R_V^{\geqslant 3}(q)^4 + 1 \geqslant \frac{1}{\kappa} |\partial_q V(q)|^{\frac{4}{3}}. \tag{3.21}$$

Furthermore, it results from Assumption 1, in particular (1.9), that for all $q \in (\text{supp } \chi_j) \cap \big(\mathbb{R}^d \setminus \Sigma(\kappa)\big)$,

$$2|\text{Hess } V(q)| + C^4 R_V^{\geqslant 3}(q)^4 + 1 \leqslant \tfrac{5}{2} |\text{Hess } V(q)|. \tag{3.22}$$

[1] As an example, we may take the function V on $\mathbb{R}^2$ equal to $\frac{r^6}{(\log r)^3}(1+\cos(\theta))$ in polar coordinates for $r > 1$.

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From (3.21) and (3.22), we get for every $q \in (\text{supp } \chi_j) \cap (\mathbb{R}^d \setminus \Sigma(\kappa))$,

$$|\text{Hess } V(q)| \geqslant \frac{2}{5\kappa} |\nabla V(q)|^{\frac{4}{3}}, \quad |\text{Hess } V(q)| \geqslant \frac{2}{5} \geqslant \frac{2}{5\kappa}.$$

Hence there exists a constant $c'' > 0$ such that

$$\langle \text{Hess } V(q) \rangle \geqslant \frac{c''}{\kappa} \langle \partial_q V(q) \rangle^{4/3} \tag{3.23}$$

for any $q \in (\text{supp } \chi_j) \cap \left(\mathbb{R}^d \setminus \Sigma(\kappa)\right)$ with $|q| \geqslant C(\kappa) \geqslant C_2(\kappa)$.

The above inequality combined with (3.19) and (3.20) leads to

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 + (1+10C) t_j^4 \|u_j\|^2 \geqslant C\big(\|O_p u_j\|^2 + \|\langle D_q\rangle^{2/3} u_j\|^2 + 9 t_j^4 \|u_j\|^2 \\ + \min\left(\frac{1}{2^{4/3}}, \frac{c''}{2\kappa}\right) \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 + \frac{1}{2} \|\langle \text{Hess } V(q)\rangle^{1/2} u_j\|^2\big) \end{aligned} \tag{3.24}$$

for all $\kappa \geqslant \kappa_2$.

Collecting estimates (3.16) and (3.24), we get

$$\begin{aligned} &(\log(t_j^4))^2 \|K_{V_j^{(2)}} u_j\|^2 \\ &\quad \geqslant C\text{''}\Big(\|O_p u_j\|^2 + \|\langle D_q\rangle^{2/3} u_j\|^2 + 9 t_j^4 \|u_j\|^2 \\ &\qquad + \min\Big(\frac{1}{2^{4/3}}, \frac{c''}{2\kappa}\Big) \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 + \frac{1}{2} \|\langle \text{Hess } V(q)\rangle^{1/2} u_j\|^2\Big). \end{aligned} \tag{3.25}$$

In order to reduce the written expressions, we denote

$$\Lambda_{1,j} = \frac{O_p}{\log(t_j^4)}, \quad \Lambda_{2,j} = \frac{\langle \text{Hess } V(q)\rangle^{1/2}}{\log(t_j^4)}, \quad \Lambda_{3,j} = \frac{\langle \partial_q V(q)\rangle^{2/3}}{\log(t_j^4)}, \quad \Lambda_{4,j} = \frac{t_j^2}{\log(t_j^4)}.$$

Estimate (3.25) can be rewritten as follows:

$$\begin{aligned} &\|K_{V_j^{(2)}} u_j\|^2 \\ &\quad \geqslant C\text{''}\Big(\|\Lambda_{1,j} u_j\|^2 + \frac{1}{2} \|\Lambda_{2,j} u_j\|^2 + \min\Big(\frac{1}{2^{4/3}}, \frac{c''}{2\kappa}\Big) \|\Lambda_{3,j} u_j\|^2 \\ &\qquad + 9 \|\Lambda_{4,j} u_j\|^2 + \Big\|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)} u_j\Big\|^2\Big). \end{aligned} \tag{3.26}$$

Using (3.17) and (3.26), we obtain

$$\begin{aligned} &(1 + C\text{''}) \|K_{V_j^{(2)}} u_j\|^2 \\ &\quad \geqslant C\text{''}\Big(\|\Lambda_{1,j} u_j\|^2 + \frac{1}{2} \|\Lambda_{2,j} u_j\|^2 + \min\Big(\frac{1}{2^{4/3}}, \frac{c''}{2\kappa}\Big) \|\Lambda_{3,j} u_j\|^2 + 9 \|\Lambda_{4,j} u_j\|^2 \\ &\qquad + \Big\|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)} u_j\Big\|^2 + \frac{c'}{2} \Lambda_{\Sigma(\kappa)}(C(\kappa)) R_V^{\geqslant 3}(q_j')^4 \|u_j\|^2\Big). \end{aligned}$$

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Therefore in both cases, that is, for all $j \in I(\kappa)$, where $\kappa \geqslant \kappa_2$,

$$\|K_{V_j^{(2)}} u_j\|^2 \geqslant C^{(3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)}u_j\|^2 + \min\big(\kappa, \Lambda_{\Sigma(\kappa)}(C(\kappa))\big) R_V^{\geqslant 3}(q_j')^4\|u_j\|^2\Big). \tag{3.27}$$

Due to the elementary inequality $u^{4/3} + v^4 \geqslant \frac{1}{c_0}(u^2+v^4)^{2/3}$ satisfied for all $u, v \geqslant 1$, we obtain for all $\kappa \geqslant \kappa_2$

$$\Big\|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)}u_j\Big\|^2 + \frac{1}{2}\min\big(\kappa, \Lambda_{\Sigma(\kappa)}(C(\kappa))\big) R_V^{\geqslant 3}(q_j')^4\|u_j\|^2 \geqslant \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2, \tag{3.28}$$

where

$$\Lambda_{5,j} = \frac{(1 + D_q^2 + R_V^{\geqslant 3}(q_j')^4)^{\frac{1}{3}}}{\log(t_j^4)}.$$

In conclusion, we get by (3.27) and (3.28) for every $j \in I(\kappa)$ with $\kappa \geqslant \kappa_2$

$$\|K_{V_j^{(2)}} u_j\|^2 \geqslant C^{(3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2 + \frac{1}{2}\min\big(\kappa, \Lambda_{\Sigma(\kappa)}(C(\kappa))\big) R_V^{\geqslant 3}(q_j')^4\|u_j\|^2\Big).$$

We now fix the choice first of $C(\kappa)$ and second of κ. Because $\lim_{\varrho\to+\infty}\Lambda_{\Sigma(\kappa)}(\varrho) = +\infty$, we can choose for any $\kappa \geqslant \kappa_2$, $C(\kappa) \geqslant \max(C_1, C_2(\kappa))$ such that $\Lambda_{\Sigma(\kappa)}(C(\kappa)) \geqslant \kappa$. We then choose $\kappa = \kappa_3 \geqslant \kappa_2$ such that

$$\frac{C^{(3)}}{8}\min\big(\kappa_3, \Lambda_{\Sigma(\kappa_3)}(C(\kappa_3))\big) = \frac{C^{(3)}\kappa_3}{8} \geqslant (c_{d,r}')^2,$$

where $c_{d,r}'$ is the constant in (3.5),

$$\sum_{j\in I(\kappa_3)}\Big(\frac{1}{4}\|K_{V_j^{(2)}}u\|^2 - (c_{d,r}')^2 R_V^{\geqslant 3}(q_j')^4\|u_j\|^2\Big) \geqslant \frac{C^{(3)}}{8}\sum_{j\in I(\kappa_3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa_3}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2\Big). \tag{3.29}$$

Step 2, $j \notin I(\kappa_3)$: The set $\mathbb{N}\setminus I(\kappa_3)$ is now a fixed finite set and we can define

$$C^{(4)} = \max_{j\in\mathbb{N}\setminus I(\kappa_3)}\Big[A_{V_j^{(2)}} + \sup_{q\in\operatorname{supp}\chi_j}\Big(\langle \mathrm{Hess}\ V(q)\rangle + \langle\partial_q V(q)\rangle^{4/3}\Big) + \frac{t_j^4}{(\log(t_j^4))^2} + (1+(c_{d,r}')^2)(1+R_V^{\geqslant 3}(q_j'))^4\Big].$$

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From the lower bound (1.5), we deduce

$$\begin{aligned}
&\frac{1}{4}\|K_{V_j^{(2)}}u_j\| + C^{(4)}\|u_j\|^2 - (c'_{d,r})^2 R_V^{\geqslant 3}(q'_j)^4\|u_j\|^2 \\
&\quad\geqslant \frac{c}{4}\left[\|O_p u_j\|^2 + \|\langle D_q\rangle^{2/3}u_j\|^2\right] + (1+R_V^{\geqslant 3}(q'_j))^4\|u_j\|^2 \\
&\qquad + \|\langle \partial_q V(q)\rangle^{2/3}u_j\|^2 + \|\langle \text{Hess }V(q)\rangle^{1/2}u_j\|^2 + \left\|\frac{t_j^2}{\log(t_j^4)}u_j\right\|^2.
\end{aligned}$$

With the quantities

$$\Lambda_{1,j} = \frac{O_p}{\log(t_j^4)},\quad \Lambda_{2,j} = \frac{\langle \text{Hess }V(q)\rangle^{1/2}}{\log(t_j^4)},\quad \Lambda_{3,j} = \frac{\langle \partial_q V(q)\rangle^{\frac{2}{3}}}{\log(t_j^4)},$$

$$\Lambda_{4,j} = \frac{t_j^2}{\log(t_j^4)},\quad \Lambda_{5,j} = \frac{(1+D_q^2+R_V^{\geqslant 3}(q'_j)^4)^{\frac{1}{3}}}{\log(t_j^4)},$$

where $t_j \geqslant 2$, we deduce

$$\begin{aligned}
&\sum_{j\notin I(\kappa_3)}\left(\frac{1}{4}\|K_{V_j^{(2)}}u_j\|^2 - (c'_{d,r})^2 R_V^{\geqslant 3}(q'_j)^4\|u_j\|^2 + C^{(4)}\|u_j\|^2\right) \\
&\quad\geqslant C^{(5)}\sum_{j\notin I(\kappa_3)}\left(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa_3}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2\right).
\end{aligned} \tag{3.30}$$

Collecting (3.5), (3.29) and (3.30), there exists a positive constant $C^{(6)} \geqslant 1$ depending on V such that

$$\begin{aligned}
\|K_V u\|_{L^2}^2 + C^{(6)}\|u\|_{L^2}^2 \geqslant &\frac{1}{C^{(6)}}\sum_{j\in\mathbb{N}}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \|\Lambda_{3,j}u_j\|^2 \\
&+ \|\Lambda_{4,j}u_j\|^2 + \|\Lambda_{5,j}u_j\|^2\Big).
\end{aligned} \tag{3.31}$$

Step 3. In this final step, set $L(s) = \frac{s+1}{\log(s+1)}$ for all $s \geqslant 1$. Note that there exists a constant $c > 0$ such that for all $x \geqslant 1$,

$$\inf_{t\geqslant 2}\frac{x}{\log(t)} + t \geqslant \frac{1}{c}L(x).$$

In view of the above estimate,

$$\begin{aligned}
\|\Lambda_{1,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 &\geqslant \frac{1}{4}\int\left(\frac{\lambda^2}{(\log(t_j^4))^2} + t_j^2\right)d\mu_{u_j}(\lambda) \\
&\geqslant \frac{1}{8}\int\left(\frac{\lambda}{\log(t_j)} + t_j\right)^2 d\mu_{u_j}(\lambda) \\
&\geqslant \frac{1}{c_3}\|L(O_p)u_j\|^2.
\end{aligned}$$

Here we recall that $d\mu_{u_j}(\lambda) = d\langle E(\lambda)u_j, u_j\rangle$, where $E(\lambda)$ is the spectral family.

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Summing over j, we obtain the first term in the desired estimation (1.10). Likewise, for the second term

$$\|\Lambda_{3,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geqslant \frac{1}{c_4}\|L(\langle\partial_q V(q)\rangle^{2/3})u_j\|^2,$$

with

$$\sum_{j\in\mathbb{N}} \|L(\langle\partial_q V(q)\rangle^{2/3})u_j\|^2 = \|L(\langle\partial_q V(q)\rangle^{2/3})u\|^2.$$

To obtain the third term in (1.10), write similarly

$$\|\Lambda_{2,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geqslant \frac{1}{c_5}\|L(\langle \text{Hess } V(q)\rangle^{1/2})u_j\|^2,$$

with

$$\sum_{j\in\mathbb{N}} \|L(\langle \text{Hess } V(q)\rangle^{1/2})u_j\|^2 = \|L(\langle \text{Hess } V(q)\rangle^{1/2})u\|^2.$$

In the same way,

$$\|\Lambda_{5,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geqslant \frac{1}{c_6}\|L((1+D_q^2+R_V^{\geqslant 3}(q_j')^4)^{\frac{1}{3}})u_j\|^2.$$

By Lemma 2.5, we get

$$\sum_{j\in\mathbb{N}} \|L((1+D_q^2+R_V^{\geqslant 3}(q_j')^4)^{\frac{1}{3}})u_j\|^2 \geqslant \frac{1}{c_6}\|L((1+D_q^2+R_V^{\geqslant 3}(q)^4)^{\frac{1}{3}})u\|^2.$$

To conclude, just remark that

$$\langle u, (1+D_q^2+R_V^{\geqslant 3}(q)^4)u\rangle \geqslant \langle u, (1+D_q^2)u\rangle \geqslant \langle u, \langle D_q^2\rangle u\rangle > \|u\|^2$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Then applying (2.15) in Lemma 2.4 with $A = (1+D_q^2+R_V^{\geqslant 3}(q)^4)$, $B = \langle D_q^2\rangle$, $\alpha_0 = \frac{2}{3}$ and $k = 2$, we obtain

$$\|L((1+D_q^2+R_V^{\geqslant 3}(q)^4)^{\frac{1}{3}})u\|^2 \geqslant \|L(\langle D_q^2\rangle^{\frac{1}{3}})u\|^2 \geqslant \frac{1}{c_7}\|L(\langle D_q\rangle^{2/3})u\|^2$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Finally collecting all terms, we have found $C_V \geqslant 1$ such that

$$\|K_V u\|_{L^2}^2 + C_V\|u\|_{L^2}^2 \geqslant \frac{1}{C_V}\Big(\|L(O_p)u\|_{L^2}^2 + \|L(\langle\nabla V(q)\rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle \text{Hess } V(q)\rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big) \tag{3.32}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Because $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ is dense in $D(K_V)$ endowed with the graph norm, the result extends to any $u \in D(K_V)$. □

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4. Applications

This section is devoted to some applications of Theorem 1.1. In each of the following examples, we examine that Assumption 1 is well fulfilled. We recall here that one knows that for a potential V satisfying Assumption 1, the resolvent of the Witten Laplacian $\Delta_V^{(0)}$ is compact (see [4, Theorem 10.16]). In the case of the Witten Laplacian, the following examples were in particular considered in [4] (cf. Propositions 10.19 and 10.21).

Example 1: Let us consider as a first example of application the case

$$V(q_1, q_2) = -q_1^2 q_2^2, \quad \text{with } q = (q_1, q_2) \in \mathbb{R}^2.$$

By direct computation,

$$\partial_q V(q) = \begin{pmatrix} -2q_1 q_2^2 \\ -2q_2 q_1^2 \end{pmatrix}, \quad |\partial_q V(q)| = 2|q_1 q_2||q|,$$

$$\text{Hess } V(q) = \begin{pmatrix} -2q_2^2 & -4q_1 q_2 \\ -4q_1 q_2 & -2q_1^2 \end{pmatrix}, \quad |\text{Hess } V(q)| = 2\sqrt{|q|^4 + 6q_1^2 q_2^2} \asymp |q|^2,$$

$$R_V^{\geqslant 3}(q) = |4q_2|^{1/3} + |4q_1|^{1/3} + 2 \times 4^{1/4}.$$

It is clear that the trace of Hess $V(q)$ given by $-2|q|^2$ is negative for all $q \in \mathbb{R}^2 \setminus \{0\}$. Hence

$$\text{Tr}_{-,V}(q) > \text{Tr}_{+,V}(q) \quad \text{for all } q \in \mathbb{R}^2, |q| \geqslant 1.$$

Moreover, for all $\kappa > 0$, the algebraic set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is not bounded since $(0, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$ for all $q_2 \in \mathbb{R}$. Furthermore, for $\kappa > 1$ chosen as we wish,

$$\lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^2 \setminus \Sigma(\kappa)}} \frac{R_V^{\geqslant 3}(q)^4}{|\text{Hess } V(q)|} = \lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^2 \setminus \Sigma(\kappa)}} \frac{|q|^{4/3}}{|q|^2} = 0$$

since $R_V^{\geqslant 3}(q)^4 \leqslant |q|^{4/3}$ when $|q| \geqslant 2^3 \times 4^{3/4}$. In Figure 1 we sketch as an example $\Sigma(800)$ in blue.

The compactness of the resolvent of K_V in this case follows from Corollary 1.2.

Example 2: Let $n \in \mathbb{N}$. The polynomial $V(q) = -q_1^2(q_1^2 + q_2^2)^n$ verifies Assumption 1 only for $n = 1$.

A straightforward computation shows that

$$\partial_q V(q) = -\begin{pmatrix} 2q_1(|q|^{2n} + nq_1^2|q|^{2(n-1)}) \\ 2nq_2 q_1^2 |q|^{2(n-1)} \end{pmatrix},$$

$$\text{Hess } V(q) = -2|q|^{2(n-2)} \begin{pmatrix} |q|^4 + 5nq_1^2|q|^2 + 2n(n-1)q_1^4 & 2nq_1q_2|q|^2 + 2n(n-1)q_1^3 q_2 \\ 2nq_1q_2|q|^2 + 2n(n-1)q_1^3 q_2 & nq_1^2|q|^2 + 2n(n-1)q_1^2 q_2^2 \end{pmatrix}.$$

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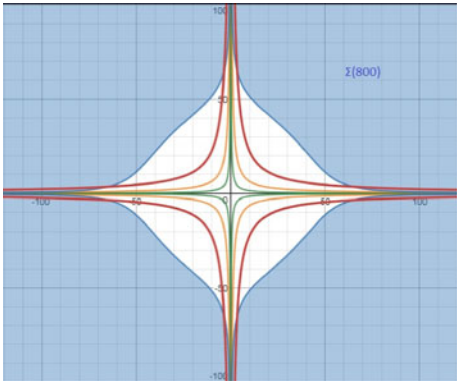

Figure 1. Contour lines of $V(q_1, q_2) = -q_1^2 q_2^2$.

Note that the trace of Hess $V(q)$ equals

$$-2|q|^{2(n-2)}\left(|q|^4 + 5nq_1^2|q|^2 + 2n(n-1)q_1^4 + nq_1^2|q|^2 + 2n(n-1)q_1^2q_2^2\right) < 0$$

for all $q \in \mathbb{R}^2$, $|q| \geqslant 1$. Hence

$$-\mathrm{Tr}_{-,V}(q) + \mathrm{Tr}_{+,V}(q) < 0, \quad \text{for any } q \in \mathbb{R}^2, \ |q| \geqslant 1.$$

In addition, for all $\kappa > 0$, the set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is not bounded since $(0, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$ for all $q_2 \in \mathbb{R}$.

For q large enough, $|\text{Hess } V(q)| \asymp |q|^{2n}$ and $|D^3 V(q)| \asymp |q|^{2n-1}$; then

$$\frac{R_V^{\geqslant 3}(q)^4}{|\text{Hess } V(q)|} \asymp \frac{(|q|^{2n-1})^{4/3}}{|q|^{2n}}.$$

Hence

$$\lim_{\substack{|q|\to+\infty \\ q\in\mathbb{R}^2\setminus\Sigma(\kappa)}} \frac{R_V^{\geqslant 3}(q)^4}{|\text{Hess } V(q)|} = 0 \quad \text{if and only if} \quad n < 2.$$

Taking as an example $\kappa = 800$, we get the shape of $\Sigma(800)$ coloured blue as in Figure 2.

In this example, the hypothesis of Theorem 1.1 is satisfied only for $n = 1$. By Corollary 1.2, we deduce that the Kramers–Fokker–Planck operator K_V with potential $V(q) = -q_1^2(q_1^2 + q_2^2)$ has a compact resolvent.

Example 3: For $\epsilon \in \mathbb{R} \setminus \{0, -1\}$, we consider $V(q_1, q_2) = (q_1^2 - q_2)^2 + \epsilon q_2^2$. For all $q \in \mathbb{R}^2$, one has

$$\partial_q V(q) = \begin{pmatrix} 4q_1(q_1^2 - q_2) \\ -2(q_1^2 - q_2) + 2\epsilon q_2 \end{pmatrix}, \quad |\partial_q V(q)| = 4|q_1(q_1^2 - q_2)| + |-2(q_1^2 - q_2) + 2\epsilon q_2|,$$

$$\text{Hess } V(q) = \begin{pmatrix} 12q_1^2 - 4q_2 & -4q_1 \\ -4q_1 & 2(1+\epsilon) \end{pmatrix}, \quad |\text{Hess } V(q)| = |12q_1^2 - 4q_2| + 8|q_1| + 4|1+\epsilon|,$$

$$R_V^{\geqslant 3}(q) = (24|q_1|)^{1/3} + 3 \times 4^{1/3} + 24^{1/4}.$$

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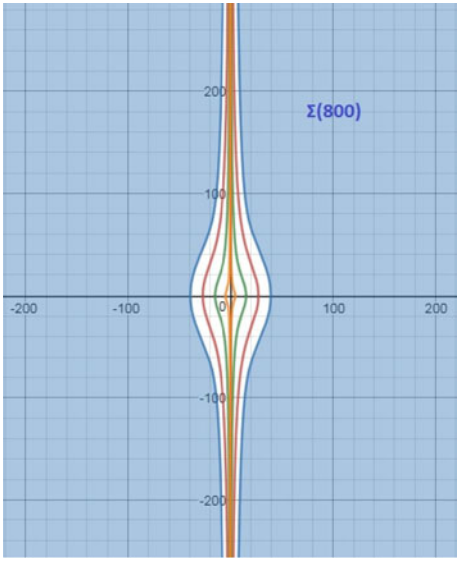

Figure 2. Contour lines of $V(q_1, q_2) = -q_1^2(q_1^2 + q_2^2)$.

In this case, we are going to show that for all $\kappa > 0$, the algebraic set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is bounded. Let $(q_1, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$; then

$$\left(|\text{Hess } V(q)| + R_V^{\geqslant 3}(q)^4 + 1\right) \geqslant \frac{1}{\kappa}|\nabla V(q)|^{\frac{4}{3}}.$$

Up to a change of coordinates $X_1 = q_1,\ X_2 = q_1^2 - q_2$, the above inequality is equivalent to

$$\left(4|2X_1^2 + X_2| + 8|X_1| + 4|1+\epsilon| + \left((24|X_1|)^{1/3} + 3 \times 4^{1/3} + 24^{1/4}\right)^4 + 1\right)$$
$$\geqslant \frac{1}{\kappa}\left(4|X_1X_2| + |-2(1+\epsilon)X_2 + 2\epsilon X_1^2|\right)^{\frac{4}{3}}.$$

Using the triangle inequality on the right-hand side and the reverse triangle inequality with the elementary inequality $(u+v)^{\frac{4}{3}} \geqslant u^{\frac{4}{3}} + v^{\frac{4}{3}}$ satisfied for all $u, v \geqslant 0$, it follows that

$$|X_1|^2 + |X_2| + |X_1| + \left(|X_1|^{\frac{1}{3}} + c\right)^4 \geqslant \frac{c'}{\kappa}\left(\left||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\right|^{\frac{4}{3}} + |X_1X_2|^{\frac{4}{3}}\right). \tag{4.1}$$

Suppose first that $|X_1| \leqslant 1$. Inequality (4.1) implies

$$|X_2| + c_1 \geqslant \frac{c'}{\kappa}\left||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\right|^{\frac{4}{3}}. \tag{4.2}$$

The right-hand part in the above inequality is bounded from above by $|X_2| + c_1$, where c_1 is some positive constant. Now we distinguish two cases.

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Case 1: If $\frac{1}{2}|2(1+\epsilon)X_2| \leqslant |2\epsilon X_1^2|$ or equivalently $|X_2| \leqslant |\frac{2\epsilon}{1+\epsilon}||X_1^2|$, then $|X_2| \leqslant |\frac{2\epsilon}{1+\epsilon}|$.
Case 2: Otherwise, if $\frac{1}{2}|2(1+\epsilon)X_2| \geqslant |2\epsilon X_1^2|$, then we get

$$|X_2| + c_1 \geqslant \frac{c'}{\kappa}|1+\epsilon||X_2|^{4/3}.$$

Using the fact that $\epsilon \neq -1$, we deduce that X_2 must be also bounded.

Now if $|X_1| \geqslant 1$, we derive from (4.1) the estimates

$$|X_1|^2 + |X_2| + c_3 \geqslant \frac{c_4}{\kappa}\Big||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\Big|^{\frac{4}{3}}, \tag{4.3}$$

$$|X_1|^2 + |X_2| + c_3 \geqslant \frac{c_4}{\kappa}|X_1X_2|^{\frac{4}{3}}. \tag{4.4}$$

Here we study three cases.

- First, if $\frac{1}{2}|2(1+\epsilon)X_2| \geqslant |2\epsilon X_1^2|$ or equivalently $|X_1| \leqslant |\frac{1+\epsilon}{2\epsilon}||X_2|$, then (4.3) gives

$$\left(1+|\frac{1+\epsilon}{\epsilon}|\right)|X_2| + c_3 \geqslant \frac{c_4}{\kappa}|(1+\epsilon)X_2|^{\frac{4}{3}}.$$

Since $\epsilon \neq -1$, it follows that X_2 is bounded and so is X_1.

- Now if $2|2(1+\epsilon)X_2| \leqslant |2\epsilon X_1^2|$ or equivalently $|X_2| \leqslant |\frac{\epsilon}{2(1+\epsilon)}||X_1^2|$, estimate (4.3) leads to

$$\left(1+\left|\frac{\epsilon}{2(1+\epsilon)}\right|\right)|X_1|^2 + c_3 \geqslant \frac{c_4}{\kappa}|\epsilon X_1|^{\frac{8}{3}}.$$

Since $\epsilon \neq 0$, it follows that X_1 is bounded and so is X_2.

- Finally, if $\frac{1}{2}|2(1+\epsilon)X_2| \leqslant |2\epsilon X_1^2| \leqslant 2|2(1+\epsilon)X_2|$, then by (4.4),

$$\left(1+\left|\frac{2\epsilon}{1+\epsilon}\right|\right)|X_1|^2 + c_3 \geqslant \frac{c_4}{\kappa}\left(|X_1||\frac{\epsilon}{2(1+\epsilon)}|X_1^2|\right)^{\frac{4}{3}}.$$

Hence, since $\epsilon \neq 0$, X_1 is bounded and then X_2 is so.

In Figure 3 we sketch as an example $\Sigma(2)$ in blue.

We conclude that for $\epsilon \in \mathbb{R} \setminus \{0, -1\}$, Assumption 1 is satisfied, and therefore by Corollary 1.2, K_V has a compact resolvent.

For $\epsilon = 0$, thanks to [4] (see Proposition 10.21, p. 111), we know that the Witten Laplacian defined by

$$\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q), \quad q = (x_1, x_2) \in \mathbb{R}^2$$

has no compact resolvent and then the Kramers–Fokker–Planck operator K_V has no compact resolvent.

This example was studied in the case of the Witten Laplacian operator by Helffer and Nier in their book [4]. A small mistake was made in [4] in Proposition 10.21. In fact, the equations $l_{11} = l_{12} = l_{111} = 0$ should be replaced by $(1+\epsilon)l_{11} = l_{12} = l_{111} = 0$. When $\epsilon = -1$, we can eventually construct a Weyl sequence for the Witten Laplacian operator

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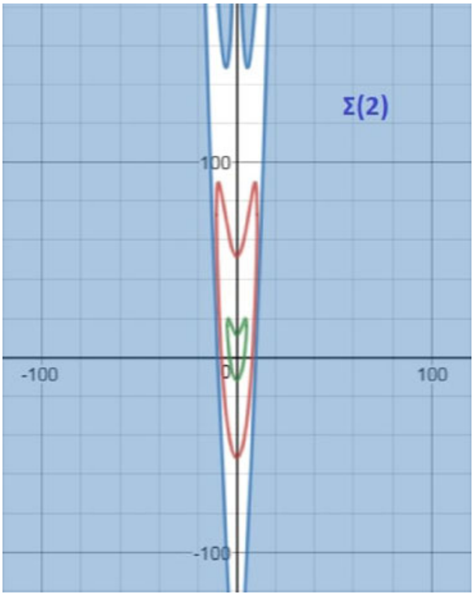

Figure 3. Contour lines of $V(q_1, q_2) = (q_1^2 - q_2)^2 + 0.5q_2^2$.

in the following way. In this case, the potential $V(q_1, q_2) = (q_1^2 - q_2)^2 - q_2^2$ is equal to $-2q_2q_1^2 + q_1^4$.

In order to construct a Weyl sequence for $\Delta_V^{(0)}$, it is sufficient to take $\chi(\frac{(q_2+n^2)}{n})$, where χ is a cutoff function supported in $[-1, 1]$, and then consider the sequence

$$u_n(q_1, q_2) = \chi\left(\frac{(q_2 + n^2)}{n}\right)\exp(-V(q_1, q_2)).$$

The support of u_n is then included in $-n^2 - n \leqslant q_2 \leqslant -n^2 + n$. Hence the u_n's have disjoint supports for large n.

Therefore we have

$$-2n^2 \leqslant q_2 \leqslant -\frac{n^2}{2} \quad \text{and} \quad -4n^2q_1^2 - q_1^4 \leqslant -V(q_1, q_2) \leqslant -n^2q_1^2 - q_1^4 \leqslant -n^2q_1^2.$$

As a result, we get for n large

$$\frac{\langle u_n, \Delta_V^{(0)}u_n\rangle}{\|u_n\|^2} = \frac{\|(\partial_q + \partial_q V(q))(u_n)\|^2}{\|u_n\|^2}$$
$$= \frac{\|(\partial_q\chi)e^{-V}\|^2}{\|u_n\|^2} = O\left(\frac{1}{n^2}\right).$$

Here, to get the lower bound of the above quantity, we restrict the integral in $q_1 = O(\frac{1}{n})$. Hence, for $\epsilon = -1$, the Witten Laplacian attached to $V(q_1, q_2) = (q_1^2 - q_2)^2 + \epsilon q_2^2$ has no

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compact resolvent and then the Kramers–Fokker–Planck operator K_V has no compact resolvent.

As a conclusion, for $\epsilon = -1$ and also for $\epsilon = 0$, the Kramers–Fokker–Planck operator K_V has no compact resolvent.

Acknowledgements. I express my sincere gratitude to Professor Francis Nier. As a PhD advisor, Professor Nier supported me in this work.

A. Slow metric and partition of unity

The purpose of this appendix is to state with references or proofs the facts concerning metrics, which are needed in the article. We first remind the following definitions.

Definitions A.1. *A metric g on $\mathbb{R}^m$ is called a slowly varying metric if there exists a constant $C \geqslant 1$ such that for all $x, y \in \mathbb{R}^m$ satisfying $g_x(x-y, x-y) \leqslant C^{-1}$, it follows that*

$$C^{-1} g_x(z,z) \leqslant g_y(z,z) \leqslant C g_x(z,z) \tag{A 1}$$

holds for all $z \in \mathbb{R}^m$.

Let g^1 and g^2 be two metrics. We say that g^1 is g^2-slow if there is a constant $c \geqslant 1$ such that for all $x, y \in \mathbb{R}^m$,

$$g_x^2(x-y, x-y) \leqslant c^{-1} \Rightarrow c^{-1} g_x^1(z,z) \leqslant g_y^1(z,z) \leqslant c g_x^1(z,z) \tag{A 2}$$

holds for all $z \in \mathbb{R}^m$.

Remark A.2. *The second statement in the above definitions is a typical application of the notion of the second microlocalization developed by Bony–Lerner (see [3]).*

Remark A.3. *Property A 1 will be satisfied if we ask only that*

$$\exists C \geqslant 1, \forall x, y, z \in \mathbb{R}^m,\ g_x(x-y, x-y) \leqslant C^{-1} \Longrightarrow g_y(z,z) \leqslant C g_x(z,z). \tag{A 3}$$

Indeed, assuming (A 3) gives that wherever $g_x(x-y, x-y) \leqslant C^{-1}$ (which is less than or equal to one since $C \geqslant 1$ from (A 3) with $x = y$), this implies $g_y(y-x, y-x) \leqslant C^{-1}$ and then $g_x(z,z) \leqslant C g_y(z,z)$ so that (A 1) is fulfilled.

Notations A.4. *For $r \in \mathbb{N}$, let E_r denote the set of polynomials with degree not greater than r:*

$$E_r = \left\{ P \in \mathbb{R}[X_1, \ldots, X_d],\ d^\circ P \leqslant r \right\}.$$

For a polynomial $P \in E_r$ of degree $r \in \mathbb{N}^*$ and for $n \in \{1, \ldots, r\}$, the function $R_P^{\geqslant n} : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$R_P^{\geqslant n}(q) = \sum_{n \leqslant |\alpha| \leqslant r} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}}. \tag{A 4}$$

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In the present article, we are mainly concerned with the metric $g^n = R_P^{\geqslant n}(q)^2\,dq^2$, where $n \in \{1,\dots,r\}$, which satisfies the following properties.

Lemma A.5. *Let $P \in E_r$, where $r \in \mathbb{N}^*$ is the degree of P, and let n be a natural number in $\{1,\dots,r\}$.*
(1) *The metric g^n is slow: There exists a uniform $C = C(n,r,d) \geqslant 1$ such that*

$$R_P^{\geqslant n}(q)|q-q'| \leqslant C^{-1} \Longrightarrow \left(\frac{R_P^{\geqslant n}(q)}{R_P^{\geqslant n}(q')}\right)^{\pm 1} \leqslant C. \tag{A 5}$$

(2) *The metric g^{n-1} is g^n-slow: There is a constant $C' = C'(n,r,d) \geqslant 1$ so that*

$$R_P^{\geqslant n}(q)|q-q'| \leqslant C'^{-1} \Longrightarrow \left(\frac{R_P^{\geqslant n-1}(q)}{R_P^{\geqslant n-1}(q')}\right)^{\pm 1} \leqslant C'. \tag{A 6}$$

Proof. The dimension d is fixed. Assume $n, r \in \mathbb{N}^*$ with $n \leqslant r$. The set

$$K_{n,r} := \left\{\overline{P} \in E_r/E_{n-1},\ R_{\overline{P}}^{\geqslant n}(0) = R_P^{\geqslant n}(0) = 1\right\}$$

is a compact set of E_r/E_{n-1}, where $\overline{P} \in E_r/E_{n-1}$ can be identified with the polynomial $\overline{P}(q) = \sum_{n\leqslant|\alpha|\leqslant r} \frac{\partial_q^\alpha P(0)}{\alpha!} q^\alpha$. For any $\varrho \geqslant 0$, the mapping

$$\begin{array}{lcl} K_{n,r} \times \overline{B(0,\varrho)} & \to & [0,+\infty) \\ (\overline{P}, t) & \mapsto & R_{\overline{P}}^{\geqslant n}(t) = \sum_{|\alpha|\geqslant n} |\partial_x^\alpha \overline{P}(t)|^{\frac{1}{|\alpha|}} \end{array}$$

is continuous because $s \mapsto s^\nu$ is continuous on $[0,+\infty)$ for any $\nu > 0$. On the compact set $K_{n,r} \times \overline{B(0,\varrho)}$, it admits a maximum $M_{n,r,\varrho}$ and a minimum $m_{n,r,\varrho}$, which cannot be 0. Actually, $R_{\overline{P}}^{\geqslant n}(t_0) = 0$ means $\partial_x^\alpha P(t_0) = 0$ for all $\alpha \in \mathbb{N}^d$, $|\alpha| \geqslant n$, and by the Taylor expansion $\partial_x^\alpha \overline{P}(t) = 0$ for all $t \in \mathbb{R}^d$, $\alpha \in \mathbb{N}^d$, $|\alpha| \geqslant n$, which contradicts $R_{\overline{P}}^{\geqslant n}(0) = 1$.
Now for a general $V \in E_r$ with $d^\circ V \geqslant n$, we know that for all $q \in \mathbb{R}^d$, $R_V^{\geqslant n}(q) \neq 0$. We thus consider for any $q \in \mathbb{R}^d$, the class $\overline{P}_q$ of $V(q + R_V^{\geqslant n}(q)^{-1}t)$ in E_r/E_{n-1}. It satisfies

$$R_{\overline{P}_q}^{\geqslant n}(t) = \sum_{n\leqslant|\alpha|\leqslant r} |\partial_t^\alpha P_q(t)|^{\frac{1}{|\alpha|}} = R_V^{\geqslant n}(q)^{-1} R_V^{\geqslant n}(q + R_V^{\geqslant n}(q)^{-1}t)\,,$$

and in particular,

$$R_{\overline{P}_q}^{\geqslant n}(0) = 1, \quad \overline{P}_q \in K_{n,r}.$$

Therefore we obtain for $\varrho = 1$

$$(|t| \leqslant 1) \Rightarrow \left(m_{n,r,1} \leqslant \left(\frac{R_V^{\geqslant n}(q + R_V^{\geqslant n}(q)^{-1}t)}{R_V^{\geqslant n}(q)}\right) \leqslant M_{n,r,1}\right),$$

which implies, with $q' = q + \frac{t}{R_V^{\geqslant n}(q)}$,

$$\left(R_V^{\geqslant n}(q)|q-q'| \leqslant 1\right) \Rightarrow \left(\frac{R_V^{\geqslant n}(q)}{R_V^{\geqslant n}(q')}\right)^{\pm 1} \leqslant \max\left\{M_{n,r,1}, \frac{1}{m_{n,r,1}}\right\}.$$

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We conclude the proof of (1) by choosing $C(n,r,d) = \max\{M_{n,r,1}, \frac{1}{m_{n,r,1}}, 1\}$ and by applying the more general result to $P \in E_r$ such that $d^\circ P = r$.
Let us prove (2). We still work in $K_{n,r} = \{\overline{P} \in E_r/E_{n-1},\ R_{\overline{P}}^{\geqslant n}(0) = R_P^{\geqslant n}(0) = 1\}$ and now with a given $\varrho \in]0,1]$. From the proof of (1), we know that there exists $M_{n,r,1}, m_{n,r,1} > 0$ such that

$$m_{n,r,1} \leqslant R_{\overline{P}}^{\geqslant n}(t) \leqslant M_{n,r,1}$$

for all t, $|t| \leqslant \varrho \leqslant 1$ and all $\overline{P} \in K_{n,r}$.
In particular, there exists a constant $\tilde{C}_{n,r}$ such that

$$\forall \overline{P} \in K_{n,r}, \forall t \in \overline{B(0,\varrho)} \subset \overline{B(0,1)}, \qquad \max_{n \leqslant |\alpha| \leqslant r} \left|\partial_q^\alpha \overline{P}(t)\right| \leqslant \tilde{C}_{n,r}. \tag{A 7}$$

For any $P \in E_r$ in the class $\overline{P} \in E_r/E_{n-1}$, we decompose $R_P^{\geqslant n-1}(t)$ into

$$R_P^{\geqslant n-1}(t) = \sum_{|\beta|=n-1} \left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}} + R_{\overline{P}}^{\geqslant n}(t).$$

By the Taylor expansion,

$$\partial_q^\beta P(t) - \partial_q^\beta P(0) = \sum_{1 \leqslant |\alpha'| \leqslant r-n+1} \frac{\partial_q^{\beta+\alpha'} P(0)}{\alpha'!} t^{\alpha'}, \quad |\beta| = n-1,$$

and owing to (A 7), there exists a constant $C_{n,r} > 0$ such that the inequality

$$\left| |\partial_q^\beta P(t)| - |\partial_q^\beta P(0)| \right| \leqslant C_{n,r}\varrho$$

holds for all $\beta \in \mathbb{N}^d$, $|\beta| = n-1$, and all $t \in \mathbb{R}^d$, $|t| \leqslant \varrho \leqslant 1$.
The uniform continuity of $s \mapsto s^{\frac{1}{n-1}}$ on $[0,+\infty[$ now implies

$$\left| \sum_{|\beta|=n-1} \left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}} - \sum_{|\beta|=n-1} \left|\partial_q^\beta P(0)\right|^{\frac{1}{n-1}} \right| \leqslant \varepsilon_{n,r}(\varrho)$$

with $\lim_{\varrho\to 0} \varepsilon_{n,r}(\varrho) = 0$ uniformly with respect to $P \in \overline{P}$, $\overline{P} \in K_{n,r}$ and $t \in \overline{B(0,\varrho)} \subset \overline{B(0,1)}$.
On one side, we write

$$\begin{aligned} R_P^{\geqslant n-1}(t) &\leqslant \sum_{|\beta|=n-1} \left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}} + M_{n,r,1} \\ &\leqslant \sum_{|\beta|=n-1} \left|\partial_q^\beta P(0)\right|^{\frac{1}{n-1}} + \varepsilon_{n,r}(\varrho) + M_{n,r,1} \\ &\leqslant \max(1, \varepsilon_{n,r}(\varrho) + M_{n,r,1}) R_P^{\geqslant n-1}(0). \end{aligned}$$

On the other side, we have

$$R_P^{\geqslant n-1}(0) \leqslant \sum_{|\beta|=n-1} \left|\partial_q^\beta P(0)\right|^{\frac{1}{n-1}} + M_{n,r,1}$$

$$\leqslant \sum_{|\beta|=n-1} \left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}} + \varepsilon_{n,r}(\varrho) + M_{n,r,1}$$
$$\leqslant \max\left(1, \frac{\varepsilon_{n,r}(\varrho) + M_{n,r,1}}{m_{n,r,1}}\right) R_P^{\geqslant n-1}(t).$$

For $\varrho_{n,r} \in]0, 1]$ chosen small enough such that $\varepsilon_{n,r}(\varrho_{n,r}) \leqslant M_{n,r,1}$, we deduce

$$\forall \overline{P} \in K_{n,r},\ \forall P \in \overline{P},\ \forall t \in \overline{B(0, \varrho_{n,r})}, \quad \left(\frac{R_P^{\geqslant n-1}(t)}{R_P^{\geqslant n-1}(0)}\right)^{\pm 1} \leqslant \max\left(2M_{n,r,1}, \frac{2M_{n,r,1}}{m_{n,r,1}}\right).$$

For $V \in E_r$ such that $d^\circ V \geqslant n$, we apply the previous estimate to $P_q(t) = V(q + R_V^{\geqslant n}(q)^{-1}t)$, with $\overline{P_q} \in K_{n,r}$, which leads to

$$\left(R_V^{\geqslant n}(q)|q - q'| \leqslant \varrho_{n,r}\right) \Rightarrow \left(\left(\frac{R_V^{\geqslant n-1}(q)}{R_V^{\geqslant n-1}(q')}\right)^{\pm 1} \leqslant \max\left(2M_{n,r,1}, \frac{2M_{n,r,1}}{m_{n,r,1}}\right)\right).$$

We conclude the proof by choosing $C'(n, r, d) = \max(2M_{n,r,1}, \frac{2M_{n,r,1}}{m_{n,r,1}}, \frac{1}{\varrho_{n,r}})$ and by applying the more general result to $P \in E_r$ such that $d^\circ P = r$. □

Remark A.6. *The proof of (1) gives a more general result than the slowness, namely when* n, r, d *are fixed: For any* $\lambda > 0$*, there exists* $C_\lambda \geqslant 1$ *such that*

$$(R_P^{\geqslant n}(q)|q - q'| \leqslant \lambda) \Rightarrow \left(\left(\frac{R_P^{\geqslant n}(q)}{R_P^{\geqslant n}(q')}\right)^{\pm 1} \leqslant C_\lambda\right),$$

without assuming that $\lambda > 0$ *is small. Actually, it is even possible to estimate* C_λ *in terms of* $\lambda \to \infty$ *by applying lemma A.5 to the polynomial* tP, $t \in [0, 1]$, *with* $t^{\frac{1}{n}} R_P^{\geqslant n}(q) \leqslant R_{tP}^{\geqslant n}(q) \leqslant t^{\frac{1}{r}} R_P^{\geqslant n}(q)$.

The main feature of a slow varying metric is that it is possible to introduce some partitions of unity related to the metric in a way made precise in the following theorem. For more details and proof, see [9] (Section 1.4, p. 25).

Theorem A.7 [9]. *For any slowly varying metric* g *in* $\mathbb{R}^m$*, one can choose a sequence* $x_\nu \in \mathbb{R}^m$ *such that the balls*

$$B_\nu = \left\{x;\ \sqrt{g_{x_\nu}(x - x_\nu, x - x_\nu)} < 1\right\}$$

form a covering of $\mathbb{R}^m$ *for which the intersection of more than* $N = (4C^3 + 1)^m$ *balls* B_ν *is always empty (*C *is the constant in (A 1)). In addition, for any decreasing sequence* d_i *with* $\sum_j d_j = 1$*, one can choose nonnegative* $\phi_\nu \in \mathcal{C}_0^\infty(B_\nu)$ *with* $\sum \phi_\nu = 1$ *in* $\mathbb{R}^m$ *so that for all* k*,*

$$|\phi_\nu^{(k)}(x; y_1, \ldots, y_k)| \leqslant (NCC_1)^k \sqrt{g_x(y_1, y_1)} \cdots \sqrt{g_x(y_k, y_k)}/d_1 \cdots d_k,$$

where C *is the constant in (A 1) and* C_1 *is a constant that depends only on* m.

Regarding the above theorem, we have the following result.

Lemma A.8. *Let $P \in E_r$, where $r \in \mathbb{N}^*$ is the degree of P and $n \in \{1, \dots, r\}$. Then there exists a partition of unity $\sum_{j\in\mathbb{N}} \Psi_j(q)^2 \equiv 1$ in $\mathbb{R}^d$ such that we have the following:*
(1) *For all $q \in \mathbb{R}^d$, the cardinality of the set $\{j, \Psi_j(q) \neq 0\}$ is uniformly bounded.*
(2) *For any natural number $j \in \mathbb{N}$,*

$$\operatorname{supp} \Psi_j \subset B(q_j, aR_P^{\geqslant n}(q_j)^{-1}) \quad \text{and} \quad \Psi_j \equiv 1 \ \text{ in } B(q_j, bR_P^{\geqslant n}(q_j)^{-1})$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of $j \in \mathbb{N}$.
(3) *For all $\alpha \in \mathbb{N}^d \setminus \{0\}$, there exists $c_\alpha > 0$ such that*

$$\sum_{j\in\mathbb{N}} |\partial_q^\alpha \Psi_j|^2 \leqslant c_\alpha R_P^{\geqslant n}(q)^{2|\alpha|}.$$

Moreover, the constants a, b and c_α can be chosen uniformly with respect to $P \in E_r$, once the degree $r \in \mathbb{N}$ and the dimension $d \in \mathbb{N}$ are fixed.

B. Around Tarski–Seidenberg theorem

In this appendix, we give an application of the Tarski–Seidenberg theorem [10], which we state in the following geometric form. We first introduce a few basic concepts needed for the statement.

Definition B.1. *A subset of $\mathbb{R}^n$ is called semialgebraic if it is a finite union of intersections of finitely many sets defined by polynomial equations or inequalities.*

Definition B.2. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be two subalgebraic sets. The function $f : A \to B$ is said to be semialgebraic if its graph $\Gamma_f = \{(x, y) \in A \times B;\ y = f(x)\}$ is a semialgebraic set of $\mathbb{R}^n \times \mathbb{R}^m$.*

Theorem B.3 [10] (Tarski–Seidenberg). *If A is a semialgebraic subset of $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$, then the projection A' of A in $\mathbb{R}^m$ is also semialgebraic.*

Proposition B.4 [10]. *If E is a semialgebraic set on $\mathbb{R}^{2+n}$, and*

$$f(x) = \inf\left\{y \in \mathbb{R};\ \exists z \in \mathbb{R}^n, (x, y, z) \in E\right\}$$

is defined and finite for large positive x, then f is identically 0 for large x or else

$$f(x) = Ax^a(1 + o(1)), \quad x \to +\infty,$$

where $A \neq 0$ and a is a rational number.

We refer to [10] (see Theorems A.2.2 and A.2.5) for detailed proofs of Theorem B.3 and Proposition B.4.

In the final part of this section, we list and recall the following notations.

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Notation B.5. *Let P be a real-valued polynomial on $\mathbb{R}^d$ with $d^\circ P = r$. For all natural numbers $n \in \{0, \dots, r\}$ and every $q \in \mathbb{R}^d$,*

$$R_P^{\geqslant n}(q) = \sum_{n \leqslant |\alpha| \leqslant r} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}}, \tag{B 1}$$

$$R_P^{=n}(q) = \sum_{|\alpha|=n} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}}. \tag{B 2}$$

Lemma B.6. *Let $\widetilde{\Sigma}$ be an unbounded semialgebraic set and V a polynomial in $\mathbb{R}[q_1, \dots, q_d]$ of degree $r \in \mathbb{N}^*$ satisfying the assumption*

$$\lim_{\substack{|q| \to +\infty \\ q \in \widetilde{\Sigma}}} \frac{R_V^{\geqslant n}(q)^\alpha}{R_V^{=m}(q)^2} = 0, \tag{B 3}$$

where $\alpha \in \mathbb{Q}$, $n, m \in \{0, 1, \dots, r-1\}$, $n > m$, are fixed numbers.

Then there exist $\delta \in (0, 1)$ and a positive nondecreasing function $\Lambda_{\widetilde{\Sigma}} : (0, +\infty) \to [0, +\infty)$ so that

$$\begin{aligned} &\forall q \in \widetilde{\Sigma},\ \forall \varrho > 0,\ |q| \geqslant \varrho, \quad \Lambda_{\widetilde{\Sigma}}(\varrho) R_V^{\geqslant n}(q)^\alpha \leqslant R_V^{=m}(q)^{2(1-\delta)} \\ \textit{and} \quad &\lim_{\varrho \to +\infty} \Lambda_{\widetilde{\Sigma}}(\varrho) = +\infty. \end{aligned}$$

Proof. Let V be a real-valued polynomial on $\mathbb{R}^d$ with degree $r \in \mathbb{N}^*$. Suppose that there are $\alpha \in \mathbb{Q}$, $n, m \in \{0, 1, \dots, r-1\}$ such that

$$\lim_{\substack{|q| \to +\infty \\ q \in \widetilde{\Sigma}}} \frac{R_V^{\geqslant n}(q)^\alpha}{R_V^{=m}(q)^2} = 0, \tag{B 4}$$

where $\widetilde{\Sigma}$ is a given unbounded semialgebraic set.

After setting $\tau = 2\,\mathrm{LCM}\big(|\beta|,\ \min(n, m) \leqslant |\beta| \leqslant r\big)$, where the abbreviation LCM stands for least common multiple, define the functions $\widetilde{R}_V^{\geqslant n}$ and $\widetilde{R}_V^{=m}$ for all $q \in \mathbb{R}^d$ by

$$\widetilde{R}_V^{\geqslant n}(q) = \sum_{n \leqslant |\alpha| \leqslant r} |\partial_q^\alpha V(q)|^{\frac{\tau}{|\alpha|}}$$

and

$$\widetilde{R}_V^{=m}(q) = \sum_{|\alpha|=m} |\partial_q^\alpha V(q)|^{\frac{\tau}{|\alpha|}}.$$

Note that one has the equivalences $R_V^{\geqslant n}(q) \asymp \big(\widetilde{R}_V^{\geqslant n}(q)\big)^{\frac{1}{\tau}}$ and $R_V^{=m}(q) \asymp \big(\widetilde{R}_V^{=m}(q)\big)^{\frac{1}{\tau}}$ for all $q \in \mathbb{R}^d$, where the functions $R_V^{\geqslant n}$ and $R_V^{=m}$ are defined respectively as in (B 1) and (B 2). Clearly, Assumption (B 4) is equivalent to

$$\lim_{\substack{|q| \to +\infty \\ q \in \widetilde{\Sigma}}} \frac{\widetilde{R}_V^{\geqslant n}(q)^\alpha}{\widetilde{R}_V^{=m}(q)^2} = 0. \tag{B 5}$$

Remark here that $\widetilde{R}_V^{\geqslant n}(q)$ and $\widetilde{R}_V^{=m}(q)$ are polynomials in $q \in \mathbb{R}^d$ variable. Furthermore, Assumption (B 5) can be written as

$$\widetilde{R}_V^{\geqslant n}(q)^\alpha \leqslant \epsilon(q)\widetilde{R}_V^{=m}(q)^2$$

for all $q \in \widetilde{\Sigma}$, where

$$\epsilon(q) = \inf\left\{\epsilon > 0,\ \epsilon\widetilde{R}_V^{=m}(q)^2 - \widetilde{R}_V^{\geqslant n}(q)^\alpha > 0\right\}, \qquad \lim_{\substack{|q|\to+\infty \\ q\in\widetilde{\Sigma}}} \epsilon(q) = 0. \tag{B 6}$$

Now, following the notations of Proposition B.4, we introduce the set

$$E = \left\{(q, \varrho, \epsilon) \in \mathbb{R}^{d+2} \text{ such that } \epsilon\widetilde{R}_V^{=m}(q)^2 - \widetilde{R}_V^{\geqslant n}(q)^\alpha > 0 \text{ and } |q|^2 \geqslant \varrho^2\right\},$$

and the function f defined in $\mathbb{R}_+$ by

$$f(\varrho) = \inf\left\{\epsilon > 0 :\ \exists q \in \mathbb{R}^d,\ \ (q, \varrho, \epsilon) \in E\right\}. \tag{B 7}$$

By the Tarski–Seidenberg theorem (see Theorem B.3), the function f is semialgebraic in ϱ. Moreover, f is defined, finite and not identically zero. Then by Proposition B.4, there exist a constant $A > 0$ and a rational number γ such that

$$f(\varrho) = A\varrho^\gamma + o_{\varrho\to+\infty}(\varrho^\gamma).$$

By definitions (B 6) and (B 7), $\lim_{\varrho\to+\infty} f(\varrho) = 0$ and then $\gamma < 0$. Hence for $\varrho \gg 1$, we know $f(\varrho) \leqslant \frac{2A}{\varrho^{|\gamma|}}$. We deduce for $|q| \gg 1$,

$$\widetilde{R}_V^{\geqslant n}(q)^\alpha \leqslant f(|q|)\widetilde{R}_V^{=m}(q)^2 \leqslant \frac{2A}{|q|^{|\gamma|}}\widetilde{R}_V^{=m}(q)^2 \tag{B 8}$$

and

$$\frac{|q|^{|\gamma|/2}}{2A}\widetilde{R}_V^{\geqslant n}(q)^\alpha \leqslant \frac{1}{|q|^{\frac{|\gamma|}{2}}}\widetilde{R}_V^{=m}(q)^2. \tag{B 9}$$

In particular, since $\widetilde{R}_V^{\geqslant n}(q) \geqslant \widetilde{R}_V^{=r}(0) > 0$, $\widetilde{R}_V^{=m}(q)$ does not vanish for $q \in \widetilde{\Sigma}$ with $|q| \geqslant 1$.

On the other hand, note

$$\forall q \in \widetilde{\Sigma},\ |q| \geqslant 1, \quad \widetilde{R}_V^{=m}(q) \leqslant c|q|^{\tau r}. \tag{B 10}$$

Inequalities (B 8) and (B 10) lead to

$$\widetilde{R}_V^{\geqslant n}(q)^\alpha \leqslant C|q|^{2\tau r - |\gamma|}$$

for every $q \in \widetilde{\Sigma}$ with $|q| \geqslant \rho \gg 1$. Therefore, since $\widetilde{R}_V^{\geqslant n}(q) \geqslant \widetilde{R}_V^{=r}(0) > 0$, we deduce $|\gamma| \leqslant 2\tau r$.

Using again (B 10), we get

$$\frac{1}{|q|^{\frac{|\gamma|}{2}}} \leqslant \frac{c^{\frac{|\gamma|}{2\tau r}}}{\widetilde{R}_V^{=m}(q)^{\frac{|\gamma|}{2\tau r}}} \tag{B 11}$$

for any $q \in \widetilde{\Sigma}$ with $|q| \geqslant 1$.

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From (B 9) and (B 11), we deduce the existence of $\varrho_0 \gg 1$ such that

$$\forall q \in \widetilde{\Sigma}, |q| \geqslant \varrho \geqslant \varrho_0 \gg 1, \quad \frac{\varrho^{|\gamma|/2}}{2A} \widetilde{R}_V^{\geqslant n}(q)^\alpha \leqslant \frac{|q|^{|\gamma|/2}}{2A} \widetilde{R}_V^{\geqslant n}(q)^\alpha \leqslant c^{\frac{|\gamma|}{2\tau r}} \widetilde{R}_V^{=m}(q)^{2(1-\frac{|\gamma|}{4\tau r})}. \tag{B 12}$$

We now take $\delta = \frac{|\gamma|}{4\tau r} \in (0,1)$ and

$$\widetilde{\Lambda}_{\widetilde{\Sigma}}(\varrho) = \begin{cases} \frac{\varrho^{|\gamma|/2}}{2Ac^{\frac{|\gamma|}{2\tau r}}} & \text{if } \varrho \geqslant \varrho_0 \\ 0 & \text{otherwise.} \end{cases}$$

The function $\widetilde{\Lambda}_{\widetilde{\Sigma}} : (0,+\infty) \to [0,+\infty)$ is clearly positive, and due to (B 12), it satisfies

$$\forall q \in \widetilde{\Sigma},\ \forall \varrho > 0,\ |q| \geqslant \varrho, \quad \widetilde{\Lambda}_{\widetilde{\Sigma}}(\varrho) R_V^{\geqslant n}(q)^\alpha \leqslant R_V^{=m}(q)^{2(1-\delta)}$$
$$\text{and} \quad \lim_{\varrho \to +\infty} \widetilde{\Lambda}_{\widetilde{\Sigma}}(\varrho) = +\infty.$$

To conclude, it is sufficient to take $\Lambda_{\widetilde{\Sigma}} : (0,+\infty) \to [0,+\infty)$ defined by

$$\Lambda_{\widetilde{\Sigma}}(\varrho) = \inf_{|q| \geqslant \varrho} \frac{R_V^{=m}(q)^{2(1-\delta)}}{R_V^{\geqslant n}(q)^\alpha},$$

which is nondecreasing and larger than $\tilde{\Lambda}_{\widetilde{\Sigma}}$. □

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