CHARACTERIZATIONS OF LINEAR DIFFERENTIAL SYSTEMS WITH A REGULAR SINGULAR POINT

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The linear differential system

$$\frac{dw}{dz} = A(z)w \tag{1}$$

where w is a vector with n components and A is an n by n matrix is said to have $z = 0$ as a regular singular point if there exists a fundamental matrix of the form

$$W(z) = S(z)z^R$$

such that S is holomorphic at $z = 0$ and R is a constant matrix ((**1**), p. 111; (**2**), p. 73). For such systems A will have at most a pole at $z = 0$ and we may write

$$A(z) = z^{-p-1}\tilde{A}(z)$$

where p is an integer, $\tilde{A}$ is holomorphic at $z = 0$, and $\tilde{A}(0) \neq 0$. However, the converse is not true. When $p \leqq -1$, A is holomorphic at $z = 0$, and every fundamental matrix is holomorphic at $z = 0$. If $p \geqq 1$, the non-negative integer p is called (after Poincaré) the *rank* of the singularity and there is a significant difference between the cases $p = 0$ and $p \geqq 1$. If $p = 0$ the linear differential system (1) is known to have $z = 0$ as a regular singular point ((**1**), p. 111); whereas, if $p \geqq 1$, $z = 0$ *may or may not* be a regular singular point.

Conceptually, a regular singular point is more intrinsic than the rank of a singular point and correspondingly more difficult to ascertain for a given system. This is easily explained by noting that a regular singular point is invariant under changes of dependent variable whereas rank is not; i.e. if T is a nonsingular matrix meromorphic at $z = 0$, the transformation $w = Tu$ carries the linear differential system (1) into the linear differential system

$$\frac{du}{dz} = B(z)u \tag{2}$$

where

$$B = T^{-1}AT - T^{-1}\frac{d}{dz}T \tag{3}$$

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and clearly, $z = 0$ is a regular singular point for $\frac{dw}{dz} = A(z)w$ if and only if $z = 0$ is a regular singular point for $\frac{du}{dz} = B(z)u$. For meromorphic T with $\det T \not\equiv 0$, equation (3) sets up an equivalence relation between the linear differential systems (1) and (2). Clearly, a necessary and sufficient condition that the linear differential system (1) have $z = 0$ as a regular singular point is that it be equivalent to a system of rank zero.

J. Moser (7) and W. B. Jurkat and D. A. Lutz (3), (4) have given algorithms to determine whether a given system with positive rank has a regular singular point. However, the application of these algorithms is lengthy, and it is desirable to have alternate criteria.

In this note we present necessary conditions for a regular singular point in terms of the symmetric functions of the matrix A. These results are modifications and extensions of results of D. A. Lutz (5) and include a singular point of rank one. Investigations of a regular singular point for linear differential systems with a singular point of rank one is important since H. L. Turrittin (8) has shown that arbitrary rank can be reduced to rank one by increasing the dimension of the system. For a discussion of this reduction in rank and further ramifications, see D. A. Lutz (6).

Our criteria are embodied in the following theorems.

Theorem 1. *Let* $A(z) = z^{-q} \sum_{k=0}^{\infty} A_k z^k$, $|z| < \delta$, $A_0 \neq 0$, $q > 0$, *and*

$$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n.$$

If $\frac{dw}{dz} = A(z)w$ *has* $z = 0$ *as a regular singular point, then*

(i) $a_k(z) = O(z^{-(k-1)q} + z^{-k}), \quad k = 1, \ldots, n,$

(ii) $A_0^k = 0$ *for some* $k \leqq n$,

(iii) $\operatorname{trace}(A_0^k A_1) = 0 \begin{cases} k = 0, 1, \ldots, n-1, & \text{if } q \geqq 3 \\ k = 1, \ldots, n-1, & \text{if } q = 2. \end{cases}$

Theorem 2. *Let* A *be as in Theorem* 1 *and let*

$$a_k(z) = O(z^{-(k-1)q} + z^{-k}), \quad k = 1, \ldots, n.$$

Then

(i) $A_0^n = 0$;

and if $A_0^{n-1} \neq 0$,

(ii) $\operatorname{trace}(A_0^k A_1) = 0 \begin{cases} k = 0, 1, \ldots, n-1, & \text{if } q \geqq 3 \\ k = 1, \ldots, n-1, & \text{if } q = 2. \end{cases}$

Corollary. *Let* $A_0^{n-1} \neq 0$. *Then* $a_k(z) = O(z^{-(k-1)q} + z^{-k})$ *if and only if there exists a transformation matrix* $P(z) = I + P_1 z + \ldots + P_m z^m$ *which has the property that*

$$P^{-1}AP - P^{-1}\frac{d}{dz}P = z^{-q}A_0 + z^{-1}R_1 + R(z)$$

where $R(z)$ *is holomorphic at* $z = 0$. *Further, if* $q \geqq 2$ *and* $a_k(z) = O(z^{-(k-1)q})$, $k = 1, \ldots, n$, *then* $R_1 = 0$.

Proof of Theorem 1. If $\dfrac{dw}{dz} = A(z)w$ has $z = 0$ as a regular singular point, there exists a fundamental matrix $W(z) = S(z)z^R$ and without loss of generality we may assume that $S(z)$ is holomorphic at $z = 0$ and $S(0) \neq 0$. Let $S(z)$ have the factorization

$$S(z) = P(z)z^D Q(z) \tag{4}$$

where $P(z)$ is a polynomial in z, $\det P(z) \equiv 1$, $Q(z)$ is holomorphic at $z = 0$, $\det Q(0) \neq 0$, and $z^D = \operatorname{diag}(z^{d_1}, z^{d_2}, \ldots, z^{d_n})$, d_i non-negative integers,

$$0 = d_1 \leqq d_2 \leqq \ldots \leqq d_n$$

((2), p. 75).

Then

$$A \sim B = P^{-1}AP - P^{-1}\frac{d}{dz}P,$$

$$B \sim C = z^{-D}Bz^D - z^{-D}(z^{-1}D)z^D,$$

$$C \sim z^{-1}R = Q^{-1}CQ - Q^{-1}\frac{d}{dz}Q.$$

Since $Q(0)$ is nonsingular, $C = O(z^{-1})$. Writing $B = z^D(C + z^{-1}D)z^{-D}$, we have

$$\det(\lambda I - B) = \lambda^n + b_1\lambda^{n-1} + \ldots + b_n,$$

where

$$b_k(z) = O(z^{-k}), \quad k = 1, \ldots, n. \tag{5}$$

We have

$$A = P\left(B + P^{-1}\frac{d}{dz}P\right)P^{-1} \tag{6}$$

and

$$\det(\lambda I - A) = \det\left(\lambda I - B - P^{-1}\frac{d}{dz}P\right)$$

$$= \det(\lambda I - B) + \Sigma\left(\lambda I - B, \; -P^{-1}\frac{d}{dz}P\right),$$

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where $\Sigma(F, G)$ represents the sum of all determinants formed from k rows of F and $n-k$ rows of G with natural ordering for $0 \leqq k<n$. Hence

$$\Sigma\left(\lambda I-B, \ -P^{-1}\frac{d}{dz}P\right)=f_1\lambda^{n-1}+\ldots+f_n,$$

where

$$f_k(z)=O(z^{-(k-1)q}), \quad k=1, \ldots, n. \tag{7}$$

Combining the order conditions (5) and (7) we have proved (i).

Write

$$A(z)=z^{-q}(A_0+zA_1+\ldots),\ B(z)=z^{-q}(B_0+zB_1+\ldots),\ P=P_0+zP_1+\ldots$$

where P_0 is nonsingular. Then from equation (6) we have ($q \geqq 2$)

$$\begin{aligned} A_0 &= P_0B_0P_0^{-1} \\ A_1 &= P_0B_1P_0^{-1}+P_1P_0^{-1}A_0-A_0P_1P_0^{-1}. \end{aligned} \tag{8}$$

From $B=z^D(C+z^{-1}D)z^{-D}$, the ijth element of B satisfies

$$b_{ij}(z)=O(z^{d_i-d_j-1}).$$

Since the d_i are nondecreasing all the elements on and below the diagonal are zero for B_0 if $q=2$ and for B_0 and B_1 if $q \geqq 3$. Thus B_0 and hence A_0 is nilpotent and trace $(B_0^kB_1)=0$, $k=0, 1, \ldots, n-1$ in case $q \geqq 3$. For $q=2$, write $B_0=(b_{ij}^0)$ and $B_1=(b_{ij}^1)$. Note that $b_{ij}^1 \neq 0$, $i>j$, implies $d_i=d_k$ for $j \leqq k \leqq i$ and hence $b_{ki}(z)=O(z^{-1})$ for $j \leqq k \leqq i$; or $b_{ki}^0=0$ for $j \leqq k \leqq i$. Since $b_{ij}^0=0$, $i \geqq j$, we have for $k=1, \ldots, n-1$,

$$\operatorname{trace}(B_0^kB_1)=\sum_{i=1}^{n}\ \sum_{i<i_1<\ldots<i_k} b_{ii_1}^0\ldots b_{i_{k-1}i_k}^0 b_{i_ki}^1=0.$$

Using equation (8) we have

$$A_0^kA_1=P_0B_0^kB_1P_0^{-1}+A_0^kP_1P_0^{-1}A_0-A_0^{k+1}P_1P_0^{-1}.$$

Thus

$$\operatorname{trace}(A_0^kA_1)=\operatorname{trace}(B_0^kB_1)=0,$$

and Theorem 1 is proved.

Proof of Theorem 2. The matrix A satisfies its characteristic equation. Hence, using the order conditions on the coefficients of the characteristic equation, we obtain for $q \geqq 2$,

$$(z^qA)^n-(\operatorname{trace} A)(z^qA)^{n-1}z^q=O(z^q).$$

Hence, $A_0^n=0$ and

$$A_0^{n-1}A_1+A_0^{n-2}A_1A_0+\ldots+A_1A_0^{n-1}=\begin{cases}0, & q \geqq 3\\ (\operatorname{trace} A_1)A_0^{n-1}, & q=2.\end{cases} \tag{9}$$

Assuming, without loss of generality, that A_0 has Jordan canonical form a simple computation shows that trace $(A_0^kA_1)=0$, $k=0, 1, \ldots, n-1$ in case $q \geqq 3$, and $k=1, \ldots, n-1$ in case $q=2$.

Remark. The equation (9) is always satisfied if the order conditions

$$a_k = O(z^{-(k-1)q}+z^{-k})$$

are satisfied. However, if A_0 is not nilpotent of maximum rank, this equation does not imply that trace $(A_0^k A_1) = 0$, $k = 1, \ldots, n-2$.

Proof of Corollary. The necessity can be proved as in Theorem 1. To prove sufficiency, let $A(z) = z^{-q}(A_0 + A_k z^k + \ldots)$ where $k \geqq 1$. In the same manner that equation (9) was derived we have

$$A_0^{n-1}A_k + \ldots + A_k A_0^{n-1} = \begin{cases} 0, & k \leqq q-2 \\ (\text{trace } A_{q-1})A_0^{n-1}, & k = q-1 \end{cases} \tag{10}$$

and hence, if $A_0^{n-1} \neq 0$, trace $(A_0^j A_k) = 0$, $j = 0, 1, \ldots, n-1$, $k \leqq q-2$.

Consider the transformation $P = I + P_k z^k$,

$$A \sim B = P^{-1}AP - P^{-1}\frac{d}{dz}P = z^{-q}(B_0 + B_k z^k + \ldots),$$

where $B_0 = A_0$, $B_k = A_0 P_k - P_k A_0 + A_k$. It is well known ((7), pp. 102-104) that the conditions trace $(A_0^j A_k) = 0$, $j = 0, 1, \ldots, n-1$, are necessary and sufficient for the existence of P_k such that $B_k = 0$. Since transformations of the type P do not affect the order conditions, the required transformation can be constructed recursively.

We now show by example that the order conditions of Theorem 1 are sharp.

Example. Let N be a maximal rank nilpotent in (super) Jordan canonical form and R a constant diagonal matrix. Then the system $\frac{dw}{dz} = B(z)w$, where $B = z^{-q}N + z^{-1}R$, has $z = 0$ as a regular singular point. This is easily seen since $B \sim C = z^{-D}Bz^D - z^{-1}D = z^{-1}(N+R-D)$, where

$$D = \text{diag}\,(0, q-1, 2(q-1), \ldots, (n-1)(q-1)).$$

For any constant matrix E, let $P(z) = \exp\{zE\}$. Then the system $\frac{dw}{dz} = A(z)w$ has $z = 0$ as a regular singular point if $A(z)$ is defined as

$$A(z) = P^{-1}(z)B(z)P(z) - P^{-1}(z)\frac{d}{dz}P(z). \tag{11}$$

Choose the first $n-1$ rows of E to be zero and the nth row to be $(1, 1, \ldots, 1, 0)$, $R = \text{diag}\,(0, \ldots, 0, -1)$, and note that

$$A(z) = P^{-1}(z)[z^{-q}N + z^{-1}R - E]P(z).$$

If $D_n(\lambda) = \det(\lambda I - A)$ where A is n by n, then considering n as a variable $(n \geqq 2)$ it follows that $D_2(\lambda) = \lambda^2 + z^{-1}\lambda + z^{-q}$, $D_{n+1}(\lambda) = \lambda D_n(\lambda) + z^{-nq}$ and

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hence by induction that

$$D_n(\lambda) = \lambda^n + z^{-1}\lambda^{n-1} + z^{-q}\lambda^{n-2} + \ldots + z^{-(n-1)q}.$$

Thus the order conditions are sharp for all k, n, and $q \geqq 2$.

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