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A quantitative computational model for complete partial metric spaces via formal balls[†]

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Given a partial metric space (X, p), we use $(\mathbf{B}X, \sqsubseteq_{d_p})$ to denote the poset of formal balls of the associated quasi-metric space (X, d_p). We obtain characterisations of complete partial metric spaces and sup-separable complete partial metric spaces in terms of domain-theoretic properties of $(\mathbf{B}X, \sqsubseteq_{d_p})$. In particular, we prove that a partial metric space (X, p) is complete if and only if the poset $(\mathbf{B}X, \sqsubseteq_{d_p})$ is a domain. Furthermore, for any complete partial metric space (X, p), we construct a Smyth complete quasi-metric q on $\mathbf{B}X$ that extends the quasi-metric d_p such that both the Scott topology and the partial order $\sqsubseteq_{d_p}$ are induced by q. This is done using the partial quasi-metric concept recently introduced and discussed by H. P. Künzi, H. Pajoohesh and M. P. Schellekens (Künzi *et al.* 2006). Our approach, which is inspired by methods due to A. Edalat and R. Heckmann (Edalat and Heckmann 1998), generalises to partial metric spaces the constructions given by R. Heckmann (Heckmann 1999) and J. J. M. M. Rutten (Rutten 1998) for metric spaces.

1. Introduction and preliminaries

Throughout this paper, we use the letters $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ to denote the set of natural numbers, the set of real numbers and the set of non-negative real numbers, respectively.

Our basic references for quasi-metric spaces are Fletcher and Lingren (1982) and Künzi (2001), and for topological notions, Engelking (1977).

Following the modern terminology, by a quasi-metric on a set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(1) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$
(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) where d is a quasi-metric on X.

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Each quasi-metric d on X induces a T_0 topology τ_d on X that has as a base the family of open balls $\{B_d(x,r) : x \in X,\ \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X, the function d^{-1} defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$ is also a quasi-metric on X, called the conjugate of d. The function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ is a metric on X.

A quasi-metric space (X,d) is said to be bicomplete if (X,d^s) is a complete metric space. In this case we say that d is a bicomplete quasi-metric on X.

A sequence $(x_n)_n$ in a quasi-metric space (X,d) is said to be left K-Cauchy (Reilly *et al.* 1982; Romaguera 1992) if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d(x_n,x_m) < \varepsilon$ whenever $m \geqslant n \geqslant n_0$. Other authors call left K-Cauchy sequences 'forward Cauchy sequences' (see, for instance, (Rutten 1998)) or simply 'Cauchy sequences' (Künzi and Schellekens 2002).

In Smyth (1988; 1991), Smyth began the study of a theory of completeness of quasi-uniform and quasi-metric spaces in order to obtain a common generalisation of dcpo's and metric spaces as used in denotational semantics. Characterisations of Smyth-completable and Smyth-complete quasi-uniform spaces were obtained by Künzi (Künzi 1993) and Sünderhauf (Sünderhauf 1995). However, for our purposes here the following well-known characterisation is sufficient: a quasi-metric space (X,d) is Smyth-complete if and only if every left K-Cauchy sequence is convergent for the topology induced by the metric d^s. In this case, we say that d is a Smyth-complete quasi-metric on X.

The partial metric spaces, and their equivalent weightable quasi-metric spaces, were introduced by Matthews in Matthews (1994) as a component in the study of the denotational semantics of dataflow networks. Matthews' work has been continued by several authors, who have studied, for instance, domain properties as well as the complexity theory of algorithms with the help of partial metrics – see, O'Neill (1996), Romaguera and Schellekens (1999), Romaguera and Schellekens (2005), Schellekens (1995), Schellekens (2004), Waszkiewicz (2001) and Waszkiewicz (2006), and so on.

A partial metric (Matthews 1994) on a set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$

(ii) $p(x,x) \leqslant p(x,y)$

(iii) $p(x,y) = p(y,x)$

(iv) $p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) where p is a partial metric on X.

Each partial metric p on X induces a T_0 topology τ_p on X that has as a base the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A quasi-metric space (X,d) is said to be weightable (Matthews 1994) if there exists a function $w : X \to \mathbb{R}^+$ such that for all $x, y \in X$, we have $d(x,y) + w(x) = d(y,x) + w(y)$. The function w is said to be a weighting function for (X,d) and the quasi-metric d is weightable by the function w.

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The next result provides the precise relationship between partial metric spaces and weightable quasi-metric spaces.

Theorem A (Matthews 1994).

(a) If (X, p) is a partial metric space, the function $d_p : X \times X \to \mathbb{R}^+$ defined by $d_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$ is a weightable quasi-metric on X with weighting function w given by $w(x) = p(x, x)$ for all $x \in X$. Furthermore $\tau_p = \tau_{d_p}$.

(b) Conversely, if (X, d) is a weightable quasi-metric space with weighting function w, then the function $p_d : X \times X \to \mathbb{R}^+$ defined by $p_d(x, y) = d(x, y) + w(x)$ for all $x, y \in X$, is a partial metric on X. Furthermore, $\tau_d = \tau_{p_d}$.

We say that a partial metric space (X, p) is sup-separable if the metric space $(X, (d_p)^s)$ is separable.

Following Matthews (1994, Definition 5.2), a sequence $(x_n)_n$ in a partial metric space (X, p) is called a Cauchy sequence if $\lim_{n,m} p(x_n, x_m)$ exists (and is finite).

Note that $(x_n)_n$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space $(X, (d_p)^s)$ (see, for instance, Matthews (1994, page 194)).

A partial metric space (X, p) is said to be complete (see Matthews (1994, Definition 5.3)) if every Cauchy sequence $(x_n)_n$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m} p(x_n, x_m)$.

It is well known (see, for instance, García-Raffi *et al.* (2002, page 8)) that every weightable bicomplete quasi-metric space is Smyth-complete. Since, a partial metric space (X, p) is complete if and only if (X, d_p) is bicomplete (Oltra *et al.* 2002), we have the following known characterisations, which will be useful later on.

Theorem B. For a partial metric space (X, p), the following are equivalent:

(1) (X, p) is complete.

(2) (X, d_p) is bicomplete.

(3) (X, d_p) is Smyth-complete.

Our basic reference for Domain Theory is Gierz *et al.* (2003).

Let us recall that a partially ordered set, or poset for short, is a set L equipped with a partial order $\leqslant$. We will use $(L, \leqslant)$ to denote it in the rest of the paper.

Example 1.1.

(a) It is well known that if (X, d) is a quasi-metric space, then the binary relation $\leqslant_d$ on X given by $x \leqslant_d y \Leftrightarrow d(x, y) = 0$, is a partial order on X, which is called the partial order induced by d. Hence $(X, \leqslant_d)$ is a poset.

(b) Similarly, if (X, p) is a partial metric space, the binary relation $\leqslant_p$ on X given by $x \leqslant_p y \Leftrightarrow p(x, y) = p(x, x)$ is a partial order on X, which is called the partial order induced by p. Hence, $(X, \leqslant_p)$ is a poset (Heckmann 1999; Matthews 1994). Note that in this case we have $\leqslant_p = \leqslant_{d_p}$.

A subset D of a poset $(L, \leqslant)$ is directed provided it is non-empty and every finite subset of D has an upper bound in D (equivalently, if for each $a, b \in D$ there is $c \in D$ such that $a \leqslant c$ and $b \leqslant c$).

A poset $(L, \leqslant)$ is said to be directed complete, and is called a dcpo, if every directed subset of L has a least upper bound.

The least upper bound of a subset D of $(L, \leqslant)$ is denoted by $\sup D$ if it exists. An element x of L is said to be maximal if the condition $x \leqslant y$ implies $x = y$. The set of all maximal elements of L is denoted by $\mathrm{Max}((L, \leqslant))$, or simply by $\mathrm{Max}(L)$ if no confusion arises.

Given a poset $(L, \leqslant)$, we say that x is way-below y, in symbols $x \ll y$, if for each directed subset D of L for which $\sup D$ exists, the relation $y \leqslant \sup D$ implies the existence of some $z \in D$ with $x \leqslant z$.

A poset $(L, \leqslant)$ is said to be continuous if it satisfies the axiom of approximation, that is, for all $x \in L$, the set $\Downarrow x = \{u \in L : u \ll x\}$ is directed and $x = \sup(\Downarrow x)$.

A continuous poset that is also a dcpo is called a domain.

A subset B of a poset $(L, \leqslant)$ is a basis for L if for each $x \in L$, the set $\Downarrow x_B = \{u \in B : u \ll x\}$ is directed and $x = \sup(\Downarrow x_B)$.

Recall that a poset has a basis if and only if it is continuous. Therefore, a dcpo has a basis if and only if it is a domain.

A dcpo having a countable basis is said to be an ω-continuous domain (Gierz *et al.* 2003). In order to simplify the terminology, in the rest of this paper ω-continuous domains will simply be called ω-domains.

The Scott topology $\sigma(L)$ of a dcpo $(L, \leqslant)$ is constructed as follows (Gierz *et al.* 2003, Chapter II): a subset U of L is open in the Scott topology provided:

(i) $U = \uparrow U$, where $\uparrow U = \{y \in X : x \leqslant y \text{ for some } x \in U\}$; and
(ii) for each directed subset D of L such that $\sup D \in U$, it follows that $D \cap U \neq \emptyset$.

If $(L, \leqslant)$ is a domain, the sets $\Uparrow x$, $x \in L$, form an open base for the Scott topology, where $\Uparrow x = \{y \in X : x \ll y\}$ (see Gierz *et al.* (2003, Proposition II-1.10)). Furthermore, the Scott topology has a countable base if and only if $(L, \leqslant)$ is an ω-domain; in this case, if B is a countable basis for $(L, \leqslant)$, the sets $\Uparrow x$, $x \in B$, are a countable base for the Scott topology (Gierz *et al.* 2003, Theorem III-4.5).

If $(L, \leqslant)$ is a continuous poset, one can also show that the sets $\Uparrow x$, $x \in L$, form an open base for a topology on L, which is also called the Scott topology of $(L, \leqslant)$ and is also denoted by $\sigma(L)$ (see, for instance, Edalat and Heckmann (1998, page 58)).

The lower (or weak) topology of a dcpo $(L, \leqslant)$ is the topology that has as a subbase the collection of sets of the form $L \setminus \uparrow x$, where $x \in L$, and we denote it by $\omega(L)$. We also recall that the supremum topology of $\sigma(L)$ and $\omega(L)$ is the Lawson topology of $(L, \leqslant)$.

In Lawson (1997), Lawson established connections between the theory of metric spaces and domain theory. In particular, he proved that a metrisable space X is a Polish space if and only if there is an ω-domain L such that:

(i) X is homeomorphic to $\mathrm{Max}(L)$ endowed with the Scott topology; and
(ii) the Scott topology coincides with the Lawson topology on $\mathrm{Max}(L)$.

Edalat and Heckmann later gave a very natural and explicit construction of an ω-domain for any Polish space satisfying conditions (i) and (ii) above using the notion of a formal ball (Edalat and Heckmann 1998). In fact, for any metric space X they constructed a partial order on the set $\mathbf{B}X$ of (closed) formal balls for which $\mathbf{B}X$ is actually a continuous poset, and X is homeomorphic to $\mathrm{Max}(\mathbf{B}X)$ equipped with the Scott topology. Moreover, $\mathbf{B}X$ is a domain if and only if X is complete, and it is an ω-domain if and only if X is a Polish space. Other contributions to the construction of computational models for several mathematical structures may be found in, for example, Edalat (1995a), Edalat (1995b), Lawson (1997), Edalat and Heckmann (1998), Heckmann (1999), Rutten (1998), Flagg and Kopperman (1997), Edalat and Sünderhauf (1999), Kopperman *et al.* (2004), Krötzsch (2006) and Waszkiewicz (2006).

On the other hand, Künzi and Vajner proved in Künzi and Vajner (1994) that every T_0 topological space having a countable base admits a compatible weightable quasi-metric and, hence, a compatible partial metric (actually, they proved a more general result using σ-point bounded bases). Since the Scott topology for each ω-domain has a countable base, it follows from Künzi and Vajner's result and Theorem A that for each ω-domain, the Scott topology is induced by a partial metric. Waszkiewicz showed the more general result that for each ω-domain both the Scott topology and partial order are induced by a partial metric (Waszkiewicz 2001). Schellekens later proved the same result using different methods (Schellekens 2003). More recently, Waszkiewicz (2006) gave characterisations of all continuous posets whose Scott topology is induced by a partial metric, in terms, variously, of measurements, of domain-theoretic bases and of exactly radially convex metrics.

In connection with the results of Edalat and Heckmann (1998), Heckmann constructed, for any metric space X, a partial metric on $\mathbf{B}X$ that extends the metric of X and induces the Scott topology on $\mathbf{B}X$ (Heckmann 1999). Working independently, Rutten obtained similar results, but using a suitable quasi-metric on the continuous poset of formal balls, instead of a partial metric (Rutten 1998, Section 7). It is interesting to note that although the notion of a formal ball proposed by Rutten is based upon certain (quasi-metric) function spaces, he proved its 'equivalence' with the notion of a formal ball given in Edalat and Heckmann (1998) by means of the co-Yoneda embedding (see Rutten (1998, Section 6)).

In this paper we present a computational model for partial metric spaces that generalises and unifies the constructions in Edalat and Heckmann (1998), Heckmann (1999) and Rutten (1998) mentioned above. To this end, we have adapted several methods in Edalat and Heckmann (1998) to our context. In particular, we observe that for each quasi-metric space (X, d) the corresponding set $\mathbf{B}X$ of formal balls can be equipped, in a similar way to what is done in Edalat and Heckmann (1998), with a partial order $\sqsubseteq_d$ and thus $(\mathbf{B}X, \sqsubseteq_d)$ is a poset. Then we characterise any complete (and any sup-separable and complete) partial metric space (X, p) in terms of domain-theoretic properties of $(\mathbf{B}X, \sqsubseteq_{d_p})$, where d_p is the quasi-metric induced by p in Theorem A. In fact, we establish the following somewhat surprising generalisations of Edalat and Heckmann's theorems cited above:

(a) (X, p) is complete if and only if $(\mathbf{B}X, \sqsubseteq_{d_p})$ is a domain.

(b) (X, p) is complete and sup-separable if and only if $(\mathbf{B}X, \sqsubseteq_{d_p})$ is an ω-domain.

Also, for any complete partial metric space (X, p), we construct a Smyth-complete quasi-metric q on $\mathbf{B}X$ that extends the quasi-metric d_p and such that both the Scott topology and

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the partial order $\sqsubseteq_{d_p}$ are induced by q. This is done using the partial quasi-metric concept recently introduced and discussed by Künzi, Pajoohesh and Schellekens (Künzi *et al.* 2006). Our construction motivates a new notion of a (quantitative) computational model, and we show that each complete partial metric space has a quantitative computational model. Finally, we apply our results to obtain a domain-theoretic proof of Matthews' fixed point theorem for complete partial metric spaces (Matthews 1994, Theorem 5.3).

Remark 1.1. While our paper was being refereed, the paper Aliakbari *et al.* (2009), by Ali Akbari, Honari, Pourmahdian and Rezaii was accepted for publication in *Mathematical Structures in Computer Science*. In their paper, the authors also investigate connections between quasi-metric spaces and domain theory *via* formal balls. Although the main focus of their work is on obtaining models in the sense of Kopperman *et al.* (2004) and Martin (1998) for quasi-metric spaces whose induced topology is T_1, they have also produced some interesting results that are related to our approach. In particular, they prove that for every Smyth-complete quasi-metric space, the poset of formal balls is a domain. Although the converse of this result does not hold in general, it is true in the realm of weightable quasi-metric spaces as the characterisation (a) mentioned in the preceding paragraph shows. In fact, that characterisation can be also deduced from results in Aliakbari *et al.* (2009) (see Section 3 below for a detailed discussion).

2. The poset of formal balls of a quasi-metric space

Given a quasi-metric space (X,d), for each $x \in X$ and $r \geqslant 0$, we define the closed ball of center x and radius r by $\overline{B}_d(x,r) = \{y \in X : d(x,y) \leqslant r\}$. It is known, and easy to see, that $\overline{B}_d(x,r)$ is a closed set with respect to $\tau_{d^{-1}}$. In particular, $\overline{B}_d(x,0) = \{y \in X : d(x,y) = 0\}$, that is, $\overline{B}_d(x,0)$ is exactly the closure of x with respect to $\tau_{d^{-1}}$.

Furthermore, as in the metric case, one clearly has, by applying the triangle inequality, that

$$d(x,y) \leqslant r - s \Longrightarrow \overline{B}_d(x,r) \supseteq \overline{B}_d(y,s).$$

However, the converse does not hold even for metric spaces, as the example in Edalat and Heckmann (1998) shows.

These facts suggest, by analogy with the metric case, the notion of a formal ball (see also Aliakbari *et al.* (2009, Definition 3.1)).

Let (X,d) be a quasi-metric space. Define

$$\mathbf{B}X := \{(x,r) : x \in X, r \in \mathbb{R}^+\}.$$

Then, each pair $(x,r) \in \mathbf{B}X$ is said to be a formal ball in (X,d).

The following easy but crucial result is also stated in Aliakbari *et al.* (2009, Section 3).

Proposition 2.1. Let (X,d) be a quasi-metric space. Define a binary relation $\sqsubseteq_d$ on $\mathbf{B}X$ by

$$(x,r) \sqsubseteq_d (y,s) \Longleftrightarrow d(x,y) \leqslant r - s.$$

Then $(\mathbf{B}X, \sqsubseteq_d)$ is a poset.

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Proof. Reflexivity of $\sqsubseteq_d$ is obvious and transitivity is an immediate consequence of the triangle inequality. In order to show antisymmetry of $\sqsubseteq_d$, we suppose that $(x,r) \sqsubseteq_d (y,s)$ and $(y,s) \sqsubseteq_d (x,r)$. Then $d(x,y) \leqslant r-s$ and $d(y,x) \leqslant s-r$. So $r=s$, and hence $d(x,y)=d(y,x)=0$. So $x=y$. □

In the rest of this section we establish several properties on the formal balls of a (weightable) quasi-metric space that will be useful later on.

The proofs of Propositions 2.2, 2.3 and 2.4 below are analogous to the proofs of Edalat and Heckmann (1998)'s Proposition 3, Lemma 4 and Theorem 5 (i)⇒(ii), respectively, so they are omitted. In particular, Proposition 2.3 is also given in Aliakbari *et al.* (2009).

Recall that a sequence $(x_n)_n$ in a poset $(L, \leqslant)$ is ascending provided $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$.

Proposition 2.2. Let (X,d) be a quasi-metric space. If $((x_n, r_n))_n$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq_d)$, then the sequence $(r_n)_n$ is decreasing and convergent, and $(x_n)_n$ is a left K-Cauchy sequence in (X,d).

Proposition 2.3. Let (X,d) be a quasi-metric space. Then we have that every left K-Cauchy sequence $(x_n)_n$ in (X,d) has a subsequence $(x_{n_k})_k$ such that $((x_{n_k}, 2^{-k}))_k$ is ascending in $(\mathbf{B}X, \sqsubseteq_d)$.

Proposition 2.4. Let (X,d) be a weightable quasi-metric space. If $((x_n, r_n))_n$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq_d)$ having a least upper bound (y,s), then $\lim_n r_n = s$ and $\lim_n d(x_n, y) = 0$.

We will prove the following result by adapting the technique used to prove (iii)⇒(i) in Edalat and Heckmann (1998, Theorem 5).

Proposition 2.5. Let (X,d) be a quasi-metric space. If $((x_n, r_n))_n$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq_d)$ such that there is $(y,s) \in \mathbf{B}X$ satisfying $\lim_n d^s(y, x_n) = 0$ and $\lim_n r_n = s$, then (y,s) is the least upper bound of $((x_n, r_n))_n$.

Proof. We first show that (y,s) is an upper bound of $((x_n, r_n))_n$. Choose $n \in \mathbb{N}$. Given $\varepsilon > 0$, there is $m > n$ such that $|r_m - s| < \varepsilon$ and $d(x_m, y) < \varepsilon$. By the triangle inequality, it follows that $d(x_n, y) < r_n - r_m + \varepsilon$, so $d(x_n, y) < r_n - s + 2\varepsilon$. Since ε is arbitrary, we deduce that $d(x_n, y) \leqslant r_n - s$, that is, $(x_n, r_n) \sqsubseteq_d (y,s)$.

Finally, let (z,t) be any upper bound of $((x_n, r_n))_n$. Then $d(x_n, z) \leqslant r_n - t$ for all $n \in \mathbb{N}$. Since $\lim_n r_n = s$ and $\lim_n d(y, x_n) = 0$, it follows from the triangle inequality that $d(y,z) \leqslant s - t$, that is, $(y,s) \sqsubseteq_d (z,t)$. □

Proposition 2.6. Let (X,d) be a quasi-metric space. If every ascending sequence in $(\mathbf{B}X, \sqsubseteq_d)$ has a least upper bound, then (X,d) is bicomplete.

Proof. Let $(x_n)_n$ be a Cauchy sequence in (X, d^s). Then $(x_n)_n$ is left K-Cauchy in (X,d). So, by Proposition 2.3, $(x_n)_n$ has a subsequence $(x_{n_k})_k$ such that $((x_{n_k}, 2^{-k}))_k$ is ascending in $(\mathbf{B}X, \sqsubseteq_d)$. By hypothesis, the sequence $((x_{n_k}, 2^{-k}))_k$ has a least upper bound (y,s), and thus $d(x_{n_k}, y) \leqslant 2^{-k} - s$ for all $k \in \mathbb{N}$, so $s = 0$. Since $d(x_{n_k}, y) \leqslant 2^{-k}$ for all $k \in \mathbb{N}$, and $(x_n)_n$ is a Cauchy sequence in (X, d^s), we deduce that $\lim_n d(x_n, y) = 0$.

We now prove that $\lim_n d(y, x_n) = 0$.

To this end, we first fix an $r > 0$. Since $(x_{n_k}, 2^{-k}) \sqsubseteq_d (y, 0)$, it follows that $(x_{n_k}, r+2^{-k}) \sqsubseteq_d (y, r)$ for all $k \in \mathbb{N}$. By hypothesis, $((x_{n_k}, r+2^{-k}))_k$ has a least upper bound (z, t). We show that, in fact, $(z, t) = (y, r)$. Indeed, since $d(x_{n_k}, z) \leqslant r + 2^{-k} - t$ for all $k \in \mathbb{N}$, it follows that $t \leqslant r$. Moreover, $d(z, y) \leqslant t - r$ because $(z, t) \sqsubseteq_d (y, r)$, so $r \leqslant t$. Hence $r = t$ and thus $d(z, y) = 0$. We also have that $d(x_{n_k}, z) \leqslant 2^{-k}$, and hence $(x_{n_k}, 2^{-k}) \sqsubseteq_d (z, 0)$ for all $k \in \mathbb{N}$. So $(y, 0) \sqsubseteq_d (z, 0)$. Consequently, $d(y, z) = 0$ and thus $y = z$. We conclude that (y, r) is the least upper bound of $((x_{n_k}, r + 2^{-k}))_k$.

Finally, choose $\varepsilon > 0$ with $\varepsilon < r$. Then there is an $n(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geqslant n(\varepsilon)$. Thus $(x_{n_k}, r+2^{-k}) \sqsubseteq_d (x_m, r - \varepsilon)$ for all $n_k, m \geqslant n(\varepsilon)$. Hence $(x_{n_k}, r+2^{-k}) \sqsubseteq_d (x_m, r - \varepsilon)$ for all $k \in \mathbb{N}$ and for all $m \geqslant n(\varepsilon)$. So $(y, r) \sqsubseteq_d (x_m, r - \varepsilon)$ for all $m \geqslant n(\varepsilon)$. It follows that $\lim_n d(y, x_n) = 0$, and we can conclude that (X, d) is bicomplete. □

The following result generalises Edalat and Heckmann (1998, Theorem 2) to weightable quasi-metric spaces.

Proposition 2.7. Let (X, d) be a weightable quasi-metric space. If D is a directed subset of $(\mathbf{B}X, \sqsubseteq_d)$, there is an ascending sequence in D that has the same upper bounds as D.

Proof. Let w be a weighting function for (X, d). Put $s = \inf\{w(x)+2r : (x, r) \in D\}$. Then, for each $n \in \mathbb{N}$, there is $(y_n, s_n) \in D$ such that $w(y_n) + 2s_n \leqslant s + 1/n$. Put $(x_1, r_1) = (y_1, s_1)$. For each $n > 1$ there is an $(x_n, r_n) \in D$ that is an upper bound of (x_{n-1}, r_{n-1}) and (y_n, s_n). Then $((x_n, r_n))_n$ is an ascending sequence in D.

Although the rest of the proof follows along similar lines to the proof of Edalat and Heckmann (1998, Theorem 2), we present it here for the sake of completeness. Thus, we shall show that any upper bound of $((x_n, r_n))_n$ is an upper bound of D. Indeed, let $(z, t) \in D$ be such that $(x_n, r_n) \sqsubseteq_d (z, t)$ for all $n \in \mathbb{N}$ and let (a, u) be an arbitrary element of D. Since D is directed, for each $n \in \mathbb{N}$ there is $(b_n, v_n) \in D$ that is an upper bound of (a, u) and (x_n, r_n).

Since for each $n \in \mathbb{N}$, $s \leqslant w(b_n)+2v_n$, and $w(x_n) \leqslant d(y_n, x_n)+w(y_n) \leqslant s_n-r_n-2s_n+s+1/n$, it follows that

$$\begin{aligned} d(a, z) &\leqslant d(a, b_n) + d(b_n, x_n) + d(x_n, z) \\ &\leqslant u - v_n + d(x_n, b_n) + w(x_n) - w(b_n) + r_n - t \\ &\leqslant u - 2v_n + 2r_n - t + w(x_n) - w(b_n) \\ &\leqslant u + w(b_n) - s + 2r_n - t - s_n - r_n + s + 1/n - w(b_n) \\ &\leqslant u - t + 1/n, \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $d(a, z) \leqslant u - t$, and we can conclude that (z, t) is an upper bound of D. □

Proposition 2.7 essentially says that, analogously to the metric case, in order to study the least upper bounds of directed subsets in the poset of formal balls of a weightable quasi-metric spaces, it is sufficient to consider ascending sequences.

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We conclude this section with an example showing that weightability of (X,d) cannot be omitted in this proposition.

Example 2.1. Let $\mathscr{A}$ be the family of all non-empty countable subsets of $\mathbb{R}$ and let d be the quasi-metric on $\mathscr{A}$ given by $d(A,B)=0$ if $A \subseteq B$, and $d(A,B)=1$ otherwise. We first observe that if $((A_n,r_n))_n$ is an ascending sequence in $(\mathbf{B}\mathscr{A}, \sqsubseteq_d)$, then $(\bigcup_n A_n, r)$ is the least upper bound of $((A_n,r_n))_n$, where $r = \lim_n r_n$ (*cf.* Proposition 2.2). However, it is clear that the subset of $\mathbf{B}\mathscr{A}$ defined by

$\{(A,0) : A$ is a non-empty finite set consisting of irrational numbers$\}$,

is directed but has no upper bound.

3. The domain of formal balls of a complete partial metric space

Edalat and Heckmann (1998, Proposition 7) proved the following crucial characterisation of the way-below relation on $\mathbf{B}X$ for any metric space X : $(x,r) \ll (y,s) \Longleftrightarrow d(x,y) < r-s$.

Although this result does not hold for partial metric spaces as Example 3.1 below shows, the following nice result from Aliakbari *et al.* (2009) will be sufficient for our purposes here.

Proposition 3.1 (Aliakbari *et al.* 2009, Corollary 3.13). If (X,d) is a Smyth-complete quasi-metric space, then:

(a) $(x,r) \ll (y,s) \Longleftrightarrow d(x,y) < r-s$.
(b) $(\mathbf{B}X, \sqsubseteq_d)$ is a domain.

From Proposition 3.1 and Theorem B we deduce the following corollary.

Corollary 3.1. If (X,p) is a complete partial metric space, then:

(a) $(x,r) \ll (y,s) \Longleftrightarrow d_p(x,y) < r-s$.
(b) $(\mathbf{B}X, \sqsubseteq_{d_p})$ is a domain.

Remark 3.1. The proof of Proposition 3.1 (a) requires the use of nets instead of sequences, which obviously involves a major degree of difficulty. Because, in our context, it is enough to work with sequences, it is interesting to give the following (relatively) simple proof of the above corollary with the help of the auxiliary propositions obtained in Section 2.

We first suppose that $(x,r) \ll (y,s)$. Let $D = \{(y, s+1/n) : n \in \mathbb{N}\}$. It is clear that $(y,s) = \sup D$. So, by hypothesis, there is $m \in \mathbb{N}$ such that $(x,r) \sqsubseteq_{d_p} (y, s+1/m)$, that is, $d_p(x,y) \leqslant r-s-1/m < r-s$.

Conversely, let $\varepsilon > 0$ be such that $d_p(x,y) < r-s-\varepsilon$. Now let D be a directed subset of $(\mathbf{B}X, \sqsubseteq_{d_p})$ such that $(y,s) \sqsubseteq_{d_p} (z,t)$, where $(z,t) = \sup D$. By Proposition 2.7, there is an ascending sequence $((z_n,t_n))_n$ in D with least upper bound (z,t). Then we have $\lim_n t_n = t$ by Proposition 2.4. Moreover, by Proposition 2.2 and Theorem B, there exists $a \in X$ such that $\lim_n (d_p)^s(a, z_n) = 0$. It follows from Proposition 2.5 that (a,t) is the least upper bound of (z_n,t_n). Consequently, $a = z$. Finally, we choose $m \in \mathbb{N}$ such that $d_p(z, z_m) < \varepsilon/2$ and

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$t_m < t + \varepsilon/2$. Then

$$\begin{aligned} d_p(x, z_m) &\leqslant d_p(x, y) + d_p(y, z) + d_p(z, z_m) \\ &< r - s - \varepsilon + s - t + \varepsilon/2 \\ &= r - (t + \varepsilon/2) < r - t_m. \end{aligned}$$

Therefore, $(x, r) \sqsubseteq_{d_p} (z_m, t_m)$, and we can conclude that $(x, r) \ll (y, s)$.

Example 3.1. Let $X = \mathbb{N} \cup \{\infty\}$ and p be the partial metric on X given by $p(\infty, \infty) = 0$, $p(\infty, n) = p(n, \infty) = 1$, and $p(n, m) = 1 + |1/n - 1/m|$ for all $n, m \in \mathbb{N}$. Let $D = \{(n, 1/n) : n \in \mathbb{N}\}$. Then it is clear that D is a directed subset of $(\mathbf{B}X, \sqsubseteq_{d_p})$ such that $(\infty, 0) = \sup D$. Since $d_p(\infty, n) > 1 - 1/n$ for all $n \in \mathbb{N}$, it follows that $(\infty, 1)$ is not way-below $(\infty, 0)$. Thus, completeness of (X, p) cannot be deleted in part (a) of Corollary 3.1.

The next example shows that the converse of Proposition 3.1 (b) does not hold in general.

Example 3.2. Let d_S be the quasi-metric on $\mathbb{R}$ given by $d_S(x, y) = y - x$ if $x \leqslant y$, and $d_S(x, y) = 1$ otherwise. Then the topological space $(\mathbb{R}, \tau_{d_S})$ is the celebrated Sorgenfrey line. Hence, $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is a domain by Corollary 3.13 and Aliakbari *et al.* (2009, Remark 4.3). Nevertheless, it is well known and almost obvious that $(\mathbb{R}, d_S)$ is not Smyth-complete (consider, for instance, the left K-Cauchy sequence $(-1/n)_n$, which does not converge with respect to τ_{d_S}).

However, for partial metric spaces, we have the following characterisations.

Theorem 3.1. For a partial metric space (X, p), the following conditions are equivalent:

(1) (X, p) is complete.
(2) Every ascending sequence in $(\mathbf{B}X, \sqsubseteq_{d_p})$ has a least upper bound.
(3) $(\mathbf{B}X, \sqsubseteq_{d_p})$ is a dcpo.
(4) $(\mathbf{B}X, \sqsubseteq_{d_p})$ is a domain.

Proof.

(1) $\Rightarrow$ (2)
Let $((x_n, r_n))_n$ be an ascending sequence in $(\mathbf{B}X, \sqsubseteq_{d_p})$. By Proposition 2.2, the sequence $(x_n)_n$ is left K–Cauchy in (X, d_p), and there is $s \in \mathbb{R}^+$ such that $\lim_n r_n = s$. It follows from Theorem B that there is a $y \in X$ such that $\lim_n (d_p)^s(y, x_n) = 0$. Therefore, (y, s) is the least upper bound of $((x_n, r_n))_n$ by Proposition 2.5.

(2) $\Rightarrow$ (1)
This follows from Proposition 2.6 and Theorem B.

(2) $\Leftrightarrow$ (3)
This follows from Proposition 2.7.

(4) $\Rightarrow$ (3)
This is obvious.

(1) $\Rightarrow$ (4)
This follows from Corollary 3.1 (b). □

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As a consequence of Theorem 3.1, we obtain the following theorem due to Edalat and Heckmann.

Corollary 3.2 (Edalat and Heckmann 1998, Theorem 6). For a metric space (X, d), the following conditions are equivalent:

(1) (X, d) is complete.
(2) Every ascending sequence in $(\mathbf{B}X, \sqsubseteq_d)$ has a least upper bound.
(3) $(\mathbf{B}X, \sqsubseteq_d)$ is a dcpo.
(4) $(\mathbf{B}X, \sqsubseteq_d)$, is a domain.

Remark 3.2. It follows from Example 2.1 that the equivalence between conditions (2) and (3) in Theorem 3.1 does not hold for quasi-metric spaces in general.

Remark 3.3. Recall (see, for instance, Künzi and Schellekens (2002)) that a quasi-metric space (X, d) is said to be sequentially Yoneda-complete if for each left K-Cauchy sequence $(x_n)_n$, there exists an $x \in X$ such that $d(x, y) = \inf_n \sup_{m \geqslant n} d(x_m, y)$ for all $y \in X$. Then (3) $\Rightarrow$ (1) in Theorem 3.1 can be also derived from Theorem B and the facts, proved in Aliakbari *et al.* (2009), that for a quasi-metric space (X, d):

(a) Smyth completeness $\Rightarrow$ sequential Yoneda completeness $\Rightarrow$ bicompleteness; and
(b) if $(\mathbf{B}X, \sqsubseteq_d)$ is a dcpo, then (X, d) is sequentially Yoneda-complete.

Note that our proof of (3) $\Rightarrow$ (1) does not use sequential Yoneda completeness and that it permits us to state, *via* Proposition 2.7, the equivalence between the condition that $(\mathbf{B}X, \sqsubseteq_d)$ is a dcpo and the more visual condition given in Theorem 3.1 (2).

We conclude this section by studying the separability of partial metric spaces from a domain-theoretic point of view. In particular, the well-known result (Edalat and Heckmann 1998) that a metric space is separable and complete if and only if $(\mathbf{B}X, \sqsubseteq_{d_p})$ is an ω-domain will be generalised to our context.

Theorem 3.2. A partial metric space (X, p) is sup-separable and complete if and only if $(\mathbf{B}X, \sqsubseteq_{d_p})$ is an ω-domain.

Proof. Suppose that $(\mathbf{B}X, \sqsubseteq_{d_p})$ is an ω-domain and let B be a countable basis for it. By Theorem 3.1, (X, p) is complete. On the other hand, by Proposition 2.7, for each $x \in X$ there is an ascending sequence $(x_n, r_n)_n$ in B with least upper bound $(x, 0)$. It follows from Proposition 2.2 that $(x_n)_n$ is a left K-Cauchy sequence in (X, d_p) and, by Theorem B, there is $y \in X$ such that $\lim_n (d_p)^s(y, x_n) = 0$. Now, Proposition 2.5 shows that $(y, 0)$ is the least upper bound of $(x_n, r_n)_n$, so $x = y$. We conclude that (X, p) is sup-separable.

Conversely, let A be a countable dense subset of $(X, (d_p)^s)$. We shall show that $B := A \times \mathbb{Q}$ is a basis for $(\mathbf{B}X, \sqsubseteq_{d_p})$, where $\mathbb{Q}$ denotes the set of all rational numbers. Let $(x, r) \in \mathbf{B}X$. Then, as in the proof of Edalat and Heckmann (1998, Theorem 8), there is a sequence $((a_n, q_n))_n$ in B such that $\lim_n q_n = r$ and $(d_p)^s(x, a_n) < q_n - r$ for all $n \in \mathbb{N}$. It follows by Corollary 3.1 that $(a_n, q_n) \in \Downarrow (x, r)_B$ for all $n \in \mathbb{N}$. Now, if $(y, s), (z, t) \in \Downarrow (x, r)_B$, then $d_p(y, x) < s - r - \varepsilon$ and $d_p(z, x) < t - r - \varepsilon$ for some $\varepsilon > 0$. Choose (a_n, q_n) such that $q_n - r < \varepsilon/2$. By the triangle inequality, $d_p(y, a_n) < s - q_n$ and $d_p(z, a_n) < t - q_n$ so

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$\Downarrow (x,r)_B$ is directed. Let (a,u) be the least upper bound of $\Downarrow (x,r)_B$. Then $(a,u) \sqsubseteq_{d_p} (x,r)$ and $(a_n, q_n) \sqsubseteq_{d_p} (a,u)$ for all $n \in \mathbb{N}$. It then follows from the triangle inequality that $d_p(x,a) < 2q_n - r - u$ for all $n \in \mathbb{N}$, so $d_p(x,a) \leqslant r - u$. We conclude that $(x,r) = (a,u)$. Hence $(\mathbf{B}X, \sqsubseteq_{d_p})$ is an ω-domain. □

Since a partial metric space (X,p) is sup-separable and complete if and only if the metric space $(X,(d_p)^s)$ is separable and complete, Theorem 3.2 leads us to the following somewhat surprising fact.

Corollary 3.3. Let (X,p) be a partial metric space. Then $(\mathbf{B}X, \sqsubseteq_{d_p})$ is an ω-domain if and only if $(X,(d_p)^s)$ is a Polish space.

4. Partial quasi-metrics on BX. Isometries from X into BX

As we mentioned in Section 1, Heckmann constructed, for any (complete) metric space (X,d), a (complete) partial metric P on $\mathbf{B}X$ that extends the metric d and such that both the partial order $\sqsubseteq_d$ and the Scott topology are induced by P (Heckmann 1999). The following partial metric P is a slight modification of Heckmann's original construction:

$$P((x,r),(y,s)) = \max\{d(x,y), |r-s|\} + r + s$$

for all $(x,r),(y,s) \in \mathbf{B}X$.

In this section we extend Heckmann's construction to (complete) partial metric spaces with the help of the notion of a partial quasi-metric.

The following notions and facts concerning partial quasi-metric spaces are given in Künzi *et al.* (2006).

A partial quasi-metric on a set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$:

(i) $x = y \Leftrightarrow p(x,x) = p(x,y)$ and $p(y,y) = p(y,x)$;
(ii) $p(x,x) \leqslant \min\{p(x,y), p(y,x)\}$;
(iii) $p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.

A partial quasi-metric space is a pair (X,p) where p is a partial quasi-metric on X.

Note that a partial quasi-metric p on X satisfying the symmetry axiom, $p(x,y) = p(y,x)$ for all $x,y \in X$ is a partial metric on X.

As with partial metrics, each partial quasi-metric p on X induces a T_0 topology τ_p on X that has as a base the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Moreover, each partial quasi-metric p on X induces a quasi-metric d_p on X given by $d_p(x,y) = p(x,y) - p(x,x)$ for all $x,y \in X$, with $\tau_p = \tau_{d_p}$ (*cf.* Theorem A).

Note also (*cf.* Example 1.1) that if (X,p) is a partial quasi-metric space, then the binary relation $\leqslant_p$ on X given by $x \leqslant_p y \Leftrightarrow p(x,y) = p(x,x)$ is a partial order on X, which is called the partial order induced by p.

Finally, we recall that a partial quasi-metric space (X,p) is said to be complete if the quasi-metric space (X,d_p) is bicomplete.

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Now suppose that (X, p) is a partial metric space. In a first attempt to extend Heckmann's construction to our context, we could define a function $P : \mathbf{B}X \times \mathbf{B}X \to \mathbb{R}^+$ by $P((x,r),(y,s)) = \max\{p(x,y), |r-s|\} + r + s$ for all $(x,r),(y,s) \in \mathbf{B}X$. Unfortunately, P is not a partial metric in general, as the following easy example shows.

Example 4.1. Let $X = \mathbb{R}^+$ and p be the partial metric on X such that $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. Then $P((1,1),(1,1)) = 3$, but $P((1,1),(0,0)) = 2$, so P is not a partial metric on $\mathbf{B}X$.

However, one can easily show that the following slight modification of the above definition of P does in fact provide a partial metric on $\mathbf{B}X$:

$$P((x,r),(y,s)) = p(x,y) + |r-s| + r + s$$

for all $(x,r),(y,s) \in \mathbf{B}X$.

Unfortunately, this partial metric has the disadvantage that the partial order $\sqsubseteq_{d_p}$ does not coincide with $\leqslant_P$ in general. Indeed, consider the partial metric space of Example 4.1. Then $(0,1) \sqsubseteq_{d_p} (1,0)$, but $P((0,1),(0,1)) = 2$, and $P((0,1),(1,0)) = 3$.

The construction given in our next result avoids this inconvenience and provides a suitable extension of Heckmann's construction to our framework.

Theorem 4.1. Let (X, p) be a partial metric space. Let $Q : \mathbf{B}X \times \mathbf{B}X \to \mathbb{R}^+$ be given by

$$Q((x,r),(y,s)) = \max\{d_p(x,y), |r-s|\} + r + s,$$

$(x,r),(y,s) \in \mathbf{B}X$, and let

$$\mathsf{i} : X \to \mathbf{B}X$$

be given by

$$\mathsf{i}(x) = (x, 0).$$

Then the following hold:

(a) Q is a partial quasi-metric on $\mathbf{B}X$.
(b) i is an isometry from (X, d_p) into $(\mathbf{B}X, Q)$.
(c) $\leqslant_Q = \sqsubseteq_{d_p}$ on $\mathbf{B}X$.
(d) If (X, p) is complete, the topology induced by Q coincides with the Scott topology inherited from $(\mathbf{B}X, \sqsubseteq_{d_p})$.
(e) $(\mathbf{B}X, d_Q)$ is Smyth-complete if and only if (X, p) is complete.

Proof.

(a) Let $(x,r),(y,s),(z,t) \in \mathbf{B}X$. Then:

(i)
$$(x,r) = (y,s) \Leftrightarrow Q((x,r),(y,s)) = 2r = Q((y,s),(x,r)) = 2s$$
$$\Leftrightarrow Q((x,r),(x,r)) = Q((x,r),(y,s))$$
and
$$Q((y,s),(y,s)) = Q((y,s),(x,r)).$$

(ii)
$$Q((x,r),(x,r)) = 2r \leqslant \min\{Q((x,r),(y,s)), Q((y,s),(x,r))\}.$$

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(iii)
$$\begin{aligned} Q((x,r),(y,s)) + Q((z,t),(z,t)) &= \max\{d_p(x,y), |r-s|\} + r + s + 2t \\ &\leqslant \max\{d_p(x,z), |r-t|\} + r + t + \\ &\quad \max\{d_p(z,y), |r-t|\} + s + t \\ &= Q((x,r),(z,t)) + Q((z,t),(y,s)). \end{aligned}$$

Therefore Q is a partial quasi-metric on $\mathbf{B}X$.

(b) *Let* $x, y \in X$. Then

$$Q(\mathsf{i}(x), \mathsf{i}(y)) = Q((x,0),(y,0)) = d_p(x,y).$$

Hence i is an isometry from (X, d_p) into $(\mathbf{B}X, Q)$.

(c) Let $(x,r),(y,s) \in \mathbf{B}X$. Then

$$\begin{aligned} (x,r) \leqslant_Q (y,s) &\Leftrightarrow Q((x,r),(x,r)) = Q((x,r),(y,s)) \\ &\Leftrightarrow 2r = \max\{d_p(x,y), |r-s|\} + r + s \\ &\Leftrightarrow d_p(x,y) \leqslant r - s \\ &\Leftrightarrow (x,r) \sqsubseteq_{d_p} (y,s). \end{aligned}$$

(d) We first show that $\sigma(\mathbf{B}X) \subseteq \tau_Q$. To do this, we fix $(x,r) \in \mathbf{B}X$, and let $(z,t) \in \mathbf{B}X$ be such that $(x,r) \in \Uparrow (z,t)$. Since $(z,t) \ll (x,r)$, we have $d_p(z,x) < t - r$. Choose $\varepsilon > 0$ such that $\varepsilon + d_p(z,x) < t - r$. We shall prove that $B_{d_Q}((x,r),\varepsilon) \subseteq \Uparrow (z,t)$. To this end, let $(y,s) \in B_{d_Q}((x,r),\varepsilon)$. Then

$$\max\{d_p(x,y), |r-s|\} + s - r < \delta\,,$$

so $d_p(x,y) < \delta + r - s$, and thus

$$\begin{aligned} d_p(z,y) &\leqslant d_p(z,x) + d_p(x,y) \\ &< (t - r - \varepsilon) + (\varepsilon + r - s) = t - s. \end{aligned}$$

Hence $(z,t) \ll (y,s)$ by Corollary 3.1. We conclude that $\sigma(\mathbf{B}X) \subseteq \tau_Q$.

Next we show that $\tau_Q \subseteq \sigma(\mathbf{B}X)$. Fix $(x,r) \in \mathbf{B}X$ and $\varepsilon > 0$. We shall prove that $\Uparrow (x, r + \varepsilon/2) \subseteq B_{d_Q}((x,r),\varepsilon)$. To this end, let $(y,s) \in \Uparrow (x, r+\varepsilon/2)$. Then $d_p(x,y) < r - s + \varepsilon/2$, so, in particular, $2(s-r) < \varepsilon$. Hence

$$\begin{aligned} d_Q((x,r),(y,s)) &= \max\{d_p(x,y), |r-s|\} + s - r \\ &\leqslant \max\{r - s + \varepsilon/2, |r-s|\} + s - r \\ &\leqslant \max\{\varepsilon/2, 2(s-r)\} < \varepsilon. \end{aligned}$$

We conclude that $\sigma(\mathbf{B}X) = \tau_Q$.

(e) We suppose that (X,p) is complete and let $((x_n,r_n))_n$ be a left K-Cauchy sequence in $(\mathbf{B}X, d_Q)$. Then, for each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $d_Q((x_n,r_n),(x_m,r_m)) < \varepsilon$ whenever $m \geqslant n \geqslant n_\varepsilon$. Thus

$$\max\{d_p(x_n,x_m), |r_n - r_m|\} + r_m - r_n < \varepsilon \qquad (*)$$

whenever $m \geqslant n \geqslant n_\varepsilon$. Hence, the sequence $(r_n)_n$ is bounded in $\mathbb{R}^+$. So there exists a subsequence $(r_{n_k})_k$ of $(r_n)_n$ and an $r \in \mathbb{R}^+$ such that $\lim_k r_{n_k} = r$. Therefore, for

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each $\varepsilon > 0$ there is a $k_\varepsilon \geqslant n_\varepsilon$ such that $|r_{n_m} - r_{n_k}| < \varepsilon$ whenever $m, k \geqslant k_\varepsilon$. Hence, for $m \geqslant k \geqslant k_\varepsilon$, it follows from $(*)$ that

$$d_p(x_{n_k}, x_{n_m}) < \varepsilon + r_{n_k} - r_{n_m} < 2\varepsilon.$$

Consequently, $(x_{n_k})_k$ is a left K-Cauchy sequence in (X, d_p) and there is $x \in X$ such that $\lim_k (d_p)^s(x, x_{n_k}) = 0$. Then it is clear that

$$\lim_k (d_Q)^s((x,r),(x_{n_k}, r_{n_k})) = 0,$$

and from left K-Cauchyness of $((x_n, r_n))_n$, it easily follows that

$$\lim_n (d_Q)^s((x,r),(x_n, r_n)) = 0.$$

Therefore (X, d_Q) is Smyth-complete.

To show the converse, we assume $(\mathbf{B}X, d_Q)$ is Smyth-complete and let $(x_n)_n$ be a left K-Cauchy sequence in (X, d_p). Then $((x_n, 0))_n$ is a left K-Cauchy sequence in $(\mathbf{B}X, d_Q)$, so there is an $x \in X$ such that $\lim_n (d_Q)^s((x,0),(x_n,0)) = 0$. Thus $\lim_n (d_p)^s(x, x_n) = 0$, and, consequently, (X, d_p) is Smyth-complete, so (X, p) is complete by Theorem B. $\square$

Remark 4.1. Note that Theorem 4.1 (e), implies that the partial quasi-metric space $(\mathbf{B}X, Q)$ is complete if and only if the partial metric space (X, p) is complete.

Indeed, if (X, p) is complete, then $(\mathbf{B}X, d_Q)$ is Smyth-complete, and thus bicomplete, that is, $(\mathbf{B}X, Q)$ is complete. Conversely, if $(\mathbf{B}X, Q)$ is complete and $(x_n)_n$ is a Cauchy sequence in $(X, (d_p)^s)$, it follows that $((x_n, 0))_n$ is a Cauchy sequence in $(\mathbf{B}X, (d_Q)^s)$, so, as in the proof of (e), there exists $x \in X$ such that $\lim_n (d_p)^s(x, x_n) = 0$, and hence (X, d_p) is bicomplete. Therefore (X, p) is complete by Theorem B.

5. Quantitative computational models of complete partial metric spaces

There exist several notions of a (computational) model in the literature – see, for example, Flagg and Kopperman (1997), Kopperman *et al.* (2004), Lawson (1997), Martin (1998), Rutten (1998) and Waszkiewicz (2006).

Following Martin (Martin 1998), when we talk of a model of a topological space (X, τ), we mean a pair (L, ϕ) such that L is a domain and $\phi : X \to \mathrm{Max}(L)$ is a homeomorphism, where $\mathrm{Max}(L)$ carries the subspace Scott topology inherited from L. If L is an ω-domain, and the Scott and Lawson topologies agree on $\mathrm{Max}(L)$, we say that the model (L, ϕ) is an ω-computational model (Flagg and Kopperman (1997) and Lawson (1997) just call them computational models).

We shall refer simply to L as a model (or a computational model) of (X, τ) if no confusion arises.

Edalat and Heckmann (1998, Theorem 13) then shows that for every complete metric space (X, d), the domain of formal balls is a model such that the Scott and Lawson topologies coincide on $\mathrm{Max}(\mathbf{B}X)$, and that for every separable complete metric space, the ω-domain of formal balls is an ω-computational model.

In the light of the results obtained in Section 4, which extend the construction of Heckmann for metric spaces (Heckmann 1999) to complete partial metric spaces, we here

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propose a new notion of a (quantitative) computational model that generalises the model defined by Rutten for complete metric spaces (Rutten 1998) to complete partial metric spaces. Then we will show that every complete partial metric space has a model in our sense, and we will study its relation to those mentioned above.

Definition 5.1. A quantitative computational model of a complete partial metric space (X, p) is a triple (L, q, ϕ) such that L is a domain, q is a Smyth-complete quasi-metric on L and ϕ is a map from X into L such that:

(i) The topology induced by q coincides with the Scott topology inherited from L.
(ii) The partial order induced by q coincides with the partial order of L.
(iii) ϕ is an isometry from (X, d_p) into (L, q).
(iv) $\phi(\mathrm{Max}(X, \leqslant_p)) = \mathrm{Max}(L)$.

Note that condition (iv) in the above definition gives $\phi(X) = \mathrm{Max}(L)$ when τ_p is a T_1 topology on X, so, in particular, when (X, p) is a metric space.

Obviously, every quantitative computational model of a complete partial metric space (X, d) is a model in the sense of Martin for $\mathrm{Max}(X, \leqslant_p)$.

When the (complete) partial metric space is a metric space, the partial quasi-metric Q of Theorem 4.1 is a partial metric, and, thus, the induced quasi-metric d_Q is weightable. This suggests the following metric version of Definition 5.1.

Definition 5.2. A quantitative computational model of a complete metric space (X, d) is a triple (L, q, ϕ) such that L is a domain, q is a Smyth-complete weightable quasi-metric on L and ϕ is a map from X into L such that:

(i) The topology induced by q coincides with the Scott topology inherited from L.
(ii) The partial order induced by q coincides with the partial order of L.
(iii) ϕ is an isometry from (X, d) into (L, q).
(iv) $\phi(X) = \mathrm{Max}(L)$.

If, in addition, L is an ω-domain and the Scott and Lawson topologies agree on $\mathrm{Max}(L)$, then (L, q, ϕ) is called a quantitative ω-computational model of (X, d).

Obviously, every quantitative ω-computational model of a separable complete metric space is an ω-computational model.

On the other hand, the construction given in Heckmann (1999) shows that for every complete metric space (X, d), the domain of formal balls is actually a quantitative computational model such that the Scott and Lawson topologies coincide on $\mathrm{Max}(\mathbf{B}X)$, and that for every separable complete metric space, the ω-domain of formal balls is actually a quantitative ω-computational model. Note also that the notion of a quantitative computational model as given in Definition 5.2 is essentially equivalent to the definition of a computational model given in Rutten (1998, Section 7).

Theorem 5.1. Each complete partial metric space has a quantitative computational model.

Proof. Let (X, p) be a complete partial metric space and Q and $\mathfrak{i}$ be the partial quasi-metric on $\mathbf{B}X$ and the isometry constructed in Theorem 4.1, respectively. By Theorem 3.1, $\mathbf{B}X (= (\mathbf{B}X, \sqsubseteq_{d_p}))$ is a domain. By Theorem 4.1 (d) and (e), the topology induced by the quasi-metric d_Q coincides with the Scott topology inherited from $\mathbf{B}X$, and it is Smyth-complete. Moreover, it follows from Theorem 4.1 (c) that the partial order induced by d_Q coincides with $\sqsubseteq_{d_p}$. Finally, one can show immediately that

$$\mathrm{Max}(\mathbf{B}X) = \{(x,0) : x \in \mathrm{Max}(X, \leqslant_p)\},$$

and thus $\mathrm{Max}(\mathbf{B}X) = \mathsf{i}(\mathrm{Max}(X, \leqslant_p))$. Consequently, $(\mathbf{B}X, d_Q, \mathsf{i})$ is a quantitative computational model of (X, p). □

We conclude this section by comparing the notion of a quantitative computational model given in Definition 5.2 with the recent notion of a quantitative domain introduced in Waszkiewicz (2006).

Recall that if μ is a measurement on a poset L, then, by definition, $\ker\mu = \{x \in L : \mu(x) = 0\}$ (see Martin (2000), Waszkiewicz (2003) and Waszkiewicz (2006) for the notion of a measurement).

According to Waszkiewicz (2006, Definition 6.2), if L is a continuous poset, we use $\mathrm{CMax}(L)$ to denote the set of elements x in L such that every Lawson neighbourhood of x contains a Scott neighbourhood of x.

The following notion is due to Waszkiewicz.

Definition 5.3 (Waszkiewicz 2006, Definition 6.3). A quantitative domain is a domain L such that there is a partial metric p on L satisfying the following conditions:

(i) $\tau_p \subseteq \sigma(L)$.
(ii) The function $\mu : L \to \mathbb{R}^+$ given by $\mu(x) = p(x,x)$ is a measurement.
(iii) $\ker\mu = \mathrm{CMax}(L)$.
(iv) The metric $(d_p)^s$ induces the Lawson topology.

It is clear that if (X, d) is a complete metric space, then the domain of formal balls with the partial metric P of Heckmann's construction (see the first paragraph of Section 4), satisfies conditions (i), (ii) and (iii) of Definition 5.3. In fact, the following are well known:

(a) $\tau_P = \sigma(\mathbf{B}X)$.
(b) The function μ given by $\mu((x,r)) = P((x,r),(x,r)) = 2r$ for all $(x,r) \in \mathbf{B}X$ is a measurement.
(c) $\ker\mu = \mathrm{Max}(\mathbf{B}X)$, and, on the other hand, $\mathrm{Max}(\mathbf{B}X) = \mathrm{CMax}(\mathbf{B}X)$ because the Scott and Lawson topology agree on $\mathrm{Max}(\mathbf{B}X)$.

Unfortunately, not every quantitative computational model satisfies condition (iv) of Definition 5.3. In fact, the domain of formal balls of any complete metric space (X, d) for which the metric $(d_P)^s$ does not induce the Lawson topology provides an example of a quantitative computational model that is not a quantitative domain. In connection with this, it is interesting to note that if p is a partial metric on a continuous poset $(L, \leqslant)$ such that the τ_p agrees with the Scott topology induced by the order $\leqslant$, then the topology induced by the metric $(d_p)^s$ is finer than the Lawson topology on L (see Waszkiewicz (2003, Theorem 23)).

However, we can obtain a positive result if (X, d) is a separable complete metric space. Indeed, in this case $\mathbf{B}X$ is an ω-domain without a bottom element. Hence, the lifting

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$\mathbf{B}X \cup \{\bot\}$ of $\mathbf{B}X$ is an ω-domain. Since every ω-domain with a bottom element is a quantitative domain (Waszkiewicz 2006, Theorem 6.5), we deduce that for every separable complete metric space (X, d), $\mathbf{B}X \cup \{\bot\}$ is a quantitative domain.

6. The functor B for partial metric spaces

Edalat and Heckmann (1998, Section 3) discussed the so-called functor $\mathbf{B}$ from a certain category of metric spaces to the category of continuous posets and continuous functions, and presented a domain-theoretic proof of the Banach fixed point theorem. In this section we analyse this functor in the partial metric framework and present a domain-theoretic proof of the partial metric version of Banach's fixed point theorem, which was obtained as Matthews (1994, Theorem 5.3).

Let (X, d) and (Y, e) be two quasi-metric spaces. A function f from X into Y is said to be a Lipschitz function (*cf.* Romaguera and Sanchis (2000) and Romaguera and Sanchis (2005)) if there is a $c \in \mathbb{R}^+$ such that $e(f(x), f(y)) \leqslant cd(x, y)$ for all $x, y \in X$. The number c is then called a Lipschitz constant for f. If $c \in [0, 1[$, we say that f is a contraction.

Let $f : (X, d) \to (Y, e)$ be a Lipschitz function with Lipschitz constant c. Analogously to the metric case, we define the function $\mathbf{B}(f, c) : (\mathbf{B}X, \sqsubseteq_d) \to (\mathbf{B}Y, \sqsubseteq_e)$ by $\mathbf{B}(f, c)((x, r)) = (fx, cr)$ for all $(x, r) \in \mathbf{B}X$.

Then, adapting *mutatis mutandis* the construction of Edalat and Heckmann given in Edalat and Heckmann (1998, page 62), we obtain the following:

— The collection of pairs (f, c), where f is a Lipschitz function from a quasi-metric space (X, d) into a quasi-metric space (Y, e) with a Lipschitz constant c, forms a category where $\mathrm{id}_X = (\mathrm{id}_X, 1)$ and $(g, c') \circ (f, c) = (g \circ f, c'c)$. The functor $\mathbf{B}$ is defined on this category by $\mathbf{B}(f, c)((x, r)) = (f(x), cr)$ for all $(x, r) \in \mathbf{B}X$.

— If $f : (X, d) \to (Y, e)$ is a Lipschitz function with Lipschitz constant c, then $\mathbf{B}(f, c)$ is order preserving.

Edalat and Heckmann also proved the crucial result that if (X, d) and (Y, e) are metric spaces, then $\mathbf{B}(f, c)$ is Scott-continuous. The situation is quite different for quasi-metric spaces. In fact, the next example shows that Scott continuity of $\mathbf{B}(f, c)$ does not even hold for weightable quasi-metric spaces in general. (Recall that a function $f : L \to M$ between posets is Scott-continuous provided f is order preserving and $f(\sup D) = \sup f(D)$ for all directed subsets D of L for which $\sup D$ exists.)

Example 6.1. Let (X, p) be the partial metric space of Example 3.1. Then $d_p(\infty, n) = 1$, $d_p(n, \infty) = 0$ and $d_p(n, m) = |1/n - 1/m|$ for all $n, m \in \mathbb{N}$.

Consider the function $f : X \to X$ defined as $f(\infty) = \infty$ and $f(n) = 1$ for all $n \in \mathbb{N}$. It is routine to check that f is a Lipschitz function with Lipschitz constant $c = 1$, so $\mathbf{B}(f, 1)$ is order preserving from $(\mathbf{B}X, \sqsubseteq_{d_p})$ into itself. We shall prove that $\mathbf{B}(f, 1)$ is not Scott-continuous. To do this, we let $D = \{(n, 1/n)) : n \in \mathbb{N}\}$. So D is directed with least upper bound $(\infty, 0)$ (see Example 3.1). Moreover, $\mathbf{B}(f, 1)((\infty, 0)) = (\infty, 0)$ and $\mathbf{B}(f, 1)((n, 1/n)) = (1, 1/n)$ for all $n \in \mathbb{N}$. From the fact that $(1, 1/n) \sqsubseteq_{d_p} (1, 0)$ for all

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$n \in \mathbb{N}$, it follows that $(1, 0)$ is an upper bound of $\mathbf{B}(f, 1)(D)$. Finally, since $d_p(\infty, 1) > 0$, we conclude that $(\infty, 0)$ is not the least upper bound of $\mathbf{B}(f, 1)(D)$.

However, if the weightable quasi-metric space (X, d) is also bicomplete or, equivalently, Smyth-complete, we can give a positive answer.

Proposition 6.1. Let $f : (X, d) \to (Y, e)$ be a Lipschitz function with Lipschitz constant c. If (X, d) is Smyth-complete and weightable, then $\mathbf{B}(f, c)$ is Scott-continuous.

Proof. Suppose D is a directed subset of $(\mathbf{B}X, \sqsubseteq_d)$ with least upper bound (y, s). Then $\mathbf{B}(f, c)(D)$ is directed, having $\mathbf{B}(f, c)((y, s))$ as an upper bound. We shall prove that $\mathbf{B}(f, c)((y, s)) = \sup \mathbf{B}(f, c)(D)$.

First note that, by Proposition 2.7, there is an ascending sequence $((x_n, r_n))_n$ in D with least upper bound (y, s). It follows from Propositions 2.2 and 2.4 that $(x_n)_n$ is a left K-Cauchy sequence and $\lim_n r_n = s$. By Smyth completeness of (X, d), there exists $x \in X$ such that $\lim_n d^s(x, x_n) = 0$. Hence, by Proposition 2.5, $x = y$, and, consequently, $\lim_n e^s(f(y), f(x_n)) = 0$.

Finally, let $(z, t) \in \mathbf{B}Y$ be such that $(f(x), cr) \sqsubseteq_e (z, t)$ for all $(x, r) \in D$. Given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$e(f(y), z) \leqslant e(f(y), f(x_k)) + e(f(x_k), z) < \varepsilon + cr_k - t < 2\varepsilon + cs - t.$$

Then $(f(y), cs) \sqsubseteq_e (z, t)$. We conclude that $\mathbf{B}(f, c)$ is Scott-continuous. □

Now suppose that f is a function from a partial metric space (X, p) into a partial metric space (Y, q). We say that f is a Lipschitz function if it is a Lipschitz function from the weightable quasi-metric space (X, d_p) into the weightable quasi-metric space (Y, d_q).

Then, Proposition 6.1 can be restated as follows. Let f be a Lipschitz function from a complete partial metric space (X, p) into a partial metric space (Y, q) with Lipschitz constant c. Then $\mathbf{B}(f, c)$ is Scott-continuous from $(\mathbf{B}X, \sqsubseteq_{d_p})$ into $(\mathbf{B}Y, \sqsubseteq_{d_q})$.

From the preceding facts and results, we deduce the following theorem (*cf.* Edalat and Heckmann (1998, Theorem 17)).

Theorem 6.1. With the definition $\mathbf{B}(f, c)((x, r)) = (f(x), cr)$, $\mathbf{B}$ becomes a functor from the category of maps with Lipschitz constants between complete partial metric spaces into the category of Scott-continuous functions between (continuous) dcpo's. This functor has the property that $\mathbf{B}(f, c) \circ \mathsf{i} = \mathsf{i} \circ f$ for all maps f with Lipschitz constant c, where $\mathsf{i}(x) = (x, 0)$ for all $x \in X$, as in Theorem 4.1 above.

In the rest of the section we will focus on the nice partial metric version of Banach's fixed point obtained in Matthews (1994).

Theorem 6.2 (Matthews). Let (X, p) be a complete partial metric space and f be a self-map on X such that there is $c \in]0, 1[$ with $p(f(x), f(y)) \leqslant cp(x, y)$ for all $x, y \in X$. Then f has a unique fixed point $a \in X$, and $p(a, a) = 0$.

Edalat and Heckmann derived the Banach fixed point theorem (Edalat and Heckmann 1998, Theorem 18) from the dcpo fixed point theorem (Gierz *et al.* 2003, Proposition II-2.4). Of course, Scott-continuity of $\mathbf{B}(f, c)$ is crucial in their proof.

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In contrast to the metric case, we give an example of a self-map f on a complete partial metric space (X, p) that satisfies the conditions of Matthews' theorem but for which $\mathbf{B}(f, c)$ is not Scott-continuous (actually, it is not even order preserving). This example reveals the great difficulty that arises when one tries to derive Matthews' fixed point theorem from the dcpo fixed point theorem with this approach. Despite all this, we shall that it is possible to give a domain-theoretic proof of that theorem based on our results.

Example 6.2. Let $X = [0, 1/2]$ and p be the partial metric on X given by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Since $d_p(x, y) = \max\{y - x, 0\}$ for all $x, y \in X$, it follows that $(d_p)^s$ is the Euclidean metric on X, and thus p is a complete partial metric on X. Now let $f : X \to X$ be given by $f(x) = x^2$ for all $x \in X$. Clearly, f satisfies the condition of Theorem 6.2 for $c = 1/2$. However, $\mathbf{B}(f, 1/2)$ is not order preserving. Indeed, it suffices to take any $\varepsilon \in]0, 1/2[$ and note that $(1/2 - \varepsilon, \varepsilon) \sqsubseteq_{d_p} (1/2, 0)$ but $d_p(f(1/2 - \varepsilon), f(1/2)) = \varepsilon(1 - \varepsilon) > \varepsilon/2$.

We conclude this section by giving the promised domain-theoretic proof of Theorem 6.2: Let $x \in X$ and $r = p(x, f(x))/(1 - c)$. Then

$$d_p(x, f(x)) \leqslant p(x, f(x)) = r - rc.$$

Thus $(x, r) \sqsubseteq_{d_p} (f(x), cr)$. By induction, we easily deduce that

$$(f^n(x), c^n r) \sqsubseteq_{d_p} (f^{n+1}(x), c^{n+1} r)$$

for all $n = 0, 1, 2, \ldots$

Therefore, $((f^n(x), c^n r))_n$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq_{d_p})$. By Theorem 3.1, $((f^n(x), c^n r))_n$ has a least upper bound (y, s) in $\mathbf{B}X$, and, by Proposition 2.4,

$$\lim_n c^n r = s \quad \text{and} \quad \lim_n (p(f^n(x), y) - p(f^n(x), f^n(x)) = 0.$$

So $s = 0$. On the other hand, since $p(f^n(x), f^n(x)) \leqslant c^n p(x, x)$ for all $n \in \mathbb{N}$, it follows that $\lim_n p(f^n(x), f^n(x)) = 0$, and thus $\lim_n p(f^n(x), y) = 0$. Consequently, $p(y, y) = 0$ because $p(y, y) \leqslant p(y, f^n(x))$ for all $n \in \mathbb{N}$.

Next we show that y is the unique fixed point of f.

We first note that $(y, 0)$ is maximal in $(\mathbf{B}X, \sqsubseteq_{d_p})$. In fact, if $(y, 0) \sqsubseteq_{d_p} (z, t)$, we have $t = 0$ and $p(y, z) - p(y, y) = 0$, so $p(y, z) = 0$, and thus $p(z, z) = 0$, that is, $y = z$.

We also have $\lim_n p(f(y), f^n(x)) = 0$, because we have $\lim_n p(f^n(x), y) = 0$ and $p(f(y), f^{n+1}(x)) \leqslant cp(y, f^n(x))$ for all $n \in \mathbb{N}$.

Since $p(y, f(y)) \leqslant p(y, f^n(x)) + p(f^n(x), f(y))$ for all $n \in \mathbb{N}$, we get $p(y, f(y)) = 0$, so $y = f(y)$.

Finally, suppose $z \in X$ satisfies $z = f(z)$. Then

$$p(y, z) = p(f(y), f(z)) \leqslant cp(y, z).$$

Consequently, $p(y, z) = 0$, and thus $d_p(y, z) = 0$, that is, $(y, 0) \sqsubseteq_{d_p} (z, 0)$. Since $(y, 0)$ is maximal we conclude that $y = z$, which completes the proof.

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7. Further work

As further work we are interested, among other things, in the following issues:

(1) A characterisation of those partial metric spaces (X, p) that satisfy the condition $(x, r) \ll (y, s) \iff d_p(x, y) < r - s$.
For such a class, the posets of formal balls will be continuous, and we could study the extension of several of our results to these spaces.

(2) A satisfactory solution to the problem posed in (1) would give us guarantees of success in attacking the attractive question of studying the relation between the bicompletion of the space of formal balls $(\mathbf{B}X, d_Q)$ and the space of formal balls of the bicompletion of a weightable quasi-metric space (X, d).

(3) In light of Proposition 3.1, the question of constructing and studying a compatible quasi-metric for the domain of formal balls of a Smyth-complete quasi-metric space also seems natural and deserves attention.

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