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# The Initial and Terminal Cluster Sets of an Analytic Curve

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*Abstract.* For an analytic curve $\gamma: (a,b) \to \mathbb{C}$, the set of values approached by $\gamma(t)$, as $t \searrow a$ and as $t \nearrow b$ can be any two continua of $\mathbb{C} \cup \{\infty\}$.

## 1 Introduction

For $-\infty \le a < b \le +\infty$, and a Riemann surface $X$, we say that $\gamma: (a,b) \to X$ is a real-analytic curve, if it is real-analytic for every local coordinate of $X$. That is, for every $t_0 \in (a,b)$ and every local coordinate $z$ at $\gamma(t_0)$, the function $z \circ \gamma$ is representable by a power series in an interval centered at $t_0$. Analytic curves in Riemann surfaces were studied in [3]; however, in this note, we consider only the case where $X$ is the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Thus, $\gamma$ is analytic if and only if, first, in an interval about each $t_0 \in (a,b)$, for which $\gamma(t_0)$ is finite, $\gamma$ can be developed in a power series and second, in an interval about each $t_0 \in (a,b)$, for which $\gamma(t_0)$ is infinite, $1/\gamma$ can be developed in a power series. A real-analytic curve $\gamma$ is said to be *regular* if its derivative never vanishes, by which we mean that $(z \circ \gamma)'$ has no zeros, for every local coordinate $z$. If $\gamma(t_0) = \infty$, this means that $(1/\gamma)'(t_0) \ne 0$. For brevity, we shall say (as many authors do) that $\gamma$ is an analytic curve to mean that $\gamma$ is a regular real-analytic curve. We denote by $C(\gamma, a)$ and $C(\gamma, b)$ respectively the initial and terminal cluster sets:

$$C(\gamma, a) = \left\{ w \in \overline{\mathbb{C}} : \exists t_n \in (a,b), t_n \to a, \gamma(t_n) \to w \right\},$$
$$C(\gamma, b) = \left\{ w \in \overline{\mathbb{C}} : \exists t_n \in (a,b), t_n \to b, \gamma(t_n) \to w \right\}.$$

Both cluster sets are continua in $\overline{\mathbb{C}}$, that is, nonempty compact connected sets. A *degenerate* continuum is a continuum consisting of a single point. Our principal result is that the initial and terminal cluster sets can be arbitrarily prescribed continua in $\overline{\mathbb{C}}$. For characterizations of other cluster sets, see [1, Theorems 4.3–4.5 and p. 165].

It is also of interest to know whether an analytic curve can be extended in some sense (which we now specify). A notion of extendability for an analytic curve was introduced by Nestoridis and Papadopoulos in [5]. Let us say that an analytic curve $\gamma(t)$, $a < t < b$, can be extended initially, if there is an analytic curve $\sigma(s)$, $L < s < R$, a value $A \in (L,R)$, and an analytic change of parameter $\phi: (A,R) \to (a,b)$, such that

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$\sigma(s) = \gamma(\phi(s))$, for $A < s < R$. We say that $\sigma$ is an *initial analytic extension* of $\gamma$. We note that if $\sigma$ has such an initial analytic extension, then

$$\lim_{t \searrow a} \gamma'(t) = \lim_{s \searrow A} \gamma'(\phi(s))\phi'(s)/\phi'(s) = \lim_{s \searrow A} \sigma'(s)/\phi'(s). \tag{1.1}$$

If $\sigma(A)$ is finite, this limit is $\sigma'(A)/\phi'(A)$. On the other hand, if $\sigma(A)$ is infinite, then $\sigma(s)$ has a simple pole at $A$ and $\sigma'(s)$ has a double pole at $A$. Thus, in a neighbourhood of $A$,

$$\sigma'(s) = (s - A)^{-2}\alpha(s), \quad \text{where } \alpha(A) \neq 0.$$

Hence,

$$\lim_{s \searrow A} \arg \sigma'(s) = \pi/2 + \arg \alpha(s).$$

It follows from (1.1) that, if $\gamma$ has an initial extension, then $\arg \gamma'(t)$ has a limit, as $t$ approaches the initial point $a$ from the right.

A terminal analytic extension for $\gamma$ is defined analogously, and if $\gamma$ has a terminal extension, then $\arg \gamma'(t)$ has a limit, as $t$ approaches the point $b$ from the left. Let us say that an analytic curve is *maximal* as an analytic curve (or *analytically maximal*) if it has neither an initial nor a terminal analytic extension.

## 2 Results and Preparatory Lemmas

**Theorem 2.1** *For any two continua $K^-$ and $K^+$ of the Riemann sphere, there exists an analytic curve $\gamma: (-\infty, +\infty) \to \mathbb{C}$ that is the restriction of an entire function such that*

$$C(\gamma, -\infty) = K^- \quad \text{and} \quad C(\gamma, +\infty) = K^+.$$

*Moreover, the curve $\gamma$ is maximal as an analytic curve.*

For distinct points $z_1$ and $z_2$ in $\mathbb{C}$, we denote by $[z_1, z_2]$ the line segment from $z_1$ to $z_2$ and we denote the arc of the Riemann sphere

$$[+i, \infty, -i] = \{iy : +1 \le y < +\infty\} \cup \{\infty\} \cup \{iy : -\infty < y \le -1\}.$$

The doubly slit plane $\mathbb{C} \setminus [+i, \infty, -i]$ is, in some sense, a maximal proper simply-connected domain containing the real line $(-\infty, +\infty)$. From the domain $\mathbb{C} \setminus [+i, \infty, -i]$, there are two natural approaches to $\infty$, and we define two corresponding cluster sets as follows. For a mapping

$$G: C \setminus [+i, \infty, -i] \longrightarrow \overline{\mathbb{C}},$$

we define the two cluster sets:

$$C(G, -\infty) = \left\{ w \in \overline{\mathbb{C}} : \exists z_n \to \infty, \Re z_n < 0, G(z_n) \to w \right\},$$
$$C(G, +\infty) = \left\{ w \in \overline{\mathbb{C}} : \exists z_n \to \infty, \Re z_n > 0, G(z_n) \to w \right\}.$$

**Corollary 2.2** *For any two continua $K^-$ and $K^+$ of the Riemann sphere, there exists an analytic curve $g: (-\infty, +\infty) \to \mathbb{C}$, that extends to a (locally) conformal mapping $G$ on the doubly-slit plane $\mathbb{C} \setminus [+i, \infty, -i]$ for which*

$$C(g, -\infty) = C(G, -\infty) = K^- \quad \text{and} \quad C(g, +\infty) = C(G, +\infty) = K^+.$$

*Moreover, the curve $g$ is maximal as an analytic curve.*

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***Lemma 2.3*** *For $t_1 < t_2 < t_3$, and non colinear points $z_1, z_2, z_3 \in \mathbb{C}$, consider the parametrisations $\sigma_1: [t_j, t_{j+1}] \to [z_j, z_{j+1}]$ of the segments $[z_j, z_{j+1}]$ given by*

$$\sigma_j(t) = z_j + \frac{t - t_j}{t_{j+1} - t_j}(z_{j+1} - z_j) \quad \textit{for } j = 1, 2.$$

*For each $\epsilon > 0$, and all sufficiently small $\delta > 0$, there is a $C^1$-smooth curve $\sigma_\delta: [t_1, t_3] \to \mathbb{C}$, with nonvanishing derivative such that*

$$\sigma_\delta(t) = \begin{cases} \sigma_1(t) & \textit{if } t_1 \le t \le t_2 - \delta, \\ \sigma_2(t) & \textit{if } t_2 + \delta < t \le t_3, \end{cases}$$

*$\sigma'_\delta(t) \ne 0$ for $t \in [t_1, t_3]$, and $|\sigma_\delta(t) - \sigma_j(t)| < \epsilon$ for $t \in [t_j, t_{j+1}]$, $j = 1, 2$.*

**Proof** A proof of the lemma can be given by constructing a circle $C$ tangent to the segments $[z_2, z_1, ]$ and $[z_2, z_3]$, whose center lies on the bisector of the acute angle formed by these segments. Denote by $w_1$ and $w_2$ the points of tangency and replace the two segments $[w_1, z_2]$ and $[z_2, w_2]$ by the smaller of the two arcs of $C \smallsetminus \{w_1, w_2\}$. We form a curve without corners: the concatenation of the segment $[z_1, w_1]$, the circular arc from $w_1$ to $w_2$, and the segment $[w_1, z_3]$. With an appropriate parametrisation of the arc, this curve is smooth and $C^1$, with nonvanishing derivative. By choosing the center close to the vertex $z_2$, we can make this curve as close to the original polygonal curve as we wish. ∎

**Step 1: A sequence with prescribed cluster set**. Let $K$ be a continuum in $\overline{\mathbb{C}}$. Of course, it is easy to construct a sequence in $\mathbb{C}$ with $K$ as cluster set, but we wish this sequence to have special properties. We begin with the following well-known fact.

***Lemma 2.4*** *Let $K$ be a connected metric space. Let $\delta > 0$ and $p, q \in K$. Then there exists $n \in \mathbb{N}$ and $\{p_1, p_2, \dots, p_n \in K\}$, with $p_1 = p$, $p_n = q$ and $\operatorname{dist}(p_j, p_{j+1}) < \delta$, $j = 1, \dots, n-1$.*

Suppose first that $K$ is a nondegenerate continuum in $\overline{\mathbb{C}}$. Let $z_j \in K \cap \mathbb{C}$, $j = 1, 2, \dots,$ be a dense sequence of distinct points in $K$. By Lemma 2.4, and by induction, there is an increasing sequence $n(j) \in \mathbb{N}$ and a sequence $p_n \in K$ such that $p_{n(j)} = z_j$ and

$$|p_n - p_{n+1}| < \tfrac{1}{j}, \quad \text{for } n(j) \le n < n(j+1).$$

By inserting nearby points (possibly not in $K$), we can assume that no three consecutive points are colinear. Moreover, by occasionally inserting at most two nearby points, we can assume that that there are subsequences $p_{n(k)}$ and $p_{n(\ell)}$, both of which approach every point of $K$, such that every segment $[p_{n(k)}, p_{n(k)+1}]$ is horizontal (with $[p_{n(k)}$ as left end point and every segment $[p_{n(\ell)}, p_{n(\ell)+1}]$ is vertical (with $p_{n(\ell)}$ as lower point).

Now, suppose $K$ is a degenerate continuum in $\overline{\mathbb{C}}$ (that is, a point). We repeat the above procedure, but we now begin with an arbitrary sequence $z_j$ of distinct points in $\mathbb{C}$ that converges to $K$.

We recapitulate this construction in the following lemma.

***Lemma 2.5*** *For any two continua $K^-$ and $K^+$ of the Riemann sphere, there exists a double sequence $\{p_n, n \in \mathbb{Z}\}$ in $\mathbb{C}$ such that the cluster set of the sequence $p_0, p_{-1}, \dots,$ is precisely $K^-$, and the cluster set of the sequence $p_0, p_1, \dots,$ is precisely $K^+$. No three consecutive points are colinear. There are subsequences $p_{n(i)}$ and $p_{n(j)}$ of $p_0, p_{-1}, \dots,$ both of which approach every point of $K^-$, such that every segment $[p_{n(i)}, p_{n(i)+1}]$ is horizontal and every segment $[p_{n(j)}, p_{n(j)+1}]$ is vertical. Similarly, there are subsequences $p_{n(k)}$ and $p_{n(\ell)}$ of $p_0, p_1, \dots,$ both of which approach every point of $K^+$, such that every segment $[p_{n(k)}, p_{n(k)+1}]$ is horizontal (with $p_{n(k)}$ as left end point) and every segment $[p_{n(\ell)}, p_{n(\ell)+1}]$ is vertical (with $p_{n(\ell)}$ as lower point).*

**Step 2: A polygonal curve with prescribed initial and terminal cluster sets**. Suppose $\eta_n$ is a linear mapping of $[n, n+1]$ onto the segment $[p_n, p_{n+1}]$ We define a curve $\eta\colon (-\infty, +\infty) \to \mathbb{C}$ by setting $\eta = \sum_{n=-\infty}^{\infty} \eta_n$, where the sum represents concatination. Such a curve is said to be a polygonal curve with nodes $p_n$.

***Lemma 2.6*** *For any two continua $K^-$ and $K^+$ of the Riemann sphere, there exists a polygonal curve $\eta\colon (-\infty, +\infty) \to \mathbb{C}$ for which*

$$C(\eta, -\infty) = K^- \quad \text{and} \quad C(\eta, +\infty) = K^+.$$

*No three consecutive nodes are colinear. There are sequences $s_{n(i)}$ and $s_{n(j)}$ of real numbers tending to $-\infty$, such that at these values $\eta$ has a non-vanishing derivative, with $\arg(\eta'(s_{n(i)})) = 0$ and $\arg(\eta'(s_{n(j)})) = \pi/2$. Moreover, the sequences $\eta(s_{n(i)})$ and $\eta(s_{n(j)})$ have $K^-$ as set of limits. There are analogous sequences $s_{n(k)}$ and $s_{n(\ell)}$ with respect to $K^+$.*

**Proof** By Lemma 2.5 there is a sequence $p_n; n = 0, 1, 2, \dots,$ associated with $K^+$ and a sequence $p_n; n = 0, -1, -2, \dots,$ associated with $K^-$. Let $\eta_n$ be a linear mapping of the interval $[n, n+1]$ onto the segment $[p_n, p_{n+1}]$ and put $\eta = \sum \eta_n$. Since the lengths of the segments $[p_n, p_{n+1}]$ tend to 0, as $n \to \infty$, it follows that the cluster set of $\eta$ at $-\infty$ is $K^-$ and the cluster set of $\eta$ at $+\infty$ is $K^+$.

We construct the sequence $s_{n(i)}$ as follows. Let $\{n(i)\}$ be the sequence from Lemma 2.5. From the previous paragraph, $\eta_{n(i)}$ is a linear mapping of the interval $[n(i), n(i+1)]$ onto the horizontal segment $[p_{n(i)}, p_{n(i)+1}]$. As $s_{n(i)}$, we choose the mid-point of the open interval $(n(i), n(i+1))$. Clearly, the sequence $s_{n(i)}$ has the required properties. The other three sequences $s_{n(j)}, s_{n(k)}$ and $s_{n(\ell)}$ are constructed similarly. ∎

**Step 3: A smooth curve with prescribed initial and terminal cluster sets**. Now we shall smooth the polygonal curve $\eta$.

***Lemma 2.7*** *For any two continua $K^-$ and $K^+$ of the Riemann sphere, there exists a smooth curve $\sigma\colon (-\infty, +\infty) \to \mathbb{C}$, for which*

$$C(\sigma, -\infty) = K^- \quad \text{and} \quad C(\sigma, +\infty) = K^+.$$

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*The curve $\gamma$ has the same values as the polygonal curve $\eta$ in a neighborhood of the values $s_{n(i)}$, $s_{n(j)}$, $s_{n(k)}$, and $s_{n(\ell)}$.*

**Proof** We begin with the polygonal curve $\eta = \sum \eta_n$ from Lemma 2.6. We replace each $\eta_n$ by a smoothing $\sigma_n$ of $\eta_n$ obtained by Lemma 2.3 such that

$$|\sigma_n(t) - \eta_n(t)| < (|n| + 1)^{-1}, \quad \text{for } t \in [n, n+1], \quad n \in \mathbb{Z}.$$

The concatination $\sigma = \sum \sigma_n$ has the required properties. Indeed, it has the required initial and terminal cluster sets, because $|\sigma(t) - \eta(t)| \to 0$, as $t \to \infty$. Each time we invoke Lemma 2.3, we may choose $\delta$ so small that $\sigma(t) = \eta(t)$ in an interval about the mid-point of the parameter interval $(n, n+1)$. The four sequences consist of such mid-points. ∎

## 3 Proof of Theorem 2.1

By a theorem of Hoischen [4] (see also [2, Cor. 1.4] for an elementary proof), for each $C^1$-smooth function $\sigma: (-\infty, +\infty) \to \mathbb{C}$ and each continuous function $\epsilon: (-\infty, +\infty) \to (0, +\infty)$, there is entire function $f$ such that

$$|f^{(j)}(t) - \sigma^{(j)}(t)| < \epsilon(t), \quad \text{for all } t \in (-\infty, +\infty), \quad j = 1, 2.$$

If $\sigma$ is the curve from Lemma 2.7 and the function $\epsilon$ tends to zero, as $t \to \infty$, then the curve $\gamma(t) = f(t)$, for $t \in (-\infty, +\infty)$, has the same initial and terminal cluster sets as the curve $\sigma$. That is, $\gamma$ has $K^-$ and $K^+$ as initial and terminal cluster sets.

Moreover, we claim that the curve $\gamma$ cannot be extended to $-\infty$ or $+\infty$ analytically for any reparametrization of the increasing parameter, provided we choose $\epsilon$ to decrease sufficiently rapidly. In fact, this is obvious in the case that the corresponding cluster set is non-degenerate. The following proof is thus only of interest if one or both of the initial and terminal cluster sets are degenerate continua (singletons).

Since $\sigma'(t) \neq 0$, it follows that $\arg \sigma'$ is uniformly continuous on compact subsets of $(-\infty, +\infty)$. Hence we can choose $\epsilon$ to decrease so rapidly that $\arg \gamma'(t)$ is close to zero for $t = s_{n(i)}$ and $t = s_{n(k)}$ and is close to $\pi/2$ for $t = s_{n(j)}$ and $t = s_{n(\ell)}$. It follows that $\arg \gamma'(t)$ diverges as $t \to -\infty$ and as $t \to +\infty$. But we have shown in the introduction, that if $\gamma$ has an initial extension, then $\arg \gamma'(t)$ has a limit as $t \searrow -\infty$, and if $\gamma$ has an terminal extension, then $\arg \gamma'(t)$ has a limit as $t \nearrow +\infty$. Thus, the parametrization is maximal. Moreover, $\gamma$ cannot be extended analytically to any larger Riemann surface by any analytic reparametrization, for such a parametrization would be conformal and (by definition) preserve angles. In particular, if $K^-$ or $K^+$ is a point $P$ on the Riemann sphere, then $\gamma$ cannot be extended analytically through $P$ by any analytic reparametrisation.

This concludes the proof of Theorem 2.1. ∎

## 4 Proof of the Corollary

Let $f$ and $\gamma$ be the entire function and analytic curve obtained from Theorem 2.1, where $\gamma$ is the restriction of $f$ to the real line. Let $\Omega$ be a neighborhood of the real line, in which $f'$ is zero-free. We can assume that $\Omega$ has the form of a "strip" $w = u + iv$ : $|v| < \varphi(u)$. We can assume that $\varphi$ decreases to zero so rapidly that $|f(u+iv) - \gamma(u)| <$

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$1/(1+u)$ for $|v| < \varphi(u)$. This assures us that $f$ has the same initial and terminal cluster sets in the strip $\Omega$ as $\gamma$ has on the real line.

Let $h$ be the conformal mapping of $\mathbb{C} \setminus [+i, \infty, -i]$ onto the strip $\Omega$, which sends $-\infty$ to $-\infty$, $+\infty$ to $+\infty$, 0 to 0, and the real line to itself. The locally conformal function $G = f \circ h$ and its restriction $g$ to the real line, have the required properties. Indeed, since $h$ is an order preserving homeomorphism of the real line, $g = \gamma \circ h$ has the same initial and terminal cluster sets as $\gamma$. Similarly, $G = f \circ h$ has the same initial and terminal cluster sets as $f$ in $\Omega$, which are the same initial and terminal cluster sets as those of $\gamma$.

There remains to check that $g$ cannot be analytically extended. We note that $g'(u) = \gamma'(h(u))h'(u)$, and $h'(u)$ is real and positive, so $\arg h'(u) = 0$. Thus, $\arg g'(u) = \arg \gamma'(h(u)) + \arg h'(u) = \arg \gamma'(h(u))$. Since $\arg \gamma'(t)$ diverges as $t \to -\infty$ and as $t \to +\infty$, the same holds for $\arg g'(u)$, as $u \to -\infty$ and as $u \to +\infty$. Thus, $g$ cannot be extended analytically and this concludes the proof of the corollary. ∎

## 5 Examples of Maximal Analytic Curves

If the initial cluster set of a curve is a singleton $\{P\}$, we call $P$ the initial end of the curve. Similarly, if the terminal cluster set of a curve is a singleton, we call it the terminal end of the curve. A particular case of Theorem 2.2 is that, for any two points (not necessarily distinct) $k^-$ and $k^+$ of the Riemann sphere, there is a maximal analytic curve, having $k^-$ and $k^+$ as initial and terminal ends respectively. We now give a few explicit examples of maximal analytic curves having both initial and terminal ends. As in the general case proved above, the reason that these curves are maximal is that the argument of the tangent $\gamma'(t)$ diverges as $t \to \pm\infty$.

***Example 1*** Both ends are finite and equal:

$$\gamma(t) = e^{-t^2+it}, \quad -\infty < t < +\infty.$$

***Example 2*** Both ends are finite and distinct. Consider the function

$$\psi(s) = s \exp\Big(\frac{1}{1-s^2}\Big), \quad -1 < s < +1.$$

The function $\psi$ is analytic with positive derivative and hence has an analytic inverse. $\eta\colon (-\infty, +\infty) \to (-1, +1)$. The analytic curve

$$\gamma(t) = \eta(t) + i(\eta^2(t) - 1)\sin\big(\exp(-(\eta^2(t)-1)^{-1})\big)$$

has $\pm 1$ as ends. As $x \searrow 0$, $\exp(-x^{-1})$ approaches 0 much faster and so the argument of $\gamma'(t)$ does not have a limit as $t \to \pm 1$.

***Example 3*** One end is finite and one is infinite:

$$\gamma(t) = e^{t+it}, \quad -\infty < t < +\infty.$$

***Example 4*** Both ends are infinite:

$$\gamma(t) = e^{t^2+it}, \quad -\infty < t < +\infty.$$

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