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ARTICLE

Expected Maximum Block Size in Critical Random Graphs

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Abstract
Let $G(n, M)$ be a uniform random graph with n vertices and M edges. Let $\wp_{n,M}$ be the maximum block size of $G(n, M)$, that is, the maximum size of its maximal 2-connected induced subgraphs. We determine the expectation of $\wp_{n,M}$ near the critical point $M = n/2$. When $n - 2M \gg n^{2/3}$, we find a constant c_1 such that

$$c_1 = \lim_{n\to\infty}\left(1 - \frac{2M}{n}\right)\mathbb{E}(\wp_{n,M}).$$

Inside the window of transition of $G(n, M)$ with $M = (n/2)(1 + \lambda n^{-1/3})$, where λ is any real number, we find an exact analytic expression for

$$c_2(\lambda) = \lim_{n\to\infty}\frac{\mathbb{E}\left(\wp_{n,(n/2)(1+\lambda n^{-1/3})}\right)}{n^{1/3}}.$$

This study relies on the symbolic method and analytic tools from generating function theory, which enable us to describe the evolution of $n^{-1/3}\,\mathbb{E}\left(\wp_{n,(n/2)(1+\lambda n^{-1/3})}\right)$ as a function of λ.

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1. Introduction

Random graph theory [2, 13, 20] is an active area of research that combines algorithmics, combinatorics, probability theory and graph theory. The uniform random graph model $G(n, M)$ studied by Erdős and Rényi [8] consists of n vertices with M edges drawn uniformly at random from the set of $\binom{n}{2}$ possible edges. Another closely related model, denoted $G(n, p)$, was introduced independently by Gilbert [15]. In the $G(n, p)$ model, each potential edge is chosen to be included in the graph, independently of the other edges, with probability p. Erdős and Rényi showed that for many properties of random graphs, graphs with a number of edges slightly less than a given threshold are unlikely to have a certain property, whereas graphs with slightly more edges are almost guaranteed to satisfy the same property, showing radical changes in their structures (referred to as *phase transition*). As shown in their seminal paper [8], when $M = cn/2$ for constant c, the largest component of $G(n, M)$ has asymptotically almost surely (a.a.s. for short) $O(\log n)$, $\Theta(n^{2/3})$ or $\Theta(n)$ vertices according to whether $c < 1$, $c = 1$ or $c > 1$. This *double-jump* phenomenon about the structures of $G(n, M)$ was one of the most spectacular results in [8] which later became a cornerstone of random graph theory. Due to such a dramatic change, researchers worked around the critical value $n/2$, and one can distinguish three different phases: *subcritical* when $(M - n/2)n^{-2/3} \to -\infty$, *critical* for $M = n/2 + O(n^{2/3})$ and *supercritical* as $(M - n/2)n^{-2/3} \to \infty$. We refer to Bollobás [2] and Janson, Łuczak and Ruciński [20] for books devoted to the random graphs $G(n, M)$ and $G(n, p)$. While the $G(n, p)$ model is the one more commonly used today, partly due to the independence of the edges, the $G(n, M)$ model has a more enumerative flavour, allowing generating-function-based approaches. By setting

$$p = \frac{1}{n} + \frac{\lambda}{n^{4/3}},$$

the stated results of this paper can be extended to the $G(n, p)$ model.

Previous work. In graph theory, a *block* is a maximal 2-connected subgraph (formal definitions are given in Section 2). The problem of estimating the maximum block size has been well studied for some classes of graphs. For a graph drawn uniformly from the class of simple labelled planar graphs with n vertices, the expectation of the number of vertices in the largest block is αn asymptotically almost surely (a.a.s.) where $\alpha \approx 0.95982$ (see Panagiotou and Steger [24] and Giménez, Noy and Rué [16]).

For the labelled connected class, these authors also proved independently that a connected random planar graph has a unique block of linear size.

When we restrict to subcritical graphs, Drmota and Noy [7] proved that the maximum block size of a random connected graph in an aperiodic[a] subcritical graph class is $O(\log n)$.

For maps (a map is a planar graph embedded in the plane), Gao and Wormald [14] proved that a random map with n edges has almost surely around $n/3$ edges. That is, the probability that the size of the largest block is about $n/3$ tends to 1 as n goes to infinity. This result has been improved by Banderier, Flajolet, Schaeffer and Soria [1].

Panagiotou [23] obtained more general results for any graph class $\mathcal{C}$. He showed that the size of the largest block of a random graph from $\mathcal{C}$ with n vertices and m edges is either linear $\Theta(n)$ or logarithmic $O(\log n)$. In particular, the author pointed out that random planar graphs with cn edges belong to the first category, while random outerplanar and series-parallel graphs with fixed average degree belong to the second category.

For the Erdős–Rényi $G(n, M)$ model, the maximum block size is implicitly a well-studied graph property when $M = cn/2$ for fixed $c < 1$. For this range, $G(n, M)$ contains only trees and unicyclic components a.a.s. [8]. So, studying maximum block size and the longest cycle are the same in this case. Let $\wp_{n,M}$ denote the maximum block size of $G(n, M)$. It is shown in [2, Corollary 5.8] that as $M = cn/2$ for fixed $c < 1$, then $\wp_{n,M}$ is a.a.s. at most ω for any function $\omega = \omega(n) \to \infty$. Pittel [25] then obtained the limiting distribution (amongst other results) for $\wp_{n,M}$ for $c < 1$. Note that the results of Pittel are extremely precise and include other parameters of random graphs with c satisfying $c < 1 - \varepsilon$ for fixed $\varepsilon > 0$.

Our results. In this paper we study the fine nature of the Erdős–Rényi phase transition, with emphasis on what happens as the number of edges is close to $n/2$: within the window of the phase transition and near to it, we quantify precisely the expectation of the maximum block size of $G(n, M)$.

For subcritical random graphs, our finding can be stated precisely as follows.[b]

Theorem 1.1. *If $n - 2M \gg n^{2/3}$, the maximum block size $\wp_{n,M}$ of $G(n, M)$ satisfies*

$$\mathbb{E}(\wp_{n,M}) \sim c_1 \frac{n}{n - 2M}, \tag{1.1}$$

[a] In the periodic case, $n \equiv 1 \bmod d$ for some $d > 1$ (see [7] for more details).

[b] Throughout the rest of this paper we use standard asymptotic notation. We write $A(n) \ll B(n)$ if $A(n) = o(B(n))$, *i.e.* $\lim_{n\to\infty} A(n)/B(n) = 0$. Similarly, we write $A(n) \sim B(n)$ if $\lim_{n\to\infty} A(n)/B(n) = 1$.

where $c_1 \approx 0.378\,911$ is the constant given by

$$c_1 = \int_0^\infty (1 - e^{-E_1(v)})\, dv \quad \text{with } E_1(x) = \frac{1}{2}\int_x^\infty e^{-t}\frac{dt}{t}. \tag{1.2}$$

Note that the constant c_1 above is up to a factor of 2 referred to as the Flajolet–Odlyzko constant in Finch's book [9, p. 290] and appeared as the proportion of the expected size of the largest connected component in a random mapping of n elements [11].

For critical random graphs, we have the following result.

Theorem 1.2. *Let λ be any real constant and $M = (n/2)(1 + \lambda n^{-1/3})$. The maximum block size $\wp_{n,M}$ of $G(n,M)$ satisfies*

$$\mathbb{E}(\wp_{n,M}) \sim c_2(\lambda)\, n^{1/3}, \tag{1.3}$$

where

$$c_2(\lambda) = \frac{1}{\alpha}\int_0^\infty \left(1 - \sqrt{2\pi}\sum_{r\geqslant 0}\sum_{d\geqslant 0} A\left(3r + \frac{1}{2}, \lambda\right) e^{-E_1(u)}\, e_{r,d}^{[\lfloor u\alpha^{-1}n^{1/3}\rfloor]}(e^{-u})\right) du, \tag{1.4}$$

$E_1(x)$ is defined in (1.2), α is the positive solution of

$$\lambda = \alpha^{-1} - \alpha, \tag{1.5}$$

the function A is defined by

$$A(y,\lambda) = \frac{e^{-\lambda^3/6}}{3^{(y+1)/3}}\sum_{k\geqslant 0}\frac{(\frac{1}{2}3^{2/3}\lambda)^k}{k!\,\Gamma((y+1-2k)/3)}, \tag{1.6}$$

and the $(e_{r,d}^{[s]}(z))$ are polynomials with rational coefficients defined recursively by (2.14).

Our results give the first-order asymptotics of the expectation of the maximum block size in the critical regime of $G(n,M)$. As is the case for similar results on the probability of planarity of the Erdős–Rényi critical random graphs [22] or on the core of large random hypergraphs [5], our results are also expressed in terms of the function A, which is related to the Airy function, as described in [19, equations (10.28)–(10.32)].

Indeed, the function $A(y,\lambda)$ given by (1.6) has been encountered in the physics of random graphs (see [10, 19]).

It is important to note that there is *no discontinuity* between Theorems 1.1 and 1.2. First, observe that as

$$M = \frac{n}{2} - \frac{\lambda(n)n^{2/3}}{2} \quad \text{with } 1 \ll \lambda(n) \ll n^{1/3},$$

equation (1.1) states that $\mathbb{E}(\wp_{n,M})$ is about $c_1 n^{1/3}/(\lambda(n))$. Next, to see that this value matches the one from (1.3), we argue as follows. In (1.5), as $\lambda(n) \to -\infty$ we have $\alpha \sim |\lambda(n)|$ and

$$A\left(3r + \frac{1}{2}, \lambda\right) \sim \frac{1}{\sqrt{2\pi}\,|\lambda(n)|^{3r}}$$

(see [19, equation (10.3)]). Thus, all the terms in the inner double summation 'vanish' except the one corresponding to $r = 0$ and $d = 0$ (this term is the coefficient for graphs without multicyclic components $e_{0,0}^{[k]} = 1$). It is then notable that as $\lambda(n) \to -\infty$, $c_2(\lambda(n))$ behaves as $c_1/|\lambda(n)|$.

Outline of the proofs and organization of the paper. In [11, Section 4], Flajolet and Odlyzko described generating-function-based methods to study extremal statistics on random mappings.

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Random graphs are obviously harder structures but, as shown in the masterful work of Janson, Knuth, Łuczak and Pittel [19], analytic combinatorics can be used to study the development of the connected components of $G(n, M)$ in depth. As in [11], we will characterize the expectation of $\wp_{n,M}$ by means of truncated generating functions.

Given a family $\mathcal{F}$ of graphs, let (F_n) denote the number of graphs of $\mathcal{F}$ with n vertices. The *exponential generating function* (EGF for short) associated with the sequence (F_n) (or family $\mathcal{F}$) is $F(z) = \sum_{n\geqslant 0} F_n(z^n/n!)$. Let $F^{[k]}(z)$ be the EGF of the graphs in $\mathcal{F}$ but *with all blocks of size at most k*. From the formula for the mean value of a discrete random variable X,

$$\mathbb{E}(X) = \sum_{k\geqslant 0} k\mathbb{P}[X = k] = \sum_{k\geqslant 0} (1 - \mathbb{P}[X \leqslant k]), \tag{1.7}$$

we get a generating function version to obtain

$$\Xi(z) = \sum_{k\geqslant 0} [F(z) - F^{[k]}(z)], \tag{1.8}$$

and the expectation of the maximum block size of graphs of $\mathcal{F}$ is[c]

$$\frac{n![z^n]\ \Xi(z)}{F_n}.$$

In this paper we apply the scheme above by counting realizations of $G(n, M)$ with all blocks of size less than a certain value k. Once we get the forms of their generating functions, we will use complex analysis techniques to obtain our results.

To get the forms of the exponential generating function, we start with the enumeration of trees of given degree specification. For sake of simplicity, suppose that we know how to count the number of labellings of a fixed tree τ with a given degree specification. We can replace each node v of degree $d(v)$ in τ with a block b_v of size at most k and with $d(v)$ distinguished vertices. Such blocks can be enumerated using the Chae–Palmer–Robinson formula [3] for the number of cubic (3-regular) block multigraphs with any given numbers of single edges and double edges. This is of great help since a.a.s. in critical random graphs, the cores are cubic: it is well known (see [19, 20]) that the typical realizations of random graphs when M is close to $n/2$ contain a set of trees, another set of unicyclic components (connected components containing exactly one cycle) and some complex components not necessarily connected but with cubic (or 3-regular) 3-cores.

By substituting each edge of the original tree τ with a sequence of rooted trees with generating series $1/(1 - T(z))$ (where $T(z) = z\, e^{T(z)}$ is the Cayley rooted tree EGF), we can now deduce from this construction the EGFs of the typical graphs whose blocks are of size almost k. Finally, using these series in the spirit of (1.8), we can quantify asymptotically the expectation of the maximum block size in critical random graphs.

This paper is organized as follows. Section 2 starts with the enumeration of trees of given degree specification. We then show how to enumerate 2-connected graphs with 3-regular 3-cores. Combining the trees and the block graphs lead to the forms of the generating functions of connected graphs under certain conditions. Section 2 ends with the enumeration of complex connected components with all blocks of size less than a parameter k. Based on the previous results and by means of analytic methods, Section 3 (resp. 4) provides the proof of Theorem 1.1 (resp. 1.2).

[c] For any power series $A(z) = \sum a_n z^n$, $[z^n]A(z)$ denotes the nth coefficient of $A(z)$, *i.e.* $[z^n]A(z) = a_n$.

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2. Enumerative tools

Trees of given degree specification. Let $U(z)$ be the exponential generating function of labelled unrooted trees and let $T(z)$ be the EGF of rooted labelled trees. It is well known that[d]

$$U(z) = \sum_{n=1}^{\infty} n^{n-2}\frac{z^n}{n!} = T(z) - \frac{T(z)^2}{2} \quad \text{and} \quad T(z) = \sum_{n=1}^{\infty} n^{n-1}\frac{z^n}{n!} = z\, e^{T(z)}. \tag{2.1}$$

For a tree with exactly m_i vertices of degree i, define its *degree specification* as the $(n-1)$-tuple $(m_1, m_2, \ldots, m_{n-1})$. We have the following.

Lemma 2.1. *The number of labelled trees with n vertices and degree specification*

$$(m_1, m_2, \ldots, m_{n-1})$$

with $\sum_{i=1}^n m_i = n$ *and* $\sum_{i=1}^n im_i = 2n-2$ *is*

$$a_n(m_1, m_2, \ldots, m_{n-1}) = \frac{(n-2)!}{\prod_{i=1}^{n-1}((i-1)!)^{m_i}}\binom{n}{m_1, m_2, \ldots, m_{n-1}}.$$

Proof. Using the Prüfer code (see [21]), the number of trees with degree sequence $d_1, d_2, \ldots, d_n$ (*i.e.* with node numbered i of degree d_i) is $(n-2)!/(\prod_{i=1}^n (d_i - 1)!)$. The result is then obtained by regrouping nodes of the same degree. □

Define the EGF associated with $a_n(m_1, m_2, \ldots, m_{n-1})$ by

$$U(\delta_1, \delta_2, \ldots; z) = \sum_{n=2}^{\infty}\sum a_n(m_1, m_2, \ldots, m_{n-1})\delta_1^{m_1}\delta_2^{m_2}\cdots\delta_{n-1}^{m_{n-1}}\frac{z^n}{n!}, \tag{2.2}$$

where the inner summation is taken over all i such that $\sum im_i = 2n-2$ and $\sum m_i = n$. Define $U_n(\delta_1, \delta_2, \ldots, \delta_{n-1})$ as

$$U_n(\delta_1, \delta_2, \ldots, \delta_{n-1}) = [z^n]U(\delta_1, \delta_2, \ldots, \delta_{n-1}; z). \tag{2.3}$$

The following result allows us to compute $U_n(\delta_1, \ldots, \delta_{n-1})$ recursively.

Lemma 2.2. *The generating functions U_n defined in (2.3) satisfy $U_2(\delta_1) = \delta_1^2/(2)$, and for any $n \geqslant 3$,*

$$U_n(\delta_1, \ldots, \delta_{n-1}) = \delta_2 U_{n-1}(\delta_1, \ldots, \delta_{n-2}) + \sum_{i=2}^{n-2}\delta_{i+1}\int_0^{\delta_1}\frac{\partial}{\partial\delta_i}U_{n-1}(x, \delta_2, \ldots, \delta_{n-2})\,dx.$$

Proof. The case $n=2$ is immediate. Let $\mathcal{U}_n$ be the family of trees of size n and let $\mathcal{U}_n^\bullet$ be the family of rooted trees of size n whose roots are of degree one. Deleting the root of the latter trees gives unrooted trees of size $n-1$. Conversely, an element of $\mathcal{U}_n^\bullet$ can be obtained from an element of $\mathcal{U}_{n-1}$, by choosing any vertex and by attaching to this vertex a new vertex which is the root of the newly obtained tree. In terms of EGF, we have

$$U_n^\bullet(\delta_1, \ldots, \delta_{n-1}) = \sum_{i=1}^{n-2}\delta_1\delta_{i+1}\frac{\partial}{\partial\delta_i}U_{n-1}(\delta_1, \ldots, \delta_{n-2}).$$

The combinatorial operator corresponds to choosing a vertex of degree i and adding the root is $\delta_1\delta_i(\partial/(\partial\delta_i))$. The multiplication by the terms $\delta_{i+1}\delta_i^{-1}$ reflects the fact that we have a vertex of

[d] We refer for instance to Goulden and Jackson [17] for combinatorial operators, to Harary and Palmer [18] for graphical enumeration and to Flajolet and Sedgewick [12] for the symbolic method of generating functions.

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degree i that becomes a vertex of degree $i+1$ after the addition of the new vertex of degree 1 (thus the term δ_1). Next, we have to unmark the root, which is by construction of degree one. After a bit of algebra, we obtain the result. $\square$

Enumerating 2-connected graphs whose kernels are 3-regular. A *bridge* or *cut-edge* of a graph is an edge whose removal increases its number of connected components. In particular, the deletion of such an edge disconnects a connected graph. Similarly, an *articulation point* or *cut-vertex* of a connected graph is a vertex whose removal disconnects a graph. A connected graph without an articulation point is called a *block* or a 2-*connected* graph.

Following the terminology of [19], a connected graph has *excess r* if it has r edges more than vertices. Trees (resp. *unicycles* or *unicyclic components*) are connected components with excess $r=-1$ (resp. $r=0$). Connected components with excess $r>0$ are called *complex connected components*. A graph (not necessarily connected) is called *complex* when all its components are complex. The *total excess* of a graph (not necessarily connected) is the number of edges plus the number of acyclic components, minus the number of vertices.

Given a graph, its 2-*core* is obtained by deleting recursively all nodes of degree one. A *smooth* graph is a graph without vertices of degree one.

The 3-*core* (also called *kernel*) of a complex graph is the graph obtained from its 2-core by repeating the following process on any vertex of degree two: for a vertex of degree two, we can remove it and splice together the two edges that it formerly touched. A graph is said to be *cubic* or 3-*regular* if all of its vertices are of degree three. Let $\mathcal{B}_r$ denote the family of 2-connected smooth graphs of excess r with 3-regular 3-cores, and let

$$\mathcal{B}=\bigcup_{r=1}^{\infty}\mathcal{B}_r. \tag{2.4}$$

In this paragraph, we aim to enumerate asymptotically the graphs of $\mathcal{B}_r$ with n vertices. In [3], Chae, Palmer and Robinson established recurrence relations for the numbers of labelled cubic multigraphs with given connectivity, number of double edges and number of loops. For instance, they were able to rederive Wormald's result about the numbers of labelled connected simple cubic graphs with $3n$ simple edges and $2n$ vertices [3, equation (24)]. They proved that the number of such objects is given by

$$\frac{(2n)!}{3n2^n}\,(t_n-2t_{n-1}),\ n\geqslant 2 \tag{2.5}$$

with

$$t_1=0,\ t_2=1 \text{ and } t_n=3nt_{n-1}+2t_{n-2}+(3n-1)\sum_{i=2}^{n-3}t_it_{n-1-i},\qquad n\geqslant 2. \tag{2.6}$$

From the sequence (t_n), they found the number of 2-connected multigraphs.

Lemma 2.3. (Chae, Palmer, Robinson). *Let $g(s,d)$ be the number of cubic block (2-connected labelled) multigraphs with s single edges and d double edges. Then, the numbers $g(s,d)$ satisfy*

$$g(s,d)=0\quad \textit{if } s<2,\ g(s,s)=(2s-1)! \textit{ and } g(3s,0)=\frac{(2s)!}{3s2^s},\quad t_s-2t_{s-1},$$

with t_s defined as in (2.6). In all other cases,

$$g(s,d)=2n(2n-1)\left(\frac{(s-1)}{d}g(s-1,d-1)+g(s-3,d)\right).$$

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Figure 1.

We are now ready to enumerate asymptotically the family $\mathcal{B}_r$. Throughout the rest of this paper, if $A(z)$ and $B(z)$ are two EGFs, we write

$$A(z) \asymp B(z) \quad \text{if and only if } [z^n]A(z) \sim [z^n]B(z) \text{ as } n \to +\infty.$$

Lemma 2.4. *For $r \geqslant 1$, let $B_r(z)$ be the EGF of smooth multigraphs of excess r whose kernels are 3-regular and 2-connected. $B_r(z)$ satisfies $B_r(z) \asymp b_r/(1-z)^{3r}$ where $b_1 = 1/12$ and, for $r \geqslant 2$,*

$$b_r = \sum_{s+2d=3r} \frac{g(s,d)}{2^d(2r)!}, \tag{2.7}$$

with the $g(s,d)$ defined as in Lemma 2.3.

Proof. For $r = 1$, the unique 3-regular kernel of graphs of excess 1 whose cores are blocks is depicted in Figure 1. It is easily seen that their EGF is given by

$$B_1(z) = \frac{z^2}{12(1-z)^3},$$

so that

$$B_1(z) \asymp \frac{1}{12(1-z)^3}.$$

The numbers $g(s,d)$ defined in the above lemma count labelled cubic multigraphs with s single edges and d double edges. If $s + 2d = 3r$, these multigraphs are exactly the 3-cores of the graphs of the family $\mathcal{B}_r$. Starting from the EGF $g(s,d)((w^{3r}z^{2r})/(2r)!)$ – with the variable w (resp. z) marking the edges (resp. vertices) – if we want to reconstruct from these multigraphs the graphs of the family $\mathcal{B}_r$, each edge w of these multigraphs is replaced by a sequence of vertices of degree two introducing the term $1/(1-z)$ (for each of the $3r$ edges of the multigraphs). Next, we have to compensate the symmetry of each double edge introducing the factor $1/2!$ d times. We then obtain the EGF of the family $\mathcal{B}_r$:

$$B_r(z) = \sum_{s+2d=3r} \frac{g(s,d)}{2^d(2r)!} \frac{z^{2r}}{(1-z)^{3r}}.$$

The proof is then completed using singularity analysis of EGFs (see [12]). □

We need to count graphs of excess r with at most k vertices so that all the blocks of such structures are of size at most k. We begin our task with the graphs with cubic and 2-connected kernels.

Lemma 2.5. *Let $\mathcal{B}_r^{[k]}$ be the family of 2-connected graphs of excess r, with at most $k - 2r$ vertices of degree two in their 2-cores and whose 3-cores are cubic. For any fixed $r \geqslant 1$, we have*

$$B_r^{[k]}(z) \asymp b_r \frac{1-z^k}{(1-z)^{3r}}.$$

Proof. Let us use temporarily two variables x and t to mark vertices of degree two and three. The 3-cores of the graphs of $\mathcal{B}_r$ have as bivariate EGF $b_r w^{3r} t^{2r}$ (with w the variable marking the

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edges). In order to reconstruct the 2-cores of $\mathcal{B}_r^{[k]}$, we insert at most $k-2r$ vertices on the $3r$ paths between the vertices of degree three. This operation can be viewed as a sequence of an edge subdivision operation. We proceed as follows. Starting with a 3-core, we have $3r$ possibilities for adding one vertex x_1 of degree two. Since adding a new vertex increases the number of edges, the obtained graph contains exactly $3r+1$ edges. We do an edge subdivision operation to this graph to obtain a graph with exactly two (x_1, x_2) vertices of degree two. This graph contains $3r+2$ edges. We repeat the same operation, and at the ith iteration we start with a graph with $3r+i-1$ edges which contains $i-1$ vertices of degree two, to obtain a graph with i vertices of degree two $(x_1, x_2, \dots, x_i)$. Therefore, the number of possibilities for adding i vertices from the 3-core is $3r(3r+1)\cdots(3r+i-1)$. Then we divide by $i!$ to have a set of vertices $\{x_1, x_2, \dots, x_i\}$. So the generating function of the 2-cores having exactly i vertices of degree two and $2r$ vertices of degree three whose 3-cores are 3-regular is $b_r\binom{3r+i-1}{i}x^i t^{2r}$. Taking into account all possible values of i, we have

$$b_r \sum_{i=0}^{k-2r} \binom{3r+i-1}{i} x^i t^{2r} = b_r \sum_{i=0}^{k-2r} \frac{(3r+i-1)(3r+i-2)\cdots(i+1)}{(3r-1)!} x^i t^{2r}$$
$$\asymp b_r \frac{1-x^{k+1-2r}}{(1-x)^{3r}} t^{2r} \asymp b_r \frac{1-x^k}{(1-x)^{3r}} t^{2r}.$$

□

Let $\mathcal{B}_r^{\bullet s}$ be the set of graphs of $\mathcal{B}_r$ such that s vertices of degree two of their 2-cores are distinguished amongst the others. In other words, an element of $\mathcal{B}_r^{\bullet s}$ can be obtained from an element of $\mathcal{B}_r$ by marking (or pointing) s unordered vertices of its 2-core. In terms of generating functions, we simply get (see [12, 17, 18])

$$B_r^{\bullet s}(z) = \frac{x^s}{s!} \frac{\partial^s}{\partial x^s} B_r(x,t) \bigg|_{t=x=z} = \frac{z^s}{s!} \frac{\partial^s}{\partial z^s} \left(b_r \frac{t^{2r}}{(1-z)^{3r}} \right) \bigg|_{t=z}, \tag{2.8}$$

where $B_r(x,t)$ is the bivariate EGF of $\mathcal{B}_r$, with x the variable for the vertices of degree two and t of the degree three. (The substitution is made after the derivations.)

Define

$$b_r^{\bullet s} = \frac{1}{s!} b_r \prod_{i=1}^{s} [3r + (s-i)]$$

so that

$$B_r^{\bullet s}(z) \asymp \frac{b_r^{\bullet s}}{(1-z)^{3r+s}}.$$

Now, if we switch to the class of graphs with blocks of size at most k, then by similar arguments, the asymptotic number of graphs of $\mathcal{B}_r^{\bullet s}$ with s distinguished vertices and at most k vertices on their 2-cores can be deduced by the formula

$$B_r^{\bullet s,[k]}(z) \asymp b_r^{\bullet s} \frac{1-z^k}{(1-z)^{3r+s}}.$$

Counting 2-cores with cubic kernels by number of bridges. In this paragraph, we aim to enumerate connected smooth graphs whose 3-cores are 3-regular according to their number of bridges (or cut-edges) and their excess. To that end, let $\mathcal{C}_r$ be the family of such graphs with excess $r \geqslant 0$, and for any $d \geqslant 0$ let

$$\mathcal{C}_{r,d} \stackrel{\text{def}}{=} \{G \in \mathcal{C}_r : G \text{ is a cycle or its 3-core is 3-regular and has } d \text{ bridges}\}.$$

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Clearly, we have $\mathcal{C}_{r,0} = \mathcal{B}_r$. If we want to mark the excess of these graphs by the variable w, we simply have

$$C_{r,d}(w,z) = w^r C_{r,d}(z).$$

Lemma 2.6. *For any $r \geqslant 1$ and $d \geqslant 1$,*

$$C_{r,d}(z) = \tag{2.9}$$
$$[w^r]U_{d+1}\big(B^{\bullet 1}(w,z),\ 2!B^{\bullet 2}(w,z),\ 3!B^{\bullet 3}(w,z) + w^{-1}z,\ 4!B^{\bullet 4}(w,z),\ldots, d!B^{\bullet d}(w,z)\big)\frac{w^d}{(1-z)^d},$$

where U_{d+1} is the EGF given by Lemma 2.2,

$$B_0(w,z) = -\frac{1}{2}\log(1-z) - \frac{z}{2} - \frac{z^2}{4},$$
$$B_0^{\bullet s}(w,z) = \frac{1}{s!}\frac{\partial^s}{\partial z^s}B_0(w,z),$$
$$B^{\bullet s}(w,z) = \sum_{r\geqslant 0} w^r B_r^{\bullet s}(z).$$

Proof. Any element of the family $\mathcal{C}_{rd}$ can be obtained from a tree with $d+1$ vertices as follows. Consider a tree $\mathcal{T}$ of size $d+1$. For each vertex v of $\mathcal{T}$ of degree s, we can replace v with an element of $\mathcal{B}^{\bullet s}$ in $s!$ manners. We distinguish two cases according to the degree of v: vertices of degree three can be left unchanged or replaced by elements of $\mathcal{B}^{\bullet 3}$. This yields the term $3!B^{\bullet s}(w,z) + w^{-1}z$ in (2.9). Next, each edge of $\mathcal{T}$ can be replaced by a path of length at least 1 with a factor w, which parametrizes the excess of the obtained graph. Since we have d edges in the tree, we add the factor $w^d/((1-z)^d)$. □

Lemma 2.7. *For $r \geqslant 1$ and $d \geqslant 1$, we have*

$$C_{r,d}(z) \asymp \frac{c_{r,d}}{(1-z)^{3r}},$$

where the coefficients $c_{r,d}$ are defined by

$$c_{r,d} = [w^r]U_{d+1}(\beta_1(w), \beta_2(w), \beta_3(w) + w^{-1}, \beta_4(w),\ldots,\beta_d(w))w^d,$$

with b_ℓ given by (2.7) and

$$\beta_s(w) = \frac{(s-1)!}{2} + \sum_{\ell=1}^{r-1} w^\ell b_\ell \prod_{i=1}^{s}[3\ell + (s-i)] \quad \text{with } s \geqslant 1.$$

Proof. Applying the operator of

$$\frac{z^s}{s!}\frac{\partial^s}{\partial z^s}$$

on unicyclic components gives

$$b_0^{\bullet s} = \frac{1}{s!}\frac{(s-1)!}{2}.$$

Define the ordinary generating function of $(b_\ell^{\bullet s})_{\ell\geqslant 0}$ as

$$b^{\bullet s}(w) = \sum_{\ell=0}^{\infty} b_\ell^{\bullet s} w^\ell = \frac{1}{s!}\left(\frac{(s-1)!}{2} + \sum_{\ell=1}^{\infty} b_\ell \prod_{i=1}^{s}[3\ell + (s-i)]\, w^\ell\right). \tag{2.10}$$

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After a bit of algebra, we obtain

$$c_{r,d} = [w^r]U_{d+1}(b^{\bullet 1}(w),\ 2!b^{\bullet 2}(w), 3!b^{\bullet 3}(w) + w^{-1},\ 4!b^{\bullet 4}(w), \ldots, d!b^{\bullet d}(w))w^d. \tag{2.11}$$

Observe that for any $d \geqslant 1$, each block used to obtain an element of $\mathcal{C}_{r,d}$ is necessarily of excess at most $r-1$. So, the summation in (2.10) can be truncated to $r-1$. □

Let us restrict our attention to elements of $\mathcal{C}_{r,d}$ with blocks of size at most k. Denote this set of graphs by $\mathcal{C}^{[k]}_{r,d}$. Since they can be obtained from a tree with $d+1$ vertices by replacing each vertex of degree s with an s-marked block (a block with a distinguished degree of degree two) of the family $\bigcup_{r=0}^{\infty} \mathcal{B}^{\bullet s,[k]}$, we infer the following lemma.

Lemma 2.8. *For fixed values of r, the EGF of graphs of $\mathcal{C}^{[k]}_{r,d}$ satisfies*

$$C^{[k]}_{r,d} \asymp c_{r,d}\frac{(1-z^k)^{d+1}}{(1-z)^{3r}}.$$

From connected components to complex components. Let $\mathcal{E}^{[k]}_r$ denote the family of complex graphs (not necessarily connected) of total excess r with all blocks of size $\leqslant k$. Let $E^{[k]}_r$ be the EGF of $\mathcal{E}^{[k]}_r$. Using the symbolic method and sprouting the rooted trees from the smooth graphs counted by $C^{[k]}_{r,d}(z)$, we obtain

$$\sum_{r=0}^{\infty} w^r E^{[k]}_r(z) = \exp\left(\sum_{r=1}^{\infty} w^r \sum_{d\geqslant 0}^{2r-1} C^{[k]}_{r,d}(\,T(z)\,)\right).$$

We now use a general scheme which relates EGFs of connected components and EGFs of complex components (see for instance [19, Section 8]). If $E(w,z) = 1 + \sum_{r\geqslant 1} w^r E_r(z)$ with $E_r(z) \asymp e_r/((1-T(z))^{3r})$ and $C_r(z) \asymp c_r/((1-T(z))^{3r})$ are EGFs satisfying

$$1 + \sum_{r\geqslant 1} w^r E_r(z) = \exp\left(\sum_{r=1}^{\infty} w^r C_r(z)\right). \tag{2.12}$$

then the coefficients (e_r) and (c_r) are related by

$$e_0 = 1 \quad \text{and} \quad e_r = c_r + \frac{1}{r}\sum_{j=1}^{r-1} j c_j e_{r-j} \text{ as } r \geqslant 1. \tag{2.13}$$

Similarly, after some algebra we obtain the following result.

Lemma 2.9. *For fixed $r \geqslant 1$,*

$$E^{[k]}_r(z) \asymp \sum_{d=0}^{2r-1} \frac{e^{[k]}_{r,d}(\,T(z)\,)}{(1-T(z))^{3r}},$$

where the functions $(e^{[k]}_{r,d})$ are defined recursively by $e^{[k]}_{0,0}(z) = 1$, $e^{[k]}_{r,d}(z) = 0$ if $d > 2r-1$ and

$$e^{[k]}_{r,d}(z) = c_{r,d}(1-z^k)^{d+1} + \frac{1}{r}\sum_{j=1}^{r-1} j c_{j,d}\, e^{[k]}_{r-j,d}(z)\,(1-z^k)^{d+1}. \tag{2.14}$$

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Remark. Note that the maximal range $2r-1$ of d appears when the 2-core is a cactus graph (each edge lies on a path or on a unique cycle), each cycle has exactly one vertex of degree three and its 3-core is 3-regular.

3. Proof of Theorem 1.1

Following the work of Flajolet and Odlyzko [11] on extremal statistics of random mappings, let us introduce the relevant EGF for the expectation of the maximum block size in $G(n,M)$.

On the one hand, if there are n vertices, M edges and with a total excess r there must be exactly $n-M+r$ acyclic components. Thus, the number of (n,M)-graphs[e] of total excess r without blocks of size larger than k is

$$n![z^n]\frac{U(z)^{n-M+r}}{(n-M+r)!}(e^{W_0(z)-\sum_{i=k+1}^{\infty}T(z)^i/(2i)})E_r^{[k]}(z),$$

where

$$W_0(z)=-\frac{1}{2}\log(1-T(z))-\frac{T(z)}{2}-\frac{T(z)^2}{4}$$

is the EGF of connected graphs of excess $r=0$ (see [19, equation (3.5)]). Note that the above expression is very similar to the right-hand side of [19, equation (13.5)]).

On the other hand, the EGF of all (n,M)-graphs is

$$G_M(z)=\sum_{n\geqslant 0}\binom{\binom{n}{2}}{M}\frac{z^n}{n!}.$$

Define

$$\Xi(z)=\sum_{k\geqslant 0}\left[G_M(z)-\sum_{r\geqslant 0}\sum_{n\geqslant 0}\left(n![z^n]\frac{U(z)^{n-M+r}}{(n-M+r)!}(e^{W_0(z)-\sum_{i=k+1}^{\infty}T(z)^i/(2i)})E_r^{[k]}(z)\right)\frac{z^n}{n!}\right] \tag{3.1}$$

so that

$$\frac{n![z^n]}{\binom{\binom{n}{2}}{M}}\Xi(z)=\sum_{k\geqslant 0}\left[1-\sum_{r\geqslant 0}\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{U(z)^{n-M+r}}{(n-M+r)!}(e^{W_0(z)-\sum_{i=k+1}^{\infty}T(z)^i/(2i)})E_r^{[k]}(z)\right] \tag{3.2}$$

is the expectation of $\wp_{n,M}$.

We know from the theory of random graphs that in the subcritical phase, *i.e.* when $n-2M\gg n^{2/3}$, $G(n,M)$ has no complex components with probability $1-O(n^2/((n-2M)^3))$ (see for instance [4, Theorem 3.2]). We can restrict our attention to the typical random graphs since we can obtain the result by using bounds for the EGF $E_r^{[k]}(z)$ in (3.1):

$$1\leqslant\sum_{r\geqslant 0}E_r^{[k]}(z)\leqslant\sum_{r\geqslant 0}E_r(z)\leqslant\sum_{r\geqslant 0}\frac{e_r}{(1-T(z))^{3r}}\,T(z).$$

The inequality between the two EGFs means that the coefficients of every power of z obeys the same relation. In the above formula, the last inequality is [19, equation (15.2)] with $e_r=(6r)!/(2^{5r}3^{2r}(3r)!\,(2r)!)$. By means of these inequalities between EGFs, the coefficients of $\Xi(z)$ are bounded from below by those of

$$\sum_{k\geqslant 0}\left[G_M(z)-\sum_{n\geqslant 0}\left(n![z^n]\frac{U(z)^{n-M}}{(n-M)!}\frac{e^{-T(z)/2-T(z)^2/4}}{(1-T(z))^{1/2}}\exp\left(-\sum_{j\geqslant k+1}\frac{T(z)^j}{2j}\right)\right)\frac{z^n}{n!}\right] \tag{3.3}$$

[e] A graph with n vertices and M edges.

and bounded from above by those of

$$\sum_{k\geqslant 0}\Bigg[G_M(z)-\sum_{r\geqslant 0}\sum_{n\geqslant 0}\Bigg(n![z^n]\frac{U(z)^{n-M}}{(n-M)!}\frac{e^{-T(z)/2-T(z)^2/4}}{(1-T(z))^{1/2}}\exp\Bigg(-\sum_{j\geqslant k+1}\frac{T(z)^j}{2j}\Bigg)$$
$$\times\frac{U(z)^r}{(n-M+r)\cdots(n-M+1)}\frac{e_rT(z)}{(1-T(z))^{3r}}\Bigg)\frac{z^n}{n!}\Bigg]. \tag{3.4}$$

We need the following lemma to obtain the asymptotics of the coefficients of (3.3) and (3.4).

Lemma 3.1. *Let a and b be any fixed rational numbers. For any sequence of integers $M(n)$ such that $\delta n < M$ for some $\delta \in [0, 1/2]$ but $n-2M \gg n^{2/3}$, define*

$$f_{a,b}(n,M)=\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{U(z)^{n-M}}{(n-M)!}\frac{U(z)^b\,e^{-T(z)/2-T(z)^2/4}}{(1-T(z))^a}.$$

We have

$$f_{a,b}(n,M)\sim 2^b\left(\frac{M}{n}\right)^b\left(1-\frac{M}{n}\right)^b\left(1-\frac{2M}{n}\right)^{1/2-a}.$$

Proof. We write $f_{a,b}(m,n)=\mathrm{St}(m,n)\cdot\mathrm{Ca}(m,n)$ with

$$\mathrm{St}(m,n)=\frac{n!}{\binom{\binom{n}{2}}{m}(n-m)!}\quad\text{and}\quad\mathrm{Ca}(m,n)=[z^n]\frac{U(z)^{n-m}}{(n-m)!}\frac{U(z)^b\,e^{-t(z)/2-t(z)^2/4}}{(1-T(z))^a}.$$

Using Stirling's formula, for the stated range of m we have

$$\frac{n!m!}{(n-m)!}=\sqrt{2\pi}\,\frac{n^{n+1/2}m^{m+1/2}}{(n-m)^{n-m+1/2}}e^{-2m}\left(1+O\left(\frac{1}{n}\right)\right).$$

We also have

$$\binom{\binom{n}{2}}{m}=\frac{n^{2m}}{2^m m!}\exp\left(-\frac{m}{n}-\frac{m^2}{n^2}+O\left(\frac{m}{n^2}\right)+O\left(\frac{m^3}{n^4}\right)\right).$$

Next, we obtain

$$\mathrm{St}(m,n)=\left(\frac{2\pi nm}{n-m}\right)^{1/2}\frac{2^m n^n m^m}{n^{2m}(n-m)^{n-m}}\exp\left(-2m+\frac{m}{n}+\frac{m^2}{n^2}\right)\left(1+O\left(\frac{1}{n}\right)\right). \tag{3.5}$$

For $\mathrm{Ca}(m,n)$, using Cauchy's integral formula and replacing z with $z\,e^{-z}$, we obtain

$$\begin{aligned}\mathrm{Ca}(m,n)&=\frac{2^{m-n}}{2\pi i}\oint(2T(z)-T(z)^2)^{n-m}\frac{U(z)^b\,e^{-T(z)/2-T(z)^2/4}}{(1-T(z))^a}\frac{dz}{z^{n+1}}\\&=\frac{2^{m-n}}{2\pi i}\oint g(z)e^{nh(z)}\frac{dz}{z},\end{aligned} \tag{3.6}$$

where

$$g(z)=\frac{(z-z^2/2)^b\,e^{-z/2-z^2/4}}{(1-z)^{a-1}},$$

and

$$h(z)=z-\frac{m}{n}\log z+\left(1-\frac{m}{n}\right)\log(2-z).$$

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We have $h'(z)=0$ for $z=1$ or $z=2m/n$. Further, $h''(1)=2m/n-1<0$ and

$$h''\left(\frac{2m}{n}\right)=\frac{n(n-2m)}{4m(n-m)}>0.$$

As in [10], we can apply the saddle-point method integrating around the circular path $|z|=2m/n$. Let $\Phi(\theta)$ be the real part of $h(2m/ne^{i\theta})$. We have

$$\Phi(\theta)=\frac{2m}{n}\cos\theta+\left(1-\frac{2m}{n}\right)\log 2-\frac{m}{n}\log\left(\frac{m}{n}\right)+\frac{(1-m/n)}{2}\log\left(1+\frac{m^2}{n^2}-\frac{2m}{n}\cos\theta\right)$$

and

$$\Phi'(\theta)=-\frac{2m}{n}\sin\theta+\frac{(1-m/n)m}{n(1+m^2/n^2-2m/n\cos\theta)}\sin\theta.$$

We note that $\Phi(\theta)$ is a symmetric function of θ. Fix a sufficiently small positive constant θ_0. Then, $\Phi(\theta)$ takes its maximum value at $\theta=\theta_0$ as $\theta\in[-\pi,-\theta_0]\cup[\theta_0,\pi]$. In fact,

$$\Phi(\theta)-\Phi(\pi)=\frac{4m}{n}+\left(1-\frac{m}{n}\right)\log\left(\frac{n-m}{n+m}\right)+O(\theta^2).$$

Therefore, if $\theta\to 0$, $\Phi(\theta)>\Phi(\pi)$. We also have $\Phi'(\theta)=0$ for $\theta=0$ and $\theta=\theta_1$ (for some θ_1 such that $0<\theta_1<\pi$). Calculations show that $\Phi(\theta)$ is decreasing from 0 to θ_1 and then increasing from θ_1 to π. We also have

$$h^{(p)}(z)=(p-1)!\left((-1)^p\frac{m}{n\,z^p}-\frac{(n-m)}{n\,(2-z)^p}\right),\quad p\geqslant 2.$$

Hence,

$$h\left(\frac{2m}{n}e^{i\theta}\right)=h\left(\frac{2m}{n}\right)+\sum_{p\geqslant 2}\xi_p(e^{i\theta}-1)^p,$$

where

$$\xi_p=\frac{(2m/n)^p}{p!}h^{(p)}\left(\frac{2m}{n}\right).$$

We observe that in the stated range we have

$$|\xi_p|=\frac{(2m)^p}{pn^p}\left|(-1)^p\frac{m}{n\,(2m/n)^p}-\frac{(n-m)}{n\,(2-2m/n)^p}\right|\leqslant\frac{m}{np}+\frac{m^p}{(n-m)^p}\frac{n-m}{np}.$$

We then have

$$\left|\sum_{p\geqslant 4}\xi_p(e^{i\theta}-1)^p\right|=O(\theta^4).$$

This allows us to write

$$h\left(\frac{2m}{n}e^{i\theta}\right)=h\left(\frac{2m}{n}\right)-\frac{m(n-2m)}{2\,n(n-m)}\theta^2-i\frac{(n^2-5nm+2m^2)m}{6n(n-m)^2}\theta^3+O(\theta^4).$$

Let $\tau=n(n-m)/(m(n-2m))$ and

$$\theta_0=\left(\frac{(n-m)}{(n-2m)m}\right)^{1/2}\cdot\omega(n)=\sqrt{\frac{\tau}{n}}\cdot\omega(n)$$

where we need a function $\omega(n)$ satisfying $n\theta_0^2\gg 1$ but $n\theta_0^3\ll 1$ as $n\to\infty$. We choose

$$\omega(n)=\frac{(n-2m)^{1/4}}{n^{1/6}}. \tag{3.7}$$

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We can now use the magnitude of the integrand at θ_0 to bound the error, and our choice of θ_0 satisfies

$$\left| g\left(\frac{2m}{n}\, e^{i\theta_0}\right)\left(\exp\left(nh\left(\frac{2m}{n}\, e^{i\theta_0}\right)\right) - \exp\left(nh\left(\frac{2m}{n}\right)\right)\right)\right| = O(e^{-\omega(n)^2/2}).$$

Thus we obtain

$$\mathrm{Ca}(m, n) = \frac{2^{m-n}}{2\pi} \int_{-\theta_0}^{\theta_0} g\left(\frac{2m}{n}\, e^{i\theta}\right) \exp\left(nh\left(\frac{2m}{n}\, e^{i\theta}\right)\right) d\theta (1 + O(e^{-\omega(n)^2/2})).$$

We replace θ with $(\tau^{1/2}/n^{1/2})t$. The integral in the above equation leads to

$$\left(\frac{\tau}{n}\right)^{1/2} \int_{-\omega(n)}^{\omega(n)} g\left(\frac{2m}{n} \exp\left(it\sqrt{\frac{\tau}{n}}\right)\right) \exp\left(nh\left(\frac{2m}{n} \exp\left(it\sqrt{\frac{\tau}{n}}\right)\right)\right) dt.$$

Expanding $g(2m/n\, e^{it\sqrt{\tau/n}})$, we obtain

$$\left(\frac{\tau}{n}\right)^{1/2} \int_{-\omega(n)}^{\omega(n)} g\left(\frac{2m}{n}\right)\left(1 - i\,\frac{2m\tau^{1/2}(n^2 - 2m^2)}{n^{5/2}(n-2m)}\, t + O\left(\frac{n^2}{(n-2m)^3} t^2\right)\right)$$
$$\times \exp\left(nh\left(\frac{2m}{n} \exp\left(it\sqrt{\frac{\tau}{n}}\right)\right)\right) dt.$$

Observe that our choice of $\omega(n)$ in (3.7) and the hypothesis $n - 2m \gg n^{2/3}$ justify such an expansion. Similarly, using the expansion of $h(2m/n\, e^{it\sqrt{\tau/n}})$ yields

$$\left(\frac{\tau}{n}\right)^{1/2} \int_{-\omega(n)}^{\omega(n)} g\left(\frac{2m}{n}\right)\left(1 - i\,\frac{2m\tau^{1/2}(n^2 - 2m^2)}{n^{5/2}(n-2m)}\, t + O\left(\frac{n^2}{(n-2m)^3} t^2\right)\right)$$
$$\times \exp\left(nh\left(\frac{2m}{n}\right) - \frac{1}{2}t^2\right)$$
$$\times \left(1 - i\,\frac{(n^2 - 5nm + 2m^2)}{6(n-m)^{1/2} m^{1/2} (n-2m)^{3/2}}\, t^3 + O\left(\frac{n}{(n-2m)^2} t^4\right)\right) dt.$$

Using the symmetry of the function, we can cancel terms such as it and it^3 (in fact all odd powers of t). Standard calculations show also that for m in the stated ranges, multiplication of the factors of it and it^3 leads to a term of order of magnitude $O(n^2/(n-2m)^3 t^4)$. Therefore we obtain

$$\mathrm{Ca}(m, n) = \frac{2^{m-n}}{2\pi} \left(\frac{\tau}{n}\right)^{1/2} g\left(\frac{2m}{n}\right) e^{nh(2m/n)} \int_{-\omega(n)}^{\omega(n)} e^{-t^2/2}\left(1 + O\left(\frac{n^2}{(n-2m)^3} t^4\right)\right) dt,$$

$$\mathrm{Ca}(m, n) = 2^{m-n}\left(\frac{\tau}{2\pi\, n}\right)^{1/2} g\left(\frac{2m}{n}\right) e^{nh(2m/n)}\left(1 + e^{-O(\omega(n)^2)} + O\left(\frac{n^2}{(n-2m)^3}\right)\right). \tag{3.8}$$

Multiplying (3.5) and (3.8) leads to the result after nice cancellations. (Note that the error terms $e^{-O(\omega(n)^2)}$ and $O(1/n)$ can be regrouped with the $O(n^2(n-2m)^{-3})$.) □

Using Lemma 3.1 with $a = 1/2$ and $b = 0$ on (3.3) after the change of variable $u = T(z)$ and approximating the sum by an integral, we obtain that $\mathbb{E}(\wp_{n,M})$ is asymptotically bounded from below by

$$\sum_{k\geqslant 0}\left(1 - \exp\left(-\frac{1}{2}\int_{(k+1)(1-2M/n)}^{\infty} e^{-v}\frac{dv}{v}\right)\right).$$

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(We use the Euler–Maclaurin summation formula and the change of variable $(k+1)(1-(2M)/n) = u$ so $dk = (1-(2M)/n)^{-1}du$.) The differences between (3.3) and (3.4) are the terms

$$\frac{e_r\, U(z)^r\, T(z)}{(n-M+r)\cdots(n-M+1)(1-T(z))^{3r}}$$

for $r \geqslant 0$, but our proof of Lemma 3.1 shows that the only contribution of these terms comes from the term with $r=0$ ($e_0 = 1$) for the considered values of M. Combining these facts, we obtain (1.1).

4. Proof of Theorem 1.2

The following technical result is essentially Lemma 3 of [19]. Here we give it in a modified version tailored to our needs (namely involving truncated series).

Lemma 4.1. *Let $M = (n/2)(1+\lambda n^{-1/3})$. Then for any natural integers a, k and r we have*

$$\begin{aligned}&\frac{n!}{\binom{\binom{n}{2}}{M}}\,[z^n]\,\frac{U(z)^{n-M+r}}{(n-M+r)!}\,\frac{T(z)^a(1-T(z)^k)}{(1-T(z))^{3r}}\,\exp\!\left(W_0(z)-\sum_{i=k}^{\infty}\frac{T(z)^i}{2i}\right)\\ &\quad=\sqrt{2\pi}\,\exp\left(-\sum_{j=k}^{\infty}\frac{e^{-j\alpha n^{-1/3}}}{2j}\right)(1-e^{-k\alpha n^{-1/3}})A\left(3r+\frac{1}{2},\lambda\right)\left(1+O\!\left(\frac{\lambda^4}{n^{1/3}}\right)\right),\end{aligned}\tag{4.1}$$

uniformly for $|\lambda| \ll n^{1/12}$, where $A(y,\mu)$ is defined by (1.6) and α is given by (1.5).

Proof. Using Stirling's formula, we obtain

$$\mathrm{St}(M,n) = \frac{n!}{\binom{\binom{n}{2}}{M}}\,\frac{1}{(n-M+r)!} = \sqrt{2\pi n}\,\frac{2^{n-M+r}}{n^r}\,\exp\left(-\frac{\lambda^3}{6}+\frac{3}{4}-n\right)\left(1+O\!\left(\frac{\lambda^4}{n^{1/3}}\right)\right).\tag{4.2}$$

Using Cauchy integral's formula and substituting z by $z\,e^{-z}$, we obtain

$$\begin{aligned}\mathrm{Ca}(M,n) &= [z^n]U(z)^{n-M+r}\,\frac{T(z)^a\,(1-T(z)^k)}{(1-T(z))^{3r}}\,e^{(V(z)-\sum_{j=k}^{\infty}T(z)^j/(2j))}\\ &= \frac{1}{2\pi i}\oint\left(T(z)-\frac{T(z)^2}{2}\right)^{n-M+r}\frac{T(z)^a\,e^{-T(z)/2-T(z)^2/4-\sum_{j=k}^{\infty}T(z)^j/2j}}{(1-T(z))^{3r+1/2}}\,\frac{dz}{z^{n+1}}\\ &= \frac{2^{M-n-r}e^n}{2\pi i}\oint g(u)\,\exp\left(nh(u)\right)\frac{du}{u},\end{aligned}\tag{4.3}$$

where the integrand has been split into

$$g(u) = \frac{u^a\,(2u-u^2)^r\,e^{-u/2-u^2/4-\sum_{j=k}^{\infty}u^j/2j}\,(1-u^k)}{(1-u)^{3r-1/2}}$$

and

$$h(u) = u-1-\log u-\left(1-\frac{M}{n}\right)\log\frac{1}{1-(u-1)^2}.$$

The contour in (4.3) should keep $|u|<1$. Precisely at the critical value $M = n/2$ we also have $h(1) = h'(1) = h''(1) = 0$. This triple zero occurs in the procedure used by Janson, Knuth, Łuczak and Pittel [19] when investigating the value of the integral for large n. Let $\nu = n^{-1/3}$, and let α be the positive solution of (1.5). Following the proof of [19, Lemma 3], we will evaluate (4.3) on the

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path $z = e^{-(\alpha+it)\nu}$, where t runs from $-\pi n^{1/3}$ to $\pi n^{1/3}$:

$$\oint f(z)\,\frac{dz}{z} = i\nu \int_{-\pi n^{1/3}}^{\pi n^{1/3}} f(e^{-(\alpha+it)\nu})\,dt.$$

The main contribution to the value of this integral comes from the vicinity of $t = 0$. The magnitude of $e^{h(z)}$ depends on the real part of $h(z)$, *i.e.* $\Re h(z)$. $\Re h(e^{-(\alpha+it)\nu})$ decreases as $|t|$ increases and $|e^{nh(z)}|$ has its maximum on the circle $z = e^{-(\alpha+it)\nu}$ when $t = 0$.

We have

$$n\,h(e^{-s\nu}) = \frac{1}{3}s^3 + \frac{1}{2}\lambda s^2 + O((\lambda^2 s^2 + s^4)\nu),$$

uniformly in any region such that $|s\nu| < \log 2$. In [19, equation (10.7)], Janson, Knuth, Łuczak and Pittel define

$$A(y,\mu) = \frac{1}{2\pi i}\int_{\Pi(1)} s^{1-y} e^{K(\mu,s)}\,ds,$$

where $K(\mu, s)$ is the polynomial

$$K(\mu,s) = \frac{(s+\mu)^2(2s-\mu)}{6} = \frac{s^3}{3} + \frac{\mu s^2}{2} - \frac{\mu^3}{6}$$

and $\Pi(\alpha)$ is a path in the complex plane that consists of the following three straight line segments:

$$s(t) = \begin{cases} -e^{-\pi i/3}\,t & \text{for} -\infty < t \leqslant -2\alpha, \\ \alpha + it\sin\pi/3 & \text{for} -2\alpha \leqslant t \leqslant +2\alpha, \\ e^{+\pi i/3}\,t & \text{for} +2\alpha \leqslant t < +\infty. \end{cases}$$

In particular, Janson, Knuth, Łuczak and Pittel proved (see [19, Section 10]) that $A(y, \mu)$ can be expressed as (1.6).

For the function $g(u)$, we have

$$\begin{aligned} g(e^{-s\nu}) &= \frac{(2e^{-s\nu} - e^{-2s\nu})^r}{(1-e^{-s\nu})^{3r-1/2}}\, e^{-as\nu - e^{-s\nu}/2 - e^{-2s\nu}/4 - \sum_{j=k}^{\infty} e^{-js\nu}/(2j)}\,(1-e^{-ks\nu}) \\ &= (s\nu)^{1/2-3r} e^{-3/4-\sum_{j=k}^{\infty} e^{-js\nu}/(2j)}\,(1-e^{-ks\nu})\,(1+O(s\nu)). \end{aligned}$$

For $g(u)e^{nh(u)}$ in the integrand of (4.3), we have

$$\begin{aligned} e^{-\lambda^3/6} f(e^{-s\nu}) = e^{-3/4-\sum_{j=k}^{\infty} e^{-js\nu}/(2j)}\,&\nu^{1/2-3r}\,(1-e^{-ks\nu})\,s^{1-(3r+1/2)} e^{K(\lambda,s)} \\ &\times (1 + O(s\nu) + O(\lambda^2 s^2 \nu) + O(s^4\nu)), \end{aligned}$$

when $s = O(n^{1/12})$. Finally,

$$\begin{aligned} \frac{e^{-\lambda^3/6}}{2\pi i}\oint g(u)e^{nh(u)}\,\frac{du}{u} &= \exp\left(-\frac{3}{4} - \sum_{j=k}^{\infty}\frac{e^{-j\alpha\nu}}{2j}\right)(1-e^{-k\alpha\nu}) \\ &\quad \times \nu^{3/2-3r} A\left(3r+\frac{1}{2},\lambda\right) + O(\nu^{5/2-3r} e^{-\lambda^3/6}\lambda^{3r/2+1/4}), \end{aligned}$$

where the error term has been derived from those already in [19]. The proof of the lemma is completed by multiplying (4.2) and (4.3). □

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Using this lemma, equation (3.2) and approximating a sum by an integral, the expectation of $\wp_{n,M}$ is about

$$\sum_{k=0}^{n}\left(1-\sum_{r}\sum_{d}\sqrt{2\pi}\,\exp\left(-\sum_{j=k}^{\infty}\frac{e^{-j\alpha n^{-1/3}}}{2j}\right)e_{r,d}^{[k]}(e^{-\alpha n^{-1/3}})A\left(3r+\frac{1}{2},\lambda\right)\right) \tag{4.4}$$
$$=\alpha^{-1}n^{1/3}\int_{0}^{\alpha n^{2/3}}\left(1-\sum_{r}\sum_{d}\sqrt{2\pi}\,\exp\left(-\int_{u}^{\infty}\frac{e^{-v}}{2v}dv\right)e_{r,d}^{[\lfloor u\alpha^{-1}n^{1/3}\rfloor]}(e^{-u})A\left(3r+\frac{1}{2},\lambda\right)\right)du$$

where the polynomials $e_{r,d}^{[s]}$ are given by (2.14).

Remark. We can give a lower bound for the expectation of $\wp_{n,M}$ with a simple expression. Observe that

$$\frac{1-T(z)^k}{(1-T(z))^{3r}}\leqslant\frac{1-T(z)^{k^2}}{(1-T(z))^{3r}}.$$

Thus, in (2.14), if we replace the factor $(1-z^k)$ by $(1-z^{k^2})$, we obtain a lower bound for $\mathbb{E}(\wp_{n,M})$. From (4.4), we can only consider the summation with values of k such that $k\gg n^{1/6}$ since the expectation of $\wp_{n,M}$ is of order $O(n^{1/3})$. For $k\gg n^{1/6}$,

$$e_{r,d}^{[k]}(e^{-k^2\alpha n^{-1/3}})=e_{r,d}\,(1-O(e^{-k^2n^{-1/3}})), \tag{4.5}$$

where the coefficients $e_{r,d}$ are defined recursively by

$$e_{r,d}=c_{r,d}+\frac{1}{r}\sum_{j=1}^{r-1}jc_{j,d}\,e_{r-j,d}. \tag{4.6}$$

Using these observations, the expectation of $\wp_{n,M}$ is bounded from below by

$$\alpha^{-1}n^{1/3}\int_{0}^{\alpha n^{2/3}}\left(1-\sum_{r}\sum_{d}\sqrt{2\pi}\,\exp\left(-\int_{u}^{\infty}\frac{e^{-v}}{2v}dv\right)e_{r,d}A\left(3r+\frac{1}{2},\lambda\right)\right)du. \tag{4.7}$$

Observe now that $\sum_d e_{r,d}=e_r$ with e_r defined by (2.13) (see also [19, equation (7.2)]) and $\sum_r\sqrt{2\pi}\,e_rA(3r+1/2,\lambda)=1$ as proved in the same paper (see [19, Section 14]). Thus, after nice cancellations, we obtain

$$\mathbb{E}(\wp_{n,M})\geqslant\frac{n^{1/3}}{\alpha}\int_{0}^{\infty}\left(1-\exp\left(-\int_{u}^{\infty}\frac{e^{-v}}{2v}\,dv\right)\right)du. \tag{4.8}$$

5. Conclusion

We have shown that the generating function approach is well suited to precisely analysing the expectation of the maximum block size of random graphs. Our analysis shows that analytic combinatorics gives access to a fine description of extremal parameters inside the window of transition of random graphs. Our work complements previous descriptions of the structures of random graphs such as the one in [6], where typical distances between the vertices and sizes of the 2-core have been investigated.

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References

[1] Banderier, C., Flajolet, P., Schaeffer, G. and Soria, M. (2001) Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Struct. Alg.* **19** 194–246.
[2] Bollobás, B. (2001) *Random Graphs*, second edition. Cambridge University Press.
[3] Chae, G. B., Palmer, E. M. and Robinson R. W. (2007) Counting labeled general cubic graphs. *Discrete Math.* **307** 2979–2992.
[4] Daudé, H. and Ravelomanana, V. (2009) Random 2-XORSAT phase transition. *Algorithmica* **59** 48–65.
[5] Dembo, A. and Montanari, A. (2008) Finite size scaling for the core of large random hypergraphs. *Ann. Appl. Probab.* **18** 1993–2040.
[6] Ding, J., Kim, J. H., Lubetzky, E. and Peres, Y. (2011) Anatomy of a young giant component in the random graph. *Random Struct. Alg.* **39** 139–178.
[7] Drmota, M. and Noy, M. (2013) Extremal parameters in sub-critical graph classes. In *10th Meeting on Analytic Algorithmics and Combinatorics (ANALCO)*, pp. 1–7.
[8] Erdős, P. and Rényi, A. (1960) On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.* **5** 17–61.
[9] Finch, S. R. (2003) *Mathematical Constants*. Cambridge University Press.
[10] Flajolet, P., Knuth, D. E. and Pittel, B. (1989) The first cycles in an evolving graph. *Discrete Math.* **75** 167–215.
[11] Flajolet, P. and Odlyzko A. (1990) Random mapping statistics. In *Advances in Cryptology (EUROCRYPT)*, Vol. 434 of Lecture Notes in Computer Science, Springer, pp. 329–354.
[12] Flajolet, P. and Sedgewick, B. (2009) *Analytic Combinatorics*. Cambridge University Press.
[13] Frieze, A. M. and McDiarmid C. (1997). Algorithmic theory of random graphs. *Random Struct. Alg.* **10** 5–42.
[14] Gao, Z. and Wormald, N. C. (1999) The size of the largest components in random planar maps. *SIAM J. Discrete Math.* **12** 217–228.
[15] Gilbert, E. (1959) Random graphs. *Ann. Math. Statist.* **30** 1141–1144.
[16] Giménez, O., Noy, M. and Rué, J. J. (2013) Graph classes with given 3-connected components: Asymptotic enumeration and random graphs. *Random Struct. Alg.* **42** 438–479.
[17] Goulden, I. P. and Jackson, D. M. (1983) *Combinatorial Enumeration*. Wiley.
[18] Harary, F. and Palmer, E. (1973) *Graphical Enumeration*. Academic Press.
[19] Janson, S., Knuth, D. E., Łuczak, T. and Pittel, B. (1993) The birth of the giant component. *Random Struct. Alg.* **4** 233–358.
[20] Janson, S., Łuczak, T. and Ruciński, A. (2000) *Random Graphs*. Wiley-Interscience.
[21] Matoušek, J. and Nešetřil, J. (2008) *An Invitation to Discrete Mathematics*, second edition. Oxford University Press.
[22] Noy, M., Ravelomanana, V. and Rué, J. J. (2015) On the probability of planarity of a random graph near the critical point. *Proc. Amer. Math. Soc.* **143** 925–936.
[23] Panagiotou, K. (2009) Blocks in constrained random graphs with fixed average degree. In *21st International Conference on Formal Power Series and Algebraic Combinatorics*.
[24] Panagiotou, K. and Steger, A. (2010) Maximal biconnected subgraphs of random planar graphs. *ACM Trans. Alg.* **6** 31.
[25] Pittel, B. (1988) A random graph with a sub-critical number of edges. *Trans. Amer. Math. Soc.* **309** 51–75.
[26] Wright E. M. (1977) The number of connected sparsely edged graphs. *J. Graph Theory* **1** 317–330.
[27] Wright E. M. (1980) The number of connected sparsely edged graphs III: Asymptotic results. *J. Graph Theory* **4** 393–407.

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