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A CHARACTERIZATION OF THE CONTINUOUS q-ULTRASPHERICAL POLYNOMIALS

BY

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ABSTRACT. In his Ph.D. thesis Allaway found all polynomials $\{A_n(x)\}_{n=0}^{\infty}$ that can be represented as

$$w(\cos\theta)A_n(\cos\theta)=\sum_{k=0}^{\infty} a_k b_{n+k}\sin(n+2k+1)\theta,$$

$$0<\theta<\pi, \qquad n=0,1,\ldots,$$

and $a_0b_n\neq 0$. We solve the essentially equivalent problem of finding all symmetric polynomials $\{A_n(x)\}_{n=0}^{\infty}$ with $a_kb_{n+k}=\int_{-\infty}^{\infty}A_n(x)U_{n+2k}(x)\,d\alpha(x)$ when $\{A_n(x)\}_{n=0}^{\infty}$ are orthogonal with respect to $d\alpha(x)$. The polynomials are the continuous q-ultraspherical polynomials and some of their limiting cases.

1. **Introduction.** The ultraspherical polynomials $\{C_n^\lambda(x)\}_{n=0}^{\infty}$ can be defined by the recurrence relation

$$nC_n^\lambda(x)=2x(n+\lambda-1)C_{n-1}^\lambda(x)-(n+2\lambda-2)C_{n-2}^\lambda(x), \tag{1.1}$$

$n\geq 2$, $C_0^\lambda(x)=1$, $C_1^\lambda(x)=2\lambda x$. Szegö [9], [10, (4.9.22)] showed that for all $n\geq 0$

$$\frac{(\sin\theta)^{2\lambda-1}\Gamma(\lambda)\Gamma(\lambda+1)n!}{2^{2-2\lambda}\Gamma(n+2\lambda)}C_n^\lambda(\cos\theta)=\sum_{k=0}^{\infty}\frac{(1-\lambda)_k(n+k)!}{k!\,(\lambda+1)_{n+k}}\sin(n+2k+1)\theta, \tag{1.2}$$

where the shifted factorial $(a)_n$ is defined by

$$(a)_n=\frac{\Gamma(n+a)}{\Gamma(a)}=a(a+1)\cdots(a+n-1). \tag{1.3}$$

The well known orthogonality of $C_n^\lambda(x)$ follows formally from (1.2), since

$$\begin{aligned}\int_0^\pi \frac{\sin(m+1)\theta}{\sin\theta}C_n^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\,d\theta\\ =d_n\sum_{k=0}^{\infty}a_kb_{n+k}\int_0^\pi \sin(n+2k+1)\theta\sin(m+1)\theta\,d\theta\\ =0,\end{aligned}$$

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where $m<n$, and where $\sin(m+1)\theta/\sin\theta$ is a polynomial of degree m in $x=\cos\theta$. Thus,

$$\int_{-1}^{1} C_n^\lambda(x)p_m(x)(1-x^2)^{\lambda-1/2}\,dx=0,$$

for all polynomials $p_m(x)$ of degree $m<n$. When $\lambda>1/2$, this argument is easily justified by uniform convergence, and it can be justified with some work when $0<\lambda\le 1/2$. When $-1/2<\lambda<0$ orthogonality continues to hold but the series (1.2) diverges. The series (1.2) suggests the following question. Find all sets of polynomials $\{A_n(x)\}_{n=0}^\infty$ that can be represented in the form

$$w(x)A_n(x)=\sum_{k=0}^\infty a_k b_{n+k} U_{n+2k}(x), \tag{1.5}$$

where $U_n(x)$ is defined by

$$U_n(x)=\sin(n+1)\theta/\sin\theta, \quad x=\cos\theta. \tag{1.6}$$

The integral (1.4) suggests a related question. Find all sets of polynomials $\{A_n(x)\}_{n=0}^\infty$ that are orthogonal and

$$\int_{-1}^{1} A_n(x)U_{n+2k}(x)w(x)\,dx=a_k b_{n+k}, \tag{1.7}$$

when $k, n=0, 1, 2, \ldots,$.

From the formal argument above $\{A_n(x)\}_{n=0}^\infty$ is orthogonal with respect to $w(x)$, so we will assume that in (1.7). The first of these questions was solved by Allaway [2]. The same methods are used to solve both problems, and a slightly wider class of polynomials arises in the solution of the second problem, so we will solve it here.

When Allaway solved the first problem the most general such polynomials were not well known, so he did not mention them. These polynomials were introduced by L. J. Rogers [8]. Their recurrence relation is

$$\begin{aligned}(1-q^n)C_n(x;\beta\mid q)&=2x(1-\beta q^{n-1})C_{n-1}(x;\beta\mid q)\\ &\quad-(1-\beta^2q^{n-2})C_{n-2}(x;\beta\mid q) \quad (n\ge 2),\\ C_0(x;\beta\mid q)=1,&\qquad C_1(x;\beta\mid q)=2(1-\beta)x/(1-q).\end{aligned} \tag{1.8}$$

When $\beta=q^\lambda$ and $q\to 1$, they converge to the ultraspherical polynomials $\{C_n^\lambda(x)\}_{n=0}^\infty$.

A more general identity than (1.5) was found in [4]. It is

$$w_\beta(x)C_n(x;\beta\mid q)=\sum_{k=0}^\infty a(k,n)C_{n+2k}(x;\gamma\mid q)w_\gamma(x), \tag{1.9}$$

where

$$a(k,n)=\frac{\beta^k(\gamma/\beta;q)_k(q^{n+1};q)_{2k}(\gamma^2q^{n+2k};q)_\infty(\beta q^{n+k+1};q)_\infty}{(q;q)_k(\gamma q^{n+k};q)_\infty(\beta^2q^n;q)_\infty(\gamma;q)_\infty}$$
$$\times(\beta:q)_\infty(1-\gamma q^{n+2k}).$$

This can also be written as

$$a(k,n)=\frac{(\gamma^2;q)_\infty(\beta;q)_\infty(\beta q;q)_\infty\beta^k(\gamma/\beta;q)_k(\beta^2;q)_n}{(\gamma;q)_\infty(\gamma q;q)_\infty(\beta^2;q)_\infty(q;q)_k(q;q)_n}$$
$$\times\frac{(\gamma;q)_{n+k}(q;q)_{n+2k}(\gamma q;q)_{n+2k}}{(\beta q;q)_{n+k}(\gamma^2;q)_{n+2k}(\gamma;q)_{n+2k}}$$

Here $|q|<1$,

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k), \tag{1.10}$$

and

$$(a;q)_n=(a;q)_\infty/(aq^n;q)_\infty=\prod_{k=0}^{n-1}(1-aq^k). \tag{1.11}$$

In (1.9) take $\gamma=q$ to obtain

$$\begin{aligned}&\frac{w_\beta(x)(q;q)_\infty(\beta^2;q)_\infty}{\sqrt{(1-x^2)}(\beta;q)_\infty(\beta q;q)_\infty}\frac{(q;q)_n}{(\beta^2;q)_n}C_n(x;\beta\mid q)\\&\quad=\sum_{k=0}^{\infty}\frac{\beta^k(q/\beta;q)_k(q;q)_{n+k}}{(q;q)_k(\beta q;q)_{n+k}}U_{n+2k}(x).\end{aligned} \tag{1.12}$$

This has the form (1.5) with

$$\begin{aligned}a_k&=\beta^k(q/\beta;q)_k/(q;q)_k\\ b_n&=(q;q)_n/(\beta q;q)_n.\end{aligned} \tag{1.13}$$

These are the values of a_k and b_n that need to be found in both (1.5) and (1.7). Since $|\beta|<1$ is necessary for convergence in (1.12) (when $|q|<1$, which we will assume most of the time), it is better to consider the second problem. However the same problem occurs here, since β^k grows geometrically when $|\beta|>1$, and it cannot grow this fast if it is given by (1.7). Thus we extend the representation slightly to

$$\int_{-\infty}^{\infty}A_n(x)U_{n+2k}(x)\,d\alpha(x)=a_kb_{n+k}, \tag{1.7$'$}$$

where $\{A_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to $d\alpha(x)$.

2. **The characterization problem and some necessary and sufficient conditions.** We will add one last restriction to the problem. The series (1.5) is even or odd depending on the parity of n. We assume the same about $A_n(x)$.

PROBLEM 1. Find all sets of orthogonal polynomials $\{A_n(x)\}_{n=0}^{\infty}$ with

$$\int_{-\infty}^{\infty}A_n(x)A_k(x)\,d\alpha(x)=\delta_{k,n}h_n, \tag{2.1}$$

$$A_n(-x)=(-1)^nA_n(x), \tag{2.2}$$

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$$\int_{-\infty}^{\infty} A_n(x)U_{n+2k}(x)\,d\alpha(x) = a_k b_{n+k}, \tag{2.3}$$

$k, n = 0, 1, \ldots.$

If $\{A_n(x)\}_{n=0}^{\infty}$ is a set of polynomials orthogonal on the real line with respect to a complex measure then it satisfies for $n = 2, 3, \ldots$

$$A_n(x) = (2\mu_n x + c_n)A_{n-1}(x) - \lambda_n A_{n-2}(x), \tag{2.4}$$

$A_0(x) = 1$, $A_1(x) = 2\mu_1 x + c_1$. If $A_n(-x) = (-1)^n A_n(x)$, then $c_n = 0$, $n = 1, 2, \ldots.$ The coefficients μ_n and λ_n satisfy $\mu_n\lambda_{n+1} \neq 0$, $n = 1, 2, \ldots.$ See [7, Chapter 1, Theorem 4.4].

For definiteness take $a_0 = 1$. Since

$$U_n(x) = 2^n x^n + \text{lower terms}$$

and

$$A_n(x) = (2x)^n \prod_{i=1}^{n} \mu_i + \text{lower terms},$$

we have

$$h_n = \int_{-\infty}^{\infty} [A_n(x)]^2\,d\alpha(x) = b_n \prod_{i=1}^{n} \mu_i.$$

The recurrence relation (2.4) gives

$$h_n = \mu_n\lambda_{n+1}h_{n-1}/\mu_{n+1}, \qquad n = 1, 2, \ldots,$$

so

$$b_n = h_0 \prod_{j=2}^{n+1} \gamma_j, \qquad n = 1, 2, \ldots, \tag{2.5}$$

where

$$\gamma_j = \lambda_j/\mu_j, \qquad j = 2, 3, \ldots.$$

Use the recurrence relation (2.4) and

$$\begin{aligned} 2xU_n &= U_{n+1}(x) + U_{n-1}(x), \qquad n = 1, 2, \ldots \\ U_0(x) &= 1,\ U_1(x) = 2x, \end{aligned} \tag{2.6}$$

in (2.3) to obtain

$$a_k b_{n+k} = \frac{a_{k-1}b_{n+k}}{\mu_{n+1}} + \frac{\lambda_{n+1}}{\mu_{n+1}} a_k b_{n+k-1} - a_{k-1}b_{n+k-1},$$

or

$$b_{n+k}\left[a_k - \frac{a_{k-1}}{\mu_{n+1}}\right] = b_{n+k-1}\left[\frac{\lambda_{n+1}}{\mu_{n+1}} a_k - a_{k-1}\right].$$

Then use (2.5) and replace n by $n-1$. The result is

$$\gamma_{n+k}[a_k - a_{k-1}/\mu_n] = \gamma_n a_k - a_{k-1}, \qquad n, k \geq 1, \tag{2.7}$$

with $\gamma_1 = 0$. This is consistent with (2.4), since $\lambda_1 = 0$ implies $A_1(x)$ is as given.

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Let $n=1$ in (2.7) to obtain

$$a_k=\left(\frac{1}{\mu_1}-\frac{1}{\gamma_{k+1}}\right)a_{k-1}. \tag{2.8}$$

Since $\lambda_{n+1}\mu_n\neq 0$, γ_n does not vanish, so if $a_n=0$, then $a_k=0$ for $k>n$. Let m be the smallest integer with $a_m=0$ if such exists, and be infinite if $a_k\neq 0$, $k=0,1,\ldots$.

Set $\alpha_k=a_k/a_{k-1}$, $k=1,2,\ldots,m$. Then

$$a_k=\prod_{i=1}^{k}\alpha_i$$

and equation (2.7) becomes

$$\gamma_{n+k}[\alpha_k-\mu_n^{-1}]=\gamma_n\alpha_k-1 \qquad n\geq 1; \qquad 1\leq k\leq m. \tag{2.9}$$

To solve Problem 1 we solve (2.9).

3. **The solution of a finite difference equation.** When $n=1$ in (2.9), we recover

$$\alpha_k=\mu_1^{-1}-\gamma_{k+1}^{-1} \qquad 1\leq k\leq m. \tag{3.1}$$

Use (3.1) with k replaced by $k+1$ and (2.9) when $n=2$ to obtain

$$\alpha_{k+1}(1-\gamma_2\alpha_k)+\left(\frac{\gamma_2}{\mu_1}-1\right)\alpha_k+\left(\frac{1}{\mu_2}-\frac{1}{\mu_1}\right)=0, \qquad 1\leq k\leq m-1. \tag{3.2}$$

From (3.2) we see that α_k is uniquely determined in terms of μ_1, μ_2 and γ_2 if $\alpha_i\gamma_2\neq 1$, $i=1,2,\ldots,k-1$. If $\alpha_i\gamma_2=1$, then $\gamma_2=\mu_2$. Thus there are two cases to consider, $\gamma_2\neq\mu_2$ and $\gamma_2=\mu_2$.

CASE I. $\gamma_2\neq\mu_2$. In this case

$$\alpha_{k+1}=(a+b\alpha_k)/(1-c\alpha_k) \tag{3.3}$$

with $a=\mu_1^{-1}-\mu_2^{-1}$, $b=1-\gamma_2/\mu_1$, and $c=\gamma_2$. The solution of (3.3) is a standard exercise in continued fractions, but we are a bit better off than that since we know what the answer should be. Since a_k is given by

$$a_k=\frac{\beta^k(q/\beta;q)_k}{(q;q)_k},$$

we want

$$\alpha_k=a_k/a_{k-1}=(\beta-q^k)/(1-q^k).$$

This is a little too much, since the constants μ_1, μ_2 and γ_2 are independent, so α_k must depend on three parameters. We can take for $n\geq 1$,

$$\alpha_n=\frac{(\beta-q^n)}{\rho(1-q^n)}. \tag{3.4}$$

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To determine how β, ρ and q are defined in terms of μ_1, μ_2 and γ_2 substitute for α_n as given by (3.4) in (3.3). The details are tedious and will be omitted. The right choices are

$$\mu_1 = \rho/(1+\beta) \tag{3.5}$$

$$\mu_2 = \rho(1-\beta q)/(1-\beta^2 q) \tag{3.6}$$

$$\gamma_2 = \rho(1-q)/(1-\beta q). \tag{3.7}$$

It is easy to show that (3.4) satisfies (3.2) with this choice of μ_1, μ_2 and γ_2. For $n \geq 1$, formula (3.1) gives

$$\gamma_n = \rho(1-q^{n-1})/(1-\beta q^{n-1}) \tag{3.8}$$

and (2.9) gives

$$\mu_n = \rho(1-\beta q^{n-1})/(1-\beta^2 q^{n-1}) \tag{3.9}$$

$$\lambda_n = \rho^2(1-q^{n-1})/(1-\beta^2 q^{n-1}). \tag{3.10}$$

The Jacobian of the transformation (3.5)–(3.7) is

$$\frac{-\rho^2(1-\beta)^2(1-q)}{(1+\beta)^2(1-\beta^2 q)^2(1-\beta q)}.$$

Thus there are potential problems when $\beta = 1$ or $q = 1$. If $\beta = 1$ and $q \neq 1$, then $\gamma_2 = \mu_2$, which is the second case. If $\beta \neq 1$, $q = 1$, then $\gamma_2 = 0$. We assume this did not happen when we assumed $\{A_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial set. The remaining case is when $\beta \to 1$ and $q \to 1$ simultaneously. In this case we can take $\beta = q^\lambda$ and obtain

$$\alpha_k = (k-\lambda)/\rho k \tag{3.11}$$

$$\gamma_k = \rho(k-1)/(k+\lambda-1) \tag{3.12}$$

$$\mu_k = \rho(k+\lambda-1)/(k+2\lambda-1) \tag{3.13}$$

$$\lambda_k = \rho^2(k-1)/(k+2\lambda-1). \tag{3.14}$$

Thus the cases that arise when $\gamma_2 \neq \mu_2$ are the continuous q-ultraspherical polynomials and the ultraspherical polynomials.

We have not mentioned the integer m except as a condition for how long these formulas hold. When $\beta = q^m$, $\alpha_m = 0$, so $a_m = 0$, and then $a_k = 0$ for $k \geq m$. This case is a limit of the case $\beta = q^\lambda$, when $\lambda \to m$, so a second method of treating the case m finite is to change the parameter slightly so m is infinite, and at the end take an appropriate limit.

CASE II. $\gamma_2 = \mu_2$. This can be handled directly, and it was in [2]. However, it is easier to change γ_2 so that $\gamma_2 \neq \mu_2$ and then take an appropriate limit. When $\gamma_2 = \mu_2$, (3.6) and (3.7) give $q(1-\beta)^2 = 0$. There is no problem in taking $q = 0$ in all the above formulas.

The other case $\beta=1$ is more subtle. It seems to imply $\alpha_n=\rho^{-1}$, $n=1,2,\ldots$, and also $\gamma_n=\mu_n=1$. However, if $q^n=1$ a limit needs to be taken in (3.4). One way this can be done is to set

(3.15) $$q=s\exp(2\pi i/k), \qquad \beta=s^{\lambda k}.$$

Then

$$\alpha_n=\lim_{s\to 1}\frac{s^{\lambda k}-s^n\exp(2\pi in/k)}{\rho(1-s^n\exp(2\pi in/k))},$$

so

(3.16) $$\alpha_{mk}=\frac{m-\lambda}{\rho m};\ \alpha_n=\rho^{-1}, \qquad n\neq mk,\ n>0,$$

(3.17) $$\gamma_{mk+1}=\frac{\rho m}{(m+\lambda)};\ \gamma_n=\rho, \qquad n\neq mk+1,\ n>1,$$

(3.18) $$\mu_{mk+1}=\frac{\rho(m+\lambda)}{(m+2\lambda)};\ \mu_n=\rho, \qquad n\neq mk+1,\ n>1,$$

(3.19) $$\lambda_{mk+1}=\frac{\rho^2 m}{(m+2\lambda)};\ \lambda_n=\rho^2, \qquad n\neq mk+1,\ n>1.$$

It is easy to show that any other kth root of unity leads to the same polynomials.

4. **Further comments.** These polynomials are orthogonal when $\lambda_{n+1}\mu_n\neq 0$, $n=1,2,\ldots.$ In general there are many measures for which

(4.1) $$\int_{-\infty}^{\infty}A_n(x)A_m(x)\,d\alpha(x)=0, \qquad m\neq n,$$

and it is not clear how to find any of them. There is one very important class of polynomials where there is a positive measure $d\alpha(x)$, and in this case the measure has been found. Positivity of some measure is equivalent to

(4.2) $$\lambda_n/\mu_{n-1}\mu_n>0, \qquad n\geq 2,$$

or when $A_n(x)$ is replaced by $i^nA_n(ix)$ by

(4.3) $$\lambda_n/\mu_{n-1}\mu_n<0, \qquad n\geq 2.$$

As far as we know all the special cases where the measure is positive have been worked out, but we may be wrong. The cases when q is real were treated in [3], [4], [5] and [6]. The complex cases of q were not treated in any of these papers. The polynomials in Case II and their orthogonality relation will be given in [1]. When one starts with the polynomials in Case I with γ_n, μ_n, and λ_n given by (3.8)–(3.10), it is possible to renormalize them in another way so that a different set of polynomials arise when q approaches a root of unity. This second set of sieved ultraspherical polynomials will also be treated in [1].

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