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Random local complex dynamics

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Abstract. The study of the dynamics of an holomorphic map near a fixed point is a central topic in complex dynamical systems. In this paper, we will consider the corresponding random setting: given a probability measure ν with compact support on the space of germs of holomorphic maps fixing the origin, we study the compositions $f_n \circ \cdots \circ f_1$, where each f_i is chosen independently with probability ν. As in the deterministic case, the stability of the family of the random iterates is mostly determined by the linear part of the germs in the support of the measure. A particularly interesting case occurs when all Lyapunov exponents vanish, in which case stability implies simultaneous linearizability of all germs in $\text{supp}(\nu)$.

Key words: complex dynamics, random dynamics, local dynamics
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1. *Introduction*

An elementary but fundamental result in the theory of local complex dynamical systems is the following.

Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ *be a neutral germ, i.e., all eigenvalues of* $df(0)$ *have norm one. Then* $\{f^n\}$ *is normal in a neighborhood of the origin if and only if* f *is locally linearizable and* $df(0)$ *is diagonalizable.*

Our goal in this paper is to generalize this statement to the random setting, i.e., when studying compositions of maps that are chosen to be independent and identically distributed. Our main result is the following theorem.

THEOREM 1.1. *Suppose that* ν *is a neutral probability measure with compact support on* $\mathcal{O}(\mathbb{C}^m, 0)$. *Then the origin lies in the random Fatou set if and only if all the germs in* $\text{supp}(\nu)$ *are simultaneously linearizable and the semigroup of differentials*

$$dS_\nu := \{df_\omega^n(0) \mid \omega \in \text{supp}(\nu)^{\mathbb{N}}, n \in \mathbb{N}\}$$

is conjugate to a sub-semigroup of $U(m)$.

As an illustration, consider the case where all maps in the support of ν are of the form $z \mapsto \lambda z + z^2$, with $|\lambda| = 1$. The fact that all the maps must be simultaneously linearizable implies that ν is supported at a single point: a germ with a Siegel disk.

Let us be more precise about our setting. Instead of considering normality of a family of iterates $\{f^n\}$, we will consider the family $\mathcal{F}_\omega = \{f^n_\omega\}$. Here $\omega = (f_n)$ is a sequence of germs each chosen independently with a probability ν and $f^n_\omega := f_n \circ \cdots \circ f_1$. An important observation in the proof of Theorem 1.1 is that the sequence $(f^n_\omega)_{n\in\mathbb{N}}$ is almost certainly normal in a neighborhood of the origin if and only if the induced semigroup S_ν, generated by the support of ν, is normal in a (possibly smaller) neighborhood of the origin. In one variable, this equivalence is a quick consequence of the Koebe quarter theorem; in higher dimensions, one uses Hurwitz's theorem for the same purpose (see Corollary 4.5).

In the random setting we may introduce Lyapunov exponents $\kappa_1 > \cdots > \kappa_s$, which play the same role as the (logarithms of absolute values of the) eigenvalues of $df(0)$ (see [**O68**] or [**GM89**] for more information on Lyapunov exponents). Analogous to the deterministic case, we say that the measure ν is *attracting* if $\kappa_1 < 0$, *repelling* if $\kappa_1 > 0$, *neutral* if the only Lyapunov exponent is $\kappa_1 = 0$ and *semi-attracting* if $\kappa_1 = 0$ and $\kappa_2 < 0$.

The equivalence between the normality of the random dynamical system and the normality of the induced semigroup breaks down when we leave the neutral setting, and the situation becomes considerably more complicated. In §3, we show that it is not possible to decide whether the origin lies in the random Fatou set just by looking at the Lyapunov exponents or at linearizability. In the two-dimensional setting, the fact that the origin is in the random Fatou set implies the existence of stable manifolds, analogous to their deterministic setting.

THEOREM 1.2. *Let ν be a semi-attracting measure on $\mathcal{O}(\mathbb{C}^2, 0)$ with compact support. If the origin lies in the random Fatou set, then almost surely every limit germ $g = \lim_{k\to\infty} f^{n_k}_\omega$ has rank one and, given z sufficiently close to the origin, its stable set $\mathbb{W}^s_\omega(z)$ is locally a one-dimensional complex manifold.*

The paper is organized as follows. In §2, we review the background on random matrices and prove that a neutral measure of linear maps is stable if and only if there is a conjugation to a sub-semigroup of $U(m)$. In §3, we give a more precise formulation of the problem and treat several examples showing that our assumptions are necessary. We will study normality of the family $\mathcal{F}_\omega$ when $\kappa_1 \neq 0$ in §4.1 and for neutral measures in §4.2. Semi-attracting measures will be considered in §5.

Throughout the paper, we use the inductive limit topology on $\mathcal{O}(\mathbb{C}^m, 0)$. Its construction and properties are discussed in the appendix.

1.1. *Historical references.* The investigation on the random dynamics of holomorphic maps began with the work of Fornaess and Sibony. In [**FS91**], they showed that a generic rational map, with attracting cycles, admits a neighborhood W such that, for almost every sequence of functions chosen to be independent and identically distributed with respect to an absolutely continuous probability measure supported in W, the Julia set of the family $\mathcal{F}_\omega$ has zero measure. Furthermore, for every $z \in \widehat{\mathbb{C}}$, the point z belongs almost certainly to

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the Fatou set of the family $\mathcal{F}_\omega$. We refer to the paper of Sumi [**Su11**] for a generalization of this result.

The local dynamics of a holomorphic germ f fixing the origin is one of the earliest problems studied in complex dynamics. Determining the linearizability of neutral and semi-attracting germs is a subtle problem. In one dimension, a complete description of this phenomenon was given in the works of Cremer [**C38**], Siegel [**Si42**], Brjuno [**Brj71**] and Yoccoz [**Y95**].

The local dynamics of germs in several complex variables has many analogies with the one-dimensional case, but also many differences. We refer to the survey of Abate [**A03**] for more details. In this setting, the presence of resonances constitute an obstacle to linearizability, as studied by Poincaré [**P28**] for attracting germs. The description of attracting germs was completed in dimension two by Lattès [**La11**] and in arbitrary dimensions independently by Sternberg [**Ste57**, **Ste58**] and Rosay and Rudin [**RR88**].

Neutral and semi-attracting germs in several complex variables have also been intensively studied, particularly in the parabolic case, where all eigenvalues of norm one are roots of unity. See, for instance, the works of Écalle [**Ec85**], Hakim [**H98**] and Abate [**A01**] for germs tangential to the identity and the works of Ueda [**U86**, **U91**] and Rivi [**Ri01**] for a description of semi-parabolic germs. For the non-parabolic case, consider, for example, the papers of Bracci and Molino [**BM04**] and Bracci and Zaitsev [**BZ13**] for neutral germs and the recent papers of Firsova, Lyubich, Radu and Tanase [**LRT16**, **FLRT16**] for semi-attracting germs.

2. *Products of random matrices*

Definition. Let $(\Omega, \mathscr{B}, \mu)$ be a probability space and let $T : \Omega \to \Omega$ be an ergodic transformation. Given a measurable function $M : \Omega \to \mathrm{Mat}(m, \mathbb{C})$, the *(linear) cocycle* defined by M over T is the skew-product transformation

$$F : \Omega \times \mathbb{C}^m \to \Omega \times \mathbb{C}^m, \quad (\omega, v) \mapsto (T\omega, M_\omega v).$$

We will also refer to the triple (Ω, T, M) simply as a (linear) cocycle.

Observe that $F^n(\omega, v) = (T^n\omega, M_\omega^n v)$, where

$$M_\omega^n := M_{T^{n-1}\omega} \cdots M_\omega.$$

By elementary properties of measurable functions, the function $\omega \mapsto \sup_n \|M_\omega^n\|$ is also measurable. Therefore the set

$$\Omega_b = \{\omega \in \Omega \mid M_\omega^n \text{ is bounded}\}$$

is measurable and it satisfies $T^{-1}\Omega_b = \Omega_b$. By ergodicity of T, we either have $\mu(\Omega_b) = 0$ or $\mu(\Omega_b) = 1$.

Problem. For which cocycles is the family $\mathcal{M}_\omega = \{M_\omega^n\}$ almost surely bounded?

A cocycle of particular importance for our work is the following.

Definition (independent and identically distributed cocycle). Given a probability measure ν on $\mathrm{Mat}(m, \mathbb{C})$, let $\Omega = \mathrm{supp}(\nu)^{\mathbb{N}}$ equipped with the product metric $\mu = \nu^\infty$, let $T : \Omega \to \Omega$ be the shift map and let $M : \Omega \to \mathrm{Mat}(m, \mathbb{C})$ be the function that returns the first element

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of the sequence ω. Then the triple (Ω, T, M) is called the *independent and identically distributed cocycle generated by* ν.

Remark. It is well known that T is an ergodic transformation. Furthermore, the map M is continuous and therefore measurable, showing that the triple (Ω, T, M) fulfils the conditions in the definition of cocycle.

We notice that $M_{T^{n-1}\omega}$ coincides with the nth element of the initial sequence and that $X_i(\omega) := M_{T^{i-1}\omega}$ is an independent and identically distributed sequence of random variables with values in $\mathrm{Mat}(m, \mathbb{C})$, each chosen with probability ν. In this case, M^n_ω is given by the product of the first n elements of the sequence ω.

The problem of the iteration of random matrices was first studied by Furstenberg and Kesten in [**FK60**]. An important generalization of their result is the multiplicative ergodic theorem of Oseledec [**O68**]. The following version of the theorem can be found in [**Rue79**].

THEOREM 2.1. (Multiplicative ergodic theorem) *Let* (Ω, T, M) *be a cocycle such that*

$$\log^+ \|M_\omega\| \in L^1(\Omega, \mu).$$

There exist numbers $+\infty > \kappa_1 > \cdots > \kappa_s \geq -\infty$*, called Lyapunov exponents, and natural numbers* $\alpha_1, \ldots, \alpha_s$*, called Lyapunov multiplicities, satisfying the equation* $\alpha_1 + \cdots + \alpha_s = m$ *such that, for almost every* $\omega \in \Omega$*, we have the following.*

(a) *There exists a Hermitian matrix* Λ_ω *with eigenvalues* $e^{\kappa_1} > \cdots > e^{\kappa_s}$*, with respective multiplicities* α_i*, such that*

$$\lim_{n\to\infty} ((M^n_\omega)^* M^n_\omega)^{1/2n} = \Lambda_\omega. \tag{1}$$

(b) *Suppose that* $\mathcal{U}_1(\omega), \ldots, \mathcal{U}_s(\omega)$ *are the eigenspaces of* Λ_ω*. Let* $\mathcal{V}_{s+1}(\omega) = \{0\}$ *and* $\mathcal{V}_i(\omega) = \mathcal{U}_i(\omega) \oplus \cdots \oplus U_s(\omega)$*. Then, given* $v \in \mathcal{V}_i(\omega) \backslash \mathcal{V}_{i+1}(\omega)$*,*

$$\lim_{n\to\infty} n^{-1} \log \|M^n_\omega v\| = \kappa_i. \tag{2}$$

The set $\sigma = \{(\kappa_1, \alpha_1), \ldots, (\kappa_s, \alpha_s)\}$ *is called the Lyapunov spectrum.*

Definition. Let (Ω, T, M) be a cocycle such that $\log^+ \|M_\omega\| \in L^1(\Omega, \mu)$, and let $\kappa_1 > \cdots > \kappa_s$ be the Lyapunov exponents. We say that the cocycle is:
attracting if the maximal Lyapunov exponent $\kappa_1 < 0$;
repelling if the maximal Lyapunov exponent $\kappa_1 > 0$;
neutral if the Lyapunov spectrum $\sigma_\mu = \{(0, m)\}$; and
semi-attracting if the maximal Lyapunov exponent $\kappa_1 = 0$ and $\alpha_1 \neq m$.

Remark. In the work of Furstenberg and Kesten [**FK60**], it is proved that, for every cocycle, we may define

$$\kappa_\mu = \lim_{n\to\infty} n^{-1} \mathbf{E} \log \|M^n_\omega\|. \tag{3}$$

Furthermore, under the assumption that $\log^+ \|M_\omega\| \in L^1(\Omega, \mu)$ for almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} n^{-1} \log \|M^n_\omega\| = \kappa_\mu. \tag{4}$$

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Under the hypothesis of Theorem 2.1, we can describe the connection between Lyapunov exponents and κ_μ as follows.

LEMMA 2.2. *Let* (Ω, T, M) *be a cocycle such that* $\log^+ \|M_\omega\| \in L^1(\Omega, \mu)$. *Then the maximal Lyapunov exponent* κ_1 *is equal to* κ_μ. *Furthermore,*

$$\alpha_1\kappa_1 + \cdots + \alpha_s\kappa_s = \mathbf{E} \log |\det(M_\omega)|.$$

Proof. Let κ_1 be the maximal Lyapunov exponent and choose $\omega \in \Omega$ such that (1), (2) and (4) hold. It is not hard to prove that $\kappa_\mu \geq \kappa_1$; thus it remains to prove that $\kappa_\mu \leq \kappa_1$. Let v_n be a unit vector such that $\|M^n_\omega\| = \|M^n_\omega v_n\|$. We take n_k such that $v_{n_k} \to v$. By (2), for some $i = 1, \ldots, s$,

$$\begin{aligned}
\kappa_i &= \lim_{k\to\infty} n_k^{-1} \log \|M^{n_k}_\omega v\| \\
&\geq \lim_{k\to\infty} n_k^{-1} \log(\|M^{n_k}_\omega v_{n_k}\| - \|M^{n_k}_\omega (v_{n_k} - v)\|) \\
&\geq \lim_{k\to\infty} n_k^{-1} \log \|M^{n_k}_\omega\| + \lim_{k\to\infty} n_k^{-1} \log(1 - \|v_{n_k} - v\|) \\
&\geq \kappa_\mu.
\end{aligned}$$

Since $\kappa_i \leq \kappa_1$, the equality $\kappa_\mu = \kappa_1$ follows.

If we apply $\log|\cdot|$ to both side of (1), we obtain that

$$\begin{aligned}
\alpha_1\kappa_1 + \cdots + \alpha_s\kappa_s &= \log |\det(\Lambda_\omega)| \\
&= \lim_{n\to\infty} \frac{1}{n} \log |\det(M^n_\omega)| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |\det(M_{T^i\omega})|.
\end{aligned}$$

By Birkhoff's ergodic theorem, the last term of this equality converges almost surely to $\mathbf{E} \log |\det(M_\omega)|$. By choosing an appropriate ω, we obtain the desired equality. □

COROLLARY 2.3. *Let* (Ω, T, M) *be a cocycle such that* $\log^+ \|M_\omega\| \in L^1(\Omega, \mu)$. *We have the following list of equivalences.*

(a) *The cocycle is attracting if and only if* $\kappa_\mu < 0$.

(b) *The cocycle is repelling if and only if* $\kappa_\mu > 0$.

(c) *The cocycle is neutral if and only if* $\kappa_\mu = 0$ *and* $\mathbf{E} \log |\det(M_\omega)| = 0$.

(d) *The cocycle is semi-attracting if and only if* $\kappa_\mu = 0$ *and* $\mathbf{E} \log |\det(M_\omega)| < 0$.

When the cocycle is attracting or repelling, (4) gives an answer to the problem of the boundedness of the family $\mathcal{M}_\omega = \{M^n_\omega\}$.

COROLLARY 2.4. *If a cocycle is attracting, then the family* $\mathcal{M}_\omega$ *is almost certainly bounded in* $\mathrm{Mat}(m, \mathbb{C})$. *If a cocycle is repelling, then* $\mathcal{M}_\omega$ *is almost certainly unbounded.*

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2.1. *Neutral measures.* The stability problem is considerably more complicated in the neutral setting. However, for independent and identically distributed cocycles generated by probability measures with compact support, we can give a precise description of the stable systems in Proposition 2.7 below. In the two examples that follow, we show that no such statement can hold for all neutral cocycles, when we drop the independent and identically distributed assumption.

Given a probability measure ν on $\mathrm{Mat}(m, \mathbb{C})$ with compact support, let (Ω, T, M) be the independent and identically distributed cocycle generated by ν. By Tychonoff's theorem, the set Ω is compact, and thus $\log^+ \|M_\omega\| \in L^1(\Omega, \mu)$.

Definition. Let ν be a measure on $\mathrm{Mat}(m, \mathbb{C})$ with compact support. We say that ν is attracting (respectively, repelling, neutral and semi-attracting) if the independent and identically distributed cocycle generated by ν is attracting (respectively, repelling, neutral and semi-attracting).

LEMMA 2.5. *Let ν be a probability measure on* $\mathrm{Mat}(m, \mathbb{C})$ *with compact support, and let* (Ω, T, M) *be the independent and identically distributed cocycle generated by ν. Then the set*

$$\Omega_a := \{\omega \in \Omega \mid \forall n \in \mathbb{N} \text{ and } \forall \alpha \in \Omega, \exists k_j : M^n_{T^{k_j}\omega} \to M^n_\alpha\}$$

is a full measure subset of Ω.

Proof. Let $n \in \mathbb{N}$ and choose a sequence $\varepsilon_j \to 0$. We define

$$\Omega_{n,j} := \{\omega \in \Omega \mid \forall \alpha \in \Omega, \exists k : \|M^n_{T^k\omega} - M^n_\alpha\| < \varepsilon_j\}.$$

Is it clear that $\bigcap_{n,j} \Omega_{n,j} = \Omega_a$. Since the sets $\Omega_{n,j}$ are countably many, it suffices to prove that, for every $n, j \in \mathbb{N}$, the set $\Omega_{n,j}$ has full measure.

Suppose that n and j are fixed. Given $\alpha \in \Omega$, using the continuity of T and M, we may find an open neighborhood $U_\alpha \ni \alpha$ so that, given $\beta \in U_\alpha$,

$$\|M^n_\alpha - M^n_\beta\| < \varepsilon_j/2.$$

By compactness of $\mathrm{supp}(\nu)$, the set Ω is also compact; therefore we may find $\alpha_1, \ldots, \alpha_N$ such that $\Omega = \cup_i U_{\alpha_i}$. For simplicity, we will write U_i for U_{α_i}. Notice that $\mathrm{supp}(\mu) = \Omega$; therefore all the sets U_i have positive measure. Using standard results from ergodic theory and the fact that the α_i are finite in number, we find that

$$\Omega'_{n,j} = \{\omega \in \Omega \mid \forall i = 1, \ldots N, \exists k_i : T^{k_i}\omega \in U_i\}$$

has full measure in Ω.

Finally, given $\omega \in \Omega'_{n,j}$ and $\alpha \in \Omega$, there exists U_i so that $\alpha \in U_i$ and k_i so that $T^{k_i}\omega \in U_i$. By the definition of U_i, we obtain that

$$\|M^n_{T^{k_i}\omega} - M^n_\alpha\| \leq \|M^n_{T^{k_i}\omega} - M^n_{\alpha_i}\| + \|M^n_{\alpha_i} - M^n_\alpha\| < \varepsilon_j,$$

which proves that $\Omega'_{n,j} \subset \Omega_{n,j}$ and thus that $\Omega_{n,j}$ has full measure. □

Remark. The previous lemma is valid in a much more general context. As a matter of fact, in the proof, we only used that (Ω, T, M) is a continuous cocycle over a compact space, meaning that both T and M are continuous functions and Ω is compact.

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The following lemma is a consequence of [**Str06**, Theorem 14.35, Remark 14.36].

LEMMA 2.6. *Let $\mathcal{G}$ be a compact subgroup of* $\mathrm{GL}(m, \mathbb{C})$. *Then $\mathcal{G}$ is conjugate to a subgroup of the standard unitary group $U(m)$.*

We define the set

$$S_\nu = \{M_\omega^n \mid \omega \in \Omega, n \in \mathbb{N}\}.$$

For the independent and identically distributed cocycles, the set $S_\nu \subset \mathrm{Mat}(m, \mathbb{C})$ has a semigroup structure, which is not necessarily true for general cocycles. In later sections, we will use the same notation S_ν for semigroups induced by maps that are not necessarily linear.

PROPOSITION 2.7. *Let ν be a neutral measure on* $\mathrm{Mat}(m, \mathbb{C})$ *with compact support, and let (Ω, T, M) be the independent and identically distributed cocycle generated by ν. Then the following are equivalent.*

(a) *The family $\mathcal{M}_\omega$ is almost certainly bounded.*
(b) *The semigroup S_ν is relatively compact in* $\mathrm{GL}(m, \mathbb{C})$ *and is conjugated to a sub-semigroup of the unitary group $U(m)$.*

Before proceeding to the proof, we want to discuss why there is no hope that a (similar) proposition holds for a generic neutral cocycle or even for a continuous neutral cocycle over a compact space. We will show this by constructing two cocycles for which $\mathcal{M}_\omega$ is almost certainly bounded but for which $S_\Omega = \{M_\omega^n \mid \omega \in \Omega, n \in \mathbb{N}\}$ is not relatively compact in $\mathrm{GL}(m, \mathbb{C})$.

Example. Let $X = \mathbb{R}/\mathbb{Z}$ be the unit circle with the Borel σ-algebra $\mathscr{B}$ and Lebesgue measure λ. Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ and define $T : X \to X$ by $Tx = x + \theta \mod 1$. It is well known that the transformation T is ergodic. Given $x_1, x_2 \in X$, we write

$$d_X(x_1, x_2) = \inf_{m \in \mathbb{Z}} |x_1 + m - x_2|.$$

Let $\varphi \in L^1(X, \lambda)$ be a non-negative and unbounded function. The function $M : X \to \mathbb{R}^+$, defined as

$$\log(M_x) = f(x) := \varphi(x) - \varphi(Tx),$$

is measurable and defines a neutral cocycle over T.

Notice that $M_x^n = e^{\varphi(x)-\varphi(T^n x)} \leq e^{\varphi(x)}$. Since $\varphi(x) < \infty$ for almost every $x \in X$, it follows that M_x^n is almost surely a bounded sequence. On the other hand, we can find $K_0 > 0$ so that the set $X_{K_0} = \{x \in X \mid \varphi(x) < K_0\}$ has positive measure. Since T is ergodic, for almost every $x \in X$, there exists n_0 such that $T^{n_0}x \in X_{K_0}$. We conclude that $M_x^{n_0} > e^{\varphi(x)-K_0}$. Since φ is unbounded, it follows that S_X is unbounded in $\mathbb{R}^+$.

Example. In the previous example, the function f may be unbounded, in which case the unboundedness of S_X is not particularly surprising. In this second example, we will construct a function φ for which f is continuous and therefore bounded.

Let $\omega_n = (n-1)\theta \mod 1 = T^{n-1}0$ and let $a_k = 2^k k$. For every $k > 0$, we may choose $\varepsilon_k > 0$ such that;

(a) $\varepsilon_k \leq 1/a_k^2$; and
(b) the sets $[\omega_1 - \varepsilon_k, \omega_1 + \varepsilon_k], \ldots, [\omega_{2a_k} - \varepsilon_k, \omega_{2a_k} + \varepsilon_k]$ are pairwise disjoint.

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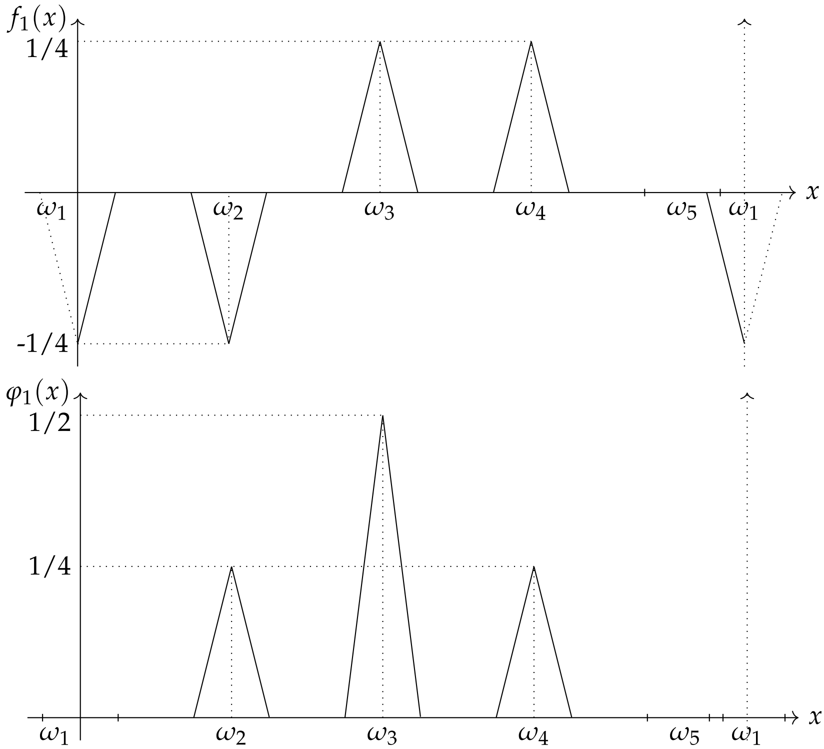

FIGURE 1. The graphs of the functions f_1 and φ_1.

Write $I_{j,k} = [\omega_j - \varepsilon_k, \omega_j + \varepsilon_k]$ and define f_k as

$$f_k(x) := \begin{cases} -\dfrac{1}{2^k}\left(1 - \dfrac{d_X(x,\omega_j)}{\varepsilon_k}\right) & \text{if } x \in I_{j,k} \text{ for } j = 1, \ldots, a_k, \\ +\dfrac{1}{2^k}\left(1 - \dfrac{d_X(x,\omega_j)}{\varepsilon_k}\right) & \text{if } x \in I_{j,k} \text{ for } j = a_k + 1, \ldots, 2a_k, \\ 0 & \text{elsewhere.} \end{cases}$$

We define a corresponding function φ_k as

$$\varphi_k(x) := \begin{cases} -\displaystyle\sum_{i=1}^{j-1} f_k(T^{-i}x) & \text{if } x \in I_{j,k} \text{ for } j = 2, \ldots, 2a_k, \\ 0 & \text{elsewhere,} \end{cases}$$

see Figure 1 for the graphs of f_1 and φ_1.

With some elementary calculations, one can verify that $f_k(x) = \varphi_k(x) - \varphi_k(Tx)$ and that

$$\int_X \varphi_k \, d\lambda = a_k^2 \varepsilon_k \frac{1}{2^k} \leq \frac{1}{2^k}.$$

Let $f(x) = \sum_{k=1}^{\infty} f_k(x)$ and $\varphi(x) = \sum_{k=1}^{\infty} \varphi_k(x)$. Since $|f_k(x)| \leq (1/2)^k$, the function $f(x)$ is continuous. On the other hand, since all the φ_k are non-negative functions, also φ is non-negative. Furthermore the above estimate on the integral of φ_k implies that

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$\varphi \in L^1(X, \lambda)$. Finally, we notice that $\varphi_k(\omega_{a_k+1}) = k$ and therefore that φ is unbounded. The function φ that we constructed satisfies all the hypotheses of the previous example and $f(x) = \varphi(x) - \varphi(Tx)$ is a continuous function.

Proof of Proposition 2.7. The implication $b. \Rightarrow a.$ is trivial. Suppose, on the other hand, that, for almost every $\omega \in \Omega$, the family $\mathcal{M}_\omega$ is bounded.

Given $\omega \in \Omega$, we define $f(\omega) = \log|\det(M_\omega)|$. By elementary properties of the determinant, we obtain that

$$\log|\det(M^n_\omega)| = \sum_{i=0}^{n-1} f(T^i\omega).$$

Let $Y_i := f(T^{i-1}\omega)$ and $X_n = \sum_{i=1}^n Y_i$. The Y_i form a sequence of independent and identically distributed random variables with expected value $\mathbf{E}(Y_i) = 0$. If $\sqrt{\mathrm{Var}(Y_i)} > 0$, almost surely there exists a sequence n_k so that $X^{n_k} \to \infty$, which contradicts the fact that $\mathcal{M}_\omega$ is almost certainly bounded. It follows that $\mathrm{Var}(Y_i) = 0$, which implies that $|\det(M^n_\omega)| = 1$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$.

Let Ω_a be as in the Lemma 2.5 and choose $\omega_0 \in \Omega_a$ for which $\mathcal{M}_{\omega_0}$ is bounded. We write $M^{-n}_{\omega_0} = (M^n_{\omega_0})^{-1}$. Since $|\det(M^n_\omega)| = 1$ for every n, the extended family $\widehat{\mathcal{M}}_{\omega_0} := \{(M^n_{\omega_0})\}_{n\in\mathbb{Z}}$ is also bounded. Let C be a bound on the norms of the matrices of this family.

Let $\omega \in \Omega$ and $n \in \mathbb{N}$. Since $\omega_0 \in \Omega_a$, there exists k_j such that $M^n_{T^{k_j}\omega_0} \to M^n_\omega$. Now

$$\|M^n_{T^{k_j}\omega_0}\| = \|M^{n+k_j}_{\omega_0} \cdot M^{-k_j}_{\omega_0}\| < C^2.$$

This proves that the set S_ν is bounded in $\mathrm{GL}(m, \mathbb{C})$. Its closure $G_\nu = \overline{S_\nu}$ is compact in $\mathrm{GL}(m, \mathbb{C})$.

Given $M \in G_\nu$, its orbit is also contained in G_ν, and thus it is bounded. Since $|\det(M)| = 1$, the matrix M is diagonalizable and every eigenvalue of M has norm one. In this case, there exists n_k such that $M^{n_k} \to \mathrm{id}$. Furthermore, $M^{n_k - 1}$ is a convergent sequence and its limit coincides with M^{-1}. Hence G_ν is a compact subgroup of $\mathrm{GL}(m, \mathbb{C})$. By the previous lemma, it follows that G_ν is conjugated to a subgroup of $U(m)$. □

2.2. *Semi-attracting measures.* Throughout the rest of this section, we assume that ν is semi-attracting and with compact support, and we consider the independent and identically distributed cocycle (Ω, T, M) generated by ν. Almost surely $M^n_\omega \neq 0$ for every n. Let $\mathbb{P}\Omega = \Omega \times \mathbb{P}^{m-1}$ and $\mathbb{P}F : \mathbb{P}\Omega \to \mathbb{P}\Omega$ be the map defined by

$$\mathbb{P}F : (\omega, [v]) \to (T\omega, [M_\omega v]).$$

Finally, let $\Phi : \mathbb{P}\Omega \to \mathbb{R}$ be the map $\Phi(\omega, [v]) = \log(\|M_\omega v\|/\|v\|)$. For every $\omega \in \Omega$ and $v \in \mathbb{C}^m$,

$$\sum_{k=0}^{n-1} \Phi \circ \mathbb{P}F^k(\omega, [v]) = \log\frac{\|M^n_\omega v\|}{\|v\|} \leq \log\|M^n_\omega\|. \tag{5}$$

Let $0 = \kappa_1 > \cdots > \kappa_s$ be the Lyapunov exponents of the measure ν and let $\mathbb{P}^{m-1} = \mathcal{V}_1(\omega) \supset \cdots \supset \mathcal{V}_s(\omega)$ be the collection of vector subspaces introduced in Theorem 2.1,

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defined for almost every $\omega \in \Omega$. We write $\mathbb{P}\Omega_1, \ldots, \mathbb{P}\Omega_s$ for the family of disjoint subsets of $\mathbb{P}\Omega$ given by

$$\mathbb{P}\Omega_j = \{(\omega, [v]) : v \in \mathcal{V}_j(\omega) \backslash \mathcal{V}_{j-1}(\omega)\}.$$

We recall the following statement.

THEOREM 2.8. [**Le84**, **V14**] *Given any $\mathbb{P}F$-invariant ergodic probability measure m on $\mathbb{P}\Omega$ that projects down to $\mu = \nu^\infty$, there exists $j \in \{1, \ldots, s\}$ such that*

$$\int \Phi \, dm = \kappa_j \quad \textit{and} \quad m(\mathbb{P}\Omega_j) = 1. \tag{6}$$

Conversely, given $j \in \{1, \ldots, s\}$, there exists a $\mathbb{P}F$-invariant ergodic probability measure m_j projecting to μ and satisfying (6).

The following lemma closely resembles [**GR85**, Lemma 3.6]. We provide a proof for the sake of completeness.

LEMMA 2.9. *Let (X, T, μ) be an ergodic dynamical system and let $f : X \to \mathbb{R}$ an integrable function. Suppose that $\int_X f \, d\mu = 0$. Then almost surely there exists a sequence n_k such that*

$$\lim_{k \to \infty} \sum_{i=0}^{n_k - 1} f \circ T^i x = 0.$$

Proof. By replacing (X, T, μ) with its natural extension, if necessary, we may assume that T is invertible. We consider the product $Y = X \times \mathbb{R}$ and the map $S : Y \to Y$ defined as

$$S(x, r) := (Tx, r + f(x)).$$

The transformation S preserves the measure $\nu = \mu \times \lambda$, where λ is the Lebesgue measure of $\mathbb{R}$. We can write

$$S^n(x, r) = \left(T^n x, r + \sum_{i=0}^{n-1} f \circ T^i x\right) = (T^n x, s_n(x, r)).$$

Suppose that there exists a *wandering* subset $A \subset Y$, i.e., $\nu(A) > 0$, and, for every couple $i \neq j$, the sets $S^i(A)$ and $S^j(A)$ are disjoint up to a set of measure zero. By ergodicity of μ, for almost every $(x, r) \in Y$,

$$\lim_{n \to \infty} \frac{s_n(x, r)}{n} = 0.$$

We can choose $B \subset A$ of finite positive measure and such that the limit above holds on B uniformly. Since $\nu(\bigcup_{0 \leq i \leq n} S^i(B)) \leq \lambda(\bigcup_{0 \leq i \leq n} s_i(B))$, it follows that $\lim_n \nu(\bigcup_{0 \leq i \leq n} S^i(B))/n = 0$. On the other hand, since A is wandering and ν is S-invariant we find that $\nu(\bigcup_{0 \leq i \leq n} S^i(B))/n = \nu(B) > 0$, which contradicts the assumption on A.

Let $\varepsilon > 0$ and $Y_\varepsilon = X \times [-\varepsilon, +\varepsilon]$. We write C_ε for the set of all points of Y_ε that do not return to Y_ε under iterations of the map S. The map S is invertible; therefore if $\nu(C_\varepsilon) > 0$, then the set C_ε is wandering. Since there are no wandering sets, it follows that $\nu(C_\varepsilon) = 0$.

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Let $X_\varepsilon \subset X$ be the set of all points x for which there exists $n > 0$ such that $-\varepsilon \leq s_n(x, 0) \leq \varepsilon$. We note that $(X \backslash X_\varepsilon) \times [-\varepsilon/2, \varepsilon/2] \subset C_{\varepsilon/2}$. Thus X_ε is a full measure subset of X. The set

$$X_0 = \bigcap_n T^{-n}\left(\bigcap_k X_{1/k}\right)$$

is the intersection of countably many full measure sets, and thus it has full measure. Take $x_0 \in X_0$. Since $X_0 \subset \cap_k X_{1/k}$, for every k, there exists a minimal $n_k > 0$ such that $-1/k \leq s_{n_k}(x_0, 0) \leq 1/k$. We notice that $n_{k+1} \geq n_k$. Thus either $n_m \to \infty$, in which case the lemma is proved, or there exists n_1' such that $s_{n_1'}(x_0, 0) = 0$. If the second case occurs, we let $x_1 = T^{n_1'} x_0 \in X_0$ and we repeat the argument for x_1. It follows that one can always construct a sequence that satisfies the requirements of the lemma. □

PROPOSITION 2.10. *Let ν be a semi-attracting measure on* $\mathrm{Mat}(n, \mathbb{C})$ *with compact support and let (Ω, T, M) be the independent and identically distributed cocycle generated by ν. Suppose that $\mathcal{M}_\omega$ is almost surely bounded. Then, for every $M_\omega^n \in S_\nu$,*

$$\|M_\omega^n\| \geq 1.$$

Proof. Given $K > 0$, let

$$\Omega_K := \left\{\omega \in \Omega \,\middle|\, \sup_n \log \|M_\omega^n\| \leq K\right\}.$$

The set $\Omega_b = \bigcup_{K>0} \Omega_K$ coincides with the set of all ω for which $\mathcal{M}_\omega$ is bounded. It then follows that $\lim_{K\to\infty} \nu(\Omega) = 1$ and that we can choose $K_0 > 0$ so that $\nu(\Omega_{K_0}) > 0$.

Suppose that there exists $M_{\omega_0}^N \in S_\nu$ with $\|M_{\omega_0}^N\| < 1$. By the semigroup structure of S_ν, we may further assume that $\|M_{\omega_0}^N\| \leq 1/(3K_0)$.

For $i = 1, \ldots, N$ we choose $U_i \subset \mathrm{Mat}(m, \mathbb{C})$ such that the set $\mathcal{C}_{U_1,\ldots,U_N}$ has positive μ-measure and $\|M_\omega^N\| \leq 1/(2K_0)$ for every $\omega \in \mathcal{C}_{U_1,\ldots,U_N}$.

We define the set

$$\mathcal{U} := \mathcal{C}_{U_1,\ldots,U_N} \cap T^{-N}(\Omega_{K_0}) = U_1 \times \cdots \times U_N \times \Omega_{K_0}.$$

It is clear that this set has positive μ-measure.

Let m_1 be the $\mathbb{P}F$-invariant ergodic measure on $\mathbb{P}\Omega$ defined in Theorem 2.8. Given $(\omega, [v]) \in \mathcal{U} \times \mathbb{P}^{m-1}$, for every $n \geq N$,

$$\begin{aligned}\sum_{k=0}^{n-1} \Phi \circ \mathbb{P}F^k(\omega, [v]) &\leq \log \|M_\omega^n\| \\ &\leq \log \|M_\omega^N\| \|M_{\sigma^N(\omega)}^{n-N}\| \\ &\leq -\log 2.\end{aligned}$$

By (6), we have $\int_\Omega \Phi \, dm_1 \neq 0$ and $m_1(\mathcal{U} \times \mathbb{P}^{m-1}) = \mu(\mathcal{U}) > 0$, which contradicts the previous lemma. It follows that, for every $M_\omega^n \in S_\nu$, we have $\|M_\omega^n\| \geq 1$. □

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3. *Compositions of random germs*

Let $\mathcal{O}(\mathbb{C}^m, 0)$ be the space of all germs of holomorphic functions fixing the origin. This set can be endowed with the so-called inductive limit topology τ_{ind} (see the appendix of this paper for more details). Let ν be a Borel probability measure on the set $\mathcal{O}(\mathbb{C}^m, 0)$. Let Ω be the space of all sequences in $\text{supp}(\nu)$ and let $\mu = \nu^\infty$. Let $T : \Omega \to \Omega$ denote the shift map and let $f : \Omega \to \mathcal{O}(\mathbb{C}^m, 0)$ be the function that returns the first element of the sequence. As we will see later, the function f defines a (nonlinear) cocycle over T. Analogous to the previous section, we will write

$$f_\omega^n = f_{T^{n-1}\omega} \circ \cdots \circ f_\omega$$

and $\mathcal{F}_\omega = \{f_\omega^n\}$. Finally, we define

$$\Omega_{\text{nor}} = \{\omega \in \Omega \mid \mathcal{F}_\omega \text{ is a normal family near the origin}\}.$$

PROPOSITION 3.1. *The set Ω_{nor} is measurable.*

Proof. Normality near the origin is equivalent to relative compactness of the set $\mathcal{F}_\omega$ with respect to τ_{ind}. Furthermore, by Corollary A.9, the set $\mathcal{F}_\omega$ is relatively compact in $\mathcal{O}(\mathbb{C}^m, 0)$ if and only if $\mathcal{F}_\omega \subset \overline{B}_n(0, r)$ for some natural number n and $r > 0$. The terminology $\overline{B}_n(0, r)$ is introduced in the appendix.

Let r_n be an increasing sequence of positive real numbers such that $r_n \to \infty$. Since $\overline{B}_n(0, r) \subset \overline{B}_{n+1}(0, r)$, it follows that

$$\begin{aligned}\Omega_{\text{nor}} &= \bigcup_n \{\omega \in \Omega \mid \mathcal{F}_\omega \subset \overline{B}_n(0, r_n)\}\\ &= \bigcup_n \bigcap_m \{\omega \in \Omega \mid f_\omega^m \in \overline{B}_n(0, r_n)\}.\end{aligned}$$

The function $\omega \mapsto f_\omega^m$ is measurable for every m. By Lemma A.5 and Corollary A.9, $\overline{B}_n(0, r_n)$ is a closed set in the inductive limit topology. This proves that Ω_{nor} is the intersection of countably many measurable sets, and thus that it is a measurable set. □

Problem. Let $\omega \in \Omega$ be a random sequence of germs. What is the probability that $\mathcal{F}_\omega$ is a normal family?

We notice that the set Ω_{nor} is backward invariant. The next proposition follows by ergodicity of T.

PROPOSITION 3.2. *Normality of $\mathcal{F}_\omega$ occurs with either probability* 1 *or* 0*, depending on the probability measure μ.*

Random Fatou set. We say that the origin belongs to the *(local) random Fatou set* if Ω_{nor} has full measure. If this is not the case, we say that the origin belongs to the *(local) random Julia set*.

Now suppose that the measure ν has compact support. By Corollary A.9, there exists $R > 0$ such that each $f \in \text{supp}(\nu)$ is holomorphic and bounded on $\mathbb{B}_R$, the open ball of $\mathbb{C}^m$ with radius R and center the origin. Given $\omega \in \Omega$, we may define the set

$$D_\omega := \bigcap_{n=0}^{\infty} (f_\omega^n)^{-1}(\mathbb{B}_R). \tag{7}$$

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Clearly, $0 \in D_\omega$, and thus D_ω is not empty. We have the invariance relation

$$f_\omega^n(D_\omega) = D_{T^n(\omega)}.$$

Given $z \in D_\omega$ its orbit $\{f_\omega^n(z) \mid n \in \mathbb{N}\}$ is well defined and bounded. By the weak Montel theorem, the family $\mathcal{F}_\omega$ is normal on $\mathrm{int}(D_\omega)$. We will call this set the *(local) Fatou set* of the sequence ω and we will denote it by F_ω. It is clear that the origin lies in the random Fatou set if and only if it almost certainly lies in the Fatou set of ω.

We define

$$D := \{(\omega, z) \in \Omega \times \mathbb{C}^m \mid z \in D_\omega\}.$$

Definition. The *(nonlinear) cocycle* defined by f over T is the skew-product transformation $F : D \to D$,

$$F(\omega, z) = (T\omega, f_\omega(z)), \quad F^n(\omega, z) = (T^n\omega, f_\omega^n(z)).$$

We will also refer to the triple (Ω, T, f) as a (nonlinear) cocycle.

By Lemma A.1, the map $d_0 : f \mapsto df(0)$ is a continuous map. Following the previous section, we introduce the following classification of probability measures on $\mathcal{O}(\mathbb{C}^m, 0)$.

Definition. Let ν be a probability measure on $\mathcal{O}(\mathbb{C}^m, 0)$ so that $(d_0)_*\nu$ has compact support. We say that the measure ν is *attracting, repelling, neutral or semi-attracting* if $(d_0)_*\nu$ is, respectively, attracting, repelling, neutral or semi-attracting.

We are now ready to state the central theorem of this paper. Notice that the neutral case coincides with Theorem 1.1 of the introduction.

THEOREM 3.3. *Let ν be a probability measure on $\mathcal{O}(\mathbb{C}^m, 0)$ with compact support, and suppose that ν is either attracting, repelling or neutral.*

Attracting: the origin lies in the random Fatou set and it is almost surely an attracting point for the system $\{f_\omega^n|_{D_\omega}\}$.

Repelling: the origin lies in the random Julia set.

Neutral: the origin lies in the random Fatou set if and only if all the germs in $\mathrm{supp}(\nu)$ *are simultaneously linearizable, and the semigroup of the differentials*

$$dS_\nu := \{df_\omega^n(0) \mid \omega \in \Omega, n \in \mathbb{N}\}$$

is conjugate to a sub-semigroup of $U(m)$.

If we drop the compactness hypothesis on $\mathrm{supp}(\nu)$, then this theorem is false, in general. We will show a counterexample, for an attracting measure ν with non-compact support.

Example. Let $0 < \lambda_1 < 1$ and let $a_i = \lambda_1^{-3\cdot 2^i}$. We consider the family

$$\mathcal{A} := \{f_i(z) = \lambda_1 z + a_i z^2\}.$$

Let ν be the probability measure on $\mathcal{O}(\mathbb{C}, 0)$ such that $\nu(f_i) = 1/2^i$. The support of ν coincides with $\mathcal{A}$ and is not compact. On the other hand, we have $\mathrm{supp}((d_0)_*\nu) = \{\lambda\}$, which implies that the measure ν is attracting. In this case, the (unique) Lyapunov exponent is $\kappa_1 = \log \lambda_1 < 0$.

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Let $\omega = (f_{i_1}, f_{i_2}, \ldots)$ be a sequence of independent and identically distributed random germs of $\mathcal{A}$. We write $\lambda_2^{(n)}$ for the coefficient relative to z^2 of f_ω^n. It follows that

$$\lambda_2^{(n)} = \lambda_1^n \sum_{k=1}^{n} \lambda_1^{k-1} a_{i_k} \geq \lambda_1^{2n} a_{i_n}.$$

Write p_n for the probability that $\lambda_1^{2n} a_{i_n} \geq \lambda_1^{-n}$. The inequality holds if and only if $i_n \geq \log n$. Thus

$$p_n = \sum_{k \geq \log n} \frac{1}{2^k} \sim \frac{2}{n}.$$

Given ω and $m > 0$, by the independence assumption, the probability that, for every $n \geq m$, the inequality $\lambda_1^{2n} a_{i_n} < \lambda_1^{-n}$ holds is equal to

$$P_m = \prod_{n \geq m} (1 - p_n) = \exp\bigg(\sum_{n \geq m} \log(1 - p_n)\bigg).$$

Since the general term of the series above is asymptotic to $-2/n$, we see that $\sum_{n \geq m} \log(1 - p_n) = -\infty$ and therefore $P_m = 0$. It follows that almost surely there exists $n_k \to \infty$ such that $\lambda_2^{(n_k)} \to \infty$ and thus that almost surely $\mathcal{F}_\omega$ is not a normal family.

We conclude this section with the discussion of a particular semi-attracting measure. The example shows that, in the semi-attracting case, $\mathrm{supp}(\nu)$ may contains germs that are not linearizable but, at the same time, the origin can be in the random Fatou set. Furthermore, the semigroup S_ν generated by $\mathrm{supp}(\nu)$ may not be bounded in $\mathcal{O}(\mathbb{C}^m, 0)$. Thus the methods developed in §4.2 for the neutral case do not work for the semi-attracting one.

Example. Suppose that f and g are given by

$$f(z, w) = (z, w/2),$$
$$g(z, w) = (z + zw, w).$$

The nth iterate of g has the form $g^n(z, w) = (z + nzw + O(|z|^3), w)$. Therefore $\{g^n\}$ is not normal on any neighborhood of the origin, and thus it cannot be linearized.

Let $\nu := \frac{1}{2}\delta_f + \frac{1}{2}\delta_g$ be a probability measure. A short computation gives $\kappa_1 = 0$ and $\kappa_2 = \frac{1}{2} \log \frac{1}{2} < 0$.

Let $\omega \in \Omega$ be a sequence of independent and identically distributed random germs and write $f_\omega^n = (f_{\omega,1}^n, f_{\omega,2}^n)$. Then

$$|f_{\omega,1}^{n+1}(z, w)| \leq \begin{cases} |f_{\omega,1}^n(z, w)| & \text{if } f_n = f, \\ |f_{\omega,1}^n(z, w)|(1 + |f_{\omega,2}^n(z, w)|) & \text{if } f_n = g, \end{cases}$$

$$|f_{\omega,2}^{n+1}(z, w)| = \begin{cases} \frac{1}{2}|f_{\omega,2}^n(z, w)| & \text{if } f_n = f, \\ |f_{\omega,2}^n(z, w)| & \text{if } f_n = g. \end{cases}$$

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Let $\alpha(n) = \#\{k \in \{1, \ldots, n\} | f_k = f\}$. Using the estimates above, we find that

$$|f^n_{\omega,2}(z, w)| = \frac{|w|}{2^{\alpha(n)}},$$
$$|f^n_{\omega,1}(z, w)| \leq |z| \prod_{i=1}^{n}\bigg(1 + \frac{|w|}{2^{\alpha(i)}}\bigg).$$

It is a well-known fact that almost surely we have $\alpha(n) \sim n/2$. It follows that

$$\log |f^n_{\omega,1}(z, w)| \leq \log |z| + \sum_{i=1}^{n} \log\bigg(1 + \frac{|w|}{2^{\alpha(i)}}\bigg).$$

As $n \to \infty$, almost certainly $a_i := \log(1 + |w|/2^{\alpha(i)}) \sim |w|/2^{i/2}$; therefore $\sum_{i=1}^{\infty} a_i < \infty$ and $f^n_\omega(z, w)$ is uniformly bounded for every $(z, w) \in \mathbb{C}^2$. We conclude that, in this case, $\mathcal{F}_\omega$ is normal for almost every sequence ω.

4. *Proof of Theorem 3.3*

We start this section with the simpler attracting and repelling cases; we will cover the neutral case afterwards.

4.1. *Attracting and repelling measures.*

If the measure ν is repelling, then the sequence of the differentials $df^n_\omega(0)$ diverges almost surely. If this is the case, the origin lies in the random Julia set.

The following statement implies the attracting case. It is likely to be well known, but we provide a proof for the completeness.

PROPOSITION 4.1. *Let ν be an attracting measure on $\mathcal{O}(\mathbb{C}^m, 0)$ with compact support. Then the origin lies in the random Fatou set. Furthermore, there exists almost surely a neighborhood of the origin on which all orbits converge to the origin.*

Proof. By Lemma 2.2 and (4), there exists $n_0 > 0$ such that $\mathbf{E} \log \|df^{n_0}_\omega(0)\| < 0$. Without loss of generality, we may assume that $n_0 = 1$.

We claim that there exists $\varepsilon > 0$ so that

$$\mathbf{E} \log(\|df(0)\| + \varepsilon) < 0. \tag{8}$$

Given $\delta > 0$, we write

$$\mathbf{E} \log \|df(0)\| = \int_{\|df(0)\| \leq \delta} \log \|df(0)\|\, d\nu + \int_{\|df(0)\| > \delta} \log \|df(0)\|\, d\nu,$$
$$= L_\delta + U_\delta.$$

Suppose $U_\delta < 0$ for some $\delta > 0$. In this case, we can easily find $\varepsilon > 0$ sufficiently small so that (8) is satisfied. On the other hand, suppose that $U_\delta \geq 0$ for every $\delta > 0$. Then

$$\int_{df(0) \neq 0} \log \|df(0)\|\, d\nu \geq 0.$$

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It follows that $M_0 := \nu\{df(0) = 0\} > 0$. We conclude that

$$\mathbf{E} \log(\|df(0)\| + \varepsilon) = M_0 \log \varepsilon + \int_{\|df(0)\| \neq 0} (\log \|df(0)\| + \varepsilon)\, d\mu(f)$$
$$\leq M_0 \log \varepsilon + \sup_{f \in \mathrm{supp}(\mu)} \log(\|df(0)\| + \varepsilon).$$

Therefore, also in this second case, we can choose $\varepsilon > 0$ sufficiently small so that (8) is satisfied.

Let $\alpha_f := \|df(0)\| + \varepsilon$. By the compactness of $\mathrm{supp}(\nu)$, there exists $0 < r < R$ such that every $f \in \mathrm{supp}(\nu)$ is holomorphic on $\mathbb{B}_R$ and

$$\|f(z)\| \leq \alpha_f \|z\| \quad \text{for all } z \in \mathbb{B}_r.$$

Since $\mathbf{E} \log \alpha_f < 0$, for almost every sequence $\omega \in \Omega$, we have $\alpha^n_\omega := \alpha_{f_{T^{n-1}\omega}} \cdots \alpha_{f_\omega} \to 0$. In particular, there exists $0 < \delta \leq r$, which depends on ω, such that if $\|z\| < \delta$, then $\alpha^n_\omega |z| < r$ for every $n > 0$.

Let $\|z\| < \delta$. A simple induction argument shows that

$$\|f^n_\omega(z)\| \leq \alpha^n_\omega \|z\| < r \quad \text{for all } n \in \mathbb{N}.$$

This shows that $\mathbb{B}_\delta \subset D_\omega$ and that the family $\mathcal{F}_\omega$ is normal at the origin. This is true for almost every $\omega \in \Omega$; therefore the origin lies in the random Fatou set. Furthermore since $\alpha^n_\omega \to 0$ as $n \to \infty$, the orbits of all points in $\mathbb{B}_\delta$ converge to the origin. □

4.2. *Neutral measures.* The following lemma is proved along the lines of Lemma 2.5.

LEMMA 4.2. *Let ν be a probability measure on $\mathcal{O}(\mathbb{C}^m, 0)$ with compact support. Then the set*

$$\Omega_a := \{\omega \in \Omega \mid \forall n \in \mathbb{N} \text{ and } \forall \alpha \in \Omega, \exists k_j : f^n_{T^{k_j}\omega} \to f^n_\alpha\}$$

is a full measure subset of Ω.

Proof. Unlike the standard topology of $\mathrm{Mat}(m, \mathbb{C})$, the inductive limit topology is not metrizable. However, given $n \in \mathbb{N}$, the image of Ω under the continuous map $\omega \mapsto f^n_\omega$ is a compact set in the inductive limit topology. By Corollary A.9, we can find a natural number N_n so that, for every $\omega \in \Omega$, the germ f^n_ω belongs to X_{N_n}.

We choose a sequence $\varepsilon_j \to 0$ and we define

$$\Omega_{n,j} = \{\omega \in \Omega \mid \forall \alpha \in \Omega, \exists k : d_{N_n}(f^n_\omega, f^n_\alpha) < \varepsilon_j\}.$$

By Theorem A.6, we have $\Omega_a = \cup_{n,j} \Omega_{n,j}$. From this point on, we follows the proof of Lemma 2.5, replacing the standard metric of $\mathrm{Mat}(m, \mathbb{C})$ with the metric d_{N_n}. □

THEOREM 4.3. (Hurwitz's theorem) *Suppose that $(f_n)_n$ is a sequence of injective holomorphic maps on a domain $U \subset \mathbb{C}^m$ converging uniformly on compact subsets to a function g. Then g is either injective or degenerate.*

Given a probability measure ν with compact support, choose $R > 0$ and define D_ω as explained in §3.

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LEMMA 4.4. *Let ν be a neutral measure with compact support. The origin lies in the random Fatou set if and only if, given $\varepsilon > 0$, there exists $\rho > 0$ such that, for every $\omega \in \Omega$, we have $\mathbb{B}_\rho \subset D_\omega$ and*

$$f_\omega^n(\mathbb{B}_\rho) \subset \mathbb{B}_\varepsilon \quad \textit{for all } n \in \mathbb{N}. \tag{9}$$

Proof. Suppose that, for every $\varepsilon > 0$, there exists such ρ. Then, by the weak Montel theorem, the origin lies in the random Fatou set.

Suppose, on the other hand, that the origin lies in the random Fatou set and let $\omega_0 \in \Omega_a \cap \Omega_{\text{nor}}$. By the Ascoli–Arzelá theorem, given $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\mathbb{B}_{\delta_0} \subset D_{\omega_0}$ and

$$f_{\omega_0}^n(\mathbb{B}_{\delta_0}) \subset \mathbb{B}_\varepsilon \quad \text{for all } n \in \mathbb{N}.$$

By the compactness of $\text{supp}(\nu)$, by choosing $\varepsilon > 0$ small enough, we may assume that every $f \in \text{supp}(\nu)$ is holomorphic and injective on $\mathbb{B}_\varepsilon$. Therefore $f_{\omega_0}^n : \mathbb{B}_{\delta_0} \to \mathbb{B}_\varepsilon$ is injective for all values of n.

Suppose that there exists n_k so that

$$\mathbb{B}_{1/k} \not\subset f_{\omega_0}^{n_k}(\mathbb{B}_{\delta_0}). \tag{10}$$

By taking a subsequence, if necessary, we may assume that $f_{\omega_0}^{n_k} \to g$ uniformly on compact subsets of $\mathbb{B}_{\delta_0}$. By Proposition 2.7, we have $|\det(df_{\omega_0}^{n_k}(0))| = 1$ for every k, and thus $|\det(dg(0))| = 1$. Thanks to Hurwitz's theorem, we can conclude that g is injective on $\mathbb{B}_{\delta_0}$.

Since g is an open function, we can choose $\rho' > 0$ such that $\mathbb{B}_{\rho'} \subset g(\mathbb{B}_{\delta_0})$ and thus such that $\mathbb{B}_{\rho'} \subset f_{\omega_0}^{n_k}(\mathbb{B}_{\delta_0})$ for big enough k, which contradicts (10).

It follows that there exists $\rho_0 > 0$ so that $\mathbb{B}_{\rho_0} \subset f_{\omega_0}^n(\mathbb{B}_{\delta_0})$ for every n. By the invariance relation (7), given $n, k \in \mathbb{N}$, we find that $\mathbb{B}_{\rho_0} \subset D_{T^k\omega_0}$ and

$$\begin{aligned} f_{T^k\omega_0}^n(\mathbb{B}_{\rho_0}) &\subset f_{T^k\omega_0}^n \circ f_{\omega_0}^k(\mathbb{B}_{\delta_0}) \\ &\subset \mathbb{B}_\varepsilon. \end{aligned}$$

Let $\rho < \rho_0$ be fixed. Given $\omega \in \Omega$ and $n \in \mathbb{N}$, there exists a sequence k_j so that $f_{T^{k_j}\omega_0}^n$ converges to f_ω^n in the inductive limit topology. On the other hand, by the weak Montel theorem, we may also assume that $f_{T^{k_j}\omega_0}^n$ converges to some g uniformly on compact subsets of $\mathbb{B}_{\rho_0}$. By Theorem A.6, the maps g and f_ω^n agree on a small ball containing the origin, and therefore, by the identity principle, they coincide as germs. We conclude that f_ω^n is holomorphic on $\mathbb{B}_\rho$ and, since ρ is independent from n and ω, that $\mathbb{B}_\rho \subset D_\omega$ for every $\omega \in \Omega$. Finally, uniform convergence on compact subsets of $\mathbb{B}_{\rho_0}$ implies (9). □

We write S_ν for the semigroup

$$S_\nu := \{f_\omega^n \mid \omega \in \Omega, n \in \mathbb{N}\}.$$

The following corollary is an immediate consequence of the previous lemma.

COROLLARY 4.5. *Let ν be a neutral measure with compact support. Then the origin lies in the random Fatou set if and only if S_ν is relatively compact in $\mathcal{O}(\mathbb{C}^m, 0)$.*

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Definition. Given $f \in \mathcal{O}(\mathbb{C}^m, 0)$, we say that f is *linearizable* if there exists $\varphi \in \mathcal{O}(\mathbb{C}^m, 0)$, locally invertible at the origin, such that

$$\varphi \circ f(z) = df(0) \cdot \varphi.$$

If this is the case, we say that f is *linearized* by φ.

COROLLARY 4.6. *Let ν be a neutral measure with compact support. If the origin lies in the random Fatou set, then every element of S_ν is linearizable.*

Proof. If the semigroup S_ν is relatively compact in $\mathcal{O}(\mathbb{C}^m, 0)$, then, given $f \in S_\nu$, the family $\{f^n\}$ is also relatively compact. It follows that the origin lies in the Fatou set of f and, therefore, that the germ is linearizable. □

We notice that linearizability of every element of S_ν does not imply that the origin lies in the random Fatou set. Consider the following example.

Example. Let $\lambda = e^{2\pi i\alpha}$ be an irrational rotation with α Brjuno number. We define the maps

$$f_1(z) = \lambda(z + z^2) \quad \text{and} \quad f_2(z) = \lambda(z - z^2)$$

and consider a measure such that $\text{supp}(\nu) = \{f_1, f_2\}$.

In this case, we see immediately that every function in the semigroup $S_\nu = \langle f_1, f_2\rangle$ is linearizable. Nevertheless, f_1 and f_2 are not simultaneously linearizable.

Let $z_0 \neq 0$ and define integers $n_i \in \{1, 2\}$ recursively as follows. If $\text{Re}(z_i) \geq 0$, then $n_{i+1} = 1$; otherwise, $n_{i+1} = 2$. Let $\omega := (f_{n_1}, f_{n_2}, \ldots)$ and $z_i = f_\omega^i(z_0)$.

We claim that the orbit of z_0 converges to infinity. To see this, suppose that $\text{Re}(z_i) \geq 0$. Then it follows that the angle between the vectors z_i and z_i^2 is at most $\pi/2$, and hence

$$|z_{i+1}| = |z_i + z_i^2| \geq |z_i|.$$

The irrational rotation guarantees that the norm increases often enough to converge to infinity, which proves the claim.

We conclude that, in this case, S_ν is not relatively compact; therefore the origin does not lie in the random Fatou set.

Suppose that ν is a neutral measure with compact support for which the origin lies in the random Fatou set. We write G_ν for the closure of S_ν in $\mathcal{O}(\mathbb{C}^m, 0)$.

LEMMA 4.7. *The set G_ν is a compact topological group. Moreover, there exists an open subset $M \subset \mathbb{C}^m$ such that every $f \in G_\nu$ belongs to* $\text{Aut}(M)$.

Proof. By Lemma 4.4, the set G_ν is compact. Furthermore, by Corollary 4.6, every $f \in S_\nu$ is linearizable and, by Proposition 2.7, the differential $df(0)$ is conjugated to an element of $U(m)$. It follows that, for some sequence n_j, we have $f^{n_j} \to \text{id}$. This proves that G_ν contains the identity element. Furthermore, the germ $g = \lim_{j\to\infty} f^{n_j - 1}$ is the inverse of f in $\mathcal{O}(\mathbb{C}^m, 0)$, and thus G_ν contains also the inverse of f.

Let $f \in G_\nu$ and $f_n \in S_\nu$ such that $f_n \to f$. Every f_n has an inverse $f_n^{-1} \in G_\nu$. By taking a subsequence, if necessary, we may assume that $f_n^{-1} \to g \in G_\nu$. It is not difficult

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to prove that g is the inverse element of f in $\mathcal{O}(\mathbb{C}^m, 0)$, which finally proves that G_ν is a compact group.

Let $\varepsilon > 0$ and $\rho > 0$, as in Lemma 4.4. If we choose ε small enough, we may further assume that all the elements of $\text{supp}(\nu)$ are univalent on $\mathbb{B}_\varepsilon$. It follows that every $f \in S_\nu$ is univalent on $\mathbb{B}_\rho$. Since every $g \in G_\nu$ is invertible, by Hurwitz's theorem, it is also univalent on $\mathbb{B}_\rho$.

By taking a smaller ε, if necessary, we may further assume that every $g \in G_\nu$ is univalent on $\mathbb{B}_\varepsilon$. We define the open set

$$M = \bigcup_{g \in G_\nu} g(\mathbb{B}_\rho) \subset \mathbb{B}_\varepsilon.$$

Given $g \in G_\nu$ and $x \in M$, then $x = \hat{g}(z)$ for some $\hat{g} \in G_\nu$ and $z \in \mathbb{B}_\rho$. It follows that $g(M) \subset M$. Furthermore, since $g^{-1} \in G_\nu$, we obtain the equality $g(M) = M$, which proves that $g \in \text{Aut}(M)$. □

The following result is known as Bochner's linearization theorem. A proof of the theorem, valid for a C^k diffeomorphism, can be found in [**DK00**]. The same proof is valid also in the holomorphic case, up to small modifications.

THEOREM 4.8. (Bochner's linearization theorem) *Let M be a complex manifold and $x_0 \in M$. Let A be a continuous homomorphism from a compact group K to* Aut(M) *such that $A_k(x_0) = x_0$ for all $k \in K$. Then there exists a K invariant open neighborhood U of x_0 in M and a biholomorphism φ from U onto an open neighborhood $V \subset T_{x_0}M$ of* 0 *such that*

$$\varphi(x_0) = 0, \quad d\varphi(x_0) = \text{id} : T_{x_0}M \to T_{x_0}M$$

and

$$\varphi \circ A_k(x) = dA_k(x_0) \cdot \varphi(x) \quad \textit{for all } k \in K, x \in U.$$

Definition. We say that two or more germs are *simultaneously linearizable* if there exists a locally invertible $\varphi \in \mathcal{O}(\mathbb{C}^m, 0)$ such that all the germs are linearized by φ.

LEMMA 4.9. *Let ν be a neutral measure on $\mathcal{O}(\mathbb{C}^m, 0)$ with compact support. If the origin lies in the random Fatou set, then all the elements $f \in$* supp(ν) *are simultaneously linearizable.*

Proof. Let G_ν be the closure of S_ν and let M be the G_ν-invariant open set described in Lemma 4.7. The compact group G_ν induces an action on M that satisfies the hypothesis of Bochner's linearization theorem. It follows that all the germs $f \in G_\nu$ can be simultaneously linearized. □

We are finally ready to conclude the proof of Theorem 3.3.

Proof of Theorem 3.3. Suppose that the origin lies in the random Fatou set. By the previous lemma, it follows that all the germs in $\text{supp}(\nu)$ are simultaneously linearizable, which implies that the semigroup S_ν is conjugated to the semigroup of the differentials dS_ν. Furthermore, by Proposition 2.7, the semigroup dS_ν is itself conjugate to a subsemigroup of $U(m)$. The other implication of the theorem is trivial. □

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5. *Discussion of semi-attracting measures*

Recall from §3 that, for every $\omega \in \Omega_{\mathrm{nor}}$, the origin belongs to the Fatou set $F_\omega := \mathrm{int}(D_\omega)$. Let U_ω be the connected component of F_ω containing the origin.

PROPOSITION 5.1. *Let ν be a semi-attracting measure on $\mathcal{O}(\mathbb{C}^m, 0)$ with compact support, so that the origin lies in the random Fatou set. Then, almost surely, every limit map $g = \lim_{k\to\infty} f_\omega^{n_k}$ is degenerate on U_ω.*

Proof. By Corollary 2.3, we have $\mathbf{E} \log|\det df(0)| < 0$. Along the lines of Proposition 4.1, we can choose $\varepsilon > 0$ such that $\mathbf{E} \log(|\det df(0)| + \varepsilon) < 0$. In particular, for almost every $\omega \in \Omega$,

$$\prod_{k=1}^{\infty} (|\det df_{T^{k-1}\omega}(0)| + \varepsilon) = 0. \tag{11}$$

By compactness of $\mathrm{supp}(\nu)$, we may choose $r > 0$ so that, for every $\|z\| < r$,

$$|\det df(z)| < |\det df(0)| + \varepsilon \quad \text{for all } f \in \mathrm{supp}(\mu).$$

Since the origin belongs to the random Fatou set, for almost every $\omega \in \Omega$, there exists $\delta_\omega > 0$ such that $\|f_\omega^n(z)\| < r$ for every $z \in \mathbb{B}_{\delta_\omega}$ and $n \geq 0$. We conclude that, for $z \in \mathbb{B}_{\delta_\omega}$,

$$\begin{aligned} |\det df_\omega^n(z)| &= \prod_{k=1}^{n} |\det df_{T^{k-1}\omega}(f_\omega^{k-1}(p))| \\ &\leq \prod_{k=1}^{n} (|\det df_{T^{k-1}\omega}(0)| + \varepsilon). \end{aligned}$$

Let n_k be a sequence such that $f_\omega^{n_k} \to g$ locally uniformly on a neighborhood U of the origin. Then, on a possibly smaller neighborhood, we have $\det dg \equiv 0$. By the identity principle, we conclude that the same is true on U_ω. □

Given $z \in U_\omega$, we define the *stable set*

$$\mathbb{W}_\omega^s(z) := \{w \in U_\omega \mid \|f_\omega^n(z) - f_\omega^n(w)\| \to 0\}.$$

When $m = 2$, we will prove that, given ν semi-attracting with compact support and such that the origin lies in the random Fatou set, the stable set through every point z sufficiently close to the origin is locally a complex submanifold. It is a natural to ask whether the same is true when $m > 2$.

Before proceeding to the proof of this result, we will present two examples of this phenomenon in the case where the germs in $\mathrm{supp}(\nu)$ are linear maps.

Example. Suppose $\nu = \frac{1}{2}\delta_g + \frac{1}{2}\delta_h$, where

$$g(z, w) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \quad h(z, w) = \begin{pmatrix} 1/2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

We notice that, given $\omega \in \Omega$,

$$f_\omega^n(z, w) = \begin{pmatrix} 1/2^n & \alpha_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \quad \text{with } \alpha_n = \begin{cases} \dfrac{\alpha_{n-1}}{2} & \text{if } f_{T^{n-1}\omega} = g, \\ \dfrac{\alpha_{n-1}}{2} + 1 & \text{if } f_{T^n\omega} = h. \end{cases}$$

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The two Lyapunov exponents of the measure ν are $\kappa_1 = 0$ and $\kappa_2 = -\log 2$; therefore the measure is semi-attracting. Furthermore, we have $0 \le \alpha_n < 2$ for every n and every ω, which implies that the origin lies in the random Fatou set. Given $(z_0, w_0) \in U_\omega = \mathbb{C}^2$, we can write its stable set as

$$\mathbb{W}^s_\omega(z_0, w_0) = \{(z, w) \in \mathbb{C}^2 \mid w = w_0\},$$

which is a one-dimensional manifold. We notice that, in this case, the stable manifolds are independent of the sequence $\omega \in \Omega$ but that this is not true in general.

Consider, for example, the measure $\tilde{\nu}$, obtained by taking instead the maps

$$g(z, w) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \quad h(z, w) = \begin{pmatrix} 1/2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

This measure has the same Lyapunov exponents and the origin lies in the random Fatou set. However, the sequence f^n_ω converges, for every ω, to a map of the form $\tilde{g}(z, w) = (z + \beta_\omega w, 0)$. Hence the stable manifolds are again parallel complex planes, but now depend on ω.

Remark. The last two examples are helpful models in order to understand the random dynamics of semi-attracting measures. We notice that, in both cases, $df(0)$ and $dg(0)$ share an eigenvector. In the theory of linear cocycles, one says that a cocycle is *strongly irreducible* if there is no finite family of proper subspaces invariant by M_ω for ν-almost every ω. This raises the following question.

Does there exist a semi-attracting measure such that M^n_ω is almost surely bounded but such that the corresponding cocycle is strongly irreducible?

LEMMA 5.2. *There exists $r > 0$ and, for almost every $\omega \in \Omega$, a sequence n_k dependent on ω, so that $f^{n_k}_\omega \to g$ locally uniformly on U_ω and so that*

$$\mathbb{W}^s_\omega(z) = \{w \in U_\omega \mid g(w) = g(z)\}$$

for every $z \in U_\omega$ that satisfies $\|g(z)\| \le r$.

Proof. Given $r > 0$, we define the measurable sets

$$\mathcal{A}_r := \{\omega \in \Omega \mid \mathbb{B}_r \Subset U_\omega\}.$$

It is clear that $\mathcal{A}_{r_1} \supset \mathcal{A}_{r_2}$ when $r_1 < r_2$ and that $\bigcup_{r>0} \mathcal{A}_r = \Omega_{\mathrm{nor}}$. Since the latter is a full measure set, we may choose $r > 0$ so that $\mu(\mathcal{A}_r) > 1/2$.

Let $\varepsilon_1 > 0$. By the Ascoli–Arzelà theorem, the collection of measurable sets

$$\mathcal{B}^{(1)}_\delta := \left\{ \omega \in \mathcal{A}_r \,\middle|\, \begin{array}{l} \forall z, w \in \mathbb{B}_r \text{ with } \|z - w\| < \delta : \\ \sup_n \|f^n_\omega(z) - f^n_\omega(w)\| < \varepsilon_1 \end{array} \right\},$$

defined for $\delta > 0$, covers $\mathcal{A}_r$. Furthermore, it satisfies $\mathcal{B}^{(1)}_\delta \supset \mathcal{B}^{(1)}_{\delta'}$ for $\delta < \delta'$, and thus we may choose $\delta_1 > 0$ so that $\mu(\mathcal{B}^{(1)}_{\delta_1}) > 1/2$. We will write $\mathcal{B}^{(1)}$ for the set $\mathcal{B}^{(1)}_{\delta_1}$.

Given a sequence of positive real numbers ε_k with $\lim_{k\to\infty} \varepsilon_k = 0$, we may recursively define a sequence δ_k of positive real numbers in such a way that

$$\mathcal{B}^{(k+1)} := \left\{ \omega \in B^{(k)} \,\middle|\, \begin{array}{l} \forall z, w \in \mathbb{B}_r \text{ with } \|z - w\| < \delta_{k+1} : \\ \sup_n \|f^n_\omega(z) - f^n_\omega(w)\| < \varepsilon_{k+1} \end{array} \right\}$$

has measure strictly greater that $1/2$.

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Since the nested sets $\mathcal{B}^{(k)}$ all have measure at least 1/2, the intersection, which we will simply denote by $\mathcal{B}$, satisfies $\mu(\mathcal{B}) \geq 1/2$.

By ergodicity of the transformation T, there exists, for almost every $\omega \in \Omega_{\text{nor}}$, a sequence n_k so that $T^{n_k}(\omega) \in \mathcal{B}$. By taking a subsequence, if necessary, we find that $f_\omega^{n_k}$ converges locally uniformly on U_ω to a function g.

Given $z \in U_\omega$ such that $\|g(z)\| < r$, it is clear that $g(w) = g(z)$ for every $w \in \mathbb{W}_\omega^s(z)$. On the other hand, given $w \in U_\omega$ that satisfies $g(w) = g(z)$, we can find k_0 so that $f_\omega^{n_k}(z), f_\omega^{n_k}(w) \in \mathbb{B}_r$ for every $k \geq k_0$. Furthermore, we may also find a sequence $k_0 \leq k_1 \leq k_2 \leq \cdots$ so that, whenever $k \geq k_i$, we have $\|f_\omega^{n_k}(z) - f_\omega^{n_k}(w)\| < \delta_i$.

Since $T^{n_k}\omega \in \mathcal{B}$ for every k, we conclude that, given $n \geq n_{k_i}$,

$$\|f_\omega^n(z) - f_\omega^n(w)\| = \|f_{T^{n_{k_i}}\omega}^{n-n_{k_i}}(f_\omega^{n_{k_i}}(z)) - f_{T^{n_{k_i}}\omega}^{n-n_{k_i}}(f_\omega^{n_{k_i}}(w))\| < \varepsilon_i.$$

This shows that $w \in \mathbb{W}_\omega^s(z)$, which proves the desired equality. □

When $m = 2$ and the measure is semi-attracting, we obtain the following (local) version of the stable manifold theorem.

COROLLARY 5.3. (Stable manifold theorem) *Let μ be a semi-attracting measure on $\mathcal{O}(\mathbb{C}^2, 0)$ with compact support, for which the origin lies in the random Fatou set. Then there exists, for almost every $\omega \in \Omega$, a constant $\rho > 0$ sufficiently small so that, given $z \in \mathbb{B}_\rho$, the set $\mathbb{W}_\omega^s(z)$ is locally a one-dimensional complex manifold.*

Proof. Let $\omega \in \Omega$ be such that the map g described in the previous lemma exists. Since $g(0) = 0$, we can choose $\rho > 0$ sufficiently small so that $\mathbb{B}_\rho \subset U_\omega$ and $g(\mathbb{B}_\rho) \subset \mathbb{B}_r$. Here the value of r is again determined by the previous lemma.

By Proposition 5.1, the function g is degenerate. Furthermore, by Proposition 2.10, we must have $\|dg(0)\| \geq 1$, and thus we may further assume, by shrinking ρ, if necessary, that g has rank one on every point of $\mathbb{B}_\rho$.

By the constant rank theorem, $g(\mathbb{B}_\rho)$ is a one-dimensional submanifold and every point in it is a regular value for the holomorphic map $g : \mathbb{B}_\rho \to g(\mathbb{B}_\rho)$. By the implicit function theorem, given $z \in \mathbb{B}_\rho$, the level set $\{w \in \mathbb{B}_\rho \mid g(w) = g(z)\}$ is a one-dimensional complex manifold. The level sets of the map g coincide with the stable sets of the sequence ω. It follows that the latter are locally one-dimensional manifolds. □

A. *Appendix. Holomorphic germs and their topology*

We write $\mathcal{O}(\mathbb{C}^m, 0)$ for the space of germs at zero of holomorphic maps from $\mathbb{C}^m$ to itself. Note that, in the main text, we required that the germs fix the origin, but this requirement plays no role here.

Our goal in this appendix is to define a topology on $\mathcal{O}(\mathbb{C}^m, 0)$ with the following property.

Local uniform convergence. A sequence of germs (f_k) converges to a germ f if there exists $r > 0$ such that f and all the f_n admit a representative that is bounded and holomorphic on $\mathbb{B}_r$ and

$$\sup_{\|z\| \leq r} \|f_k(z) - f(z)\| \to 0 \quad \text{as } n \to \infty.$$

The construction of such a topology, given in Theorem A.6 below, resembles very closely the so-called inductive limit topology of a sequence of nested Fréchet spaces $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$. See [**Rud91**] or [**Gr09**] for an example of this construction in the case of $C_0^\infty(\Omega)$. For a discussion of inductive limit topology for holomorphic germs, see, for instance, [**M79**], where germs in possibly infinite-dimensional spaces are considered. While the results presented here are undoubtedly known to experts on the topic, we include them for the sake of completeness.

Suppose that $\varepsilon_n \to 0$ is a strictly decreasing sequence. Let X_n be the space of all bounded and holomorphic functions on $\mathbb{B}_{\varepsilon_n}$ equipped with the distance

$$d_n(f, g) := \sup_{\|z\| \le \varepsilon_n} \|f(z) - g(z)\|.$$

There exists a natural injection of X_n into $\mathcal{O}(\mathbb{C}^m, 0)$, which we will denote by π_n. Furthermore, it is clear that $X_n \subset X_{n+1}$.

We write τ_n for the standard metric topology on X_n. We note that τ_n is not equivalent to the compact-open topology. However, it will be clear that the inductive limit topology on $\mathcal{O}(\mathbb{C}^m, 0)$ obtained using either sequence of topologies is identical.

Inductive limit topology. The *inductive limit topology*, which we will denote by τ_{ind}, is the finest topology on $\mathcal{O}(\mathbb{C}^m, 0)$ such that each injection $\pi_n : X_n \to \mathcal{O}(\mathbb{C}^m, 0)$ is a continuous function.

In other words, a set U is open in the inductive limit topology if and only if, for each n, $X_n \cap \pi_n^{-1}(U)$ is open with respect to the topology τ_n.

Since each map π_n is injective, from now on, we will avoid writing the map π_n and we will consider each X_n as a subset of $\mathcal{O}(\mathbb{C}^m, 0)$.

Remark. In the classical construction of the inductive limit topology of a nested sequence of Fréchet spaces, $X_1 \subset X_2 \subset \cdots$, it is assumed that the topology on X_n is induced by the topology on X_{n+1}, which fails in our setting. Consider, for example, the germs $f_n = n(z/\varepsilon_1)^n$. This sequence converges uniformly to zero on $\mathbb{B}_r$ for $r < \varepsilon_1$; in particular, we have $d_2(f_n, 0) \to 0$. On the other hand, we see that $d_1(f_n, 0) \to \infty$. This shows that τ_1 is different from $\tau_2|_{X_1}$.

LEMMA A.1. *Let $\alpha \in \mathbb{N}^m$. The derivation $D_\alpha : f \mapsto \partial_\alpha f(0)$ is a continuous map from $\mathcal{O}(\mathbb{C}^m, 0)$, endowed with the inductive limit topology, to $\mathbb{C}$.*

Proof. If D_α were not continuous, there would exist $U \subset \mathbb{C}$ open and a natural number n such that $V := D_\alpha^{-1}(U) \cap X_n$ is not open with respect to the metric topology of X_n. If so, there exists $(f_k) \subset X_n \backslash V$ such that $f_k \to g \in V$ with respect to the metric d_n. By Weierstrass, we have $D_\alpha f_k \to D_\alpha g$, which is not possible since $f_k \notin V$. □

COROLLARY A.2. *The inductive limit topology on $\mathcal{O}(\mathbb{C}^m, 0)$ is Hausdorff.*

Proof. Given two germs $f_1 \neq f_2$, there exists $\alpha \in \mathbb{N}^m$ such that $D_\alpha f \neq D_\alpha f_2$. The Hausdorff property of τ_{ind} follows from the continuity of D_α. □

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Recall that a subset U of a topological space X is *sequentially open* if each sequence (x_k) in X converging to a point of U is eventually in U. We say that X is a *sequential space* if every sequentially open subset of X is open.

LEMMA A.3. *$\mathcal{O}(\mathbb{C}^m, 0)$, endowed with the inductive limit topology, is a sequential space.*

Proof. First, we notice that if $f_k \to f$ with respect to the metric topology of X_n, then the convergence is valid in τ_{ind}. If this was not the case, we could find an open set $U \ni f$ in the inductive limit topology such that (f_k) is not eventually contained in U. However, this is not possible since $U \cap X_n$ is open with respect to the metric topology of X_n, and f_k converges to f in this topology.

As a consequence, given $U \subset \mathcal{O}(\mathbb{C}^m, 0)$ sequentially open, the set $U \cap X_n$ is sequentially open in the metric topology of X_n. Metric spaces are sequential, and thus $U \cap X_n$ is open in X_n. By the definition of τ_{ind}, it follows that U is open, which concludes the proof. □

Recall that a topological space X is countably compact if, given an open countable cover of X, there exists a finite subcover. A proof of the following proposition can be found in [**En89**, p. 209].

PROPOSITION A.4. *Sequential compactness and countable compactness are equivalent in the class of sequential Hausdorff spaces.*

Given $r > 0$ and $f \in X_n$, we write $B_n(f, r)$ (respectively, $\overline{B}_n(f, r)$) for the open (respectively, closed) ball of radius r and center f with respect to the metric d_n.

LEMMA A.5. *The closed ball $\overline{B}_n(f, r)$ is compact in (X_{n+1}, d_{n+1}).*

Proof. It is sufficient to prove the lemma for the case $f = 0$. Suppose (f_k) is a sequence in $\overline{B}_n(0, r)$. Then, by the weak Montel theorem, there exists a subsequence k_m such that $f_{k_m} \to f_\infty$ uniformly on every compact subset $K \subset \mathbb{B}_{\varepsilon_n}$. In particular, $d_{n+1}(f_{k_m}, f_\infty) \to 0$.

The function f_∞ is holomorphic on $\mathbb{B}_{\varepsilon_n}$. Furthermore, by uniform convergence on compact subsets of $\mathbb{B}_{\varepsilon_n}$, we have $\|f_\infty(z)\| \le r$ for every $z \in B_{\varepsilon_n}(0)$. This proves that $f_\infty \in \overline{B}_n(0, r)$, which concludes the proof of the lemma. □

THEOREM A.6. *A sequence (f_k) is a convergent sequence with respect to τ_{ind} if and only if there exists N such that $(f_k) \subset X_N$ and (f_k) is convergent with respect to the metric d_N.*

Proof. We have already proved the only if part in Lemma A.3. Suppose now that (f_k) is a convergent sequence in the inductive limit topology. Write $f_\infty = \lim_{k\to\infty} f_k$ and let n_0 so that $f_\infty \in X_{n_0}$. First, we prove that there exists N_0 such that $(f_k) \subset X_{N_0}$. If this were not the case, by taking a subsequence, if necessary, we may assume that, for every n, $X_n \cap (f_k)$ is a finite set and that the intersection is empty when $n \le n_0$.

Let $U = \mathcal{O}(\mathbb{C}^m, 0)\backslash(f_k)$. The set $X_n\backslash U$ is finite for every $n \in \mathbb{N}$; therefore U is open in the inductive limit topology. It is clear that $f_\infty \in U$ but that the sequence (f_k) is not eventually contained in U, which contradicts $f_\infty = \lim_{k\to\infty} f_k$. The existence of N_0 follows.

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Suppose now that, for every $N \geq N_0$, we have $d_N(f_k, f_\infty) \not\to 0$. If $\varepsilon > 0$ is fixed, then the sequence f_k is not eventually contained in $B_N(f_\infty, \varepsilon)$. This follows from the fact that $B_N(f_\infty, \varepsilon)$ are relatively compact in X_{N+1}, plus the fact that convergence in X_{N+1} implies convergence with respect to τ_{ind}.

Let $(k_n^N) \subset \mathbb{N}$ be the subsequence obtained by removing all the indices k for which $f_k \in B_N(f_\infty, \varepsilon)$. Since $B_N(f_\infty, \varepsilon) \subset B_{N+1}(f_\infty, \varepsilon)$, it follows that

$$(k_n^{N+1}) \subset (k_n^N) \subset \cdots \subset (k_n^{N_0}).$$

Let $k_n := k_n^{n+N_0}$. Then, for every $n \geq N - N_0$, we have that $f_{k_n} \notin B_N(f_\infty, \varepsilon)$. Furthermore, we notice that

$$d_N(f_{k_n}, f_\infty) \to \infty \quad \text{for all } N \geq N_0.$$

Suppose, on the contrary, that there exists $(k'_n) \subset (k_n)$ such that $d_N(f_{k'_n}, f_\infty) \to M$. By the previous lemma, by taking a subsequence of k'_n, if necessary, we may assume that $f_{k'_n} \to g$ in X_{N+1} with $g \neq f_\infty$. Convergence in X_{N+1} implies convergence in τ_{ind}; therefore this is not possible.

The set $U = \mathcal{O}(\mathbb{C}^m, 0) \backslash (f_{k_n})$ contains $f_\infty \in U$. Since $(f_k) \subset X_{N_0}$, given $N \geq N_0$, we have that $U \cap X_N = X_N \backslash (f_{k_n})$. The sequence (f_{k_n}) is divergent in X_N, and thus $U \cap X_N$ is open in X_N for $N \geq N_0$. Finally, if V is open in X_{N+1}, then $V \cap X_N$ is open in X_N. This proves that U is an open neighborhood of f_∞ in the inductive limit topology, which gives a contradiction. It follows that there exists N_1 such that $d_{N_1}(f_k, f_\infty) \to 0$. □

COROLLARY A.7. *A set $K \subset \mathcal{O}(\mathbb{C}^m, 0)$ is sequentially compact in τ_{ind} if and only if there exists N such that $K \subset X_N$ and it is compact with respect to the metric topology of X_N.*

Proof. Suppose that K is sequentially compact in τ_{ind}. First, there exists N_0 such that $K \subset X_{N_0}$. If this were not the case, we could find a sequence $(f_k) \subset K$ such that $f_k \notin X_k$. By the previous theorem, such a sequence does not have any convergent subsequence with respect to τ_{ind}, which contradicts sequential compactness.

Suppose that K is unbounded in X_N for every $k \geq N_0$. We can find $f_k \in K$ such that $d_k(f_k, 0) > N$. The previous theorem proves that this sequence does not have a convergent subsequence, which contradicts sequential compactness. This proves that K is bounded in X_{N_1} for some $N_1 \geq N_0$. By Lemma A.5, the set K is relatively compact in X_{N_1+1}. Since the inductive limit topology is sequential, it follows that K is a closed set in τ_{ind}. This shows that K is a compact set in X_{N_1+1}.

On the other hand, since convergence in X_N implies convergence in τ_{ind}, a compact set $K \subset X_N$ is sequentially compact in τ_{ind}. □

COROLLARY A.8. *Compactness and sequential compactness are equivalent in $\mathcal{O}(\mathbb{C}^m, 0)$.*

Proof. Since $\mathcal{O}(\mathbb{C}^m, 0)$ is a Hausdorff sequential space, we only have to prove that every sequentially compact set is compact. By the previous corollary, if K is sequentially compact, then there exists N such that $K \subset X_N$ and K is compact in X_N. Let $\{U_\alpha\}$ be an open cover of K. By the definition of the inductive limit topology, it follows that $\{U_\alpha \cap X_N\}$ is an open cover of K in X_N. Using the compactness of K in X_N, we can extract a finite open subcover $\{U_1, \ldots, U_n\}$. This proves that K is compact with respect to the topology τ_{ind}. □

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The following corollary is an immediate consequence of the previous ones.

COROLLARY A.9. *A set $K \subset \mathcal{O}(\mathbb{C}^m, 0)$ is compact in τ_{ind} if and only if there exists N such that $K \subset X_N$ is compact with respect to the metric topology of X_N.*

We conclude this appendix by showing that the inductive limit topology is not metrizable. Notice that a similar proof also shows that this topology is not even first countable.

PROPOSITION A.10. *The inductive limit topology is not metrizable.*

Proof. Suppose, on the contrary, that there exists a metric d_{ind} on $\mathcal{O}(\mathbb{C}^m, 0)$ such that τ_{ind} coincides with the metric topology of d_{ind}. We will write $B_{\text{ind}}(f, r)$ for the open ball of center f and radius r with respect of this metric.

We note that, for every n, we can construct a sequence $(f_k^n)_{k \in \mathbb{N}} \subset X_n \backslash X_{n-1}$ such that $\lim_{k\to\infty} d_n(f_k^n, 0) = 0$. Since convergence in X_n implies convergence in τ_{ind}, for every n, we can find k_n such that

$$f_{k_n}^n \in B_{\text{ind}}(0, 1/n).$$

Now let $g_n = f_{k_n}^n$. It is clear that $d_{\text{ind}}(g_n, 0) \to 0$ as $n \to \infty$. By Theorem A.6, it follows that (g_n) is contained in some X_N, which is not possible by the definition of the f_k^n. This contradicts the fact that τ_{ind} is metrizable. □

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