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Abstract

We show that there is an essentially unique S-algebra structure on the Morava K-theory spectrum $K(n)$, while $K(n)$ has uncountably many MU or $\widehat{E(n)}$-algebra structures. Here $\widehat{E(n)}$ is the $K(n)$-localized Johnson–Wilson spectrum. To prove this we set up a spectral sequence computing the homotopy groups of the moduli space of A_∞ structures on a spectrum, and use the theory of S-algebra k-invariants for connective S-algebras found in the work of Dugger and Shipley [*Postnikov extensions of ring spectra*, Algebr. Geom. Topol. **6** (2006), 1785–1829 (electronic)] to show that all the uniqueness obstructions are hit by differentials.

1. Introduction

We study the moduli space of S-algebra structures on the Morava K-theory spectrum $K(n)$. Recall that, given a prime p and an integer $n \geqslant 1$, $K(n)$ is the spectrum carrying the Honda formal group of height n over $\mathbb{F}_p$, and that $K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2p^n - 2$. Robinson [Rob89] found that there are uncountably many ways to build an A_∞ structure on $K(n)$, but he did not ask if these A_∞ structures might all be equivalent. The point is that there are two distinct definitions of the moduli space of S-algebra structures, and in this paper we use the version where we allow automorphisms of the underlying spectrum. We prove the following theorem.

Theorem A. *There is an essentially unique S-algebra structure on $K(n)$, in the sense that the moduli space of S-algebra structures on $K(n)$ is connected.*

This should be compared to the situation where we study the moduli space of R-algebra structures on $K(n)$ for some other commutative S-algebra R.

Theorem B. *Let $R = MU$ or $R = \widehat{E(n)}$. Then there are uncountably many R-algebra structures on $K(n)$, in the sense that the moduli space of R-algebra structures on $K(n)$ has uncountably many path components.*

If BP is a commutative S-algebra, Theorem B remains true with $R = BP$.

We will use two approaches to study the moduli space of S-algebra or R-algebra structures on a spectrum A. For our first approach, we use the equivalence between S-algebras and A_∞ ring spectra, and study how to build an A_∞ structure on A by induction on the A_m structure. This is the approach taken by Robinson [Rob89] and later by the current author [Ang08]. We need to modify this approach slightly to get the right notion of equivalence of A_∞ structures; this amounts to allowing maps $(A, \phi) \to (A, \psi)$ of A_∞ ring spectra where the underlying map $A \to A$ of spectra is not the identity but merely a weak equivalence.

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We will define the appropriate moduli space of S-algebra structures on A, which we denote by $B\mathscr{A}^S(A)$, and set up a spectral sequence $\{E_r^{s,t}\}$ which contains the obstructions to $B\mathscr{A}^S(A)$ being non-empty and, given a basepoint, computes the homotopy groups of this space. The spectral sequence is similar to the one found in [Rez98] based on derived functors of derivations.

Using this approach, the uniqueness obstructions for $K(n)$ lie in $E_\infty^{s,s}$ for $s \geqslant 1$. On the E_2 term, everything in positive filtration is concentrated in even total degree, so every class in $E_2^{s,s}$ for $s \geqslant 1$ is a permanent cycle. However, $E_1^{0,1}$ is large, in fact $E_1^{0,1}$ is closely related to the Morava stabilizer group, and there are potential differentials $d_r : E_r^{0,1} \to E_r^{r,r}$ for all $r \geqslant 1$ killing the uniqueness obstructions.

This should be compared to the situation for the Morava E-theory spectrum E_n. See [Rez98] for a spectral sequence which computes the space of A_∞ structures on E_n and [GH04] for a spectral sequence which computes the space of E_∞ structures on E_n. In both cases the E_2 term is trivial in positive filtration, so there is no need to compute any differentials.

The other approach, which works only if A is connective, is to study how to build A as a Postnikov tower in the category of S-algebras. For this we use a result of Dugger and Shipley [DS06] which tells us that the set of ways to build $P_m A$ from $P_{m-1}A$ as an S-algebra can be calculated using $THH_S^{m+2}(P_{m-1}A; H\pi_m A)$.

These topological Hochschild cohomology groups can be calculated when $A = k(n)$ is the connective Morava K-theory spectrum, and this lets us identify the uniqueness obstructions for building $k(n)$ as an S-algebra. Once again the obstructions are non-trivial, but something interesting happens. Each of the obstructions we found using the first approach also live in the E_2 term of the canonical spectral sequence converging to $THH_S^*(P_{m-1}k(n); H\mathbb{F}_p)$ for some m, but in every case the obstruction is killed by a differential. Hence the corresponding S-algebra structures on $P_m k(n)$ are equivalent, and this equivalence can be lifted first to $k(n)$ and then to $K(n)$.

We emphasize that both approaches are necessary to prove Theorem A. Using only the first approach is insufficient because we do not know how to calculate the differentials in the spectral sequence converging to $\pi_* B\mathscr{A}^S(K(n))$ directly. Using only the second approach is insufficient because the connective Morava K-theory spectrum $k(n)$ does *not* have a unique S-algebra structure. While the obstructions we found in the first approach are killed in the spectral sequence converging to $THH_S^*(P_{m-1}k(n); H\mathbb{F}_p)$ for suitable m, there are other uniqueness obstructions here and we do not have a direct way to show that those obstructions become trivial when inverting v_n.

1.1 Organization

In § 2 we define the moduli space of A_∞ structures on a spectrum A and construct a spectral sequence converging to the homotopy groups of this moduli space. Because we need to allow maps of A_∞ ring spectra which commute with the operad structure only up to homotopy and higher homotopies, we use a certain multicategory with r colors to define $(r-1)$-fold composites, and as a result the moduli space is (the geometric realization of) an ∞-category, regarded as a simplicial set.

In § 3 we compute the E_2 term of this spectral sequence for $K(n)$, with $\widehat{E(n)}$, MU and S as the ground ring in §§ 3.1–3.3 respectively. In the first two cases the spectral sequence collapses at the E_2 term, and in the last case there are potential differentials. Counting the classes that are left in $E_2^{s,s}$ with $\widehat{E(n)}$ or MU as the ground ring then proves Theorem B.

In § 4 we recall the theory of k-invariants for connective S-algebras, which live in topological Hochschild cohomology, due to Dugger and Shipley [DS06], and discuss the relationship with additive k-invariants.

In § 5 we compute the relevant topological Hochschild cohomology groups for Postnikov sections of connective Morava K-theory, with $BP\langle n\rangle_p$, MU and S as the ground ring in §§ 5.1–5.3 respectively. The calculation with $BP\langle n\rangle_p$ as the ground ring requires optimistic assumptions about the commutativity of the multiplication on $BP\langle n\rangle_p$; we include it because it is parallel to the situation of $K(n)$ as an $\widehat{E(n)}$-algebra and it gives a clearer conceptual picture of what is going on.

In § 6 we put the pieces together to prove Theorem A.

Finally, in § 7 we discuss the moduli space of S-algebra structures on the 2-periodic version K_n of Morava K-theory. In this case we do not have a unique S-algebra structure on K_n, but we conjecture that there are only finitely many such structures.

2. The moduli space of A_∞ structures

Recall that in a good category of spectra, such as [EKMM97], any A_∞ ring spectrum can be replaced with a weakly equivalent S-algebra. Moreover, the functor from the multicategory describing n-fold composition of A_∞ ring spectra to the multicategory describing n-fold composition of S-algebras is a weak equivalence on all Hom sets, and this implies that the moduli space of A_∞ structures on A we define below, which only depends on the homotopy type of A, is weakly equivalent to the moduli space of S-algebra structures on A.

Other approaches to studying the moduli space of A_∞ structures on a spectrum A, such as the one found in [Rez98], assumes that A comes with a fixed homotopy commutative multiplication. At $p=2$ the Morava K-theory spectrum $K(n)$ does not have a homotopy commutative multiplication [Nas02], and in any case we prefer to fix as little data as possible, so instead of following [Rez98] we will set up a similar spectral sequence based on the obstruction theory in [Ang08, Rob89].

We take an A_∞ ring spectrum to mean an algebra over the Stasheff associahedra operad $\mathcal{K}=\{K_n\}_{n\geqslant 0}$. For $0\leqslant n\leqslant\infty$ an A_n structure on X is a compatible family of maps

$$(K_m)_+\wedge X^{(m)}\to X$$

for $m\leqslant n$, where $X^{(m)}$ denotes the m-fold smash product of X with itself. If we work in the category of R-modules for some commutative S-algebra R, all smash products are over R.

Using only maps $X\to Y$ of A_n ring spectra which commute strictly with the operad action is too restrictive, so following Boardman and Vogt [BV73] we define a map of A_n ring spectra to be a family of maps $(L_m)_+\wedge X^{(m)}\to Y$ for $m\leqslant n$, where L_m is a certain polyhedron of dimensions $m-1$. Here L_m can be defined in terms of the W-construction on the multicategory (colored operad, colored PRO) with two objects 0 and 1 and $\mathrm{Hom}(\epsilon_1,\ldots,\epsilon_n;\epsilon)$ a point if $\epsilon_1+\cdots+\epsilon_n\leqslant\epsilon$ and empty otherwise, or more concretely as a certain space of metric trees with two colors. We think of $(L_m)_+\wedge X^{(m)}\to Y$ as a homotopy between the maps $(K_m)_+\wedge X^{(m)}\to X\to Y$ and $(K_m)_+\wedge X^{(m)}\to(K_m)_+\wedge Y^{(m)}\to Y$.

As observed in [BV73, ch. 4], while it is possible to 'compose' the maps we just defined, composition is not associative. Instead, we get an ∞-category (quasi-category, restricted Kan complex) of A_n ring spectra encoding the various ways of composing multiple maps, where an r-simplex is a 'composite of $r-1$ maps' defined in terms of a multicategory with r colors.

This is not actually a problem for us, because we can take the geometric realization of an ∞-category just as easily as we can take the geometric realization of (the nerve of) a category.

If R is a commutative S-algebra and A is an R-module, let $B\mathscr{A}_n^R(A)$ be the moduli space of A_n structures on A in the category of R-modules. To be precise, we let $B\mathscr{A}_n^R(A)$ be the geometric realization of the ∞-category $\mathscr{A}_n^R(A)$ defined as follows. An object (0-simplex) in $\mathscr{A}_n^R(A)$ is a pair (X, ϕ) where X is weakly equivalent to A and $\phi = \{\phi_m\}_{0 \leqslant m \leqslant n}$ is an A_n structure on X in the category of R-modules. For convenience we will assume X is cofibrant as an R-module. A morphism (1-simplex) $(X, \phi) \to (Y, \psi)$ is a map $X \to Y$ of A_n ring spectra, where the underlying map $X \to Y$ of spectra is a weak equivalence. An r-simplex is defined similarly, as in [BV73, Definition 4.7]. A choice of weak equivalence $X \to A$ is not part of the data. Some care is needed to make sure that we end up with a small (∞-)category, which we need to apply geometric realization; we refer the reader to [DK84] for one possible solution.

A general argument due to Dwyer and Kan [DK84] shows that the moduli space $B\mathscr{A}_n^R(A)$ decomposes as

$$B\mathscr{A}_n^R(A) \simeq \coprod_{[X]} B\,\mathrm{Aut}_{\mathscr{A}_n^R(A)}(X),$$

where the coproduct runs over one representative from each path component of $B\mathscr{A}_n^R(A)$ and $\mathrm{Aut}_{\mathscr{A}_n^R}(X)$ is the topological monoid of self-equivalences of a cofibrant–fibrant model for X.

In particular, an A_1 structure consists only of the unit map $R \to A$ and the identity map $A \to A$, and an automorphism of A as an A_1-algebra is a unit-preserving weak equivalence $A \to A$ of R-modules. Let $\mathrm{Aut}_R(A)_1$ denote the space of unit-preserving R-module automorphisms of a cofibrant–fibrant model of A. Then $B\mathscr{A}_1^R(A) \simeq B\,\mathrm{Aut}_R(A)_1$.

Given a tower of fibrations

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_0$$

with inverse limit X, recall [BK72, ch. IX, § 4] that we get a 'fringed' spectral sequence (called 'the (extended) homotopy spectral sequence' in *loc. cit.*)

$$E_1^{s,t} = \pi_{t-s} F_s \Longrightarrow \pi_{t-s} X,$$

where F_s is the fiber of $X_s \to X_{s-1}$. This is not quite a spectral sequence in the usual sense, for the following reasons. First, X might be empty, and the spectral sequence only exists as long as we can lift a given basepoint up the tower. The terms $E_1^{s,s+1}$ on the superdiagonal, contributing to $\pi_1 X$, are in general non-abelian, and the terms $E_1^{s,s}$ on the diagonal, contributing to $\pi_0 X$, are only sets. The fringing refers to the lack of negative dimensional terms to receive differentials.

This spectral sequence has good convergence properties, it converges completely as long as there are no $\lim^1$ terms [BK72, Lemma IX.5.4].

Also recall [Bou89] that if the tower of fibrations comes from the Tot-tower of a (simple, fibrant) cosimplicial space, the above spectral sequence has (some) negative dimensional terms. In particular, $E_1^{s,s-1}$ exists and serves both as the target of differentials from the diagonal and as the place where obstructions to lifting a basepoint up the tower lie.

In our case the nth space in the tower of fibrations will be the space $B\mathscr{A}_{n+1}^R(A)$, and although this tower does not come from a cosimplicial space we will describe sets $E_1^{s,s-1}$ containing the obstructions to lifting a basepoint up the tower. Moreover, the only non-abelian group on the superdiagonal is $E_1^{0,1}$ and while $E_1^{s,s}$ is not a group, it is a torsor over an abelian group that can be described in the same way as $E_1^{s,t}$ for $t - s \geqslant 1$.

We wish to identify the fiber of $B\mathscr{A}_{n+1}^R(A) \to B\mathscr{A}_n^R(A)$ with the space of extensions of a given A_n structure on A to an A_{n+1} structure. If $B\mathscr{A}_n^R(A)$ was the classifying space of a category we could use Quillen's Theorem B [Qui73]. Instead we use the following version, with notation from [Lur09].

LEMMA 2.1. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a map of ∞-categories with the property that for every $f : d \to d'$ in $\mathcal{D}$ the maps*

$$\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/} \xleftarrow{\simeq} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{f/} \xrightarrow{\simeq} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d'/}$$

are weak equivalences. Then the homotopy fiber of $\mathcal{C} \to \mathcal{D}$ is weakly equivalent to $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$.

Sketch proof. The homotopy fiber of $\mathcal{C} \to \mathcal{D}$ is the fiber of

$$p : \mathcal{C} \times_{\mathcal{D}} \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}.$$

The hypothesis implies that the inverse image of any 0-simplex or 1-simplex in $\mathcal{D}$ is weakly equivalent to $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$, and the case for a general simplex in $\mathcal{D}$ follows. □

Let $\bar{A}$ denote the cofiber of the unit map $R \to A$ (assuming A is cofibrant) and let $\bigvee^n A$ denote the 'fat wedge'

$$\bigvee{}^n A = \bigvee_{1 \leqslant i \leqslant n} A^{(i-1)} \wedge R \wedge A^{(n-i)}.$$

Then the canonical map $\bigvee^n A \to A^{(n)}$ is a cofibration, with cofiber $\bar{A}^{(n)}$.

Now consider the forgetful functor $F : \mathscr{A}_{n+1}^R(A) \to \mathscr{A}_n^R(A)$. Given $(X, \phi) \in \mathscr{A}_n^R(A)$ and $(Y, \psi) \in \mathscr{A}_n^R(A)_{(X,\phi)/}$, the fiber over (Y, ψ) in $\mathscr{A}_{n+1}^R(A) \times_{\mathscr{A}_n^R(A)} \mathscr{A}_n^R(A)_{(X,\phi)/}$ is the space of extensions of the A_n structure ψ on Y to an A_{n+1} structure.

An A_{n+1} structure on Y extending ψ is a map

$$m_{n+1} : (K_{n+1})_+ \wedge Y^{(n+1)} \to Y$$

satisfying two conditions. First, m_{n+1} is determined by ψ on $(\partial K_{n+1})_+ \wedge Y^{(n+1)}$, and second, m_{n+1} is determined by the unitality condition on $(K_{n+1})_+ \wedge \bigvee^{n+1} Y$.

The cofiber of the map

$$(\partial K_{n+1})_+ \wedge Y^{(n+1)} \coprod_{(\partial K_{n+1})_+ \wedge \bigvee^{n+1} Y} (K_{n+1})_+ \wedge \bigvee{}^{n+1} Y \to (K_{n+1})_+ \wedge Y^{(n+1)}$$

is $\Sigma^{n-1}\bar{Y}^{n+1}$, and hence the space of extensions of ψ to an A_{n+1} structure is weakly equivalent to $\mathrm{Hom}(\Sigma^{n-1}\bar{Y}^{(n+1)}, Y)$, which is weakly equivalent to

$$\mathrm{Hom}(\Sigma^{n-1}\bar{A}^{(n+1)}, A).$$

Similarly, given $f : (X, \phi) \to (Y, \psi)$ in $\mathscr{A}_n^R(A)$ and an element $(Z, \xi) \in \mathscr{A}_n^R(A)_{f/}$, the fiber over (Z, ξ) is the space of extensions of the A_n structure ξ on Z, and the maps in the Lemma 2.1 are clearly weak equivalences. Hence we can conclude that the fiber of $F : B\mathscr{A}_{n+1}^R(A) \to B\mathscr{A}_n^R(A)$ is the space of extensions of a given A_n structure to an A_{n+1} structure, as we wanted.

THEOREM 2.2. *There is a spectral sequence $\{E_r^{s,t}\}$ with $E_1^{s,t}$ defined for $s \geqslant 0$ and $t - s \geqslant -1$ converging to $\pi_{t-s} B\mathscr{A}^R(A)$ with the obstructions to $B\mathscr{A}^R(A)$ being non-empty on the* *subdiagonal* $t-s=-1$. *We have* $E_1^{0,-1}=\varnothing$, $E_1^{0,0}=0$, $E_1^{0,1}\cong \pi_0 \operatorname{Aut}_R(A)_1$, *and*

$$E_1^{s,t}\cong [\Sigma^{t-1}\bar{A}^{(s+1)}, A]$$

otherwise. Here $E_1^{s,t}$ *is a group for* $t-s\geqslant 1$, *a torsor over the corresponding group for* $t-s=0$, *and a set for* $t-s=-1$.

Proof. From the obstruction theory developed in [Ang08] we conclude that we get a tower of fibrations

$$B\mathscr{A}^R(A)\simeq B\mathscr{A}^R_\infty(A)\to\cdots\to B\mathscr{A}^R_2(A)\to B\mathscr{A}^R_1(A),$$

and the spectral sequence is the one associated with this tower.

The above discussion identifies $E_1^{s,t}$ for $t-s\geqslant 0$. The obstruction theory in [Ang08] also identifies the obstruction to extending an A_n structure to an A_{n+1} structure with an element in $E_1^{n,n-1}$. □

We would like to compare this to topological Hochschild cohomology, in particular to the E_2 term of the topological Hochschild cohomology spectral sequence, because that is something we can compute. Let $\{\tilde{E}_r^{p,q}\}$ be the spectral sequence with E_1 term $\tilde{E}_1^{p,q}=\pi_q F_S(\bar{A}^{(p)}, A)$ converging to $\pi_{q-p}THH_R(A)$ if A is an R-algebra so that topological Hochschild cohomology is defined.

Theorem 2.3. *Suppose* $B\mathscr{A}^R(A)$ *is non-empty, and choose an* R*-algebra structure on* A. *Then*

$$E_2^{s,t}\cong \tilde{E}_2^{s+1,t-1}$$

for $s\geqslant 2$ *and* $t-s\geqslant -1$. *This isomorphism of* E_2 *terms is an isomorphism of abelian groups for* $t-s\geqslant 1$, *of torsors for* $t-s=0$, *and of sets for* $t-s=-1$.

Proof. The E_1 terms are isomorphic for $s\geqslant 1$ and $t-s\geqslant -1$, and the argument for why the d_1 differential on $E_1^{*,*}$ is isomorphic to the Hochschild differential is contained in [Rob89] or [Ang08]. □

3. The spectral sequence for Morava K-theory

In this section we prove Theorem B by explicitly calculating the E_∞ term of the spectral sequence converging to $\pi_* B\mathscr{A}^R(K(n))$ for $R=\widehat{E(n)}$ and $R=MU$. We also calculate the E_2 term for $R=S$.

3.1 Ground ring $R=\widehat{E(n)}$

Let $R=\widehat{E(n)}$ be the $K(n)$-localization of the Johnson–Wilson spectrum, with homotopy groups

$$\widehat{E(n)}_*=\mathbb{Z}_{(p)}[v_1,\ldots,v_{n-1},v_n^{\pm 1}]^\wedge_I.$$

Here $I=(p,v_1,\ldots,v_{n-1})$ and $(-)^\wedge_I$ denotes I-completion. Then $\widehat{E(n)}$ can be given the structure of a commutative S-algebra [RW02], and $K(n)\simeq \widehat{E(n)}/I$. As in [Ang08] we find that the spectral sequence converging to $\pi_* THH_{\widehat{E(n)}}(K(n))=THH^{-*}_{\widehat{E(n)}}(K(n))$ collapses at the E_2 term (there are interesting extensions) because everything is concentrated in even total degree, with

$$\tilde{E}_2=\tilde{E}_\infty=K(n)_*[q_0,\ldots,q_{n-1}].$$

Here q_i is in filtration 1 and total homological degree $-2p^i$. There can obviously be no $\lim^1$ terms, so the spectral sequence converges completely.

This gives us the positive filtration part of the spectral sequence converging to $\pi_* B\mathscr{A}^{\widehat{E(n)}}(K(n))$. In particular, there are no obstructions to the existence of an A_∞ structure on $K(n)$, and the part contributing to $\pi_0 B\mathscr{A}^{\widehat{E(n)}}(K(n))$ is the homological degree -2 part of $K(n)_*[q_0, \ldots, q_{n-1}]$ of degree at least two in the q_i. We also know that

$$\pi_* F_{\widehat{E(n)}}(K(n), K(n)) \cong \Lambda_{K(n)_*}(Q_0, \ldots, Q_{n-1}),$$

where Q_i is the Bockstein corresponding to v_i, and the degree zero part of this is $\mathbb{F}_p$. Only one of these p maps commutes with the unit $\widehat{E(n)} \to K(n)$, so we find that $E_1^{0,1} = E_2^{0,1} = 0$. Hence there are no possible differentials originating from $E_2^{0,1}$. Everything in positive filtration is concentrated in even total degree, so the spectral sequence collapses at the E_2 term with infinitely many classes on the diagonal. This proves Theorem B for $R = \widehat{E(n)}$.

3.2 Ground ring $R = MU$

A similar argument shows that $K(n)$ has uncountably many MU-algebra structures. We first consider the connective Morava K-theory spectrum $k(n)$ with $k(n)_* = \mathbb{F}_p[v_n]$. We choose x_i such that $MU_* = \mathbb{Z}[x_1, x_2, \ldots]$ and

$$k(n) = MU/(p, x_1, \ldots, x_{p^n-2}, x_{p^n}, \ldots).$$

We can also choose these generators in such a way that x_{p^i-1} maps to v_i for $0 \leqslant i \leqslant n$ and x_j maps to 0 otherwise, under a suitable map $MU \to \widehat{E(n)}$ (which can be chosen to be H_∞, although it is an open question whether or not it can be chosen to be E_∞).

In this case, $E_1^{0,1}$ is non-trivial, but not large enough to kill all the obstructions. To be more precise, the E_2 term for topological Hochschild cohomology of $k(n)$ with ground ring MU looks like

$$\tilde{E}_2^{*,*} = k(n)_*[\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_{p^n-2}, \tilde{q}_{p^n}, \ldots]$$

with $\tilde{q}_i$ in filtration 1 and total homological degree $-2i-2$.

The term $E_1^{0,1}$ consists of infinite sums $1 + \sum v_I Q_I$, where $v_I \in k(n)_*$ is in the appropriate degree and $Q_I = Q_{i_1} \cdots Q_{i_k}$ is a product of Bocksteins. Here Q_i is the Bockstein corresponding to x_i in MU_*, or to $\tilde{q}_i$ in $\tilde{E}_2^{*,*}$.

Similarly, the term $E_1^{1,1}$ consists of infinite sums $1 \wedge 1 + \sum v_{IJ} Q_I \wedge Q_J$. The d_1 differential $d_1 : E_1^{0,1} \to E_1^{1,1}$ is given by

$$d_1(v_{ij} Q_i Q_j) = v_{ij} Q_i \wedge Q_j - v_{ij} Q_j \wedge Q_i,$$

and more generally $d_1(v_I Q_I)$ is given by the sum of all ways to write $I = J \cup K$ of $\pm v_I Q_J \wedge Q_K$. In particular, d_1 is injective, so $E_2^{0,1} = 0$ is trivial.

We also find that $v_{ij} Q_i \wedge Q_j = v_{ij} Q_j \wedge Q_i$ in $E_2^{1,1}$, and, as in [Ang08, Theorem 3.9], the kernel of $d_1 : E_1^{1,1} \to E_1^{2,1}$ picks out the homotopy associative multiplications, and this identifies $E_2^{1,1}$ with $\tilde{E}_2^{2,0}$. Again there can be no $\lim^1$ terms, so the spectral sequence converges completely. This gives a complete description of all the A_∞ structures on $k(n)$ as an MU-module. We get the same result for $K(n)$.

LEMMA 3.1. *The canonical map $B\mathscr{A}^{MU}(k(n)) \to B\mathscr{A}^{MU}(K(n))$ is a weak equivalence.*

Proof. This is clear because

$$\tilde{E}_2^{*,*}(K(n)) \cong v_n^{-1} \tilde{E}_2^{*,*}(k(n)),$$

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and these groups are isomorphic in the degrees contributing to $E_2^{*,*}(k(n))$ and $E_2^{*,*}(K(n))$, and the same holds for $E_2^{0,*}$. □

This proves Theorem B for $R=MU$. If BP is a commutative S-algebra then the same argument shows that $K(n)$ has uncountably many BP-algebra structures.

3.3 Ground ring $R=S$

By [Ang08], we have an equivalence $THH_{\widehat{E(n)}}(K(n)) \to THH_S(K(n))$ (which is visible on $\tilde{E}_2$), and this shows that the E_2 term of the spectral sequence converging to $\pi_* B\mathscr{A}^S(K(n))$ is isomorphic to the E_2 term of the spectral sequence converging to $\pi_* B\mathscr{A}^{\widehat{E(n)}}(K(n))$ in filtration $s \geqslant 2$. If p is odd this also gives an isomorphism in filtration $s=1$; if $p=2$ there is a possible differential $d_1 : E_1^{0,1} \to E_1^{1,1}$ killing the class $v_n q_{n-1}^2$.

As in [Rav04], let

$$\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_* \cong K(n)_*[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i)$$

be the nth Morava stabilizer algebra. Here $|t_i| = 2(p^i - 1)$. Recall [Nas02] that, for any choice of multiplication on $K(n)$, we have

$$K(n)_*K(n) \cong \Sigma(n) \otimes \Lambda(\alpha_0, \ldots, \alpha_{n-1})$$

as a ring for p odd, while $\alpha_i^2 = t_{i+1}$ for $0 \leqslant i \leqslant n-2$ and $\alpha_{n-1}^2 = t_n + v_n$ for $p=2$.[1] Here $|\alpha_i| = 2p^i - 1$.

Also recall that if we consider $K(n)_*K(n)^{\mathrm{op}}$ instead we get the same result except that we get to replace the relation $\alpha_{n-1}^2 = t_n + v_n$ by $\alpha_{n-1}^2 = t_n$ at $p=2$.[2]

We have that

$$K(n)^*K(n) \cong \mathrm{Hom}_{K(n)_*}(K(n)_*K(n), K(n)_*) \cong \mathrm{Hom}_{K(n)_*}(\Sigma(n), K(n)_*) \otimes \Lambda(Q_0, \ldots, Q_{n-1}),$$

where Q_i is the Bockstein dual to α_i. In particular, this means that

$$E_1^{0,1} \cong [\mathrm{Hom}_{K(n)_*}(\Sigma(n), K(n)_*) \otimes \Lambda(Q_0, \ldots, Q_{n-1})]_1^{\times},$$

which is large enough to potentially kill all the uniqueness obstructions.

Again there can be no $\lim^1$ terms, so the spectral sequence converges completely. This is clear in positive filtration; for the groups in filtration 0 this relies on observing that $E_1^{0,t}$ is p-torsion.

At $p=2$, a result by Nassau [Nas02] gives us our first differential. He shows that if ϕ is one multiplication (A_2 structure) on $K(n)$ and ϕ^{op} is the other, then the automorphism Ξ of $K(n)$ given by $t_n \mapsto v_n$ is an antiautomorphism of the multiplication. Hence ϕ and ϕ^{op} are in the same path component in $B\mathscr{A}_2^S(K(n))$. The difference $\phi - \phi^{\mathrm{op}}$ is represented by $v_n q_{n-1}^2$, so $d_1(\Xi) = v_n q_{n-1}^2$.

4. S-algebra k-invariants

For connective spectra we can build the S-algebra structure by induction on the Postnikov sections. Given a connective spectrum A, let $P_m A$ denote the Postnikov section of A with homotopy groups only up to (and including) degree m. If R is a connective commutative S-algebra

[1] If the reader prefers a unified description of $K(n)_*K(n)$ at all primes it is the above ring with p-fold Massey products (2-fold Massey products being products) $\langle \alpha_i, \ldots, \alpha_i \rangle = t_{i+1}$ for $0 \leqslant i \leqslant n-2$ and $\langle \alpha_{n-1}, \ldots, \alpha_{n-1} \rangle = t_n + v_n$ with no indeterminacy.

[2] Or replace $\langle \alpha_{n-1}, \ldots, \alpha_{n-1} \rangle = t_n + v_n$ with $\langle \alpha_{n-1}, \ldots, \alpha_{n-1} \rangle = v_n$ at any prime.

then Postnikov sections can be defined in the category of R-algebras, so if A is an R-algebra then this gives an R-algebra structure on P_mA as well. Conversely, $B\mathscr{A}^R(A) = \varprojlim B\mathscr{A}^R(P_mA)$, so we can understand $B\mathscr{A}^R(A)$ by understanding $B\mathscr{A}^R(P_mA)$ for all m.

A theory of k-invariants for connective R-algebras has been developed by Dugger and Shipley [DS06]. Suppose C is an R-algebra with homotopy groups only up to degree $m-1$, and suppose M is a π_0C module. Let $\mathcal{M}(C,(M,m))$ be the category of Postnikov extensions of C of type (M,m). The objects are R-algebras Y together with a map $Y \to C$ satisfying $\pi_iY = 0$ for $i > m$, $\pi_mY \cong M$ and $P_{m-1}Y \simeq C$. The morphism are maps over C inducing an isomorphism on homotopy.

THEOREM 4.1 (Dugger and Shipley [DS06, Proposition 1.5]). *With $\mathcal{M}(C,(M,n))$ as above,*

$$\pi_0\mathcal{M}(C,(M,m)) \cong THH_R^{m+2}(C;M)/\mathrm{Aut}(M).$$

Now suppose $C = P_{m-1}A$, $M = \pi_mA$, and we want to make sure that $Y \in \mathcal{M}(C,(M,m))$ has the homotopy type of P_mA. Then Y has to have the correct additive k-invariant, which is a map $C \to \Sigma^{m+1}HM$. Recall that the topological Hochschild cohomology spectral sequence converging to $THH_R^*(C;M)$ has $\tilde{E}_1^{s,t} = [\Sigma^tC^{(s)}, M]$, contributing to $\pi_{t-s}THH_R(C;M) = THH_R^{s-t}(C;M)$. In particular, the additive k-invariant of Y is an element in $\tilde{E}_1^{1,-m-1}$, contributing to $THH_R^{m+2}(C;M)$.

If the additive k-invariant k_m is trivial then $Y \simeq C \vee \Sigma^m HM$ as a spectrum, and Y always has at least one S-algebra structure, namely the square zero extension. If k_m is non-trivial, it might or might not survive the topological Hochschild cohomology spectral sequence. If $d_r(k_m) = y \neq 0$ then y represents the obstruction to extending the S-algebra structure on C to an S-algebra structure on Y. If k_m survives then Y has at least one S-algebra structure.

5. k-invariants for Morava K-theory

Again we study the moduli problem over each ground ring separately. First we use $BP\langle n\rangle_p$, which has homotopy groups

$$(BP\langle n\rangle_p)_* = \mathbb{Z}_p[v_1, \ldots, v_n]$$

and is the appropriate connective version of $\widehat{E(n)}$, as the ground ring, assuming it can be given the structure of a commutative S-algebra. Then we use MU, and finally we use the sphere spectrum S. First we recall the following change-of-rings result.

LEMMA 5.1 [AHL10, Corollary 2.5]. *Suppose $A \to B$ is a map of S-algebras and M is an A–B-bimodule, given an A–A-bimodule structure by pullback. Then there is a spectral sequence*

$$\tilde{E}_2^{*,*} = \mathrm{Ext}^{**}_{\pi_*B\wedge_R A}(B_*, M_*) \Longrightarrow \pi_*THH_R(A;M).$$

In particular, when $B = M = H\mathbb{F}_p$ we get a spectral sequence

$$\tilde{E}_2^{*,*} = \mathrm{Ext}_{H_*^R(A;\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*THH_R(A;H\mathbb{F}_p),$$

where $H_*^R(A;\mathbb{F}_p)$ denotes $\pi_*A \wedge_R H\mathbb{F}_p$.

Since $k(n)$ has homotopy in degrees that are multiples of $2p^n-2$, let $q = 2p^n-2$. Each additive k-invariant $k_m \in H_R^{mq+1}(P_{(m-1)q}k(n);\mathbb{F}_p)$ is non-trivial; this follows by considering $H_*(P_{mq}k(n);\mathbb{F}_p)$, which is different from $H_*(P_{(m-1)q}k(n) \vee \Sigma^{mq}H\mathbb{F}_p;\mathbb{F}_p)$.

5.1 Ground ring $BP\langle n\rangle_p$

Since we are planning to use Lemma 5.1, we start by calculating the $BP\langle n\rangle_p$-homology of the Postnikov sections of $k(n)$.

PROPOSITION 5.2. *The $BP\langle n\rangle_p$-homology of $H\mathbb{F}_p$, $P_{mq}k(n)$ and $k(n)$ is as follows:*

(i) $H_*^{BP\langle n\rangle_p}(H\mathbb{F}_p;\mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(\alpha_0,\ldots,\alpha_n)$;

(ii) $H_*^{BP\langle n\rangle_p}(P_{mq}k(n);\mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(\alpha_0,\ldots,\alpha_{n-1},a_{m+1})$;

(iii) $H_*^{BP\langle n\rangle_p}(k(n);\mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(\alpha_0,\ldots,\alpha_{n-1})$.

Here α_i is in degree $2p^i-1$ and a_{m+1} is in degree $(m+1)q+1$, $a_1=\alpha_n$.

Proof. This is clear, using the fact that we can write

$$\begin{aligned} H\mathbb{F}_p &= BP\langle n\rangle_p/(p,v_1,\ldots,v_n),\\ P_{mq}k(n) &= BP\langle n\rangle_p/(p,v_1,\ldots,v_{n-1},v_n^{m+1}),\\ k(n) &= BP\langle n\rangle/(p,v_1,\ldots,v_{n-1}). \end{aligned}$$ □

PROPOSITION 5.3. *Assuming that $BP\langle n\rangle_p$ is a commutative S-algebra, topological Hochschild cohomology of $H\mathbb{F}_p$, $P_{mq}k(n)$ and $k(n)$ over $BP\langle n\rangle_p$ with coefficients in $H\mathbb{F}_p$ is as follows:*

(i) $THH^*_{BP\langle n\rangle_p}(H\mathbb{F}_p;H\mathbb{F}_p)\cong\mathbb{F}_p[q_0,\ldots,q_n]$;

(ii) $THH^*_{BP\langle n\rangle_p}(P_{mq}k(n);H\mathbb{F}_p)\cong\mathbb{F}_p[q_0,\ldots,q_{n-1},b_{m+1}]$;

(iii) $THH^*_{BP\langle n\rangle_p}(k(n);H\mathbb{F}_p)\cong\mathbb{F}_p[q_0,\ldots,q_{n-1}]$.

Here q_i is in cohomological degree $2p^i$ and b_{m+1} is in degree $(m+1)q+2$, $b_1=q_n$.

Proof. We use Lemma 5.1. In each case there can be no differentials, because the E_2 term is concentrated in even total degree. □

The additive k-invariant of $k(n)$ dictates that we choose the k-invariant in

$$THH^{mq+2}_{BP\langle n\rangle_p}(P_{(m-1)q}k(n);H\mathbb{F}_p)$$

as $b_m+f(q_0,\ldots,q_{n-1})$ where f has degree at least two in the q_i.

Next we compare this with the moduli space of $\widehat{E(n)}$-algebra structures on $K(n)$.

LEMMA 5.4. *Assuming that $BP\langle n\rangle_p$ is a commutative S-algebra, the canonical maps*

$$B\mathscr{A}^{BP\langle n\rangle_p}(k(n))\to B\mathscr{A}^{BP\langle n\rangle_p}(K(n))\to B\mathscr{A}^{\widehat{E(n)}}(K(n))$$

are weak equivalences.

Proof. This is similar to the proof of Lemma 3.1. □

Now we can compare the two methods of studying the set of equivalence classes of $BP\langle n\rangle_p$-algebra structures on $k(n)$. We find that in the spectral sequence converging to $\pi_*B\mathscr{A}^{BP\langle n\rangle_p}(k(n))$, each uniqueness obstruction is represented by a class

$$v_n^m f(q_0,\ldots,q_{n-1})$$

for some $m\geqslant 1$, where $f(q_0,\ldots,q_{n-1})$ has homological degree $-mq-2$. If $f(q_0,\ldots,q_{n-1})=q_{i_1}\cdots q_{i_j}$ has degree j in the q_i this represents changing the A_j structure by the map

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$\Sigma^{j-2}k(n)^{(j)} \to k(n)$ given by first applying

$$Q_f = Q_{i_1} \wedge \cdots \wedge Q_{i_j}$$

and then multiplying the factors and multiplying by v_n^m.

On the other hand, we can interpret the polynomial $f(q_0, \ldots, q_{n-1})$ as being an element of $THH^{mq+2}(P_{(m-1)q}k(n); H\mathbb{F}_p)$, represented in the topological Hochschild cohomology spectral sequence by the composite

$$(P_{(m-1)q}k(n))^{(j)} \xrightarrow{Q_f} \Sigma^{mq-j+2}(P_{(m-1)q}k(n))^{(j)} \to \Sigma^{mq-j+2}H\mathbb{F}_p.$$

Lemma 5.5. *Given a uniqueness obstruction $v_n^m f(q_0, \ldots, q_{n-1})$ of degree j in the q_i represented by $v_n^m Q_f : \Sigma^{j-2}k(n)^{(j)} \to k(n)$, we get a commutative diagram as follows.*

$$\begin{array}{ccccc}
 & \nearrow & \Sigma^{j-2}(P_{(m-1)q}k(n))^{(j)} & \xrightarrow{Q_f} & \Sigma^{mq}H\mathbb{F}_p \\
 & & & & \downarrow \\
\Sigma^{j-2}k(n)^{(j)} & \xrightarrow{v_n^m Q_f} & k(n) & \longrightarrow & P_{mq}k(n) \\
 & & & & \downarrow \\
 & & & & P_{(m-1)q}k(n)
\end{array}$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccc}
\Sigma^{j-2}(P_{mq}k(n))^{(j)} & \xrightarrow{Q_f} & \Sigma^{mq}P_{mq}k(n) & \xrightarrow{v_n^m} & P_{mq}k(n) \\
\downarrow & & \downarrow & \nearrow & \downarrow \\
\Sigma^{j-2}(P_{(m-1)q}k(n))^{(j)} & \xrightarrow{Q_f} & \Sigma^{mq}P_{(m-1)q}k(n) & \xrightarrow{v_n^m=0} & P_{(m-1)q}k(n)
\end{array}$$

This gives us a map $\Sigma^{j-2}(P_{(m-1)q}k(n))^{(j)} \to P_{mq}k(n)$, and this map is trivial on $P_{(m-1)q}k(n)$ so it factors through $\Sigma^{mq}H\mathbb{F}_p$. $\square$

The upshot of this is that we can translate from obstructions in the spectral sequence converging to $\pi_* B\mathscr{A}^{BP\langle n\rangle}(k(n))$, which by Lemma 5.4 and the equivalence between topological Hochschild cohomology over $\widehat{E(n)}$ and S are the obstructions in the spectral sequence converging to $\pi_* B\mathscr{A}^S(K(n))$ to obstructions in the spectral sequence converging to $THH^*_{BP\langle n\rangle}(P_{(m-1)q}k(n); H\mathbb{F}_p)$.

5.2 Ground ring MU

Next we do the same with MU as the ground ring. If we knew that $BP\langle n\rangle_p$ was a commutative S-algebra then this section would not be necessary. The corresponding results are as follows.

Proposition 5.6. *The MU-homology of $H\mathbb{F}_p$, $P_{mq}k(n)$ and $k(n)$ is as follows:*

(i) $H_*^{MU}(H\mathbb{F}_p; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots)$;

(ii) $H_*^{MU}(P_{mq}k(n); \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\tilde{\alpha}_i : i \neq p^n - 1, a_{m+1})$;

(iii) $H_*^{MU}(k(n); \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\tilde{\alpha}_i : i \neq p^n - 1)$.

Here $\tilde{\alpha}_i$ is in degree $2i+1$ and a_{m+1} is in degree $(m+1)q+1$, $a_1 = \tilde{\alpha}_{p^n-1}$.

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Proof. This is clear, using the fact that we can write

$$\begin{aligned} H\mathbb{F}_p &= MU/(p, x_1, \ldots, x_{p^n-2}, x_{p^n-1}, x_{p^n}, \ldots), \\ P_{mq}k(n) &= MU/(p, x_1, \ldots, x_{p^n-2}, x_{p^n-1}^m, x_{p^n}, \ldots), \\ k(n) &= MU/(p, x_1, \ldots, x_{p^n-2}, x_{p^n}, \ldots). \end{aligned}$$

□

Now we can calculate topological Hochschild cohomology.

PROPOSITION 5.7. *Topological Hochschild cohomology of $\mathbb{F}_p$, $P_{mq}k(n)$ and $k(n)$ over MU with coefficients in $H\mathbb{F}_p$ is as follows:*

(i) $THH^*_{MU}(H\mathbb{F}_p; H\mathbb{F}_p) \cong \mathbb{F}_p[\tilde{q}_0, \tilde{q}_1, \ldots]$;

(ii) $THH^*_{MU}(P_{mq}k(n); H\mathbb{F}_p) \cong \mathbb{F}_p[\tilde{q}_i : i \neq p^n - 1, b_{m+1}]$;

(iii) $THH^*_{MU}(k(n); H\mathbb{F}_p) \cong \mathbb{F}_p[\tilde{q}_i : i \neq p^n - 1]$.

Here $\tilde{q}_i$ is in cohomological degree $2i+2$ and b_{m+1} is in degree $(m+1)q+2$, $b_1 = \tilde{q}_{p^n-1}$.

Proof. Again this follows from Lemma 5.1. □

Recall from Lemma 3.1 that $B\mathscr{A}^{MU}(k(n)) \to B\mathscr{A}^{MU}(K(n))$ is a weak equivalence. Just as with $BP\langle n\rangle_p$ as the ground ring, we can translate from obstructions in the spectral sequence converging to $B\mathscr{A}^S(K(n))$ to obstructions in the spectral sequence converging to $THH^*_{MU}(P_{(m-1)q}k(n); H\mathbb{F}_p)$. In this case, only $\mathbb{F}_p[\tilde{q}_0, \tilde{q}_{p-1}, \ldots, \tilde{q}_{p^{n-1}-1}]$ correspond to obstructions in the spectral sequence converging to $\pi_* B\mathscr{A}^S(K(n))$.

By this we mean that the MU-algebra k-invariant for building $P_{mq}k(n)$ from $P_{(m-1)q}k(n)$ lives in $THH^{mq+2}(P_{(m-1)q}k(n); H\mathbb{F}_p)$ and looks like $b_m + f(\tilde{q}_i : i \neq p^n - 1)$ where f has degree at least two in the $\tilde{q}_i$. This corresponds to the uniqueness obstruction $v_n^m f(\tilde{q}_i : i \neq p^n - 1)$ in the E_2 term of the spectral sequence converging to $\pi_* B\mathscr{A}^{MU}(K(n))$, and the canonical map $B\mathscr{A}^{MU}(K(n)) \to B\mathscr{A}^S(K(n))$ induces a map on E_2 terms, under which $\tilde{q}_{p^i-1}$ maps to q_i. This is clear, because both $\tilde{q}_{p^i-1}$ and q_i are represented by the Bockstein corresponding to v_i.

5.3 Ground ring S

Finally, we do the same with S as the ground ring. Let $\bar{A}_*$ denote the dual Steenrod algebra with $\bar{\tau}_n$ missing, or with $\bar{\xi}_{n+1}$ missing but with $\bar{\xi}_{n+1}^2$ present if $p = 2$. In the following we will state all results at odd primes and leave the standard modifications, replacing $\bar{\tau}_i$ with $\bar{\xi}_i$ and $\bar{\xi}_i$ with $\bar{\xi}_i^2$ at $p = 2$ to the reader.

PROPOSITION 5.8. *The* mod p *homology of $\mathbb{F}_p$, $P_{mq}k(n)$ and $k(n)$ is as follows:*

(i) $H_*(\mathbb{F}_p; \mathbb{F}_p) \cong A_*$;

(ii) $H_*(P_{mq}k(n); \mathbb{F}_p) \cong \bar{A}_* \otimes \Lambda_{\mathbb{F}_p}(a_{m+1})$;

(iii) $H_*(k(n); \mathbb{F}_p) \cong \bar{A}_*$.

Here a_{m+1} is in degree $(m+1)q+1$, $a_1 = \bar{\tau}_n$.

Proof. Only part (2) is not well-known. Consider the long exact sequence obtained by taking the mod p homology of the (co)fiber sequence

$$\Sigma^{mq} H\mathbb{F}_p \to P_{mq}k(n) \to P_{(m-1)q}k(n) \to \Sigma^{mq+1} H\mathbb{F}_p.$$

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By induction, $H_*(P_{(m-1)q}k(n);\mathbb{F}_p) \cong \bar{A}_* \otimes \Lambda_{\mathbb{F}_p}(a_m)$, and the map to $H_*(\Sigma^{mq+1}H\mathbb{F}_p;\mathbb{F}_p)$ is determined by being $\bar{A}_*$-linear and sending 1 to 0 and a_m to $\Sigma^{mq+1}q$. The result follows by combining the kernel and cokernel of this map. □

THEOREM 5.9. *Topological Hochschild cohomology of $H\mathbb{F}_p$, $P_{mq}k(n)$ and $k(n)$ with coefficients in $H\mathbb{F}_p$ is as follows:*

(i) $THH_S^*(H\mathbb{F}_p;H\mathbb{F}_p) \cong P_p(\delta\bar{\tau}_0,\delta\bar{\tau}_1,\ldots)$;

(ii) $THH_S^*(P_{mq}k(n);H\mathbb{F}_p) \cong \Lambda(\delta\bar{\xi}_{n+1}) \otimes P_p(\delta\bar{\tau}_i : i \neq n) \otimes \mathbb{F}_p[b_{m+1}]$;

(iii) $THH_S^*(k(n);H\mathbb{F}_p) \cong \Lambda(\delta\bar{\xi}_{n+1}) \otimes P_p(\delta\bar{\tau}_i : i \neq n)$.

Proof. Part (i) is dual to Bökstedt's original calculation of topological Hochschild homology of $\mathbb{F}_p$ [Bok]. For part (ii), consider the spectral sequence

$$E_2 = \Lambda(\delta\bar{\xi}_i : i \geqslant 1) \otimes \mathbb{F}_p[\delta\bar{\tau}_i : i \neq n] \otimes \mathbb{F}_p[b_{m+1}] \Longrightarrow THH_S^*(P_{mq}k(n);\mathbb{F}_p)$$

from Lemma 5.1. The map $P_{mq}k(n) \to H\mathbb{F}_p$ induces a map on topological Hochschild cohomology in the opposite direction, inducing differentials $d_{p-1}(\delta\bar{\xi}_{i+1}) = (\delta\bar{\tau}_i)^p$ for $i \neq n$.

The class b_{m+1} is the next additive k-invariant for $k(n)$, and because we know that $k(n)$ can be given an S-algebra structure, b_{m+1} has to survive the spectral sequence. The class $\delta\bar{\xi}_{n+1}$ survives for degree reasons, so each generator is a permanent cycle. Using the multiplicative structure, the spectral sequence collapses at the E_p term and part (ii) of the theorem follows. Part (iii) is similar. □

We note that the pth powers of $\delta\bar{\tau}_i$ for $0 \leqslant i \leqslant n-1$ all die, and we make the following simple but crucial observation.

LEMMA 5.10. *Consider the S-algebra k-invariant for $k(n)$ in $THH_S^{mq+2}(P_{(m-1)q}k(n);\mathbb{F}_p)$. There are no polynomials $f(\delta\bar{\tau}_0,\ldots,\delta\bar{\tau}_{n-1}) \in P_p(\delta\bar{\tau}_0,\ldots,\delta\bar{\tau}_{n-1})$ in this degree.*

Proof. This is clear because the element in highest degree is $(\delta\bar{\tau}_0)^{p-1}\cdots(\delta\bar{\tau}_{n-1})^{p-1}$ in degree $2p^n-2$, which is less than $mq+2$. □

Of course the generators $\delta\bar{\tau}_i$ are related to the generators q_i and $\tilde{q}_j$ from the previous sections.

LEMMA 5.11. *The canonical map*

$$THH_{MU}^*(P_{mq}k(n);\mathbb{F}_p) \to THH_S^*(P_{mq}k(n);\mathbb{F}_p)$$

maps $\tilde{q}_j$ to $\delta\bar{\tau}_i$ if $p^i-1=j$ and 0 otherwise.

Similarly, if $BP\langle n\rangle_p$ is a commutative S-algebra, the canonical map

$$THH_{BP\langle n\rangle_p}^*(P_{mq}k(n);\mathbb{F}_p) \to THH_S^*(P_{mq}k(n);\mathbb{F}_p)$$

maps q_i to $\delta\bar{\tau}_i$.

Proof. This follows by the description of all of the E_2 terms in terms of Bocksteins. □

6. Proof of Theorem A

We are now in a position to prove Theorem A. As we have seen, each uniqueness obstruction looks like $v_n^m f(q_0,\ldots,q_{n-1})$ for some $m \geqslant 1$ and monomial $f(q_0,\ldots,q_{n-1})$, and we can find

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these uniqueness obstructions in the corresponding topological Hochschild cohomology spectral sequence converging to $THH^*_{MU}(P_{(m-1)q}k(n); H\mathbb{F}_p)$.

In the corresponding spectral sequence converging to $THH^*_S(P_{(m-1)q}k(n); \mathbb{F}_p)$, $f(q_0, \ldots, q_{n-1})$ is killed by a differential, which means that the corresponding S-algebra structures on $P_{mq}k(n)$ are equivalent. By considering the pullback square

$$\begin{array}{ccc} PB \simeq k(n) & \longrightarrow & k(n) \\ \downarrow & & \downarrow \\ P_{mq}k(n) & \longrightarrow & P_{mq}k(n) \end{array}$$

of S-algebras, we see that the equivalence can be lifted to $k(n)$. Now we can invert v_n by $K(n)$-localizing, so this gives an equivalence between the corresponding S-algebra structures on $K(n)$ as well.

We claim that this is enough to conclude that the obstructions are also killed in the spectral sequence converging to $\pi_* B\mathscr{A}^S(K(n))$. To see this, consider $k(n)$ and $K(n)$ as MU-modules, and consider the following commutative diagram.

$$\begin{array}{ccc} B\mathscr{A}^{MU}(k(n)) & \longrightarrow & B\mathscr{A}^{S}(k(n)) \\ \downarrow \simeq & & \downarrow \\ B\mathscr{A}^{MU}(K(n)) & \longrightarrow & B\mathscr{A}^{S}(K(n)) \end{array}$$

We showed in Lemma 3.1 that $B\mathscr{A}^{MU}(k(n)) \to B\mathscr{A}^{MU}(K(n))$ is a weak equivalence, and we understand the E_2 terms of the spectral sequences converging to the homotopy groups of all the spaces in the diagram except for $B\mathscr{A}^S(k(n))$. The spectral sequences converging to $\pi_* B\mathscr{A}^{MU}(k(n))$ and $\pi_* B\mathscr{A}^{MU}(K(n))$ collapse, and from the E_2 terms we can read off that the map $\pi_0 B\mathscr{A}^{MU}(k(n)) \to \pi_0 B\mathscr{A}^S(K(n))$ is surjective.

In $\pi_0 B\mathscr{A}^{MU}(k(n))$, there are classes that map surjectively onto the E_2 term of the spectral sequence converging to $\pi_* B\mathscr{A}^S(K(n))$ which are all hit by differentials in the spectral sequence converging to $THH^*_S(P_{(m-1)q}k(n); H\mathbb{F}_p)$ for some m (Lemmas 5.10 and 5.11); hence the same must happen in the spectral sequence converging to $\pi_* B\mathscr{A}^S(K(n))$.

Our argument would be simplified by the existence of a commutative S-algebra structure on $BP\langle n\rangle_p$, in which case it follows that all the uniqueness obstructions for building $k(n)$ as a $BP\langle n\rangle_p$-algebra are hit by differentials in the spectral sequence converging to $THH^*_S(P_{(m-1)q}k(n); H\mathbb{F}_p)$ for some m. In particular, when $n=1$ using ℓ_p instead of MU gives a simpler argument.

7. 2-periodic Morava K-theory

There is a 2-periodic version of Morava K-theory, given by

$$K_n = E_n/(p, u_1, \ldots, u_{n-1}),$$

where E_n is the Morava E-theory spectrum associated to a formal group of height n over a perfect field k of characteristic p. The spectrum E_n is a commutative S-algebra [GH04], and K_n has homotopy groups

$$(K_n)_* \cong k[u, u^{-1}]$$

with $|u| = 2$. We can also ask about the space of S-algebra structures on K_n. When $p = 2$ and $n = 1$, $K_n = K(n)$; if $p > 2$ or $n > 1$ the current author [Ang08] found that $THH_S(K_n)$ varies over the moduli space of S-algebra structures on K_n, so there can be no unique S-algebra structure on K_n.

CONJECTURE 7.1. There are only finitely many S-algebra structures on K_n, in the sense that the moduli space of S-algebra structures on K_n has finitely many components.

Outline of possible proof. The spectral sequence converging to $\pi_* B\mathscr{A}^S(K_n)$ is very similar to the one converging to $\pi_* B\mathscr{A}^S(K(n))$, but now each of the n polynomial generators are in degree -2 instead of degree $-2p^i$ for $0 \leqslant i \leqslant n-1$.

If we try to build the connective version k_n using its Postnikov tower we need to understand the topological Hochschild cohomology spectral sequence. Since $H_*(k_n; \mathbb{F}_p) \cong H_*(k(n); \mathbb{F}_p) \otimes P_{p^n-1}(u)$, and similarly for the Postnikov sections, we get some extra classes in the E_2 term. Assuming that these classes are permanent cycles, we find that we have more choices than before. To build $P_2 k_n$ from Hk, we need a class in

$$THH_S^4(Hk; Hk),$$

and for p odd we are free to choose $(\delta\bar{\tau}_0)^2$. If $p = 2$ and $n > 1$ we can choose $\delta\bar{\xi}_2$. In each case this corresponds to a non-commutative multiplication. Next, to build $P_4 k_n$ from $P_2 k_n$ we need a class in $THH_S^6(P_2 k_n; Hk)$. If $p > 3$ we can choose the class we need for u to square to something non-trivial plus $(\delta\bar{\tau}_0)^3$; if $n \geqslant 2$ and $p = 2$ or $p = 3$ there are similar choices.

However, assuming that the additional classes do not change the behavior of the spectral sequence, the pth powers of $\delta\bar{\tau}_0, \ldots, \delta\bar{\tau}_{n-1}$ still die, so for m sufficiently large there are no such classes in $THH_S^{2m+2}(P_{2m-2} k_n; Hk)$. □

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