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On critical exponents of a k-Hessian equation in the whole space

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In this paper, we study negative classical solutions and stable solutions of the following k-Hessian equation

$$F_k(D^2V) = (-V)^p \quad \text{in} \quad R^n$$

with radial structure, where $n \geqslant 3$, $1 < k < n/2$ and $p > 1$. This equation is related to the extremal functions of the Hessian Sobolev inequality on the whole space. Several critical exponents including the Serrin type, the Sobolev type, and the Joseph-Lundgren type, play key roles in studying existence and decay rates. We believe that these critical exponents still come into play to research k-Hessian equations without radial structure.

Keywords: k-Hessian equation; stable solution; critical exponent; Liouville theorem; decay rate

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1. Introduction

In 1990, Tso [**34**] studied the relation between the value of exponent p and the existence results for the k-Hessian equation $F_k(D^2V) = (-V)^p$ in bounded domains. The critical exponent $p = (((n+2)k)/(n-2k))$ plays a key role. Those results are associated with the extremal functions of the Hessian Sobolev inequality for all k-admissible functions which was introduced by Wang in [**37**]. Such an inequality with the critical exponent still holds in the whole space R^n, and the extremal functions are radially symmetric (cf. [**6**], [**33**]).

Consider the Euler–Lagrange equation

$$F_k(D^2V) = (-V)^p, \quad V < 0 \quad \text{in } R^n, \tag{1.1}$$

with a general exponent $p > 1$, where $n \geqslant 3$, $1 < k < n/2$. Here $F_k[D^2V] = S_k(\lambda(D^2V))$, $\lambda(D^2V) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with λ_i being eigenvalues of the Hessian matrix $(D^2 V)$, and $S_k(\cdot)$ is the k-th symmetric function:

$$S_k(\lambda) = \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

According to the conclusions in [2], we always consider the k-admissible solutions in the following cone to ensure that the main part of (1.1) is elliptic

$$\Phi^k := \{u \in C^2(R^n); F_s(D^2u) \geqslant 0, s = 1, 2, \ldots, k\}.$$

Such an equation does not only come into play to study the extremal functions of the Hessian Sobolev inequality, but also is helpful to investigate the global existence and blow-up in finite time span for the fully nonlinear parabolic equations (such as the equations studied in [18, 30, 38]).

A special case is $F_1[D^2V] = \Delta V$, and (1.1) becomes the Lane–Emden equation

$$-\Delta u = u^p, \quad u > 0 \quad \text{in } R^n. \tag{1.2}$$

The existence results of the solutions of this equation have provided an important ingredient in the study of conformal geometry, such as the extremal functions of the Sobolev inequalities and the prescribing scalar curvature problem. It was studied rather extensively. According to theorem 3.41 in [27], (1.2) has no positive solution even on exterior domains when p is not larger than the Serrin exponent (i.e. $p \in (1, ((n)/(n-2)))$). The Liouville theorem in [15] shows that (1.2) has no positive classical solution in the subcritical case (i.e. $p \in [1, ((n+2)/(n-2)))$). In the critical case (i.e. $p = ((n+2)/(n-2))$), the positive classical solutions of (1.2) must be of the form

$$u(x) = c\left(\frac{t}{t^2 + |x - x^*|^2}\right)^{((n-2)/(2))} \tag{1.3}$$

with constants c, $t > 0$, and $x^* \in R^n$ (cf. [3] and the references therein). In supercritical case (i.e. $p > ((n+2)/(n-2))$), existence and asymptotic behaviour of positive solutions are much complicated and not completely understood. In fact, we can find cylindrically shaped solutions which do not decay along some direction. In addition, there are radial solutions with the slow decay rates solving (1.2) (cf. [15, 19, 36] and many others). Furthermore, those radial solutions are of the form

$$u(x) = \mu^{((2)/(p-1))} U(\mu|x|), \quad x \in R^n,$$

where $\mu = u^{p-1/2}(0)$, and $U(r)$ is the unique solution of

$$\begin{cases} -(U'' + \dfrac{n-1}{r}U') = U^p, & U(r) > 0,\ r > 0 \\ U'(0) = 0, \quad U(0) = 1. \end{cases}$$

For the study of 'stable' positive solutions of (1.2), the Joseph–Lundgren exponent

$$p_{jl}(n) := 1 + \frac{4}{n - 4 - 2\sqrt{n-1}}$$

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plays an important role (cf. [**16**]). Such an exponent is also essential to describe how the radial solutions intersect with the singular radial solution and with themselves (cf. [**19**]). In addition, this Joseph–Lundgren exponent can be used to study the Morse index for the sign-changed solutions of the Lane–Emden equation (cf. [**13**]) and other nonlinear elliptic equations with supercritical exponents (cf. [**7**, **8**, **17**]).

In this paper, our purpose is to study the relation between the critical exponents and the existence of kinds of solutions of k-Hessian equation (1.1). As the beginning of the study, we are concerned about the increasing negative solution of (1.1) with the radial structure as in [**6**, **26**]. Thus, (1.1) is reduced to the following equation

$$-\frac{1}{k}C_{n-1}^{k-1}(r^{n-k}|u'|^{k-1}u')' = r^{n-1}u^p, \quad u(r)>0 \quad as\ r>0. \tag{1.4}$$

Here $u(r)=u(|x|)=-V(x)$, $n\geqslant 3$, $1<k<n/2$ and $p>1$. In fact, in the critical case (i.e. $p=(((n+2)k)/(n-2k)))$, the extremal functions of the Hessian Sobolev inequality are radially symmetric (cf. [**6**, **33**, **37**]). In the noncritical case, it is clearer and more concise to study the critical exponents of the radial solutions. We believe that the ideas are helpful to investigate the corresponding problems of the solutions with general form, and those critical exponents still come into play in the study of k-Hessian equations without radial structure.

1.1. Regular solutions

Clearly, (1.4) has a singular solution

$$u_s(r) = Ar^{-((2k)/(p-k))}, \quad \text{with} \tag{1.5}$$
$$A := \left(\frac{1}{k}C_{n-1}^{k-1}\right)^{((1)/(p-k))}\left(\frac{2k}{p-k}\right)^{((k)/(p-k))}\left(n-\frac{2pk}{p-k}\right)^{((1)/(p-k))}.$$

If we write $V(x)=-u_s(|x|)$, then $V(x)$ only belongs to $C^2(R^n\setminus\{0\})$ (even it does not belong to $L^\infty_{loc}(R^n)$).

We are mainly concerned with the k-admissible solutions of (1.1). Consider the following boundary values problem

$$\begin{cases}-\frac{1}{k}C_{n-1}^{k-1}(r^{n-k}|u'|^{k-1}u')' = r^{n-1}u^p, & u(r)>0,\ r>0\\ u'(0)=0, \quad u(0)=\rho(:=\mu^{((2k)/(p-k))})>0. & \end{cases} \tag{1.6}$$

Definition 1.1. *If a solution $u(r)$ of (1.6) satisfies $u(|x|)\in C^2(R^n)$, then $u(r)$ is called a regular solution.*

Recall two critical exponents: Serrin exponent $p_{se} := ((nk)/(n-2k))$, and Sobolev exponent $p_{so} := (((n+2)k)/(n-2k))$.

When p is not larger than the Serrin exponent, (1.1) has no negative k-admissible solution (cf. [**21**, **28**, **29**]). Thus, we always assume in this paper that p is larger than

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the Serrin exponent

$$p > p_{se}. \tag{1.7}$$

In the critical case (i.e. $p = p_{so}$), all the regular solutions of (1.6) can be written as the explicit form (cf. remark 1.4 in [**26**])

$$u_\rho(r) = \left(\frac{1}{k}C_{n-1}^{k-1}\right)^{((1)/(p-k))} \rho(1 + \frac{k}{n^{1/k}(n-2k)}(\rho^{((k+1)/(n-2k))}r)^2)^{-((n-2k)/(2k))}. \tag{1.8}$$

Therefore, we will be concerned with the noncritical cases.

THEOREM 1.1. *When $p < p_{so}$, (1.6) has no regular solution.*

REMARK 1.1. By a direct calculation, when $p_{se} < p < p_{so}$, besides u_s given by (1.5), (1.4) has other singular solutions $U_s(r)$ satisfying $U_s(r)/u_s(r) \to 1$ as $r \to 0$ and $U_s(r)r^{((n-2k)/(k))} \to \lambda > 0$ as $r \to \infty$. When $k = 1$, this result can be found in [**15**, **19**, **36**].

THEOREM 1.2. *When $p > p_{so}$, all the positive regular solutions u_μ of (1.6) satisfy $u_\mu(r) \simeq r^{-((2k)/(p-k))}$ for large r. Furthermore, they are the forms of*

$$u_\mu(r) = \mu^{((2k)/(p-k))} u_1(\mu r), \quad r \geqslant 0, \tag{1.9}$$

where $u_1(r)$ is the solution of

$$\begin{cases} -\dfrac{1}{k}C_{n-1}^{k-1}(r^{n-k}|u'|^{k-1}u')' = r^{n-1}u^p, \quad u(r) > 0,\ r > 0 \\ u'(0) = 0, \quad u(0) = 1. \end{cases} \tag{1.10}$$

Here, $u(r) \simeq r^{-\theta}$ means that there exists $C > 1$ such that $1/C \leqslant u(r)r^\theta \leqslant C$ for large r.

REMARK 1.2. Problem (1.10) has a entire solution when $p > p_{so}$. In fact, by a standard argument of contraction, (1.10) has a unique local positive solution u (cf. proposition 2.1 in [**26**]). There holds $u' < 0$ as long as $u > 0$ (see the proof of Lemma 2.1). Extend this local solution rightwards. Then $u > 0$ for all $r > 0$. Otherwise, it contradicts with the Liouville theorem in [**34**].

1.2. Stable solutions

DEFINITION 1.2. *We say that a positive solution $u \in C^1(0, \infty)$ of (1.4) is stable if*

$$\int_0^\infty \left[\frac{1}{k}C_{n-1}^{k-1}r^{n-k}|u'|^{k-1}u'\varphi' - r^{n-1}u^p\varphi\right] \mathrm{d}r = 0; \tag{1.11}$$

$$Q_u(\varphi) := C_{n-1}^{k-1}\int_0^\infty r^{n-k}|u'|^{k-1}(\varphi')^2 \mathrm{d}r - p\int_0^\infty r^{n-1}u^{p-1}\varphi^2 \mathrm{d}r \geqslant 0 \tag{1.12}$$

for all $\varphi \in W_*$, where $W_* = \{\varphi(r); \varphi(r) = \phi(x) \in C_c^\infty(R^n),\ r = |x|\}$.

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Similarly, a positive solution $u \in C^1(0,\infty)$ *of (1.4) is stable on a set* (R,∞) *for some* $R>0$, *if (1.11) holds for all* $\varphi \in W_*$, *and (1.12) holds for all* $\varphi \in C_c^\infty(R,\infty)$.

Indeed, the fact that the first order Fréchet derivative of the functional $J(u)$ is equal to zero and the second order Fréchet derivative is nonnegative can lead to this definition, where

$$J(u) = \frac{C_{n-1}^{k-1}}{k(k+1)} \int_0^\infty |u'|^{k+1} r^{n-k} \mathrm{d}r - \frac{1}{p+1} \int_0^\infty u^{p+1} r^{n-1} \mathrm{d}r.$$

In addition, $Q_u(\varphi) \geqslant 0$ can also be obtained by linearizing (1.4).

It is not difficult to verify that the regular solutions u_ρ given by (1.8) and u_μ given by (1.9) satisfy (1.11). For the singular solution u_s expressed by (1.5), $p > p_{se}$ implies that 0 is not the singular point in integral terms of (1.11) (see the proof of Theorem 1.4). Therefore, u_s also satisfies (1.11).

Stable solutions of elliptic equations are important in the qualitative theory of PDEs. For example, stable solutions of the semilinear equation $\Delta u + f(u) = 0$ can be very simple for f satisfying some mild assumptions. The related consequences are helpful to understand the behaviour of large or small solutions on bounded domains (cf. [**1**]), the small diffusion problems and the De Giorgi conjecture (cf. [**9**, **10**]). In 2007, Farina [**13**] classified the stable solutions and the finite Morse index solutions of the Lane–Emden equation$-\Delta u = |u|^{p-1}u$. Afterwards, those results were extended to the equations with the negative exponents (cf. [**12**]) and with weight (cf. [**11**]), and also to the γ-Laplace equations [**7**]. In addition, the stability of positive solutions of the Brezis-Nirenberg model $-\Delta u = u^p + \lambda u$ and the analogous equation $-\Delta u = \lambda(1+u)^p$ can be applied to study the bifurcation theory (cf. [**17**, **19**]). Recently, the results of the higher order fractional Lane–Emden equations were obtained by Fazly and Wei (cf. [**14**]).

Recall other two critical exponents: the Joseph–Lundgren exponent

$$p_{jl} = \begin{cases} \infty, & \text{if } n \leqslant 2k+8, \\ \dfrac{k[n^2 - 2(k+3)n + 4k] + 4k\sqrt{2(k+1)n - 4k}}{(n-2k)(n-2k-8)}, & \text{if } n > 2k+8; \end{cases}$$

and

$$p^* = k\frac{n+2k}{n-2k}.$$

Clearly, $p_{se} < p_{so} < p_{jl}$. In addition, $p_{so} < p^*$ by virtue of $1 < k < n/2$. In view of $2k(k^2+6k+1)/(k-1)^2 > 2k+8$, we can deduce the relation between p^* and p_{jl} as follows

$$\begin{aligned} p^* \geqslant p_{jl}, &\quad \text{if } \ n \geqslant 2k(k^2+6k+1)/(k-1)^2; \\ p^* < p_{jl}, &\quad \text{if } \ n < 2k(k^2+6k+1)/(k-1)^2. \end{aligned}$$

Under the scaling transformation, $p = p_{so}$ ensures that equation (1.1) and energy $\|\cdot\|_{p+1}$ are invariant (cf [**21**]), and $p = p^*$ ensures that equation (1.1) and energy $\|\cdot\|_{p+k}$ are invariant (cf [**23**]). In addition, p^* is essential to study the separation property of solutions (see the following remark).

REMARK 1.3. Let $u_\mu(r)$ be a regular solution of (1.6). Corollary 1.7 in [26] implies that, when $p \geqslant \max\{p^*, p_{jl}\}$, $u_\mu(r) < u_s(r)$ for $r > 0$, and $u_{\mu_1}(r) < u_{\mu_2}(r)$ for $r > 0$ as long as $\mu_1 < \mu_2$.

The exponent p^* also appears in the study of γ-Laplace equations (cf. [22, 26]) and integral equations involving Wolff potentials (cf. [4, 25, 32, 35]). In particular, it plays an important role to investigate integrability, decay rates and intersection properties of the positive entire solutions. In addition, this exponent ensures that equation and energy $\|\cdot\|_{p+\gamma-1}$ are invariant under the scaling transformation (cf. [23]).

In particular, for the γ-Laplace equation

$$-\operatorname{div}(|\nabla u|^{\gamma-2}\nabla u) = K(x)u^p, \quad u > 0 \quad \text{in} \quad R^n, \tag{1.13}$$

we write $p_{se}(\gamma) = ((n(\gamma-1))/(n-\gamma))$, $p_{so}(\gamma) = ((n\gamma)/(n-\gamma)) - 1$, $p^*(\gamma) = ((n+\gamma)/(n-\gamma))(\gamma-1)$, $p_{jl} = \gamma - 1 + \gamma^2[n-\gamma-2-2\sqrt{(n-1)/(\gamma-1)}]^{-1}$ as $n > ((\gamma(\gamma+3))/(\gamma-1))$, and $p_{jl} = \infty$ as $n \leqslant ((\gamma(\gamma+3))/(\gamma-1))$.

If $\gamma \in (1, 2)$, $p_{se}(\gamma) < p^*(\gamma) < p_{so}(\gamma)$. When $K(x) \equiv 1$, according to the Liouville theorem in [31], (1.13) has no positive solution as $p < p_{so}(\gamma)$, and $p^*(\gamma)$ does not make sense. When $K(x)$ is a double bounded function, according to the result in [23], (1.13) has positive radial solutions as long as $p > p_{se}(\gamma)$. Now, p^* comes into play in studying integrability and decay rates of positive solutions.

Now, we state the results about the stable solutions.

THEOREM 1.3. *When $p < p_{jl}$, (1.4) has no stable solution.*

THEOREM 1.4. *When $p \geqslant p_{jl}$, the singular solution u_s given by (1.5) is a stable solution of (1.4).*

THEOREM 1.5. *When $p = p_{so}$ or $p \geqslant \max\{p^*, p_{jl}\}$, all the regular solutions of (1.6) are stable solutions of (1.4) on (R, ∞) for some $R > 0$. When $p_{se} < p < p_{so}$, the singular solutions introduced in remark 1.1 are stable solution of (1.4) on (R, ∞) for some $R > 0$.*

REMARK 1.4. Theorem 1.4 shows that u_s is also a stable solution of (1.4) on (R, ∞) for some $R > 0$ when $p \geqslant p_{jl}$. Combining with theorem 1.5, we know that (1.4) has stable solutions on (R, ∞) for some $R > 0$ when $p \in (p_{se}, p_{so}] \cup [p_{jl}, \infty)$. To our knowledge, it is unknown whether (1.4) has no stable solution on (R, ∞) for some $R > 0$ when p belongs to the gap (p_{so}, p_{jl}).

2. Regular solutions

LEMMA 2.1. *Let u be a regular solution of (1.6). Then, $u' < 0$ for $r > 0$, and $u(r) \to 0$ as $r \to \infty$. Moreover, there are positive constants C_1, C_2 such that for large r,*

$$C_1 r^{-((n-2k)/(k))} \leqslant u(r) \leqslant C_2 r^{-((2k)/(p-k))}. \tag{2.1}$$

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Proof. Step 1. Since u is a positive solution of (1.4),

$$-\frac{1}{k}C_{n-1}^{k-1}(r^{n-k}|u'|^{k-1}u')' > 0, \quad r > 0.$$

Integrating from 0 to R with $R > 0$, we obtain

$$R^{n-k}|u'(R)|^{k-1}u'(R) < 0$$

and hence $u' < 0$ is verified.

Step 2. By $u > 0$ and $u' < 0$ for $r > 0$, we know that $\lim_{r\to\infty} u(r)$ exists and hence is nonnegative. Suppose that $\lim_{r\to\infty} u(r) > 0$, then there exists a constant $c > 0$ such that $u \geqslant c$, and hence

$$-\frac{1}{k}C_{n-1}^{k-1}(r^{n-k}|u'|^{k-1}u')' \geqslant c^p r^{n-1}.$$

Integrating from 0 to R, we obtain

$$R^{n-k}|u'(R)|^{k-1}u'(R) \leqslant -CR^n.$$

Here $C > 0$ is independent of R. This result, together with $u' < 0$, implies $u'(R) \leqslant -CR$. Integrating again yields

$$u(r) \leqslant u(0) - Cr^2.$$

Letting $r \to \infty$, we see a contradiction with $u > 0$. This shows that $u(r) \to 0$ as $r \to \infty$.

Step 3. According to the results in [**20**] or [**29**], the regular solution of (1.6) satisfies

$$c_1 W_{((2k)/(k+1)),k+1}(u^p)(x) \leqslant u(|x|) \leqslant c_2[\inf_{x\in R^n} u(|x|) + W_{((2k)/(k+1)),k+1}(u^p)(x)], \tag{2.2}$$

where c_1, c_2 are positive constants, and $W_{((2k)/(k+1)),k+1}(u^p)$ is the Wolff potential of u^p. Namely,

$$W_{((2k)/(k+1)),k+1}(u^p)(x) = \int_0^\infty \left(\frac{\int_{B_t(x)} u^p(|y|)\mathrm{d}y}{t^{n-2k}}\right)^{1/k} \frac{\mathrm{d}t}{t}.$$

Therefore, for large $|x|$,

$$u(|x|) \geqslant c\int_{|x|+1}^\infty \left(\frac{\int_{B_1(0)} u^p(|y|)\mathrm{d}y}{t^{n-2k}}\right)^{1/k} \frac{\mathrm{d}t}{t} \geqslant c\int_{|x|+1}^\infty t^{2k-n/k}\frac{\mathrm{d}t}{t} = c|x|^{2k-n/k}.$$

Since u is radially symmetric and decreasing, we can also get

$$u(|x|) \geqslant c\int_0^{|x|/2} \left(\frac{\int_{B_{|x|}(0)\cap B_t(x)} u^p(|y|)\mathrm{d}y}{t^{n-2k}}\right)^{1/k} \frac{\mathrm{d}t}{t} \geqslant cu^{p/k}(|x|)|x|^2,$$

which implies that $u(|x|) \leqslant c|x|^{((2k)/(k-p))}$ for large $|x|$. □

The proof is complete.

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2.1. Proof of theorem 1.1

Let $p < p_{so}$. Assume that (1.6) has a positive regular solution u, we will deduce a contradiction.

Step 1. By (2.1), there exists $R > 0$ such that $u(r) \leqslant Cr^{-((2k)/(p-k))}$ for $r > R$. Thus,

$$\int_0^\infty r^{n-1}u^{p+1}\mathrm{d}r \leqslant C(R) + \int_R^\infty r^{n-((2(p+1)k)/(p-k))}\frac{dr}{r} < \infty. \tag{2.3}$$

Step 2. Let $\varphi \in C^\infty(0, \infty)$ satisfy $\varphi(r) = 1$ when $r \in (0, 1]$, $\varphi(r) = 0$ when $r \in [2, \infty)$, and $0 \leqslant \varphi \leqslant 1$. Write $\varphi_R(r) = \varphi(r/R)$. Multiply (1.4) by $u\varphi_R^{k+1}$ and integrate on $(0, \infty)$. By the initial value condition in (1.6), we get

$$\begin{aligned}\int_0^\infty r^{n-k}|u'|^{k+1}\varphi_R^{k+1}\mathrm{d}r &= \frac{k}{C_{n-1}^{k-1}}\int_0^\infty r^{n-1}u^{p+1}\varphi_R^{k+1}\mathrm{d}r\\ &\quad - (k+1)\int_0^\infty r^{n-k}u\varphi_R^k|u'|^{k-1}u'\varphi_R'\mathrm{d}r.\end{aligned} \tag{2.4}$$

By the Young inequality and the Hölder inequality, for a small $\epsilon > 0$, there holds that

$$\begin{aligned}\left|\int_0^\infty r^{n-k}u\varphi_R^k|u'|^{k-1}u'\varphi_R'\mathrm{d}r\right| &\leqslant \epsilon\int_0^\infty r^{n-k}|u'|^{k+1}\varphi_R^{k+1}\mathrm{d}r\\ &\quad + \frac{C_\epsilon}{R^{k+1}}\left(\int_0^\infty r^{n-1}u^{p+1}\mathrm{d}r\right)^{((k+1)/(p+1))}\\ &\quad \times\left(\int_R^{2R} r^{\theta+1}\frac{\mathrm{d}r}{r}\right)^{((p-k)/(p+1))},\end{aligned} \tag{2.5}$$

where $((p-k)/(p+1))\theta = n-k-(n-1)((k+1)/(p+1))$. Therefore, $(\theta+1)((p-k)/(p+1)) - (k+1) = n((p-k)/(p+1)) - 2k < 0$ by virtue of $p < p_{so}$. Letting $R \to \infty$, we deduce from (2.3), (2.4) and (2.5) that

$$\int_0^\infty r^{n-k}|u'|^{k+1}\mathrm{d}r < \infty. \tag{2.6}$$

Step 3. Multiplying (1.4) by u and integrating on $(0, R)$, we obtain that

$$\int_0^R r^{n-k}|u'|^{k+1}\mathrm{d}r - R^{n-k}u(R)|u'(R)|^{k-1}u'(R) = \frac{k}{C_{n-1}^{k-1}}\int_0^R r^{n-1}u^{p+1}\mathrm{d}r. \tag{2.7}$$

By (2.6) and (2.3), there exists $R_j \to \infty$ such that

$$R_j^{n-k+1}|u'(R_j)|^{k+1} + R_j^n u^{p+1}(R_j) \to 0. \tag{2.8}$$

Therefore, by $p < p_{so}$,

$$R_j^{n-k}u(R_j)|u'(R_j)|^{k-1}u'(R_j) \to 0, \quad as\ R_j \to \infty.$$

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Inserting this result into (2.7) and letting $R = R_j \to \infty$, we obtain

$$\int_0^\infty r^{n-k}|u'|^{k+1}\mathrm{d}r = \frac{k}{C_{n-1}^{k-1}} \int_0^\infty r^{n-1}u^{p+1}\mathrm{d}r \tag{2.9}$$

Step 4. Multiplying (1.4) by ru' and integrating on $(0, R)$, we have the Pohozaev type equality

$$\begin{aligned} &-\frac{n-2k}{k+1}\int_0^R r^{n-k}|u'|^{k+1}\mathrm{d}r + \frac{k}{C_{n-1}^{k-1}}\frac{n}{p+1}\int_0^R r^{n-1}u^{p+1}\mathrm{d}r \\ &\quad = \frac{k}{k+1}R^{n-k+1}|u'(R)|^{k+1} + \frac{k}{(p+1)C_{n-1}^{k-1}}R^n u^{p+1}(R). \end{aligned} \tag{2.10}$$

By (2.8), the right-hand side of (2.10) converges to zero when $R = R_j \to \infty$. Letting $R = R_j \to \infty$ in (2.10) and using (2.9), we can see $((n-2k)/(k+1)) = ((n)/(p+1))$, which contradicts with $p < p_{so}$.

2.2. Proof of theorem 1.2

When $k = 1$, the proof of the slow decay is based on the comparison principle (cf. lemma 2.20 and theorem 2.25 in [**24**]). For the quasilinear equation (1.4), we use the monotony inequality to replace the comparison principle.

LEMMA 2.2. *Let $u(r)$ be a regular solution of (1.6). If $u(r) = O(r^{-((2k)/(p-k))-\varepsilon})$ with some $\varepsilon \in (0, n-2k/k - ((2k)/(p-k)))$ for large r, then $u(r) = O(r^{(2k-n)/k})$ for large r.*

Proof. If $u(r) = O(r^{-((2k)/(p-k))-\varepsilon})$ for large r, we can find a large $R > 0$ such that as $r > R$,

$$u(r) \leqslant Cr^{-((2k)/(p-k))-\varepsilon}. \tag{2.11}$$

By lemma 2.1, $\inf_{[0,\infty)} u(r) = 0$. Using (2.2), we have

$$u(|x|) \leqslant C(I_1 + I_2 + I_3),$$

where

$$I_1 = \int_0^{((|x|)/(2))} \left(\frac{\int_{B_t(x)} u^p(|y|)\mathrm{d}y}{t^{n-2k}}\right)^{1/k} \frac{\mathrm{d}t}{t},$$

$$I_2 = \int_{((|x|)/(2))}^\infty \left(\frac{\int_{B_t(x)\cap B_R(0)} u^p(|y|)\mathrm{d}y}{t^{n-2k}}\right)^{1/k} \frac{\mathrm{d}t}{t},$$

$$I_3 = \int_{((|x|)/(2))}^\infty \left(\frac{\int_{B_t(x)\setminus B_R(0)} u^p(|y|)\mathrm{d}y}{t^{n-2k}}\right)^{1/k} \frac{\mathrm{d}t}{t}.$$

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For sufficiently large $|x|$, we can deduce from (2.11) that

$$I_1 \leqslant C|x|^{-p/k(((2k)/(p-k))+\varepsilon)} \int_0^{|x|/2} \left(\frac{\int_{B_t(x)} \mathrm{d}y}{t^{n-2k}} \right)^{1/k} \frac{\mathrm{d}t}{t} \leqslant C|x|^{-((2k)/(p-k))-\varepsilon p/k},$$

$$I_2 \leqslant C(|B_R(0)|u^p(0))^{1/k} \int_{((|x|)/(2))}^{\infty} t^{2k-n/k} \frac{dt}{t} \leqslant C|x|^{-((n-2k)/(k))},$$

$$I_3 \leqslant \int_{|x|/2}^{\infty} \left(\frac{\int_{B_{t+|x|}(0)\setminus B_R(0)} u^p(|y|)\mathrm{d}y}{t^{n-2k}} \right)^{1/k} \frac{\mathrm{d}t}{t} \leqslant C|x|^{-((2k)/(p-k))-\varepsilon p/k}.$$

These estimates show that $u(r) \leqslant C(r^{-n-2k/k} + r^{-((2k)/(p-k))-\varepsilon p/k}) \leqslant Cr^{-((2k)/(p-k))-\varepsilon p/k}$. Replacing (2.11) by this result to estimate I_1, I_2 and I_3 as we have done above, we get

$$u(r) \leqslant C(r^{-((n-2k)/(k))} + r^{-((2k)/(p-k))-\varepsilon(p/k)^2}) \leqslant Cr^{-((2k)/(p-k))-\varepsilon(p/k)^2}.$$

By iterating m times, we can obtain

$$u(r) \leqslant C(r^{-((n-2k)/(k))} + r^{-((2k)/(p-k))-\varepsilon(p/k)^m}).$$

Clearly, there exists a sufficiently large m_0 such that $n - 2k/k \leqslant ((2k)/(p-k)) + \varepsilon(p/k)^{m_0}$. Thus, after m_0 steps, we derive that,

$$u(r) \leqslant Cr^{-((n-2k)/(k))} \quad \textit{for large } r.$$

Lemma 2.2 is proved. □

LEMMA 2.3. *Let $u(r)$ be a regular solution of (1.6). If $u(r) = o(r^{-((2k)/(p-k))})$ for large r, then $u(r) = O(r^{(2k-n)/k})$ for large r.*

Proof. Step 1. Let $\varphi(r) \in C^1(0, \infty)$ satisfy $\lim_{r\to\infty} r^{n-((2pk)/(p-k))}\varphi(r) = 0$. Integrating (1.4) from 0 to r, we have

$$|u'|^{k-1}u' = -\frac{k}{C_{n-1}^{k-1}} r^{k-n} \int_0^r s^{n-1}u^p(s)\mathrm{d}s. \tag{2.12}$$

Thus, by $u(r) = o(r^{-((2k)/(p-k))})$ when $r \to \infty$, it follows that

$$r^{n-k}|u'(r)|^k\varphi(r) \to 0. \tag{2.13}$$

Multiply (1.4) by φ and integrate from R to ∞. By (2.13), we obtain

$$\begin{aligned} &\int_R^\infty r^{n-k}|u'|^{k-1}(u')\varphi'\mathrm{d}r \\ &= -R^{n-k}|u'(R)|^{k-1}u'(R)\varphi(R) + \frac{k}{C_{n-1}^{k-1}} \int_R^\infty r^{n-1}u^p\varphi\mathrm{d}r. \end{aligned} \tag{2.14}$$

Write $h(r) := c_* r^{-\theta}$, where c_* is a positive constant determined later, and $\theta := ((2k)/(p-k)) + \epsilon_0$ with suitably small $\epsilon_0 > 0$. By simply calculating and integrating by parts, we get

$$\int_R^\infty r^{n-k}|h'|^{k-1}h'\varphi'\mathrm{d}r = -(c_*\theta)^k \int_R^\infty r^{n-k(\theta+2)}\varphi'\mathrm{d}r$$

$$= (c_*\theta)^k[n-k(\theta+2)] \int_R^\infty r^{n-1-k(\theta+2)}\varphi\mathrm{d}r$$

$$+ (c_*\theta)^k R^{n-k(\theta+2)}\varphi(R).$$

Subtracting this result from (2.14) yields

$$\begin{aligned} &\int_R^\infty r^{n-k}[|u'|^{k-1}u' - |h'|^{k-1}h']\varphi'\mathrm{d}r \\ &\quad = [(c_*\theta)^k R^{n-k(\theta+2)} + R^{n-k}|u'(R)|^{k-1}u'(R)]\varphi(R) \\ &\qquad + \int_R^\infty r^{n-1}\left[\frac{ku^p}{C_{n-1}^{k-1}} - \frac{(c_*\theta)^k[n-k(\theta+2)]}{r^{k(\theta+2)}}\right]\varphi\mathrm{d}r. \end{aligned} \tag{2.15}$$

Step 2. In view of $k(\theta+2) = ((2pk)/(p-k)) + k\epsilon_0$, we can find $\eta_0 \in (0,\, k\epsilon_0/p)$ such that

$$k(\theta+2) > \frac{2pk}{p-k} + p\eta_0. \tag{2.16}$$

Since $u \in C^2$ is decreasing and $u(r) = o(r^{-((2k)/(p-k))})$ for large r, then either there exist positive constants c_1, c_2 such that

$$c_1 r^{-((2k)/(p-k))} \geqslant u(r) \geqslant c_2 r^{-((2k)/(p-k))-\eta_0} \tag{2.17}$$

when r is suitably large, or $\lim_{r\to\infty} u(r)r^{((2k)/(p-k))+\eta_0} = 0$, which implies that there exists $\eta \in (0,\, \eta_0)$ such that for large r,

$$u(r) \leqslant cr^{-((2k)/(p-k))-(\eta_0-\eta)}. \tag{2.18}$$

If (2.18) is true, lemma 2.3 can be proved easily by lemma 2.2.

In the following, we assume that (2.17) is true. Take $\varphi = r^{-m}(u-h)_+$ in (2.15), where $m > n - ((2pk)/(p-k))$ is sufficiently large. Then,

$$\begin{aligned} &\int_R^\infty r^{n-k-m}[|u'|^{k-1}u' - |h'|^{k-1}h'][(u-h)_+]'\mathrm{d}r \\ &\quad = [(c_*\theta)^k R^{n-k(\theta+2)-m} + R^{n-k-m}|u'(R)|^{k-1}u'(R)][u(R)-h(R)]_+ \\ &\qquad + \int_R^\infty r^{n-m-1}\left[\frac{ku^p}{C_{n-1}^{k-1}} - \frac{(c_*\theta)^k[n-k(\theta+2)]}{r^{k(\theta+2)}}\right](u-h)_+\mathrm{d}r \\ &\qquad + m\int_R^\infty r^{n-k-m-1}[|u'|^{k-1}u' - |h'|^{k-1}h'](u-h)_+\mathrm{d}r. \end{aligned} \tag{2.19}$$

By (2.12), (2.17) and (2.16), for any $\delta \in (0,\, 1)$, we can find $R_0 > 0$ such that as $r \geqslant R_0$, $|h'|^k \leqslant \delta|u'|^k$. Therefore, the last term of the right-hand side of (2.19)

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with $R = R_0$ is not larger than $m(1-\delta)\int_{R_0}^{\infty} r^{n-k-m-1}|u'|^{k-1}u'(u-h)_+\mathrm{d}r$. Choose $c_* = u(R_0)R_0^{\theta}$ to ensure $u(R_0) = h(R_0)$. Therefore, the first term of the right-hand side of (2.19) with $R = R_0$ is zero. Thus, from (2.19) with $R = R_0$ it follows that

$$\begin{aligned}&\int_{R_0}^{\infty} r^{n-k-m}[|u'|^{k-1}u' - |h'|^{k-1}h'][(u-h)_+]'\mathrm{d}r \\ &\leqslant \int_{R_0}^{\infty} r^{n-m-1}[\frac{ku^p}{C_{n-1}^{k-1}} - m(1-\delta)r^{-k}|u'|^k](u-h)_+\mathrm{d}r.\end{aligned} \tag{2.20}$$

By (2.12) and the monotonicity of $u(r)$, there holds

$$r^{-k}|u'(r)|^k \geqslant \frac{k}{C_{n-1}^{k-1}} r^{-n}u^p(r)\int_0^r s^{n-1}\mathrm{d}s \geqslant \frac{ku^p(r)}{nC_{n-1}^{k-1}}.$$

Taking m suitably large, we obtain that the right-hand side of (2.20) is not larger than zero. In view of the monotony inequality $(|a|^{k-1}a - |b|^{k-1}b)(a-b) \geqslant 2^{k-1}$ $|a-b|^{k+1}$, we obtain from (2.20) that

$$\int_{R_0}^{\infty} r^{n-k}([(u-h)_+]')^{k+1}\mathrm{d}r \leqslant 0,$$

which implies $[u(r) - h(r)]_+ \equiv Constant$ for $r \geqslant R_0$. In view of $u(R_0) = h(R_0)$, it follows $Constant = 0$, which implies $u(r) \leqslant h(r)$ for $r \geqslant R_0$. Applying lemma 2.2, we can also see the conclusion of lemma 2.3. □

Proof of Theorem 1.2. Let $p > p_{so}$.

Step 1. By lemma 2.1, we see that $u(r) \leqslant Cr^{-((2k)/(p-k))}$ for large r. We claim that there exists $c > 0$ such that $u(r) \geqslant cr^{-((2k)/(p-k))}$ for large r.

Otherwise, $\lim_{r\to\infty} u(r)r^{((2k)/(p-k))} = 0$. By lemma 2.3 it follows that $u(r) = O(r^{2k-n/k})$ for large r. Thus, $V(x) \in L^{p+1}(R^n) \cap C^2(R^n)$ (here $V(x) = -u(|x|)$). According to theorem 4.4 in [**21**], we know $p = p_{so}$, which contradicts with $p > p_{so}$.

Step 2. We define by scaling a new function

$$w(r) = \mu^{((2k)/(p-k))}u(\mu r), \quad \mu > 0.$$

By a direct calculation, we see that w still satisfies (1.4). Applying the initial value conditions, we can obtain the second conclusion of theorem 1.2. □

REMARK 2.1. Let $u_\mu(r)$ be a regular solution of (1.6) with $p > p_{so}$. When $p \geqslant p^*$, Miyamoto used the technique of phase plane analysis to show that $u_\mu(r)/u_s(r) \to 1$ as $r \to \infty$ (cf. lemma 2.5 in [**26**]). When $p \geqslant p_{so}$, theorem 1.2 shows that the decay rate of u_μ is the same as that of u_s. Furthermore, if $\lim_{r\to\infty} u(r)r^{-((2k)/(p-k))}$ exists, then it must be A which is introduced in (1.5). In fact, integrating (1.4) twice yields

$$u(r) = u(0) - \left(\frac{k}{C_{n-1}^{k-1}}\right)^{1/k} \int_0^r \left[t^{k-n}\int_0^t s^{n-1}u^p(s)ds\right]^{1/k} \mathrm{d}t.$$

Write $B := \lim_{r\to\infty}((u(r))/(r^{((2k)/(p-k))}))$. Using the L'Hospital principle twice, we get

$$B^k = \frac{k}{C_{n-1}^{k-1}}\left(\frac{2k}{p-k}\right)^{-k}\frac{\int_0^r s^{n-1}u^p(s)\mathrm{d}s}{r^{n-2pk/p-k}} = \frac{k}{C_{n-1}^{k-1}}\left(\frac{2k}{p-k}\right)^{-k}\left(n-\frac{2pk}{p-k}\right)^{-1}B^p,$$

which implies $B = A$.

3. Stable solutions

3.1. Proof of Theorem 1.3

Step 1. We claim that for every $\gamma \in [1,\, 2p + 2\sqrt{p(p-k)} - k/k)$ and any integer $m \geqslant max\{((p+\gamma)/(p-k)),\, 2\}$, there exists a constant $C > 0$ such that for any $\psi \in W_*$, there holds

$$\begin{aligned}&\int_0^R r^{n-1}u^{p+\gamma}\psi^{m(k+1)}\mathrm{d}r\\ &\quad\leqslant C\int_0^R (r^{(((n-k)(p+\gamma)-(n-1)(\gamma+k))/(p+\gamma))}|\psi'|^{k+1})^{((p+\gamma)/(p-k))}\mathrm{d}r.\end{aligned}\tag{3.1}$$

Proof of (3.1). Let $\psi \in W_*$ be a cut-off function such that $0 \leqslant \psi \leqslant 1$ and

$$\psi(r) = \begin{cases}1, & \text{if } r \leqslant R/2,\\ 0, & \text{if } r \geqslant R.\end{cases}$$

Clearly, there exists a constant $C > 0$ such that $|\psi'| \leqslant C/R$.

Taking $\varphi = u^\gamma\psi^{m(k+1)}$ in (1.11), we get

$$\begin{aligned}&\frac{\gamma}{k}C_{n-1}^{k-1}\int_0^R r^{n-k}|u'|^{k+1}u^{\gamma-1}\psi^{m(k+1)}\mathrm{d}r\\ &\quad\leqslant \frac{k+1}{k}C_{n-1}^{k-1}\int_0^R r^{n-k}|u'|^k u^\gamma\psi^{mk}|(\psi^m)'|\mathrm{d}r + \int_0^R r^{n-1}u^{p+\gamma}\psi^{m(k+1)}\mathrm{d}r.\end{aligned}$$

Using the Young inequality to the first term of the right-hand side, we can obtain that for any small $\varepsilon > 0$,

$$\begin{aligned}&\left(\frac{\gamma}{k}C_{n-1}^{k-1} - \varepsilon^2\right)\int_0^R r^{n-k}u^{\gamma-1}|u'|^{k+1}\psi^{m(k+1)}\mathrm{d}r\\ &\quad\leqslant C_\varepsilon\int_0^R r^{n-k}u^{\gamma+k}|(\psi^m)'|^{k+1}\mathrm{d}r + \int_0^R r^{n-1}u^{p+\gamma}\psi^{m(k+1)}\mathrm{d}r.\end{aligned}\tag{3.2}$$

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Taking $\varphi = u^{\gamma+1/2}\psi^{m(k+1)/2}$ in (1.12), we have

$$
\begin{aligned}
p\int_0^R & r^{n-1}u^{p+\gamma}\psi^{m(k+1)}\mathrm{d}r \\
&\leqslant \frac{C_{n-1}^{k-1}(\gamma+1)^2}{4}\int_0^R r^{n-k}u^{\gamma-1}|u'|^{k+1}\psi^{m(k+1)}\mathrm{d}r \\
&\quad + \frac{C_{n-1}^{k-1}(k+1)^2}{4}\int_0^R r^{n-k}|u'|^{k-1}u^{\gamma+1}\psi^{m(k-1)}|(\psi^m)'|^2\mathrm{d}r \\
&\quad + \frac{C_{n-1}^{k-1}(\gamma+1)(k+1)}{2}\int_0^R r^{n-k}|u'|^k u^\gamma\psi^{mk}|(\psi^m)'|\mathrm{d}r.
\end{aligned} \tag{3.3}
$$

Using the Young inequality to the second and the third terms of the right-hand side of (3.3), we get

$$
\begin{aligned}
p\int_0^R & r^{n-1}u^{p+\gamma}\psi^{m(k+1)}\mathrm{d}r \\
&\leqslant \left(\frac{C_{n-1}^{k-1}(\gamma+1)^2}{4}+\varepsilon^2\right)\int_0^R r^{n-k}|u'|^{k+1}u^{\gamma-1}\psi^{m(k+1)}\mathrm{d}r \\
&\quad + C_\varepsilon\int_0^R r^{n-k}|(\psi^m)'|^{k+1}u^{\gamma+k}\mathrm{d}r.
\end{aligned} \tag{3.4}
$$

Combining (3.2) and (3.4), we obtain by the Hölder inequality that

$$
\begin{aligned}
&\left[p-\left(\frac{C_{n-1}^{k-1}(\gamma+1)^2}{4}+\varepsilon^2\right)\frac{1}{(\gamma/k)C_{n-1}^{k-1}-\varepsilon^2}\right]\int_0^R r^{n-1}u^{p+\gamma}\psi^{m(k+1)}\mathrm{d}r \\
&\leqslant C\int_0^R r^{n-k}|(\psi^m)'|^{k+1}u^{\gamma+k}\mathrm{d}r \\
&\leqslant C\left[\int_0^R \left(r^{(n-1)((\gamma+k)/(p+\gamma))}u^{\gamma+k}\psi^{(m-1)(k+1)}\right)^{((p+\gamma)/(\gamma+k))}\mathrm{d}r\right]^{((\gamma+k)/(p+\gamma))} \\
&\quad\cdot\left[\int_0^R \left(r^{(((n-k)(p+\gamma)-(n-1)(\gamma+k))/(p+\gamma))}|\psi'|^{k+1}\right)^{((p+\gamma)/(p-k))}\mathrm{d}r\right]^{((p-k)/(p+\gamma))}.
\end{aligned} \tag{3.5}
$$

In view of $\gamma\in[1,\,((2p+2\sqrt{p(p-k)}-k)/(k)))$, $\lim\limits_{\varepsilon\to 0}[p-(C_{n-1}^{k-1}(\gamma+1)^2/4+\varepsilon^2)((1)/((\gamma/k)C_{n-1}^{k-1}-\varepsilon^2)] = p-((k(\gamma+1)^2)/(4\gamma))>0$. Therefore, the coefficient of the left-hand side of (3.5) is positive as long as ε is sufficiently small. Therefore, noting $(m-1)(k+1)((p+\gamma)/(\gamma+k))\geqslant m(k+1)$ which is implied by $m\geqslant max\{((p+\gamma)/(p-k)),\,2\}$, we can deduce (3.1) from (3.5) by the Young inequality.

Step 2. By the definition of ψ, from (3.1) we can deduce that

$$
\int_0^R r^{n-1}u^{p+\gamma}\psi^{m(k+1)}\mathrm{d}r \leqslant CR^{n+1-(((2k+1)(p+\gamma)-(\gamma+k))/(p-k))}. \tag{3.6}
$$

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When $n+1-(((2k+1)(p+\gamma)-(\gamma+k))/(p-k))<0$, the desired claim follows by letting $R\to\infty$.

Proof of (3.6). Consider a real-valued function

$$f(t)=\frac{(2k+1)(t+\gamma(t))-(\gamma(t)+k)}{t-k},\quad t\in(k,\infty),$$

where $\gamma(t)=2t+2\sqrt{t(t-k)}-k/k$. Clearly, we know $f(t)$ is a strictly decreasing function (by virtue of $f'(t)<0$ on $(k,\,\infty)$), satisfying $\lim_{t\to k}f(t)=\infty$ and $\lim_{t\to\infty}f(t)=2k+9$. Therefore, we consider separately two cases: $n\leqslant 2k+8$, and $n\geqslant 2k+9$.

Case I: $n\leqslant 2k+8$. In view of $p>p_{se}$, there exists $\gamma\in[1,\,2p+2\sqrt{p(p-k)}-k/k)$ such that $n+1-(((2k+1)(p+\gamma)-(\gamma+k))/(p-k))<0$ is true.

Case II: $n\geqslant 2k+9$. In view of $p>p_{se}$, there exists a unique $p_0>k$ such that $n+1=f(p_0)$ since $f(t)$ is decreasing in $(k,\,\infty)$. Therefore, p_0 satisfies

$$(n-2k)(n-2k-8)p_0^2-2k[n^2-2(k+3)n+4k]p_0+k^2(n-2)^2=0, \tag{3.7}$$

and

$$(n-2k-4)p_0-(n-2)k>4(p_0-k). \tag{3.8}$$

The roots of equation (3.7) are

$$p_1=\frac{k[n^2-2(k+3)n+4k]+4k\sqrt{2(k+1)n-4k}}{(n-2k)(n-2k-8)}, \tag{3.9}$$

$$p_2=\frac{k[n^2-2(k+3)n+4k]-4k\sqrt{2(k+1)n-4k}}{(n-2k)(n-2k-8)}. \tag{3.10}$$

Inequality (3.8) implies $p_0>p_2$, and hence we take $p_0=p_1$ (it equals exactly p_{jl}). Thus, when $p<p_{jl}$, there exists $\gamma\in[1,\,2p+2\sqrt{p(p-k)}-k/k)$ satisfying $n+1-(((2k+1)(p+\gamma)-(\gamma+k))/(p-k))<0$.

No matter in Case I or Case II, letting $R\to\infty$ in (3.6), we can deduce $\int_0^R r^{n-1}u^{p+\gamma}\mathrm{d}r\to 0$. This contradiction shows that (1.4) has no positive stable solution as long as $p<p_{jl}$.

3.2. Proof of Theorem 1.4

Let u_s be the singular solution of (1.4) given by (1.5). We will prove that the singular solution $u_s(r)$ is stable when $n\geqslant 2k+9$ and $p\geqslant p_{jl}$.

First, we claim that u_s satisfies (1.11). In fact, by (1.7), the improper integral $\int_0^\infty r^{n-1}u_s^p\varphi\mathrm{d}r\leqslant C\int_0^R r^{n-1-((2pk)/(p-k))}\mathrm{d}r<\infty$. Similarly, the left-hand side of (1.11) also makes sense. In addition, u_s solves (1.4). Multiply by the test function $\varphi\in W_*$ and integrate from 0 to ∞. Noting $r^{n-k}|u_s'(r)|^k\to 0$ as $r\to 0$, we know that the claim is true.

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To prove that u_s satisfies (1.12), we observe firstly that

$$
\begin{aligned}
&p\left(\frac{2}{p-k}\right)\left(n-\frac{2pk}{p-k}\right) \leqslant \frac{(n-2-((2p(k-1))/(p-k)))^2}{4}\\
&\Leftrightarrow 8n(p^2-kp)-16kp^2 \leqslant (n-2)^2(p^2-2kp+k^2)\\
&\quad +4(k-1)^2p^2-4(k-1)(n-2)(p^2-kp)\\
&\Leftrightarrow (n-2k)(n-2k-8)p^2-2k(n^2-2(k+3)n+4k)p\\
&\quad +k^2(n-2)^2 \geqslant 0\\
&\Leftrightarrow p\in(-\infty,p_2]\bigcup[p_{jl},+\infty)
\end{aligned}
\tag{3.11}
$$

where p_2 is defined in (3.10). On the contrary, by definition 1.2, we have that for any $\phi \in C_c^\infty(R^n)$,

$$
\begin{aligned}
&C_{n-1}^{k-1}\int_{R^n}\frac{1}{|x|^{k-1}}|u_s'|^{k-1}|\nabla\phi|^2\mathrm{d}x - p\int_{R^n}u_s^{p-1}\phi^2\mathrm{d}x\\
&= C_{n-1}^{k-1}\int_{R^n}\left(\frac{1}{k}C_{n-1}^{k-1}\right)^{((k-1)/(p-k))}\left(\frac{2k}{p-k}\right)^{(((k-1)p)/(p-k))}\\
&\quad\times\left(n-\frac{2pk}{p-k}\right)^{((k-1)/(p-k))}\frac{1}{|x|^{((2p(k-1))/(p-k))}}|\nabla\varphi|^2\mathrm{d}x\\
&\quad -p\int_{R^n}\left(\frac{1}{k}C_{n-1}^{k-1}\right)^{((p-1)/(p-k))}\left(\frac{2k}{p-k}\right)^{(((p-1)k)/(p-k))}\\
&\quad\times\left(n-\frac{2pk}{p-k}\right)^{((p-1)/(p-k))}\frac{1}{|x|^{((2(p-1)k)/(p-k))}}\varphi^2\mathrm{d}x\\
&= C_0\left(\int_{R^n}\frac{1}{|x|^{((2p(k-1))/(p-k))}}|\nabla\phi|^2\mathrm{d}x - p\left(\frac{2}{p-k}\right)\right.\\
&\quad\left.\times\left(n-\frac{2pk}{p-k}\right)\int_{R^n}\frac{1}{|x|^{((2(p-1)k)/(p-k))}}\phi^2\right)\mathrm{d}x,
\end{aligned}
$$

where

$$
\begin{aligned}
C_0 &= C_{n-1}^{k-1}\left(\frac{1}{k}C_{n-1}^{k-1}\right)^{((k-1)/(p-k))}\left(\frac{2k}{p-k}\right)^{(((k-1)p)/(p-k))}\\
&\quad\times\left(n-\frac{2pk}{p-k}\right)^{((k-1)/(p-k))}.
\end{aligned}
\tag{3.12}
$$

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By $p \geqslant p_{jl}$, (3.11) implies that

$$\int_{R^n} \frac{1}{|x|^{((2p(k-1))/(p-k))}} |\nabla\phi|^2 \mathrm{d}x - p\left(\frac{2}{p-k}\right)\left(n - \frac{2pk}{p-k}\right) \times \int_{R^n} \frac{1}{|x|^{((2(p-1)k)/(p-k))}} \phi^2 \mathrm{d}x$$
$$\geqslant \int_{R^n} \frac{1}{|x|^{((2p(k-1))/(p-k))}} |\nabla\phi|^2 \mathrm{d}x - \frac{(n-2-((2p(k-1))/(p-k)))^2}{4} \times \int_{R^n} \frac{1}{|x|^{((2(p-1)k)/(p-k))}} \phi^2 \mathrm{d}x.$$

It follows that

$$Q_{u_s}(\varphi) > 0, \quad \forall \varphi \in W_* \tag{3.13}$$

by the Caffarelli-Kohn-Nirenberg inequality (cf. [5])

$$\int_{R^n} \frac{|\nabla\phi|^2}{|x|^{2a}} \mathrm{d}x \geqslant C \int_{R^n} \frac{\phi^2}{|x|^{2b}} \mathrm{d}x, \quad \forall \phi \in D_a^{1,2}(R^n), \tag{3.14}$$

where $n \geqslant 3$, $0 \leqslant a < n-2/2$ and $a \leqslant b \leqslant a+1$, where $C < C_{a,b}$ and the best constant $C_{a,b}$ is given by $C_{a,b} := (n-2-2a)^2/4$. Here we take $a = ((p(k-1))/(p-k))$ and $b = a+1$. This result shows that u_s is a stable solution of (1.4) when $n \geqslant 2k+9$ and $p \geqslant p_{jl}$. The proof of Theorem 1.4 is complete.

3.3. Proof of theorem 1.5

Step 1. When $p = p_{so}$, all regular solutions u_ρ of (1.6) can be written as the form given by (1.8). When r is suitably large,

$$u_\rho(r) \leqslant D_1 r^{-((n-2k)/(k))}, \quad |u'_\rho| \geqslant D_2 r^{-((n-k)/(k))}, \tag{3.15}$$

where D_1, D_2 are positive constants independent of r. Thus,

$$pu^{p-1}(r) = O(r^{-(((k-1)n)/(k))-4}), \quad as\ r \to \infty.$$

Therefore, we can find some $R > 0$ such that for all $|x| > R$ and $\phi \in C_c^\infty(R^n \setminus \overline{B_R(0)})$, there holds

$$pu_\rho^{p-1}(|x|)\phi^2(x) < C^*|x|^{-(((k-1)n)/(k))-2}\phi^2(x),$$

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where $C^* = 4(((n-2-((k-1)/(k))n^2))/(4))D_2^{k-1}C_{n-1}^{k-1}(n-2k/k)^{k-1}$. Thus,

$$\begin{aligned}
&C_{n-1}^{k-1}\int_{R^n}\frac{1}{|x|^{k-1}}|u'_\rho|^{k-1}|\nabla\phi|^2\mathrm{d}x - p\int_{R^n}u_\rho^{p-1}\phi^2\mathrm{d}x\\
&\geqslant D_2^{k-1}C_{n-1}^{k-1}\left(\frac{n-2k}{k}\right)^{k-1}\int_{R^n}\frac{1}{|x|^{((k-1)/(k))n}}|\nabla\phi|^2\mathrm{d}x\\
&\quad -C^*\int_{R^n}\frac{1}{|x|^{((k-1)/(k))n+2}}\phi^2\mathrm{d}x \qquad (3.16)\\
&= D_2^{k-1}C_{n-1}^{k-1}\left(\frac{n-2k}{k}\right)^{k-1}\left(\int_{R^n}\frac{1}{|x|^{((k-1)/(k))n}}|\nabla\phi|^2\mathrm{d}x\right.\\
&\quad\left. -\frac{(n-2-((k-1)/(k))n)^2}{4}\int_{R^n}\frac{1}{|x|^{((k-1)/((k))n+2}}\phi^2\right)\mathrm{d}x,
\end{aligned}$$

and the right-hand side is nonnegative by the Caffarelli-Kohn-Nirenberg inequality (3.14) with $a = k-1/2kn$ and $b = a+1$. Therefore, $Q_{u_\rho}(\varphi) \geqslant 0$ for every $\varphi \in C_c^\infty(R, \infty)$. In addition, u_ρ also satisfies (1.11). So the regular solution u_ρ is stable on (R, ∞).

Step 2. Let u_μ (see (1.9)) be a regular solution of (1.6) with $p \geqslant \max\{p^*, p_{jl}\}$. We claim that u_μ is stable on (R, ∞) for some $R > 0$.

We at first prove $\lim_{r\to\infty} u'_\mu(r)/u'_s(r) = 1$ when $p \geqslant \max\{p^*, p_{jl}\}$.

Clearly, $u'_s = -(1/kC_{n-1}^{k-1})^{((1)/(p-k))}(((2k)/(p-k)))^{((p)/(p-k))}(n-((2pk)/(p-k)))^{((1)/(p-k))}r^{-((p+k)/(p-k))}$.

Combining with (2.12) and using the L'Hospital principle, we get

$$\begin{aligned}
\lim_{r\to\infty}\left(\frac{u'_\mu}{u'_s}\right)^k &= \lim_{r\to\infty}\frac{\int_0^r s^{n-1}u_\mu^p(s)ds}{(1/kC_{n-1}^{k-1})^{((p)/(p-k))}(((2k)/(p-k)))^{((pk)/(p-k))}(n-((2pk)/(p-k)))^{((k)/(p-k))}r^{n-((2pk)/(p-k))}}\\
&= \lim_{r\to\infty}\frac{r^{n-1}u_\mu^p(r)}{(((1)/(kC_{n-1}^{k-1})^{((p)/(p-k))}(((2k)/(p-k)))^{((pk)/(p-k))}(n-((2pk)/(p-k)))^{((p)/(p-k))}r^{n-((2pk)/(p-k))-1}}\\
&= \lim_{r\to\infty}\frac{u_\mu^p(r)}{u_s^p(r)}.
\end{aligned}$$

By remark 2.1, there holds $\lim\limits_{r\to\infty} u'_\mu(r)/u'_s(r) = 1$ when $p \geqslant \max\{p^*, p_{jl}\}$. Thus, there exists sufficiently large $R > 0$ such that as $r > R$,

$$|u'_\mu(r)|^{k-1} = |u'_s(r)|^{k-1} + o(1)r^{-(((k-1)(p+k))/(p-k))}.$$

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Therefore, by the strict inequality (3.13), we can find a suitably small $\delta_0 > 0$ such that for any $\psi \in C_c^\infty(R^n \setminus \overline{B_R(0)})$,

$$\begin{aligned} &C_{n-1}^{k-1} \int_{R^n} \frac{|u_\mu'(|x|)|^{k-1}}{|x|^{k-1}} |\nabla\phi|^2 \mathrm{d}x \\ &= C_{n-1}^{k-1} \int_{R^n} \frac{|u_s'(|x|)|^{k-1} + o(1)|x|^{-(((k-1)(p+k))/(p-k))}}{|x|^{k-1}} |\nabla\phi|^2 \mathrm{d}x \\ &\geqslant C_0 \left[p\left(\frac{2}{p-k}\right)\left(n - \frac{2pk}{p-k}\right) + \delta_0 + o(1)\right] \int_{R^n} \frac{1}{|x|^{((2(p-1)k)/(p-k))}} \phi^2 \mathrm{d}x \\ &\geqslant p \int_{R^n} u_s(|x|)^{p-1}\phi^2 \mathrm{d}x. \end{aligned}$$

Here C_0 is the constant in (3.12). In view of $u_s(r) > u_\mu(r)$ for $r > R$ (see remark 1.3), we can see $Q_{u_\mu}(\varphi) \geqslant 0$ for any $\varphi \in C_c^\infty(R, \infty)$. In addition, u_μ satisfies (1.11). Thus, u_μ is stable on (R, ∞) for some $R > 0$.

Step 3. Let U_s be a singular solution of (1.4) with $p \in (p_{se}, p_{so})$ introduced in remark 1.1. By the same argument in the proof of Theorem 1.4, U_s still satisfies (1.11) since 0 is not the singular point in the improper integrals of (1.11) which is implied by $\lim_{r\to 0} U_s(r)/u_s(r) = 1$.

In addition, by an analogous argument in Step 1, U_s still satisfies (1.12). In fact, $\lim_{r\to\infty} U_s(r) r^{n-2k/k} = \lambda$ implies

$$U_s(r) \leqslant C r^{-((n-2k)/(k))} \quad \text{for large } r. \tag{3.17}$$

On the contrary, by (2.12), the monotonicity of U_s, and (3.17), there holds

$$|U_s'|^k \geqslant c r^{k-n} U_s^p(r) \int_0^r s^{n-1} \mathrm{d}s \geqslant c r^{k-p((n-2k)/(k))}$$

for large r. Therefore, applying the Caffarelli-Kohn-Nirenberg inequality (3.14) with $a = pn - 2k/kk - 1/2k$ and $b = a + 1$, we obtain by (3.17) and $p > p_{se}$ that

$$\begin{aligned} \int_{R^n} \frac{|U_s'(|x|)|^{k-1}}{|x|^{k-1}} |\nabla\phi|^2 \mathrm{d}x &\geqslant c \int_{R^n} \frac{\phi^2 \mathrm{d}x}{|x|^{p((n-2k)/(k))((k-1)/(k))+2}} \\ &\geqslant p \int_{R^n} U_s^{p-1}(|x|)\phi^2 \mathrm{d}x \end{aligned}$$

for any $\phi \in C_c^\infty(R^n \setminus \overline{B_R(0)})$ with suitably large R.

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