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PRESERVATION OF SUSLIN TREES AND SIDE CONDITIONS

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Abstract. We show how to force, with finite conditions, the forcing axiom PFA(T), a relativization of PFA to proper forcing notions preserving a given Suslin tree T. The proof uses a Neeman style iteration with generalized side conditions consisting of models of two types, and a preservation theorem for such iterations. The consistency of this axiom was previously known using a standard countable support iteration and a preservation theorem due to Miyamoto.

§**1. Introduction.** In this article,[1] using the techniques introduced by Neeman in [2], we give a consistency proof of the Forcing Axiom for the class of proper forcings that preserve a given Suslin tree T, i.e., PFA(T)[2]. The consistency of this axiom is already known ([3]), using a preservation result due to Miyamoto ([1]), who showed that the property "is proper and preserves every ω_1-Suslin tree" is preserved under a countable support iteration of proper forcings. The novelty of this proof is that PFA(T) is forced with finite conditions, using a forcing that acts like an iteration.

The main preservation theorem presented here, Theorem 4.13, can be seen as a general preservation schema for properties, like being a Suslin tree, that have formulations similar to Lemma 2.2, in terms of the possibility to construct a generic condition for a product forcing, by means of conditions that, individually, are generic for their respective forcings. As a matter of fact, in the proof of Theorem 4.13, no use is made of the fact that T is a tree.

In Section 2 we review some basic results connecting the property of being Suslin and properness. In Section 3 we show, as a warm up, that the method of side conditions—with just countable models—does not influence the fact that a proper forcing preserves a Suslin tree T. Then in Section 4 we use the method of generalized side conditions with models of two types to construct a model where PFA(T) holds and T remains Suslin. We refer to [2] and [4] for a detailed presentation of a pure side conditions poset with both countable and uncountable models.

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[1] A slightly different version of this article appeared as the last chapter of the PhD thesis of the author: *Contributions towards the generalization of Forcing Axioms*, defended on February 2014.

[2] See [3], for a survey of interesting applications of PFA(T).

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§2. Suslin trees and properness. We will use the following reformulation of the definition of Suslin tree.

LEMMA 2.1. *A tree T is Suslin iff for every countable $M \prec H(\theta)$, with θ sufficiently large such that $T \in M$, and for every $t \in T_{\delta_M}$, where $\delta_M = M \cap \omega_1$,*

$$t \text{ is an } (M, T)\text{-generic condition},$$

i.e., for every maximal antichain $A \subseteq T$ in M, there is a $\xi < M \cap \omega_1$ such that $t \upharpoonright \xi \in A$.

PROOF. On the one hand, let T be a Suslin tree, $M \prec H(\theta)$ as above, $t \in T_{\delta_M}$ and $A \in M$ a maximal antichain of T. Since T is Suslin, A is countable. Then there is a $\alpha < \delta_M$ such that for all $\beta \geq \alpha$, the set $A \cap T_\beta$ is empty. Hence there is an element $h \in A$ compatible with $t \upharpoonright \alpha$. Then $t \upharpoonright ht(h) = s \in A$. Moreover $s \in A \cap M$, since all levels of T are countable.

For the other direction if $A \in M$ is an uncountable maximal antichain of T, then $A \setminus M$ is not empty. For $x \in A \setminus M$, let $t = x \upharpoonright \delta_M$. If there is a $\xi < \delta_M$ such that $t \upharpoonright \xi \in A$, then x and $t \upharpoonright \xi$ would be compatible and both in A: a contradiction. ⊣

The following lemma connects preservation of Suslin trees and properness.

LEMMA 2.2 (Miyamoto, Proposition 1.1 in [1]). *Fix a Suslin tree T, a proper poset $\mathbb{P}$ and some regular cardinal θ, large enough. Then the following are equivalent:*

(1) $\Vdash_{\mathbb{P}}$ *"T is Suslin.",*
(2) *given $M \prec H(\theta)$ countable, containing $\mathbb{P}$ and T, if $p \in \mathbb{P}$ is an $(M, \mathbb{P})$-generic condition and $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, then (p, t) is an $(M, \mathbb{P} \times T)$-generic condition,*
(3) *given $M \prec H(\theta)$ countable, containing $\mathbb{P}$ and T and given $q \in \mathbb{P} \cap M$, there is a condition $p \leq q$ such that for every condition $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, we have that (p, t) is an $(M, \mathbb{P} \times T)$-generic condition.*

§3. Preservation of T and countable models. We define the scaffolding operator from an idea of Veličković.

DEFINITION 3.1. Given a proper poset $\mathbb{P}$ and a sufficiently large cardinal θ such that $\mathbb{P} \in H(\theta)$, let $\mathbb{M}(\mathbb{P})$ be the poset consisting of conditions $p = (\mathcal{M}_p, w_p)$ such that

(1) $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary substructures of $H(\theta)$,
(2) $w_p \in \mathbb{P}$,
(3) w_p is an $(M, \mathbb{P})$-generic condition for every M in $\mathcal{M}_p$.

Moreover, we let $q \leq p$ iff $\mathcal{M}_p \subseteq \mathcal{M}_q$ and $w_q \leq_{\mathbb{P}} w_p$.

REMARK 3.2. Notice that the definition of $\mathbb{M}(\mathbb{P})$ depends on θ. However this notation causes no confusion as long as θ depends on $\mathbb{P}$ and its choice is a standard negligible part of all arguments involving properness. Then, without any specification, θ will always denote a cardinal that makes possible the definition of $\mathbb{M}(\mathbb{P})$.

REMARK 3.3. By abuse of notation we will identify an $\in$-chain $\mathcal{M}_p$ with its range.

Our aim now is to show that properness is preserved by the scaffolding operator.

LEMMA 3.4. *Let $\mathbb{P}$ be a proper poset, $M \prec H(\theta)$ and $p \in \mathbb{M}(\mathbb{P}) \cap M$. Then there is a condition $p^M = (\mathcal{M}_{p^M}, w_{p^M}) \in \mathbb{M}(\mathbb{P})$ that is weaken than p and such that $M \in \mathcal{M}_{p^M}$.*

Proof. First of all notice that since $p \in M$, we have $\mathcal{M}_p \subseteq M$. In particular the largest model in $\mathcal{M}_p$ belongs to M. So $\mathcal{M}_p \cup \{M\}$ is a finite $\in$-chain of elementary substructures of $H(\theta)$. Moreover $w_p \in M \cap \mathbb{P}$ and, by properness, there is a $w_q \leq w_p$ that is $(M, \mathbb{P})$-generic. Now, since $w_q \leq w_p$ and w_p is $(N, \mathbb{P})$-generic, for every $N \in \mathcal{M}_p$, so is w_q. Then we have that w_q is a generic condition for every model in $\mathcal{M}_p \cup \{M\}$. Finally set $\mathcal{M}_{p^M} = \mathcal{M}_p \cup \{M\}$ and $w_{p^M} = w_q$ to see that the conclusion of the lemma holds. ⊣

Theorem 3.5. *Let $\mathbb{P}$ be a proper poset. Then $\mathbb{M}(\mathbb{P})$ is proper.*

Proof. Let M^* be a countable elementary submodel of $H(\theta^*)$, for some $\theta^* > \theta$, where θ is the corresponding cardinal in the definition of $\mathbb{M}(\mathbb{P})$. If p is a condition in $\mathbb{M}(\mathbb{P}) \cap M^*$ we need to find a condition $q \leq p$ that is $(M^*, \mathbb{M}(\mathbb{P}))$-generic. Fix then a dense open $D \subseteq \mathbb{M}(\mathbb{P})$ in M^* and let $M = M^* \cap H(\theta)$. We claim that $p^M = (\mathcal{M}_p \cup \{M\}, w_p^M)$ is an $(M, \mathbb{M}(\mathbb{P}))$-generic condition.

Thanks to Lemma 3.4 we have that p^M is a condition. We now prove its genericity. Let $r \leq p^M$ and without loss of generality assume it to be in D. Define

$$E = \{w_s \in \mathbb{P} : \exists \mathcal{M}_s \text{ such that } (\mathcal{M}_s, w_s) \in D \wedge \mathcal{M}_r \cap M \subseteq \mathcal{M}_s\}$$

and notice that $E \in M^*$ and $w_r \in E$.

The set E may not be dense in $\mathbb{P}$, but

$$E_0 = \{w_t \in \mathbb{P} : \exists w_s \in E \text{ such that } w_t \leq w_s \text{ or } \forall w_s \in E(w_t \perp w_s)\}$$

is a dense subset of $\mathbb{P}$ that belongs to M^*.

Then thanks to the $(M^*, \mathbb{P})$-genericity of w_p^M and the fact that $w_r \leq w_p^M$, we have that there is a condition $w_t \in M^* \cap E_0$ that is compatible with w_r. Since $w_r \in E$ there is a condition $w_s \in E$ such that $w_t \leq w_s$. By elementarity we can find w_s in M^*. Moreover, by definition of E, there is an $\mathcal{M}_s$ such that $(\mathcal{M}_s, w_s) \in D$ and such that $\mathcal{M}_r \cap M \subseteq \mathcal{M}_s$. Again by elementarity we can find $\mathcal{M}_s$ in M. Hence $(\mathcal{M}_s, w_s) \in D \cap M^*$.

Finally notice that w_s is compatible with w_r, because w_t is so and $w_t \leq w_s$; let t^* be the witness of it, i.e., $t^* \leq w_s, w_r$. Besides $\mathcal{M}_s \subseteq M$ and it extends $\mathcal{M}_r \cap M$, so we have that $\mathcal{M} = \mathcal{M}_s \cup \{M\} \cup \mathcal{M}_r \setminus M$ is a finite $\in$-chain of elementary submodel of $H(\theta)$. Then, in order to show that $(\mathcal{M}, t^*)$ is a condition in $\mathbb{M}(\mathbb{P})$ we need to show that t^* is $(N, \mathbb{P})$-generic, for every $N \in \mathcal{M}$. But this is true because on one hand $s \in \mathbb{M}(\mathbb{P})$ and so w_s is $(N, \mathbb{P})$-generic for every $N \in \mathcal{M}_s$ and on the other hand $r \in \mathbb{M}(\mathbb{P})$ and so w_r is $(N, \mathbb{P})$-generic for every $N \in \mathcal{M}_r$. Since t^* extends both w_s and w_r, we have that t^* is generic for all the models in $\mathcal{M}$. Hence $(\mathcal{M}, t^*)$ extends both s and r, in $\mathbb{M}(\mathbb{P})$, and witnesses their compatibility. ⊣

We now want to show that the scaffolding operation does not effect the preservation of a Suslin tree T. In order to show this fact we will use the characterization of Lemma 2.2.

Lemma 3.6. *Let T be a Suslin tree and let $\mathbb{P}$ be a proper forcing such that $\Vdash_{\mathbb{P}}$ "T is Suslin". Moreover let M^* be a countable elementary submodel of $H(\theta^*)$, for some $\theta^* > \theta$, where θ is the corresponding cardinal in the definition of $\mathbb{M}(\mathbb{P})$. If $p \in \mathbb{M}(\mathbb{P})$, $M = M^* \cap H(\theta) \in \mathcal{M}_p$ and $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, then (p, t) is an $(M^*, \mathbb{M}(\mathbb{P}) \times T)$-generic condition.*

Proof. Fix a set $D \subseteq \mathbb{M}(\mathbb{P}) \times T$ dense and open in M^* and fix a condition $(r, t') \leq (p, t)$, that without loss of generality we can assume to be in D. Then define

$$E = \bigcup \{(w_q, h) : (q, h) \in D, \mathcal{M}_r \cap M \subseteq \mathcal{M}_q\}$$

and notice that $E \in M$ and $(w_r, t') \in E$. Again the set E may not be dense, but the set $\bar{E} = E^{\leq} \cup E^{\perp}$, where

$$E^{\leq} = \bigcup \{(w_s, u) : \exists (w_q, h) \in E \text{ such that } (w_s, u) \leq (w_q, h)\} \text{ and}$$
$$E^{\perp} = \bigcup \{(w_s, u) : \forall (w_q, h) \in E (w_s, u) \perp (w_q, h)\},$$

is a dense subset of $\mathbb{P} \times T$ that belongs to M^*.

Now, since $M \in \mathcal{M}_r$, the condition w_r is $(M, \mathbb{P})$-generic, by definition of $\mathbb{M}(\mathbb{P})$. Moreover since $\Vdash_{\mathbb{P}}$ "T is Suslin" we have that (w_r, t') is $(M^*, \mathbb{P} \times T)$-generic. Then there is a $(w_s, u) \in \bar{E} \cap M^*$ that is compatible with (w_r, t'). This latter fact then implies that $(w_s, u) \in E^{\leq} \cap M^*$ and so there is a condition $(w_q, h) \in E$ such that $(w_s, u) \leq (w_q, h)$. By elementarity we can find $(w_q, h) \in M^*$ and again, by elementarity we can assume $q = (\mathcal{M}_q, w_q)$ to be in M^* and so $(q, h) \in D \cap M^*$. Finally letting $\mathcal{M}_e = \mathcal{M}_q \cup \{M\} \cup \mathcal{M}_r \setminus M$, and w_e be the witness of the compatibility between w_q and w_r, we have that $e = (\mathcal{M}_e, w_e) \in \mathbb{M}(\mathbb{P})$ and that (e, t') extends both (r, t') and (q, h). ⊣

Corollary 3.7. *Let T be a Suslin tree and let $\mathbb{P}$ be a proper forcing. Then $\Vdash_{\mathbb{P}}$ "T is Suslin" implies $\Vdash_{\mathbb{M}(\mathbb{P})}$ "T is Suslin".*

§4. PFA(T) with finite conditions.

We now show that it is possible to force an analog of the Proper Forcing Axiom for proper posets that preserve a given Suslin tree T. We will follow Neeman's presentation of the consistency of PFA with finite conditions, from [2], arguing that a slight modification of his method is enough for our purposes. Then we will argue that in the model we build T remains Suslin

Recall Neeman's definition of the forcing $\mathbb{A}$ (Definition 6.1 from [2]). Fix a supercompact cardinal θ and a Laver function $F : \theta \to H(\theta)$ as a book-keeping for choosing the proper posets that preserve T. Moreover define Z as the set of ordinals α, such that $(H(\alpha), F \restriction \alpha)$ is elementary in $(H(\theta), F)$. Then let $\mathcal{Z}^\theta = \mathcal{Z}_0^\theta \cup \mathcal{Z}_1^\theta$, where $\mathcal{Z}_0^\theta$ is the collection of all countable elementary substructures of $(H(\theta), F)$ and $\mathcal{Z}_1^\theta$ is the collection of all $H(\alpha)$, such that $\alpha \in Z$ has uncountable cofinality - hence $H(\alpha)$ is countably closed. Moreover, for $\alpha \in Z$, let $f(\alpha)$ be the least cardinal such that $F(\alpha) \in H(f(\alpha))$. Notice that, by elementarity, $f(\alpha)$ is smaller than the next element of Z above α.

Definition 4.1. If $\mathcal{M}$ is a set of models in $\mathcal{Z}^\theta$, let $\pi_0(\mathcal{M}) = \mathcal{M} \cap \mathcal{Z}_0^\theta$ and $\pi_1(\mathcal{M}) = \mathcal{M} \cap \mathcal{Z}_1^\theta$.

With an abuse of notation we will identify an $\in$-chain of models with the set of models that belong to it.

Definition 4.2. Let $\mathbb{M}_\theta^2$ be the poset whose conditions $\mathcal{M}_p$ are $\in$-chains of models in $\mathcal{Z}^\theta$ closed under intersection. If $p, q \in \mathbb{M}_\theta^2$, we define $p \leq q$ iff $\mathcal{M}_q \subseteq \mathcal{M}_p$.

See Claim 4.1 in [2] for the proof that $\mathbb{M}_\theta^2$ is $\mathcal{Z}^\theta$-strongly proper.

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DEFINITION 4.3. Let T be a Suslin tree and let $\mathbb{A}(T)$ be the poset consisting of pairs $p = (\mathcal{M}_p, w_p)$ so that:

(1) $\mathcal{M}_p \in \mathbb{M}^2_\theta$.

(2) w_p is a partial function on θ, with domain contained in the (finite) set $\{\alpha < \theta : H(\alpha) \in \mathcal{M}_p$ and $\Vdash_{\mathbb{A}(T)\cap H(\alpha)}$ "$F(\alpha)$ is a proper poset, that preserves $T\}$.

Moreover, for $\alpha \in dom(w_p)$,

(3.1) $w_p(\alpha) \in H(f(\alpha))$;

(3.2) $\Vdash_{\mathbb{A}(T)\cap H(\alpha)} w_p(\alpha) \in F(\alpha)$;

(3.3) If $M \in \pi_0(\mathcal{M}_p)$ and $\alpha \in M$, then $(\mathcal{M}_p \cap H(\alpha), w_p \restriction \alpha) \Vdash_{\mathbb{A}(T)\cap H(\alpha)}$ "$w_p(\alpha)$ is an $(M[\dot{G}_\alpha], F(\alpha))$-generic condition", where $\dot{G}_\alpha$ is the canonical name for the generic filter on $\mathbb{A}(T) \cap H(\alpha)$.

The ordering on $\mathbb{A}(T)$ is the following: $q \leq p$ iff $\mathcal{M}_p \subseteq \mathcal{M}_q$ and for every $\alpha \in dom(w_p)$, $(\mathcal{M}_q \cap H(\alpha), w_q \restriction \alpha) \Vdash_{\mathbb{A}(T)\cap H(\alpha)}$ "$w_q(\alpha) \leq_{F(\alpha)} w_p(\alpha)$".

REMARK 4.4. This inductive definition makes sense, since $\mathbb{A}(T) \cap H(\alpha)$ is definable in any $M \in \mathcal{Z}^\theta_0$, with $\alpha \in M$.

REMARK 4.5. Condition (5) holds for α and M iff it holds for α and $M \cap H(\gamma)$, whenever $\gamma \in Z \cup \{\theta\}$, is larger than α.

DEFINITION 4.6. Let β be an ordinal in $Z \cup \{\theta\}$. The poset $\mathbb{A}(T)_\beta$ consists of conditions $p \in \mathbb{A}(T)$ such that $\mathrm{dom}(w_p) \subseteq \beta$.

REMARK 4.7. In order to simplify the notation, if $p \in \mathbb{A}(T)$, then we define $(p)_\alpha$ to be $(\mathcal{M}_p, w_p \restriction \alpha)$, while by $p \restriction H(\alpha)$ we denote $(\mathcal{M}_p \cap H(\alpha), w_p \restriction \alpha)$. Notice that $(p)_\alpha \in \mathbb{A}(T)_\alpha$ and $p \restriction H(\alpha) \in \mathbb{A}(T) \cap H(\alpha)$.

Following Neeman it is possible to prove the following facts. See [2] for their proofs in the case of the forcing $\mathbb{A}$, i.e., the poset that forces PFA with finite conditions. Indeed, the only difference between $\mathbb{A}$ and $\mathbb{A}(T)$ is that the Laver function F picks up posets from a smaller class; namely the class of proper posets that preserve T.

THEOREM 4.8 (Neeman, Lemma 6.7 in [2]). *Let $\beta \in Z \cup \{\theta\}$. Then $\mathbb{A}(T)_\beta$ is $\mathcal{Z}^\theta_1$-strongly proper.*

CLAIM 4.9 (Neeman, Claim 6.10 in [2]). *Let $p, q \in \mathbb{A}(T)$. Let $M \in \pi_0(\mathcal{M}_p)$ and suppose that $q \in M$. Suppose that for some $\delta < \theta$, p extends $(q)_\delta$ and $dom(w_q) \setminus \delta$ is disjoint from $dom(w_p)$. Suppose further that $(\mathcal{M}_p \cap M) \setminus H(\delta) \subseteq \mathcal{M}_q$. Then there is $w_{p'}$ extending w_p so that $dom(w_{p'})$ =$dom(w_p) \cup (dom(w_q) \setminus \delta)$ and so that $p' = (\mathcal{M}_p, w_{p'})$ is a condition in $\mathbb{A}(T)$ extending q.*

THEOREM 4.10 (Neeman, Lemma 6.11 in [2]). *Let $\beta \in Z \cup \{\theta\}$. Let p be a condition in $\mathbb{A}(T)_\beta$. Let $\theta^* > \theta$ and let $M^* \prec H(\theta^*)$ be countable with $F, \beta \in M^*$. Let $M = M^* \cap H(\theta)$ and suppose that $M \in \pi_0(\mathcal{M}_p)$. Then:*

(1) *for every $D \in M^*$ which is dense in $\mathbb{A}(T)_\beta$ there is $q \in D \cap M^*$ which is compatible with p. Moreover there is $r \in \mathbb{A}(T)_\beta$ extending both p and q, so that $\mathcal{M}_r \cap M \setminus H(\beta) \subseteq \mathcal{M}_q$, and every model in $\pi_0(\mathcal{M}_r)$ above β and outside M is either a model in $\mathcal{M}_p$ or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$ and $W \in \pi_1(\mathcal{M}_q)$.*

(2) *p is an $(M^*, \mathbb{A}(T)_\beta)$-generic condition.*

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Theorem 4.11 (Neeman, Lemma 6.13 in [2]). *After forcing with $\mathbb{A}(T)$, PFA(T) holds.*

In order to show that $\mathbb{A}(T)$ preserves T, we need the following claim.

Claim 4.12. *If $\Vdash_{\mathbb{A}(T)_\alpha}$ "T is Suslin", then $\Vdash_{\mathbb{A}(T)_\alpha \cap H(\alpha)}$ "T is Suslin".*

Proof. In order to show that $\mathbb{A}(T)_\alpha \cap H(\alpha)$ preserves T, we use the equivalent formulation in Lemma 2.2. For this, fix a countable $M^* \prec H(\theta^*)$, with $\theta^* > \theta$ and $\alpha, T \in M^*$. Then, following Remark 4.4, both $\mathbb{A}(T)_\alpha \cap H(\alpha)$ and $\mathbb{A}(T)_\alpha$ are definable in M^*. If $p \in (\mathbb{A}(T)_\alpha \cap H(\alpha)) \cap M^*$, then we want to show that there is a condition $p' \leq p$ such that for every $t \in T_{\delta_{M^*}}$, with $\delta_{M^*} = M^* \cap \omega_1$, the condition (p', t) is $(M^*, (\mathbb{A}(T)_\alpha \cap H(\alpha)) \times T)$-generic.

Let $M = M^* \cap H(\theta)$ and $\mathcal{M}_{p^M}$ be the closure under intersection of $\mathcal{M}_p \cup \{M\}$. We can find a function w_{p^M} with the same domain as w_p such that $p^M = (\mathcal{M}_{p^M}, w_{p^M})$ is a condition in $\mathbb{A}(T)_\alpha$ and such that $p^M \restriction H(\alpha) \leq p$. We claim that $p^M \restriction H(\alpha)$ is the condition we need: i.e., $(p^M \restriction H(\alpha), t)$ is an $(M^*, (\mathbb{A}(T)_\alpha \cap H(\alpha)) \times T)$-generic condition, for every $t \in T_{\delta_{M^*}}$.

To this aim fix a set $D \in M^*$ dense in $(\mathbb{A}(T)_\alpha \cap H(\alpha)) \times T$, let $t \in T_{\delta_{M^*}}$ and let $(p^M \restriction H(\alpha), t^*) \in D$, for some t^* that might extends t properly. By Theorem 4.10, p^M is an $(M^*, \mathbb{A}(T)_\alpha)$-generic condition. Then, thanks to our hypothesis, (p^M, t) is an $(M, \mathbb{A}(T)_\alpha \times T)$-generic condition and so is (p^M, t^*).

Now define E to be the set of conditions $(q, h) \in \mathbb{A}(T)_\alpha \times T$ such that $(q \restriction H(\alpha), h) \in D$ and such that $\mathcal{M}_{p^M} \cap M \subseteq \mathcal{M}_q$. Notice that $(p^M, t^*) \in E$ and $E \in M^*$. The set E may not be dense, but $E_0 = E_0^{\leq} \cup E_0^{\perp}$, where

$$E_0^{\leq} = \{(q_0, h_0) : \exists (q, h) \in E \text{ such that } (q_0, h_0) \leq (q, h)\},$$

and

$$E_0^{\perp} = \{(q_0, h_0) : \forall (q, h) \in E \ (q_0, h_0) \perp (q, h)\},$$

is a dense subset of $\mathbb{A}(T)_\alpha \times T$ belonging to M^*.

Then there is $(q_0, h_0) \in E_0 \cap M^*$ that is compatible with (p^M, t^*). Since $(p^M, t^*) \in E$, by definition of E_0, there is a condition $(q, h) \in E$ that is compatible with (p^M, t^*). By elementarity we can assume $(q, h) \in E \cap M^*$. Now, the key observation is that by strong genericity of the pure side conditions if (r, t^*) witnesses that (p^M, t^*) and (q, h) are compatible, then $(r \restriction H(\alpha), t^*)$ witnesses that $(p \restriction H(\alpha), t^*)$ and $(q \restriction H(\alpha), h)$ are compatible. This is sufficient for our claim, because by definition of E and since q is finite, $(q \restriction H(\alpha), h) \in D \cap M^*$. ⊣

We can now state and proof the main preservation theorem of this section.

Theorem 4.13. *If G is a generic filter for $\mathbb{A}(T)$, then in $V[G]$ the tree T is Suslin.*

Proof. We proceed by induction on $\beta \in Z \cup \{\theta\}$, proving that $\mathbb{A}(T)_\beta$ preserves T. If β is the first element of Z, then $\mathbb{A}(T)_\beta = \mathbb{M}^2_\theta$. ⊣

Claim 4.14. *The forcing $\mathbb{M}^2_\theta$ preserves T.*

Proof. Let $M^* \prec H(\theta^*)$ be a countable model with $\theta^* > \theta$, containing $\mathbb{M}^2_\theta$ and T, and let $\mathcal{M}_p \in \mathbb{M}^2_\theta$ be an $(M^*, \mathbb{M}^2_\theta)$-generic condition, with $M = M^* \cap H(\theta) \in \mathcal{M}_p$. Moreover, let $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$. Thanks to Lemma 2.2, it is sufficient to show that $(\mathcal{M}_p, t)$ is an $(M^*, \mathbb{M}^2_\theta \times T)$-generic condition.

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To this aim, let $D \in M^*$ be a dense subset of $\mathbb{M}^2_\theta \times T$ and assume, by density of D, that $(\mathcal{M}_p, t) \in D$. Then define

$$E = \{h \in T : \exists \mathcal{M}_q \in \mathbb{M}^2_\theta \text{ such that } (\mathcal{M}_q, h) \in D \wedge \mathcal{M}_p \cap M \subseteq \mathcal{M}_q\}.$$

Since $\mathbb{M}^2_\theta, D, \mathcal{M}_p \cap M \in M^*$, we have $E \in M^*$. The set E may not be dense in T but

$$\bar{E} = \{\bar{h} \in T : \exists h \in E(\bar{h} \leq h) \vee \forall h \in E(\bar{h} \perp h)\}.$$

belongs to M^* and it is dense in T.

By (M^*, T)-genericity of t, there is an $\bar{h} \in \bar{E} \cap M$ that is compatible with t. Moreover, since $(\mathcal{M}_p, t) \in D$, we have that $t \in E$. Since $t \in E$ and $\bar{h} \in \bar{E}$ are compatible, by definition of $\bar{E}$, there is $h \in E$, with $\bar{h} \leq h$. By elementarity pick such an h in M^*. Then, by definition of E, there is $\mathcal{M}_q \in \mathbb{M}^2_\theta$, with $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$, such that $(\mathcal{M}_q, h) \in D$. By elementarity we can find $\mathcal{M}_q \in M^*$. Then, since $\mathcal{M}_p$ is $(M, \mathbb{M}^2_\theta)$-strong generic and $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$, we have that $\mathcal{M}_p$ and $\mathcal{M}_q$ are compatible. Finally, t and $\bar{h}$ are compatible because $t \leq \bar{h}$ and $\bar{h} \leq h$. Hence $(\mathcal{M}_p, t)$ and $(\mathcal{M}_q, h)$ are compatible in $\mathbb{M}^2_\theta \times T$ and this compatibility, together with the fact that $(\mathcal{M}_q, h) \in D \cap M^*$, witnesses that $(\mathcal{M}_p, t)$ is $(M^*, \mathbb{M}^2_\theta \times T)$-generic.

If β is the successor of α in Z, then, by inductive hypothesis $\mathbb{A}(T)_\alpha$, preserves T. In order to show that $\mathbb{A}(T)_\beta$ also preserves T, we use the characterization of Lemma 2.2. For this, let $M^* \prec H(\theta^*)$ be a countable model, with $\theta^* > \theta$, containing β, F, and T. Notice that $\mathbb{A}(T)_\beta$ is definable in M^*, with β as a parameter. Moreover let $p \in \mathbb{A}(T)_\beta$ be an $(M^*, \mathbb{A}(T)_\beta)$-generic condition, with $M = M^* \cap H(\theta) \in \mathcal{M}_p$, and let $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$. Then we want to show that (p, t) is an $(M^*, \mathbb{A}(T)_\beta \times T)$-generic condition.

By elementarity of M^*, $\alpha \in M^*$. Now, fix a V-generic filter G over $\mathbb{A}(T)_\alpha$, with $(p)_\alpha \in G$. By Theorem 4.10 $(p)_\alpha$ is an $(M^*, \mathbb{A}(T)_\alpha)$-generic condition for M^* and so $M^*[G] \cap V = M^*$.

If $H(\alpha) \notin \mathcal{M}_p$ and p cannot be extended to a condition containing $H(\alpha)$, then $\mathbb{A}(T)_\beta$ below p, is equivalent to $\mathbb{A}(T)_\alpha$ below p. Then, forcing below p, the conclusion follows by inductive hypothesis. Hence, we may assume $H(\alpha) \in \mathcal{M}_p$.

Let $G_\alpha = G \cap H(\alpha)$. Then, by Theorem 4.8, we have that G_α is a V-generic filter on $\mathbb{A}(T) \cap H(\alpha)$, because $\mathbb{A}(T)_\alpha \cap H(\alpha) = \mathbb{A}(T) \cap H(\alpha)$. Without loss of generality, we can assume $\Vdash_{\mathbb{A}(T) \cap H(\alpha)}$ "$F(\alpha)$ is a proper poset that preserves T", because otherwise $\mathbb{A}(T)_\beta$ is equal to $\mathbb{A}(T)_\alpha$ and again the conclusion follows by inductive hypothesis. Let $\mathbb{Q} = F(\alpha)[G_\alpha]$. Then, by properness of $\mathbb{Q}$ in $V[G_\alpha]$, modulo extending p, we can assume $\alpha \in \mathrm{dom}(w_p)$.

Fix $D \subseteq \mathbb{A}(T)_\beta \times T$ dense and in M^*. Moreover, let $(p, t^*) \in D$, for some t^* that might extends t properly. Since we will work in $V[G_\alpha]$, we need to ensure that $\Vdash_{\mathbb{A}(T) \cap H(\alpha)}$ "T is Suslin". But this is true, by inductive hypothesis, as the Claim 4.12 shows.

Now, in $V[G_\alpha]$, define E to be the set of couples $(u, h) \in \mathbb{Q} \times T$ for which there is a condition $(q, h) \in \mathbb{A}(T)_\beta \times T$ such that

(1) $w_q(\alpha)[G_\alpha] = u$,
(2) $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$,
(3) $(q, h) \in D$, and
(4) $q \restriction H(\alpha) \in G_\alpha$.

Notice that $E \in M^*[G_\alpha]$ and that $(w_p(\alpha)[G_\alpha], t^*) \in E$. The set E may not be dense, but if we define $E_0 = E_0^{\leq} \cup E_0^{\perp}$, with

$$E_0^{\leq} = \{(u_0, h_0) \in \mathbb{Q} \times T : \exists (u, h) \in E\ (u_0, h_0) \leq (u, h)\}$$

and

$$E_0^{\perp} = \{(u_0, h_0) \in \mathbb{Q} \times T : \forall (u, h) \in E\ (u_0, h_0) \perp (u, h)\},$$

we have that E_0 is dense in $\mathbb{Q} \times T$. Moreover, notice that by elementarity E_0 is in $M^*[G_\alpha]$.

Now, since $M \in \pi_0(\mathcal{M}_p)$ and $\alpha \in M^* \cap H(\theta) = M$, we have that $\Vdash_{\mathbb{A}(T) \cap H(\alpha)}$ "$w_p(\alpha)$ is an $(M^*[\dot{G}_\alpha], F(\alpha))$-generic condition", where $\dot{G}_\alpha$ is a $\mathbb{A}(T) \cap H(\alpha)$-name for G_α. Moreover, $\Vdash_{\mathbb{A}(T) \cap H(\alpha)}$ "$F(\alpha)$ is a proper poset that preserves T" and, by inductive hypothesis and Lemma 4.12, $\Vdash_{\mathbb{A}(T) \cap H(\alpha)}$ "T is Suslin". Then by Lemma 2.2 applied in $V[G_\alpha]$ we have that $(w_p(\alpha)[G_\alpha], t)$ is an $(M^*[G_\alpha], \mathbb{Q} \times T)$-generic condition and so is $(w_p(\alpha)[G_\alpha], t^*)$.

Hence, there is a condition $(u_0, h_0) \in E_0 \cap M^*[G_\alpha]$ that is compatible with $(w_p(\alpha)[G_\alpha], t^*)$. Moreover, since $(w_p(\alpha)[G_\alpha], t^*) \in E$ we have that $(u_0, h_0) \in E_0^{\leq}$. This means that there is $(u, h) \in E$ such that $(u_0, h_0) \leq (u, h)$. By construction (u, h) is compatible with $(w_p(\alpha)[G_\alpha], t^*)$ and by elementarity we can find such a condition in $M^*[G_\alpha]$. Let $u_\alpha \in \mathbb{Q}$ be a witness of the compatibility between $w_p(\alpha)[G_\alpha]$ and u. Notice that u_α is an $(N[G_\alpha], \mathbb{Q})$-generic condition for all $N \in \pi_0(\mathcal{M}_p)$ with $\alpha \in N$, because $u_\alpha \leq w_p(\alpha)[G_\alpha]$. Since $(u, h) \in E$, there is a condition $q \in \mathbb{A}(T)_\beta$ with $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$ and $w_q(\alpha)[G_\alpha] = u$ such that $(q, h) \in D$. By elementarity let $q \in M^*[G_\alpha]$ and so $(q, h) \in M^*[G_\alpha] \cap D$. Since $M^*[G_\alpha] \subseteq M^*[G]$ and $M^*[G] \cap V = M^*$, we have $(q, h) \in D \cap M^*$. Now, by strong genericity of the pure side conditions, letting $\mathcal{M}_r$ be the closure under intersection of $\mathcal{M}_p \cup \mathcal{M}_q$, we have that $\mathcal{M}_r$ witnesses that $\mathcal{M}_p$ and $\mathcal{M}_q$ are compatible. Moreover every model in $\pi_0(\mathcal{M}_r)$ above β and outside M is either a model in $\mathcal{M}_p$ or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$ and $W \in \pi_1(\mathcal{M}_q)$. Then u_α is an $(N[G_\alpha], \mathbb{Q})$-generic condition, for all $N \in \pi_0(\mathcal{M}_r)$, with $\alpha \in N$, because of Remark 4.5 together with the fact that u_α extends both $w_p(\alpha)[G_\alpha]$ and u.

Finally, back in V, let $\dot{u}$ and $\dot{u}_\alpha$ be $\mathbb{A}(T)_\alpha \cap H(\alpha)$-names for u and u_α. Moreover, let $e \in \mathbb{A}(T)_\alpha \cap H(\alpha)$ be sufficiently strong to force all the properties we showed for q, $\dot{u}$, and $\dot{u}_\alpha$. We can also assume that e extends both $q \restriction H(\alpha)$ and $p \restriction H(\alpha)$. Now notice that $\mathcal{M}_e \cup \mathcal{M}_r$ is already an $\in$-chain closed under intersection and so if $\mathcal{M}_s = \mathcal{M}_e \cup \mathcal{M}_r$ and $w_s = w_e \cup (\alpha, \dot{u}_\alpha)$, we have that s is a condition in $\mathbb{A}(T)_\beta$. Hence (s, t^*) witnesses that (p, t) and (q, h) are compatible.

If β is a limit point of Z, let again $M^* \prec H(\theta^*)$ be a countable model containing $\mathbb{A}(T)_\beta$ and F. Then if $p \in \mathbb{A}(T)_\beta$, with $M^* \cap H(\theta) = M \in \mathcal{M}_p$, and $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, then, thanks to Lemma 2.2, it is sufficient to show that (p, t) is an $(M^*, \mathbb{A}(T)_\beta \times T)$-generic condition, in order to prove that $\mathbb{A}(T)_\beta$ preserves that T is Suslin.

To this aim, let $\bar{\beta} = sup(\beta \cap M^*)$ and let $\delta < \bar{\beta}$, in $Z \cap M^*$, be such that $\mathrm{dom}(w_p) \subseteq \delta$. Moreover fix $D \in M^*$ dense in $\mathbb{A}(T)_\beta \times T$ and let $(p, t^*) \in D$, for some t^* that might extends t properly.

Now, define E as the set of conditions $((q)_\delta, h) \in \mathbb{A}(T)_\delta \times T$ that extend to conditions $(q, h) \in D$, with $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$. The set E belongs to M^*, but it may

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not be dense in $\mathbb{A}(T)_\delta \times T$. However the set $E_0 = E_0^{\leq} \cup E_0^{\perp}$ is dense in $\mathbb{A}(T)_\delta \times T$ and belongs to M^*; where

$$E_0^{\leq} = \{(q_0, h_0) \in \mathbb{A}(T)_\delta \times T : \exists((q)_\delta, h) \in E \text{ such that } (q_0, h_0) \leq ((q)_\delta, h)\},$$

and

$$E_0^{\leq} = \{(q_0, h_0) \in \mathbb{A}(T)_\delta \times T : \forall((q)_\delta, h) \in E \ (q_0, h_0) \perp ((q)_\delta, h)\}.$$

Then, by the inductive hypothesis, find a condition $(q_0, h_0) \in E_0 \cap M^*$ that is compatible with $((p)_\delta, t^*)$. Moreover, since $((p)_\delta, t^*) \in E$ and it is compatible with (q_0, h_0), we have that $(q_0, h_0) \in E_0^{\leq}$. Then, by definition of $E_0^{\leq}$, there is a condition $((q)_\delta, h) \in E$ such that $(q_0, h_0) \leq ((q)_\delta, h)$ and which therefore is compatible with $((p)_\delta, t^*)$. By elementarity pick such a condition in M^*. Moreover, thanks to the fact that $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$ and that $\mathcal{M}_p \cap M$ witnesses the M-strong genericity of $\mathcal{M}_p$, we have that the compatibility between $((p)_\delta, t^*) = ((p)_\beta, t^*)$ and $((q)_\delta, h)$ is witnessed by a condition $\big((\mathcal{M}_r, w_1), t^*\big)$, where $\mathcal{M}_r$ is the closure under intersection of $\mathcal{M}_p \cup \mathcal{M}_q$. Then we have that $\mathcal{M}_r \cap M \setminus H(\beta) \subseteq \mathcal{M}_q$, and that every model in $\pi_0(\mathcal{M}_r)$ above β and outside M is either a model in $\mathcal{M}_p$ or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$ and $W \in \pi_1(\mathcal{M}_q)$.

Now, let $(q, h) \in D$ witness that $((q)_\delta, h) \in E$. By elementarity, we can assume $(q, h) \in D \cap M^*$. Then, thanks to the fact that $\mathcal{M}_r \cap M \setminus H(\beta) \subseteq \mathcal{M}_q$ we can apply Claim 4.9 and find a function w_2, extending w_1, defined as $\text{dom}(w_2) = \text{dom}(w_1) \cup (\text{dom}(w_q) \setminus \delta)$, such that $\big((\mathcal{M}_r, w_2), t^*\big)$ extends (q, h). Setting $w_r = w_2 \cup w_p \restriction [\bar{\beta}, \beta)$, we claim that r belongs to $\mathbb{A}(T)_\beta$.

In order to show that this latter claim holds, it is sufficient to show that if $\alpha \in \text{dom}(w_p) \restriction [\bar{\beta}, \beta)$, then $p \restriction H(\alpha)$ forces that $w_r(\alpha) = w_p(\alpha)$ is an $(N[\dot{G}_\alpha], F(\alpha))$-generic condition, where $\dot{G}_\alpha$ is the canonical name for a V-generic filter over $\mathbb{A}(T) \cap H(\alpha)$ and $N \in \pi_0(r)$, with $\alpha \in N$. Notice that $\alpha \in N$ implies $N \notin M$. Then, since p is a condition, the claim follows thanks to Remark 4.5 and the fact that every model in $\pi_0(\mathcal{M}_r)$ above β and outside M is either a model in $\mathcal{M}_p$ or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$.

Hence, finally we have that (r, t^*) belongs to $\mathbb{A}(T)_\beta \times T$ and that, by construction, it extends both (q, h) and (p, t). ⊣

§**5. Conclusions.** As stated in the introduction, Theorem 4.13 could be in principle generalized to other forcing notions. Indeed the argument patterns of all new results of this paper are similar and in proving them we just used, on the one hand, the fact that the product of a strong proper forcing with a proper forcing produces a proper forcing (see Claim 3.8 in [2]), while on the other hand we did not use essential properties of the tree T, except the characterization of Lemma 2.2. As a consequence we can state a more general theorem that extends both Theorem 4.11 and Theorem 4.13.

Definition 5.1. Let $\mathbb{Q}$ be a proper forcing. We define the class $\Gamma_{\mathbb{Q}}$ as the collection of all proper forcing notions $\mathbb{P}$ such that for every countable $M \prec H(\kappa)$, with κ sufficiently large and such that $\mathbb{Q}, \mathbb{P} \in M$, if $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic and $q \in \mathbb{Q}$ is $(M, \mathbb{Q})$-generic, then (p, q) is $(M, \mathbb{P} \times \mathbb{Q})$-generic.

Definition 5.2. If $\mathbb{Q}$ is a proper forcing we let $PFA(\mathbb{Q})$ be the forcing axiom, relative to collections of $\aleph_1$-many dense sets, for the class $\Gamma_{\mathbb{Q}}$.

THEOREM 5.3. *Given a proper forcing $\mathbb{Q}$, there is a proper forcing $\mathbb{A}(\mathbb{Q})$ that, modulo the existence of a supercompact cardinal, forces $PFA(\mathbb{Q})$. Moreover, the forcing notion $\mathbb{A}(\mathbb{Q})$ belongs to the class $\Gamma_{\mathbb{Q}}$.*

Indeed if we define $\mathbb{A}(\mathbb{Q})$ appropriately along the lines of Definition 4.3, then a slight modification of Neeman's consistency proof for PFA ([2]) shows that $\mathbb{A}(\mathbb{Q})$ is proper and that it forces $PFA(\mathbb{Q})$.

Moreover, we can summarize the proof of Theorem 4.13 as showing that the following property holds for all $\alpha \in Z \cup \{\theta\}$.

$(*)_\alpha$: if M is a countable elementary substructure of $H(\theta)$ such that $\mathbb{A}(T)_\alpha$, $T \in M$, then if p is $(M, \mathbb{A}(T)_\alpha)$-generic and t is (M, T)-generic, then (p, t) is $(M, \mathbb{A}(T)_\alpha \times T)$-generic.

Theorem 4.13 then shows that $(*)_\theta$ holds. Hence, we can also conclude that $\mathbb{A}(T)$ belongs to the class Γ_T. Since Theorem 4.13 makes no use of the tree structure of T—nor of the fact that T is c.c.c.—but only of its properness, the same argument can be adapted in order to show that $\mathbb{A}(\mathbb{Q})$ belongs to the class $\Gamma_{\mathbb{Q}}$.

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