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# ON KÄHLER NILMANIFOLDS WITH TOP HOMOLOGY IN CODIMENSION TWO

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Suppose $G$ is a connected, complex, nilpotent Lie group and $\Gamma$ is a discrete subgroup of $G$ such that $G/\Gamma$ is Kähler and the top nonvanishing homology group of $G/\Gamma$ (with coefficients in $\mathbb{Z}_2$) is in codimension two or less. We show that $G$ is then Abelian. We also note that an example from [12] shows that this fails if the top nonvanishing homology is in codimension three.

## 1. INTRODUCTION

Consider a complex homogeneous manifold $X = G/H$, where $G$ is a connected complex Lie group and $H$ is a closed, complex subgroup. There have been a number of results concerning the structure of such $X$ that are Kähler. If $X$ is compact, see the work of Matsushima [9] and Borel and Remmert [5] and if the metric is $G$-invariant see [6]. If $X$ is not compact and the metric is not necessarily $G$-invariant, then the situation appears to be much more complicated. There are some results, but usually under some restrictions on the structure of the group $G$. For example, if $G$ is semisimple, then $X$ is Kähler if and only if $H$ is an algebraic subgroup of $G$, see [4, 3], and if $G$ is solvable, then the fibre of its holomorphic reduction is a Cousin group [12].

This note presents an observation about Kähler homogeneous manifolds with Klein form $G/\Gamma$ with $\Gamma$ a discrete subgroup of a complex, nilpotent group $G$ under the topological assumption $d_{G/\Gamma} \leqslant 2$, that is, the top nonvanishing homology group of $G/\Gamma$ with coefficients in $\mathbb{Z}_2$ is in codimension at most two; see the next section for definitions. The group $G$ must be Abelian in this setting. We rely heavily upon Lie algebra calculations from [12]. We also note that an example from [12] shows that without the topological assumption $d_{G/\Gamma} \leqslant 2$ there exist Kähler nilmanifolds where no Abelian group can act transitively.

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## 2. Topological Preliminaries

The purpose of this section is to introduce the topological machinery that we shall need.

DEFINITION: Suppose $X$ is a connected, smooth manifold $X$. Define

$$h_X := \min\{r \mid H_k(X, \mathbb{Z}_2) = 0 \text{ for all } k > r\}$$

and set

$$d_X := \dim X - h_X.$$

Note that $d_X = 0$ if and only if $X$ is compact.

In order to deal with fibre bundles we need the following Fibration Lemma from [**1**]. The proofs of these statements involve spectral sequence arguments and are straightforward; see [**1**, Lemma 1 and Lemma 2]. Note that $X, F$, and $B$ are manifolds in our setting and a manifold always has the homotopy type of a $CW$-complex, see [**10**].

LEMMA 1. (Fibration Lemma) *Suppose $X \xrightarrow{F} B$ is a fibre bundle, where $X, F$, and $B$ are smooth manifolds with $X$ connected.*

(a) *Let $B$ have the homotopy type of a $CW$-complex of dimension $q$. Then*

$$d_X \geqslant d_F + (\dim B - q).$$

(b) *Moreover, if the bundle is simple, then*

$$d_X = d_F + d_B.$$

*In particular, $d_X \geqslant d_F$.*

## 3. Kähler Nilmanifolds

For $G$ a connected, simply connected, complex nilpotent Lie group the exponential map $\exp : \mathfrak{g} \to G$ is one-to-one and onto. Let $\Gamma$ be a discrete subgroup of $G$ and consider the (real) Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ that is spanned by the lattice $\exp^{-1}(\Gamma)$. Since $G$ is simply connected, the corresponding connected Lie subgroup $G_0 := \exp(\mathfrak{g}_0)$ is closed in $G$ and $G_0/\Gamma$ is compact. We use the notation $\langle\Gamma\rangle_G$ to denote the connected complex subgroup of $G$ that has Lie algebra $\mathfrak{g}_0 + i\mathfrak{g}_0$. (This group is also closed in $G$.) Since $G/\Gamma$ is biholomorphic to $G/\langle\Gamma\rangle_G \times \langle\Gamma\rangle_G/\Gamma$ and $G/\langle\Gamma\rangle_G$ is biholomorphic to $\mathbb{C}^k$ for some nonnegative integer $k$, the most interesting case occurs when $\langle\Gamma\rangle_G = G$. Then $G_0/\Gamma$ is a *generic* $CR$-submanifold of $G/\Gamma$. If, in addition, $G/\Gamma$ is Kähler, then we call the triple $(G, G_0, \Gamma)$ a Kähler Cauchy–Riemann nilmanifold; see [**12**] for Kähler Cauchy–Riemann solvmanifolds.

Given a Kähler Cauchy–Riemann nilmanifold $(G, G_0, \Gamma)$, we set

$$\mathfrak{m} := \mathfrak{g}_0 \cap i \cdot \mathfrak{g}_0.$$

and let $\mathfrak{z}$ denote the centre of $\mathfrak{g}$. Using detailed Lie algebra computations Oeljeklaus and Richthofer made two observations about Kähler Cauchy–Riemann nilmanifolds in [**12**] that we shall use in the following. In the proof of Theorem 2' on pp. 409 - 410 they showed that

$$\mathfrak{m} \subset \mathfrak{z}. \tag{1}$$

(An assumption in the statement of their Theorem 2' is that $\mathcal{O}(G/\Gamma) = \mathbb{C}$. However, this assumption is not used in the first steps of the proof which involve showing that (1) holds, but rather in a later part of the proof in order to show that $G$ is Abelian.) Oeljeklaus and Richthofer also noted in [**12**, Remark 4(b)] that

$$\mathfrak{g}_0' \cap \mathfrak{m} = (0) \tag{2}$$

in the Kähler Cauchy–Riemann nilmanifold setting. We shall use (1) and (2) later on.

**Lemma 2.** *Let $\mathfrak{g}$ be a complex Lie algebra with centre $\mathfrak{z}$ and assume* $\operatorname{codim}_{\mathbb{C}}$ $\mathfrak{g}/\mathfrak{z} \leqslant 1$. *Then $\mathfrak{g}$ is Abelian.*

Proof: We assume that $\mathfrak{g}$ is not Abelian, and thus $\operatorname{codim}_{\mathbb{C}}\mathfrak{g}/\mathfrak{z} = 1$, and derive a contradiction from this. Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$ be the projection and pick a generator $X \in \mathfrak{g}/\mathfrak{z}$. Choose any $X' \in \mathfrak{g}$ with $\pi(X') = X$. Then for any elements $X_1, X_2 \in \mathfrak{g}$ one has $\pi(X_i) = \alpha_i X$ with $\alpha_i \in \mathbb{C}$ for $i = 1, 2$. Thus $X_i = \alpha_i X' + Y_i$, where $Y_i \in \mathfrak{z}$ for $i = 1, 2$. Since the $Y_i$ are central, this gives

$$[X_1, X_2] = [\alpha_1 X' + Y_1, \alpha_2 X' + Y_2] = \alpha_1\alpha_2[X', X'] = 0.$$

This shows that $\mathfrak{g}$ is Abelian, a contradiction to our assumption and completes the proof. ∎

The next observation was pointed out to us by K. Oeljeklaus.

**Proposition 1.** *Let $(G, G_0, \Gamma)$ be a Kähler Cauchy–Riemann nilmanifold with $\langle\Gamma\rangle_G = G$ and let $G/\Gamma \to G/J$ be its holomorphic reduction. Then $J^\circ \subset Z$, where $Z$ is the centre of $G$. At the Lie algebra level one has*

$$\mathfrak{m} \subset \mathfrak{j} \subset \mathfrak{z}.$$

Proof: Since $G/J$ is holomorphically separable, $J^\circ$ is the smallest connected closed complex subgroup of $G$ that contains $M$, the connected group corresponding to the Lie algebra $\mathfrak{m}$; see [**7**, the proof of Theorem 7, p. 46]. By (1) one has $\mathfrak{m} \subset \mathfrak{z}$. But the subgroup $Z \cdot \Gamma$ is closed in $G$, see [**7**, Theorem 4]. Thus $J^\circ \subset Z \cdot \Gamma$ and, because $J^\circ$ is connected, $J^\circ \subset Z$. The corresponding statement about the inclusions of the Lie algebras follows from this inclusion. ∎

**Theorem 1.** *Let $(G, G_0, \Gamma)$ be a Kähler Cauchy–Riemann nilmanifold with $d_{G/\Gamma} \leqslant 2$. Then $G$ is Abelian.*

PROOF: If $\mathcal{O}(G/\Gamma) \cong \mathbb{C}$, then $G/\Gamma$ is a Cousin group by [**12**] and thus $G$ is Abelian. In particular, this handles the case when $G/\Gamma$ is compact.

Next we assume that $\mathcal{O}(G/\Gamma) \not\cong \mathbb{C}$ and let $G/\Gamma \to G/J$ be its holomorphic reduction, see [**7**]. Note that $\dim G/J > 0$ by assumption. Since $G/J$ is Stein [**7**], one has

$$d_{G/J} \geqslant \dim_{\mathbb{C}} G/J.$$

From the Fibration Lemma one gets $d_{G/\Gamma} \geqslant d_{G/J}$. Thus

$$2 \geqslant d_{G/J} \geqslant \dim_{\mathbb{C}} G/J. \tag{3}$$

CASE 1 $\dim_{\mathbb{C}} G/J = 1$. Since $J^\circ \subset Z$, one has $\operatorname{codim}_{\mathbb{C}} Z \leqslant \operatorname{codim}_{\mathbb{C}} J = 1$. So $G$ is Abelian by Lemma 2.

CASE 2 $\dim_{\mathbb{C}} G/J = 2$. One must have equality in equation (3). Hence $G/J$ is a Stein solvmanifold with $d_{G/J} = 2$. Every solvmanifold $X$ admits a fibration $X \to Y$, where $Y$ is a compact solvmanifold and the fibre is a real vector space of real dimension $d_X$ [**2, 11**]. So one has a fibration

$$G/J \xrightarrow{\mathbb{R}^2} Y,$$

where $Y$ is a compact solvmanifold with real dimension two. In particular, $G/J$ has the homotopy type of a CW-complex of dimension 2. Since $d_{G/\Gamma} = 2$, it follows from the Fibration Lemma that $d_{J/\Gamma} = 0$ and the fibre $J/\Gamma$ of the holomorphic reduction of $G/\Gamma$ is compact. Since the base of the holomorphic reduction of $G/\Gamma$ is Stein and its fibre is a torus, one sees that $\mathfrak{m} = \mathfrak{j} \subset \mathfrak{z}$. So the complex codimension of $\mathfrak{m}$ in $\mathfrak{g}$ is two.

Let's consider the possibilities for how $\mathfrak{z}$ can fit into the following inclusions

$$\mathfrak{m} \subset \mathfrak{z} \subset \mathfrak{g}.$$

First suppose that $\mathfrak{m} = \mathfrak{z}$. In a nilpotent Lie algebra the centre $\mathfrak{z}$ meets every ideal nontrivially. In particular, $\mathfrak{z}$ meets $\mathfrak{g}_0'$ nontrivially, since the latter is not zero, if $G$ is not Abelian. Thus

$$(0) \neq \mathfrak{g}_0' \cap \mathfrak{z} = \mathfrak{g}_0' \cap \mathfrak{m}.$$

But this contradicts (2). Hence $\mathfrak{m}$ is a proper subalgebra of $\mathfrak{z}$. By Lemma 2 it is not possible that the codimension of $\mathfrak{z}$ in $\mathfrak{g}$ be exactly equal to one. Thus $\mathfrak{z} = \mathfrak{g}$. Therefore, if $G/\Gamma$ is Kähler with $d_{G/\Gamma} \leqslant 2$, then $G$ is Abelian. This completes the proof of the Theorem. ∎

EXAMPLE 1. Then [**12**, Example 6(a)] illustrates "how one can fit" both $\mathfrak{m}$ and $\mathfrak{g}_0$ into $\mathfrak{z}$ so that they do not meet in an example of a Kähler nilmanifold $X$ with $d_X = 3$. In this example, no Abelian group can act transitively on $X$, since $\pi_1(X)$ is not Abelian. Here the complex codimension of $\mathfrak{z}$ in $\mathfrak{g}$ is two and the complex codimension of $\mathfrak{m}$ in $\mathfrak{z}$ is one.

EXAMPLE 2. Thus [**12**, Example 6(b)] is an example of a $G/\Gamma$, where $\mathfrak{m} = \mathfrak{z}$ has codimension two in $\mathfrak{g}$. So there is "no room" for a nontrivial intersection of $\mathfrak{g}_0'$ with $\mathfrak{z}$ without meeting $\mathfrak{m}$. Since $G$ is not Abelian, there is no Kähler structure on $G/\Gamma$.

EXAMPLE 3. For solvable groups this result no longer holds. The coset space of $G = \mathbb{C}^2$ (taken with its structure of a *solvable*, complex Lie group) by the discrete subgroup

$$\Gamma = \big\langle (\pi i, 0), (0, 2\pi i) \big\rangle_G$$

yields a homogeneous space $X := G/\Gamma$ that is a nontrivial $\mathbb{C}^*$-bundle over $\mathbb{C}^*$. A two-to-one covering of $X$ is a product, but no Abelian complex Lie group can act holomorphically and transitively on the space $X$ itself. See [**8**, p. 1102].

REMARK. The reader should note that we have used some essential facts about nilpotent Lie groups in the proof of the Theorem. For example, a positive dimensional nilpotent Lie group has a positive dimensional centre. This is no longer the case for solvable Lie groups. Whether there always exists an Abelian complex Lie group that acts transitively on a finite covering of a Kähler solvmanifold $G/\Gamma$ with $d_{G/\Gamma} \leqslant 2$ is beyond the scope of the present paper. As far as we can tell, another approach to this problem will be needed.

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