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Locations of interior transition layers to inhomogeneous transition problems in higher -dimensional domains

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We consider the following inhomogeneous problems

$$\begin{cases} \epsilon^2 \mathrm{div}(a(x)\nabla u(x)) + f(x,u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth and bounded domain in general dimensional space $\mathbb{R}^N$, $\epsilon > 0$ is a small parameter and function a is positive. We respectively obtain the locations of interior transition layers of the solutions of the above transition problems that are L^1-local minimizer and global minimizer of the associated energy functional.

Keywords: Local minimizer; interior transition layer; interface location; inhomogeneous transition problems

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1. Introduction

We study the following inhomogeneous transition problems

$$\begin{cases} \epsilon^2 \mathrm{div}(a(x)\nabla u(x)) + f(x,u) = 0 & \text{in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, ν is the outer unit normal to $\partial\Omega$, $\epsilon > 0$ is a small parameter and function $a \in C^1(\bar{\Omega})$ is positive. The nonlinear term f satisfies

(f_1) $f(x,\cdot)$ has two zeros $b_1(x)$, $b_2(x)$ such that $b_1, b_2 \in C^1(\Omega)$ and $b_1(x) < b_2(x)$ for all $x \in \bar{\Omega}$;
(f_2) $\partial_2 f(x, b_1(x)) < 0$ and $\partial_2 f(x, b_2(x)) < 0$ for all $x \in \bar{\Omega}$;
(f_3) For any given $x \in \bar{\Omega}$, $F(x,\cdot) \geqslant 0$. The function $\sqrt{a(\cdot)F(\cdot,\cdot)}$ is Lipschitz continuous. Here $F(x,u) := -\int_{b_1(x)}^{u} f(x,\tau)\mathrm{d}\tau$.

A typical example of a function f satisfying (f_1)–(f_3) is

$$f(x,\tau) = V(x)\tilde{f}(\tau), \tag{1.2}$$

where V is a strictly positive function and $\tilde{f}$ has precisely three zeros $\tilde{b}_1 < 0 < \tilde{b}_2$, and $\int_{\tilde{b}_1}^{\tilde{b}_2} \tilde{f}(\tau)\mathrm{d}\tau = 0$, moreover, $\tilde{f}(\tau)/\tau > \tilde{f}'(\tau)(\tau \neq 0)$.

Another typical example of a function f satisfying (f_1)–(f_3) is

$$f(x,\tau) = -(\tau - b_1(x))(\tau - b(x))(\tau - b_2(x)), \tag{1.3}$$

where $b_1(x) < b(x) < b_2(x)$ for all $x \in \Omega$.

The above two examples are related to inhomogeneous Allen–Cahn problem, which has its origin in the theory of phase transitions, see [**5**].

The corresponding energy functional of (1.1) is

$$\bar{J}_\epsilon(u) = \int_\Omega \frac{\epsilon}{2} a(x)|\nabla u|^2 + \frac{1}{\varepsilon} F(x,u)\mathrm{d}x.$$

For a smooth $(N-1)$-dimensional closed hypersurface Σ contained in Ω, we denote the domain enclosed by Σ as Ω_Σ. We denote by $\chi(A)$ the characteristic function related to set A.

Definition 1.1. *A family u_ϵ of solutions to (1.1) is said to develop an interior transition layer, as $\epsilon \to 0$, with interface at some $(N-1)$-dimensional closed hypersurface $\Sigma_0 \subset \Omega$ if*

$$u_\epsilon \to u_0 := b_1\chi(\bar{\Omega}_{\Sigma_0}) + b_2\chi(\bar{\Omega}\backslash\bar{\Omega}_{\Sigma_0}) \ \textit{in } L^1(\Omega) \ \textit{as } \epsilon \to 0. \tag{1.4}$$

We introduce the set

$$\Omega_- := \left\{ x \in \Omega : \int_{b_1(x)}^{b_2(x)} f(x,\tau)\mathrm{d}\tau = 0 \right\}.$$

We call that f satisfies the equal-area condition at the points in Ω_-. Note that $\Omega_- = \{x \in \Omega : F(x, b_2(x)) = 0\}$. It is well known that if Σ_0 is the interface of a family of solutions to (1.1) developing interior transition layer, then $\Sigma_0 \subset \Omega_-$ (see [**12**]). Plainly, if f is given by (1.2) then $\Omega_- = \Omega$. If f is given by (1.3), we have $F(x,\tau) = \frac{1}{4}[(b_1(x) - b(x))^2 - (\tau - b(x))^2]^2$ for those x satisfying $2b(x) = b_1(x) + b_2(x)$, and so $\Omega_- = \{x \in \Omega : b(x) = \frac{1}{2}(b_1(x) + b_2(x))\}$. We denote $\Omega_+ := \Omega\backslash\bar{\Omega}_-$.

The following quantity plays an important role in determining location of interior layer

$$\Lambda(x) := \int_{b_1(x)}^{b_2(x)} \sqrt{a(x)F(x,\tau)}\mathrm{d}\tau. \tag{1.5}$$

We first recall some known results of transition layers of (1.1). For the case that $a(x) \equiv 1$ and f is given by (1.2), in one-dimensional case, [**25**] shows that for an arbitrary subset of the local minimum points of $\Lambda(x)$, (1.1) admits a solution which has one layer near each point in the subset. Du and Gui [**13**] generalized the results of [**25**] to a two-dimensional case. Precisely, for a closed, non-degenerate geodesic

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Σ_0 relative to the integral $\int_\Sigma \Lambda$, (1.1) admits a solution whose layer locates near Σ_0. The corresponding results in general dimensional cases are established in [**15**, **16**, **21**, **32**, **33**]. For the corresponding fractional Laplacian, layer solutions are constructed in [**14**]. For $a(x) \equiv 1$ and f given by (1.3), there are many known existence results of transition layer solutions, see [**2**–**4**, **6**–**12**, **17**, **18**, **22**, **31**].

To construct layer solutions of a differential equation, the information of the location of the interface of a family of solutions is obviously very important and, in general, is not an easy task to find it.

In the homogeneous case, namely $a(x) \equiv 1$ and $f(x, u) \equiv f(u)$, classical theory of Γ-convergence developed in the 1970s and 1980s, showed a deep connection between this problem and the theory of minimal surfaces. By Γ-convergence theory, Modica [**23**] (see also [**20**, **24**]) proved that a family $\{u_\epsilon\}$ of local minimizers of the energy functional with uniformly bounded energy must converge as $\epsilon \to 0$, up to subsequences, in L^1-sense to a function of the form $\chi_E - \chi_{E^c}$, where χ_E denotes characteristic function of a set E, and also that ∂E has minimal perimeter.

For the inhomogeneous case, such as $a(x) \equiv 1$ and f is given by (1.2), in one-dimensional case, transition layers of solutions to (1.1) can appear only near extremum points of $\Lambda(x)$ [**26**], and, in higher-dimensional cases, the authors in [**21**] establish a necessary condition for a closed hypersurface in Ω to support layers. For $a(x) \equiv 1$, $\Omega_- = \Omega$ and general f satisfying assumptions (f_1)–(f_3), in one-dimensional case, among other things, the authors in [**27**] proved the existence of solutions to (1.1) with interior transition layer and that the layer occurs only near some extremum point of $\Lambda(x)$.

Recently, in one-dimensional domain case ($\Omega = (0, 1)$), for general $a(x)$ and f satisfying conditions (f_1)–(f_3), [**28**] obtains the following results.

Proposition 1.2. *Suppose that a family u_ϵ of solutions to (1.1) develop an interior transition layer at $\bar{x} \in \mathcal{Q}$, where $\mathcal{Q} \subset \Omega_-$ is the connected component of Ω_- that $\bar{x}$ belongs to. Then*

(i) *if u_ϵ is a family of L^1-local minimizer of $\hat{J}_\epsilon$, $\bar{x}$ is a local minimum point of $\Lambda(x)$ in $\mathcal{Q}$, where*

$$\hat{J}_\epsilon(u) := \begin{cases} \bar{J}_\epsilon(u), & u \in H^1(0,1), \\ \infty, & u \in L^1(0,1) \backslash H^1(0,1). \end{cases}$$

(ii) *if u_ϵ is a family of global minimizer of $\bar{J}_\epsilon$, $\Lambda(\bar{x}) = \min\{\Lambda(x) : x \in \mathcal{Q}\}$.*

What is the location of interior transition layer of minimizer (L^1-local or global) of the associated functional in general dimensional space? We will give a definite answer in this paper.

For small positive constant δ_0, we define

$$S := \{x \in \Omega : \mathrm{dist}(x, \Sigma_0) < 2\delta_0\}, \quad \Upsilon := [-2\delta_0, 2\delta_0].$$

We parameterize elements $x \in S$ using their closest point z in Σ_0 and their distance t (with sign, negative in the dilation of Ω_{Σ_0}). Precisely, we choose a system of coordinates z on Σ_0, and denote by $\mathbf{n}(z)$ the unique unit normal vector to Σ_0

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(at the point with coordinates z) pointing towards $\Omega\backslash\Omega_{\Sigma_0}$. Define the diffeomorphism $\Gamma : \Sigma_0 \times \Upsilon \to S$ by

$$\Gamma(z,t) = z + t\mathbf{n}(z).$$

We let the upper-case indices I, J, ... run from 1 to N, and the lower-case indices i, j, ... run from 1 to $N-1$. Using some local coordinates $(z_i)_{i=1,\dots,N-1}$ on Σ_0, and letting φ be the corresponding immersion into $\mathbb{R}^N$, we have

$$\begin{cases} \dfrac{\partial\Gamma}{\partial z_i}(z,t) = \dfrac{\partial\varphi}{\partial z_i}(z) + t\kappa_i^j(z)\dfrac{\partial\varphi}{\partial z_j}(z) \quad \text{for } i = 1,\dots,N-1, \\ \dfrac{\partial\Gamma}{\partial t}(z,t) = \mathbf{n}(z), \end{cases}$$

where (κ_i^j) are the coefficients of the mean-curvature operator on Σ_0. Let also $(\bar{g}_{ij})_{ij}$ be the coefficients of the metric on Σ_0 in the above coordinates z. Then, letting g denote the metric on Ω induced by $\mathbb{R}^N$, we have

$$g_{IJ} = \begin{pmatrix} \{g_{ij}\} & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} g_{ij} &= \left(\frac{\partial\varphi}{\partial z_i}(z) + t\kappa_i^m(z)\frac{\partial\varphi}{\partial z_m}(z), \frac{\partial\varphi}{\partial z_j}(z) + t\kappa_j^n(z)\frac{\partial\varphi}{\partial z_n}(z)\right) \\ &= \bar{g}_{ij} + t(\kappa_i^m\bar{g}_{mj} + \kappa_j^n\bar{g}_{in}) + t^2\kappa_i^m\kappa_j^n\bar{g}_{mn}. \end{aligned}$$

We have, formally

$$\det g = \det\bar{g}[1 + t\mathrm{Tr}(\bar{g}^{-1}\alpha)] + \mathrm{o}(t),$$

where

$$\alpha_{ij} = \kappa_i^m\bar{g}_{mj} + \kappa_j^n\bar{g}_{in}.$$

There holds

$$(\bar{g}^{-1}\alpha)_{il} = \bar{g}^{lj}\alpha_{ij} = \bar{g}^{lj}(\kappa_i^m\bar{g}_{mj} + \kappa_j^n\bar{g}_{in}),$$

and hence

$$\mathrm{Tr}(\bar{g}^{-1}\alpha) = \bar{g}^{ij}(\kappa_i^m\bar{g}_{mj} + \kappa_j^n\bar{g}_{in}) = 2\bar{g}^{ij}\kappa_i^m\bar{g}_{mj} = 2\kappa_i^i.$$

We recall that the quantity κ_i^i represents the mean curvature of Σ_0, we abbreviate κ_i^i as κ, and in particular it is independent of the choice of coordinates.

We have

$$\mathrm{d}V_g = \sqrt{\det g}\mathrm{d}z\mathrm{d}t = (1 + t\kappa + \mathrm{o}(t))\sqrt{\det \bar{g}}\mathrm{d}z\mathrm{d}t = (1 + t\kappa + \mathrm{o}(t))\mathrm{d}V_{\bar{g}}\mathrm{d}t.$$

For h satisfying $\|h\|_{L^\infty(\Sigma_0)} \leqslant 2\delta_0$, we define the perturbed closed $(N-1)$-dimensional hypersurface of Σ_0 as

$$\Sigma_h := \{\Gamma(z, h(z)) : z \in \Sigma_0\}.$$

We also introduce

$$J_\epsilon(u) = \begin{cases} \bar{J}_\epsilon(u), & u \in H^1(\Omega), \\ \infty, & u \in L^1(\Omega)\backslash H^1(\Omega). \end{cases}$$

We call that u_ϵ is a L^1-local minimizer of J_ϵ if there exists $\mu > 0$ such that $J_\epsilon(u_\epsilon) \leqslant J_\epsilon(u)$ for any u satisfying $\|u_\epsilon - u\|_{L^1(\Omega)} \leqslant \mu$. Each L^1-local minimizer of J_ϵ is a H^1-local minimizer of $\bar{J}_\epsilon$ as well, that is to say, it is a weak solution of (1.1). By the theory of regularity, it is a classical solution of (1.1).

Our main results are the followings.

THEOREM 1.3. *Suppose that a family u_ϵ of solutions to (1.1) develop an interior transition layer at $\Sigma_0 \subset \mathcal{Q}$, where $\mathcal{Q} \subseteq \Omega_-$ is the connected component of Ω_- that Σ_0 belongs to. If u_ϵ is a family of L^1-local minimizer of J_ϵ, then Σ_0 is a 'local minimum' surface of $\int_{\Sigma_h} \Lambda(x)$ in $\mathcal{Q}$ in the sense that there exists a $0 < \sigma(\leqslant 2\delta_0)$ such that*

$$\int_{\Sigma_0} \Lambda = \min\left\{\int_{\Sigma_h} \Lambda : \|h\|_{L^\infty(\Sigma_0)} \leqslant \sigma \text{ and } \|\nabla_{\bar{g}} h\|_{L^\infty(\Sigma_0)} = o(\epsilon^{1/4})\right\}.$$

THEOREM 1.4. *Besides the conditions of theorem 1.3, furthermore if u_ϵ is a family of global minimizer of $\bar{J}_\epsilon$, then*

$$\int_{\Sigma_0} \Lambda = \min\Big\{\int_{\Sigma} \Lambda : \text{for any closed smooth } (N-1)\text{-dimensional nontrivial surface } \Sigma \subset \mathcal{Q} \text{ with } \Omega_+\backslash\Omega_\Sigma = \Omega_+\backslash\Omega_{\Sigma_0}\Big\}.$$

REMARK 1.5. If $\mathcal{Q}$ is a simply connected domain, then, for any closed $(N-1)$-dimensional surface $\Sigma \subset \mathcal{Q}$, both $\Omega_+\backslash\Omega_\Sigma$ and $\Omega_+\backslash\Omega_{\Sigma_0}$ are equal to Ω_+, so the result of theorem 1.4 becomes

$$\int_{\Sigma_0} \Lambda = \min\left\{\int_{\Sigma} \Lambda : \text{for any closed smooth nontrivial surface } \Sigma \subset \mathcal{Q}\right\}.$$

2. Preliminaries

We first recall definition of functions with bounded variation and a property to be used. The interested reader is referred to [**19**].

DEFINITION 2.1. *A function $\phi \in L^1(\Omega)$ is said to have bounded variation in Ω if*

$$\int_\Omega |D\phi| := \sup\left\{\int_\Omega \phi \operatorname{div} v \mathrm{d}x : v = (v_1, \ldots, v_N) \in C_0^1(\Omega, \mathbb{R}^N),\right.$$
$$\left.|v(x)| \leqslant 1 \text{ for } x \in \Omega\right\} < \infty.$$

We define $BV(\Omega)$ as the space of all functions in $L^1(\Omega)$ with bounded variation. If $\phi \in BV(\Omega)$, then for any positive continuous function v, we have

$$\int_\Omega v(x)|D\phi| = \sup\left\{\int_\Omega \phi \operatorname{div} w \mathrm{d}x : w = (w_1, \ldots, w_N) \in C_0^1(\Omega, \mathbb{R}^N),\right.$$
$$\left.|w(x)| \leqslant v(x) \text{ for } x \in \Omega\right\}. \tag{2.1}$$

Consider the following initial value problem

$$\begin{cases} \partial_\tau W(x,\tau) = \sqrt{\dfrac{2F(x, W(x,\tau))}{a(x)}}, \\ W(x,0) = W_0(x), \end{cases} \tag{2.2}$$

where $W_0 \in C^1(\mathcal{Q})$ satisfies $b_1(x) \leqslant W_0(x) \leqslant b_2(x)$ for $x \in \mathcal{Q}$. This problem admits a unique solution $W(x, \tau)$ in $\mathcal{Q} \times \mathbb{R}$ and

$$b_1(x) \leqslant W(x,\tau) \leqslant b_2(x), \quad \forall (x,\tau) \in \mathcal{Q} \times \mathbb{R}.$$

Moreover, $|\nabla_x W(x, \tau)| \in L^\infty(\mathcal{Q} \times \mathbb{R})$ and $\lim_{\tau\to-\infty} W(x, \tau) = b_1(x)$, $\lim_{\tau\to+\infty} W(x, \tau) = b_2(x)$. More precisely, there exists positive constants q, α depending on F such that

(W_1) for τ large enough, $|W(x, \tau) - b_2(x)| \leqslant q\mathrm{e}^{-\alpha\tau}$;

(W_2) for $-\tau$ large enough, $|W(x, \tau) - b_1(x)| \leqslant q\mathrm{e}^{\alpha\tau}$.

The above properties of W can be seen in [**29**] (see also [**1**, **30**]).

3. Local minimum

We first establish a lower bound for $J_\epsilon(u_\epsilon)$.

LEMMA 3.1. *Suppose that a family u_ϵ of solutions to (1.1) develop an interior transition layer at $\Sigma_0 \subset \mathcal{Q}$, where $\mathcal{Q} \subseteq \Omega_-$ is the connected component of Ω_-. If u_ϵ is a family of L^1-local minimizer of J_ϵ, then*

$$J_\epsilon(u_\epsilon) \geqslant \sqrt{2}\int_{\Sigma_0} \Lambda \mathrm{d}V_{\bar{g}} + \int_{\Omega_+\setminus\Omega_{\Sigma_0}} \frac{1}{\epsilon} F(x, b_2(x))\mathrm{d}x + o(1), \tag{3.1}$$

where $o(1)$ means a quantity with limit 0 as $\epsilon \to 0$.

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Proof. First we assume that $\mathcal{Q}$ is simply connected. We have

$$J_\epsilon(u_\epsilon) = J_\epsilon(u_\epsilon, \Omega_+) + J_\epsilon(u_\epsilon, \Omega_-) \geqslant J_\epsilon(u_\epsilon, \Omega_+) + J_\epsilon(u_\epsilon, \mathcal{Q}) \geqslant \int_{\Omega_+} \frac{1}{\epsilon} F(x, b_2(x))\mathrm{d}x + J_\epsilon(u_\epsilon, \mathcal{Q}). \tag{3.2}$$

Set

$$\mathbb{U} := \{v \in C_0^1(\mathcal{Q}, \mathbb{R}^N) : |v(x)| \leqslant 1\},$$

then we have

$$\begin{aligned} J_\epsilon(u_\epsilon, \mathcal{Q}) &= \int_{\mathcal{Q}} \frac{\epsilon}{2} a(x)|\nabla u_\epsilon|^2 + \frac{1}{\epsilon} F(x, u_\epsilon)\mathrm{d}x \\ &\geqslant \sqrt{2}\int_{\mathcal{Q}} |\nabla u_\epsilon| \sqrt{a(x)F(x, u_\epsilon)} \\ &= \sup_{v\in\mathbb{U}} \left\{ \sqrt{2}\int_{\mathcal{Q}} \nabla u_\epsilon \cdot v\sqrt{a(x)F(x, u_\epsilon)} \right\}. \end{aligned}$$

If we denote

$$\psi_\epsilon(x) := \int_{b_1(x)}^{u_\epsilon(x)} \sqrt{a(x)F(x, \tau)}\mathrm{d}\tau,$$

then

$$\begin{aligned} &J_\epsilon(u_\epsilon, \mathcal{Q}) \\ &\quad\geqslant \sup_{v\in\mathbb{U}} \left\{ \sqrt{2}\int_{\mathcal{Q}} \left[\nabla\psi_\epsilon \cdot v - \int_{b_1(x)}^{u_\epsilon(x)} \nabla\left(\sqrt{a(x)F(x, \tau)}\right) \cdot v\mathrm{d}\tau \right] \mathrm{d}x \right\} \\ &\quad= \sup_{v\in\mathbb{U}} \left\{ -\sqrt{2}\int_{\mathcal{Q}} \int_{b_1(x)}^{u_\epsilon(x)} \left[\sqrt{a(x)F(x, \tau)}\,\mathrm{div}\, v \right.\right. \\ &\qquad \left.\left. +\nabla\left(\sqrt{a(x)F(x, \tau)}\right) \cdot v \right] \mathrm{d}\tau\mathrm{d}x \right\}. \end{aligned}$$

Combining the limit $u_\epsilon \to u_0$ in $L^1(\Omega)$ and the L^∞ boundedness of the several quantities $\sqrt{a(x)F(x, \tau)}$, $\nabla(\sqrt{a(x)F(x, \tau)})$, v, $\mathrm{div}\, v$, we have

$$\begin{aligned} &\lim_{\epsilon\to 0} J_\epsilon(u_\epsilon, \mathcal{Q}) \\ &\quad\geqslant \sup_{v\in\mathbb{U}} \left\{ -\sqrt{2}\int_{\mathcal{Q}} \int_{b_1(x)}^{u_0(x)} [\sqrt{a(x)F(x, \tau)}\,\mathrm{div}\, v \right. \\ &\qquad \left. +\nabla\left(\sqrt{a(x)F(x, \tau)}\right) \cdot v]\mathrm{d}\tau\mathrm{d}x \right\} \\ &\quad= \sup_{v\in\mathbb{U}} \left\{ -\sqrt{2}\int_{\mathcal{Q}} \chi(u_0(x) = b_2(x)) \right. \\ &\qquad \left. \times \int_{b_1(x)}^{b_2(x)} \left[\sqrt{a(x)F(x, \tau)}\,\mathrm{div}\, v + \nabla\left(\sqrt{a(x)F(x, \tau)}\right) \cdot v \right] \mathrm{d}\tau\mathrm{d}x \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{v\in\mathbb{U}}\left\{-\sqrt{2}\int_{\mathcal{Q}}\chi(u_0(x)=b_2(x))\right.\\
&\qquad\left.\times\mathrm{div}\left[\int_{b_1(x)}^{b_2(x)}\sqrt{a(x)F(x,\tau)}v\mathrm{d}\tau\right]\mathrm{d}x\right\}\\
&= \sqrt{2}\int_{\mathcal{Q}}\int_{b_1(x)}^{b_2(x)}|\nabla\chi(u_0(x)=b_2(x))|\sqrt{a(x)F(x,\tau)}\mathrm{d}\tau\mathrm{d}x\\
&= \sqrt{2}\int_{\Sigma_0}\int_{b_1(x)}^{b_2(x)}\sqrt{a(x)F(x,\tau)}\mathrm{d}\tau\mathrm{d}V_{\bar{g}}\\
&= \sqrt{2}\int_{\Sigma_0}\Lambda(x)\mathrm{d}V_{\bar{g}}.
\end{aligned}\tag{3.3}
$$

Note that $\Omega_+\backslash\Omega_{\Sigma_0}=\Omega_+$, since $\mathcal{Q}$ is a simply connected domain. From this and (3.2), (3.3), we obtain (3.1).

For the case that $\mathcal{Q}$ is multiply connected, (3.2) becomes

$$
J_\epsilon(u_\epsilon)\geqslant\int_{\Omega_+\backslash\Omega_{\Sigma_0}}\frac{1}{\epsilon}F(x,b_2(x))\mathrm{d}x+J_\epsilon(u_\epsilon,\mathcal{Q}),
$$

and the same argument as that of the simply connected domain case gives the desired inequality (3.1). □

We further establish an upper bound for $J_\epsilon(u_\epsilon)$.

LEMMA 3.2. *Under the conditions of lemma 3.1, then for any h satisfying $\|h\|_{L^\infty(\Sigma_0)}\leqslant\sigma$ for some $\sigma\leqslant 2\delta_0$ and $\|\nabla_{\bar{g}}h\|_{L^\infty(\Sigma_0)}=o(\epsilon^{1/4})$, we have*

$$
J_\epsilon(u_\epsilon)\leqslant\sqrt{2}\int_{\Sigma_h}\Lambda\mathrm{d}V_{\bar{g}}+\int_{\Omega_+\backslash\Omega_{\Sigma_0}}\frac{1}{\epsilon}F(x,b_2(x))\mathrm{d}x+o(1). \tag{3.4}
$$

Proof. First we also assume that $\mathcal{Q}$ is simply connected. We borrow the idea of [**30**] (see also [**28**]) to define a sequence of functions $b_\epsilon(z,t;\tau):\Sigma_0\times\Upsilon\times\Upsilon\to\mathbb{R}$

$$
b_\epsilon(z,t;\tau)=\begin{cases}
b_2(z,t), & 2\sqrt{\epsilon}\leqslant\tau<2\delta_0,\\
[b_2(z,t)-W(z,t;1/\sqrt{\epsilon})]\dfrac{\tau-2\sqrt{\epsilon}}{\sqrt{\epsilon}}+b_2(z,t), & \sqrt{\epsilon}<\tau<2\sqrt{\epsilon},\\
W(z,t;\tau/\epsilon), & |\tau|\leqslant\sqrt{\epsilon},\\
[W(z,t;-1/\sqrt{\epsilon})-b_1(z,t)]\dfrac{\tau+2\sqrt{\epsilon}}{\sqrt{\epsilon}}+b_1(z,t), & -2\sqrt{\epsilon}<\tau<-\sqrt{\epsilon},\\
b_1(z,t), & -2\delta_0<\tau\leqslant-2\sqrt{\epsilon},
\end{cases}
$$

where W is the solution of (2.2). Given h satisfying $\|h\|_{L^\infty(\Sigma_0)}\leqslant\sigma$ and $\|\nabla_{\bar{g}}h\|_{L^\infty(\Sigma_0)}=\mathrm{o}(\epsilon^{1/4})$, we define $\rho_\epsilon:\Omega\to\mathbb{R}$ by

$$
\rho_\epsilon(x)=\begin{cases}
b_2(x), & x\in\Omega\backslash\Omega_{\Sigma_{\delta_0}},\\
b_\epsilon(z,t;t-h(z)), & x=\varphi(z)+t\mathbf{n}(z)\in\Omega_{\Sigma_{\delta_0}}\backslash\Omega_{\Sigma_{-\delta_0}},\\
b_1(x), & x\in\Omega_{\Sigma_{-\delta_0}}.
\end{cases}
$$

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Claim: For any given $\mu > 0$, there exist $\epsilon_0(\mu) > 0$ and $\sigma(\mu) > 0$, such that for all $\epsilon < \epsilon_0$ we have $\|u_\epsilon - \rho_\epsilon\|_{L^1(\Omega)} \leqslant \mu$.

Indeed, if we introduce

$$\rho_0 := b_1\chi(\bar{\Omega}_{\Sigma_h}) + b_2\chi(\bar{\Omega}\backslash\bar{\Omega}_{\Sigma_h}),$$

then we know that there exists $\sigma(\mu)$ less than $2\delta_0$ such that for $\|h\|_{L^\infty(\Sigma_0)} \leqslant \sigma(\mu)$, the following inequality holds $\|u_0 - \rho_0\|_{L^1(\Omega)} < \frac{\mu}{2}$. Hence, to prove the claim it is only need to show that $\rho_\epsilon \to \rho_0$ in $L^1(\Omega)$ as $\epsilon \to 0$.

By the definitions of ρ_ϵ and ρ_0, we have that

$$\begin{aligned}\int_\Omega |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x &= \int_{\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-2\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x \\ &= \int_{\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h+\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)| + \int_{\Omega_{\Sigma_{h-\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-2\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)| \\ &\quad + \int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x.\end{aligned}$$

For the first integral of the right hand side we have

$$\begin{aligned}&\int_{\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h+\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x \\ &\quad = \frac{1}{\sqrt{\epsilon}}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}}\int_{\Sigma_0} [b_2(z, h(z)+\mu) - W(z, h(z)+\mu; 1/\sqrt{\epsilon})] \\ &\qquad \times |\mu - 2\sqrt{\epsilon}|[1 + (h(z)+\mu)\kappa + \mathrm{o}(h(z)+\mu)]\mathrm{d}V_{\bar{g}}\mathrm{d}\mu \\ &\quad \leqslant \frac{C}{\sqrt{\epsilon}}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} (2\sqrt{\epsilon} - \mu)\mathrm{d}\mu \\ &\quad = \mathrm{O}(\sqrt{\epsilon}).\end{aligned}$$

Analogously,

$$\int_{\Omega_{\Sigma_{h-\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-2\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x = \mathrm{O}(\sqrt{\epsilon}).$$

We have

$$\begin{aligned}&\int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x \\ &\quad = \int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_h}} |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x + \int_{\Omega_{\Sigma_h}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}} |\rho_\epsilon(x) - \rho_0(x)|\mathrm{d}x \\ &\quad = \int_0^{\sqrt{\epsilon}}\int_{\Sigma_{h+\mu}} |b_2(z, h(z)+\mu) - W(z, h(z)+\mu; \mu/\epsilon)|\mathrm{d}V_{\bar{g}}\mathrm{d}\mu\end{aligned}$$

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$$+\int_{-\sqrt{\epsilon}}^{0}\int_{\Sigma_{h+\mu}}|b_1(z,h(z)+\mu)-W(z,h(z)+\mu;\mu/\epsilon)|\mathrm{d}V_{\bar{g}}\mathrm{d}\mu$$
$$=\mathrm{O}(\sqrt{\epsilon}).$$

All in all we obtain that $\rho_\epsilon \to \rho_0$ in $L^1(\Omega)$ as $\epsilon \to 0$.

We decompose

$$\begin{aligned}J_\epsilon(\rho_\epsilon) &= J_\epsilon(\rho_\epsilon,\Omega\backslash\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}) + J_\epsilon(\rho_\epsilon,\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h+\sqrt{\epsilon}}})\\ &\quad + J_\epsilon(\rho_\epsilon,\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}) + J_\epsilon(\rho_\epsilon,\Omega_{\Sigma_{h-\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-2\sqrt{\epsilon}}})\\ &\quad + J_\epsilon(\rho_\epsilon,\Omega_{\Sigma_{h-2\sqrt{\epsilon}}}).\end{aligned}$$

From the definition of ρ_ϵ, we have

$$\begin{aligned}J_\epsilon(\rho_\epsilon,\Omega\backslash\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}) &= \int_{\Omega\backslash\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}}\frac{\epsilon}{2}a(x)|\nabla b_2|^2+\frac{1}{\epsilon}F(x,b_2(x))\mathrm{d}x\\ &= \frac{1}{\epsilon}\int_{\Omega\backslash\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}}F(x,b_2(x))\mathrm{d}x+\mathrm{O}(\epsilon)\\ &= \frac{1}{\epsilon}\int_{\Omega_+}F(x,b_2(x))\mathrm{d}x+\mathrm{O}(\epsilon),\end{aligned}\tag{3.5}$$

where in the last equality we used the facts that $F(x, b_2(x)) = 0$ in Ω_-, and $\Omega_+\backslash\Omega_{\Sigma_{h+2\sqrt{\epsilon}}} = \Omega_+$ in virtue of the simply connectedness of $\mathcal{Q}$.

Similarly, we have

$$J_\epsilon(\rho_\epsilon,\Omega_{\Sigma_{h-2\sqrt{\epsilon}}}) = \mathrm{O}(\epsilon).\tag{3.6}$$

We have

$$J_\epsilon(\rho_\epsilon,\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h+\sqrt{\epsilon}}}) = \frac{1}{\epsilon}\int_{\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h+\sqrt{\epsilon}}}}F(x,\rho_\epsilon(x))\mathrm{d}x+\mathrm{o}(\epsilon).$$

Recalling that $F(x,\, b_2(x)) = 0 = F_u(x,\, b_2(x))$ and $F_{uu}(x,\, b_2(x)) > 0$, we have

$$\begin{aligned}&\frac{1}{\epsilon}\int_{\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h+\sqrt{\epsilon}}}}F(x,\rho_\epsilon(x))\mathrm{d}x\\ &\quad= \frac{1}{\epsilon}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}}\int_{\Sigma_{h+\mu}}F\left(z,h(z)+\mu,\left[b_2(z,h+\mu)-W\left(z,h+\mu;\frac{1}{\sqrt{\epsilon}}\right)\right]\right.\\ &\quad\quad\left.\times\frac{\mu-2\sqrt{\epsilon}}{\sqrt{\epsilon}}+b_2(z,h+\mu)\right)\mathrm{d}V_{\bar{g}}\mathrm{d}\mu\\ &\quad\leqslant \frac{1}{\epsilon}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}}\int_{\Sigma_{h+\mu}}F\left(z,h+\mu,q\mathrm{e}^{-\alpha/\sqrt{\epsilon}}+b_2(z,h+\mu)\right)\mathrm{d}V_{\bar{g}}\mathrm{d}\mu\\ &\quad= \frac{1}{\epsilon}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}}\int_{\Sigma_{h+\mu}}\left[F\left(z,h+\mu,q\mathrm{e}^{-\alpha/\sqrt{\epsilon}}+b_2(z,h+\mu)\right)\right.\\ &\quad\quad\left.-F(z,h+\mu,b_2(z,h+\mu))\right]\mathrm{d}V_{\bar{g}}\mathrm{d}\mu\end{aligned}$$

$$\leqslant \frac{1}{\epsilon}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \mathrm{e}^{-\alpha/\sqrt{\epsilon}}\gamma_\epsilon \mathrm{d}\mu$$
$$= \frac{\sqrt{\epsilon}}{\epsilon}\mathrm{e}^{-\alpha/\sqrt{\epsilon}}\gamma_\epsilon,$$

where

$$\gamma_\epsilon := q_1 \sup\left\{\int_{\Sigma_t} F_u(z,t,\tau)\mathrm{d}V_{\bar{g}} :\right.$$
$$\left. h(z)+\sqrt{\epsilon} < t < h(z)+2\sqrt{\epsilon},\ b_2(z,t) < \tau < b_2(z,t)+q\mathrm{e}^{-\alpha/\sqrt{\epsilon}}\right\}.$$

Note that γ_ϵ is uniformly bounded in ϵ. Therefore we have

$$J_\epsilon(\rho_\epsilon, \Omega_{\Sigma_{h+2\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h+\sqrt{\epsilon}}}) = \mathrm{o}(\epsilon). \tag{3.7}$$

Similarly we have

$$J_\epsilon(\rho_\epsilon, \Omega_{\Sigma_{h-\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-2\sqrt{\epsilon}}}) = \mathrm{o}(\epsilon). \tag{3.8}$$

Finally, we consider the integral $J_\epsilon(\rho_\epsilon,\, \Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}})$. We have

$$\begin{aligned}
&J_\epsilon(\rho_\epsilon, \Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}})\\
&\quad= \int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}} \frac{\epsilon}{2}a\left|\nabla_g W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right|^2\\
&\qquad+\frac{1}{\epsilon}F\left(z,t,W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right)\mathrm{d}V_{\bar{g}}\mathrm{d}t\\
&\quad= \int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}} \frac{\epsilon}{2}a\left\{\left|\nabla_{\bar{g}} W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right|^2(1+\mathrm{O}(t))\right.\\
&\qquad\left.+\left[\partial_2 W+\frac{1}{\epsilon}\partial_3 W\right]^2\right\}+\frac{1}{\epsilon}F\left(z,t,W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right)\mathrm{d}V_{\bar{g}}\mathrm{d}t,
\end{aligned}$$

where we used the formula $|\nabla_g v(z,\,t)|^2 = |\nabla_{\bar{g}} v(z,\,t)|^2(1+\mathrm{O}(t)) + (\partial_t v(z,\,t))^2$. Then

$$\begin{aligned}
&J_\epsilon(\rho_\epsilon, \Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}})\\
&\quad= \int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}} \frac{\epsilon}{2}a(z,t)\left\{\left|\nabla_{\bar{g}} W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right|^2(1+\mathrm{O}(t))\right.\\
&\qquad\left.+\left[\partial_2 W+\frac{1}{\epsilon}\sqrt{\frac{2F\left(z,t,W(z,t,\frac{t-h(z)}{\epsilon})\right)}{a(z,t)}}\right]^2\right\}
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{\epsilon}F\left(z,t,W\Big(z,t,\frac{t-h(z)}{\epsilon}\Big)\right)\mathrm{d}V_{\bar g}\mathrm{d}t\\
&=\int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}}\frac{1}{2}\epsilon a\left[\left|\nabla_{\bar g}W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right|^2(1+\mathrm{O}(t))+(\partial_2 W)^2\right]\\
&+\partial_2 W\sqrt{2aF\left(z,t,W(z,t,\frac{t-h(z)}{\epsilon})\right)}\\
&+\frac{2}{\epsilon}F\left(z,t,W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right)\mathrm{d}V_{\bar g}\mathrm{d}t.
\end{aligned}
$$

Note that

$$
\int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}}\frac{1}{2}\epsilon a\left[\left|\nabla_{\bar g}W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right|^2(1+\mathrm{O}(t))+(\partial_2 W)^2\right]=\mathrm{o}(1),
$$

in virtue of the properties of the solution W of (2.2) and the fact that $\|\nabla_{\bar g}h\|_{L^\infty(\Sigma_0)}=\mathrm{o}(\epsilon^{1/4})$. The term $\partial_2 W\sqrt{2aF}$ is bounded in $\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}$. Now, letting $\mu=(t-h(z))/\epsilon$ and so $t=t(z,\,\mu)=h(z)+\epsilon\mu$, we have

$$
\begin{aligned}
&J_\epsilon(\rho_\epsilon,\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}})\\
&\quad=\int_{\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}}\frac{2}{\epsilon}F\left(z,t,W\left(z,t,\frac{t-h(z)}{\epsilon}\right)\right)\mathrm{d}V_{\bar g}\mathrm{d}t+\mathrm{o}(1)\\
&\quad=\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}}\int_{\Sigma_0}2F\left(z,h(z)+\epsilon\mu,W(z,h(z)+\epsilon\mu,\mu)\right)\\
&\qquad\times(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\mathrm{d}V_{\bar g}\mathrm{d}\mu+\mathrm{o}(1).
\end{aligned}
\tag{3.9}
$$

One has

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mu}\int_{\Sigma_0}\int_{b_1(z,h+\epsilon\mu)}^{W(z,h+\epsilon\mu;\mu)}\sqrt{\frac{1}{2}a(z,h+\epsilon\mu)F\left(z,h(z)+\epsilon\mu,\tau\right)}\\
&\qquad\times(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\mathrm{d}\tau\mathrm{d}V_{\bar g}\\
&\quad=\int_{\Sigma_0}\int_{b_1(z,h+\epsilon\mu)}^{W(z,h+\epsilon\mu;\mu)}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\sqrt{\frac{1}{2}a(z,h+\epsilon\mu)F\left(z,h(z)+\epsilon\mu,\tau\right)}\right.\\
&\qquad\left.\times(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\right]\mathrm{d}\tau\mathrm{d}V_{\bar g}\\
&\qquad+\int_{\Sigma_0}\sqrt{\frac{1}{2}a(z,h+\epsilon\mu)F\left(z,h(z)+\epsilon\mu,W(z,h+\epsilon\mu;\mu)\right)}\\
&\qquad\times(\epsilon\partial_2 W+\partial_3 W)(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\mathrm{d}V_{\bar g}
\end{aligned}
\tag{3.10}
$$

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Note that

$$\sqrt{\frac{1}{2}a(z,h+\epsilon\mu)F\left(z,h(z)+\epsilon\mu,W(z,h+\epsilon\mu;\mu)\right)}\partial_3 W(z,h+\epsilon\mu;\mu)$$
$$= F\left(z,h(z)+\epsilon\mu,W(z,h+\epsilon\mu;\mu)\right). \tag{3.11}$$

By (3.10) and (3.11) we have

$$\begin{aligned}
&\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}}\int_{\Sigma_0} 2F\left(z,h(z)+\epsilon\mu,W(z,h(z)+\epsilon\mu,\mu)\right)\\
&\quad\times(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\mathrm{d}V_{\bar g}\mathrm{d}\mu\\
&\quad=\int_{\Sigma_0}\int_{b_1(z,h+\sqrt{\epsilon})}^{W(z,h+\sqrt{\epsilon};1/\sqrt{\epsilon})}\sqrt{2a(z,h+\sqrt{\epsilon})F\left(z,h(z)+\sqrt{\epsilon},\tau\right)}\\
&\quad\times(1+(h(z)+\sqrt{\epsilon})\kappa+\mathrm{o}(h(z)+\sqrt{\epsilon}))\mathrm{d}\tau\mathrm{d}V_{\bar g}\\
&\quad-\int_{\Sigma_0}\int_{b_1(z,h-\sqrt{\epsilon})}^{W(z,h-\sqrt{\epsilon};-1/\sqrt{\epsilon})}\sqrt{2a(z,h-\sqrt{\epsilon})F\left(z,h(z)-\sqrt{\epsilon},\tau\right)}\\
&\quad\times(1+(h(z)-\sqrt{\epsilon})\kappa+\mathrm{o}(h(z)-\sqrt{\epsilon}))\mathrm{d}\tau\mathrm{d}V_{\bar g}\\
&\quad-2\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} I_{1,\epsilon}\mathrm{d}\mu-2\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} I_{2,\epsilon}\mathrm{d}\mu,
\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}
I_{1,\epsilon}&=\int_{\Sigma_0}\int_{b_1(z,h+\epsilon\mu)}^{W(z,h+\epsilon\mu;\mu)}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\sqrt{\frac{1}{2}a(z,h+\epsilon\mu)F\left(z,h(z)+\epsilon\mu,\tau\right)}\right.\\
&\quad\left.\times(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\right]\mathrm{d}\tau\mathrm{d}V_{\bar g},\\
I_{2,\epsilon}&=\epsilon\int_{\Sigma_0}\sqrt{\frac{1}{2}a(z,h+\epsilon\mu)F\left(z,h(z)+\epsilon\mu,W(z,h+\epsilon\mu;\mu)\right)}\partial_2 W\\
&\quad\times(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\mathrm{d}V_{\bar g}.
\end{aligned}$$

Plainly

$$\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} I_{2,\epsilon}\mathrm{d}\mu=\mathrm{O}(\sqrt{\epsilon}). \tag{3.13}$$

Recalling $t=t(z,\,\mu)=h(z)+\epsilon\mu$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\sqrt{\frac{1}{2}a(z,h+\epsilon\mu)F\left(z,h(z)+\epsilon\mu,\tau\right)}\right.\\
&\qquad\left.\times(1+(h(z)+\epsilon\mu)\kappa+\mathrm{o}(h(z)+\epsilon\mu))\right]\\
&\quad=\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{\frac{1}{2}a(z,t)F\left(z,t,\tau\right)}(1+t\kappa+\mathrm{o}(t))\right].
\end{aligned}$$

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Hence

$$\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} I_{1,\epsilon}\mathrm{d}\mu = \mathrm{O}(\sqrt{\epsilon}). \tag{3.14}$$

From (3.9), (3.12), (3.13) and (3.14) we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0} J_\epsilon(\rho_\epsilon, \Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}})\\
&\quad = \int_{\Sigma_0}\int_{b_1(z,h(z))}^{b_2(z,h(z))} \sqrt{2a(z,h)F\left(z,h(z),\tau\right)}(1+h(z)\kappa + \mathrm{o}(h(z)))\mathrm{d}\tau\mathrm{d}V_{\bar{g}}\\
&\quad = \sqrt{2}\int_{\Sigma_0} \Lambda(z,h(z))(1+h(z)\kappa+\mathrm{o}(h(z)))\mathrm{d}V_{\bar{g}}\\
&\quad = \sqrt{2}\int_{\Sigma_h} \Lambda\mathrm{d}V_{\bar{g}}.
\end{aligned} \tag{3.15}$$

Combining the above claim and the assumption that u_ϵ is a family of L^1-local minimizer of J_ϵ, we obtain

$$J_\epsilon(u_\epsilon) \leqslant J_\epsilon(\rho_\epsilon). \tag{3.16}$$

The upper bound estimate (3.4) follows from (3.16), (3.5), (3.6), (3.7), (3.8) and (3.15), where the relation $\Omega_+\backslash\Omega_{\Sigma_0} = \Omega_+$ is used again, since $\mathcal{Q}$ is simply connected.

For the case that $\mathcal{Q}$ is multiply connected, (3.5) becomes

$$J_\epsilon(\rho_\epsilon, \Omega\backslash\Omega_{\Sigma_{h+2\sqrt{\epsilon}}}) = \frac{1}{\epsilon}\int_{\Omega_+\backslash\Omega_{\Sigma_0}} F(x, b_2(x))\mathrm{d}x + \mathrm{O}(\epsilon),$$

and the same argument as that of the simply connected domain case gives the upper bound estimate (3.4). □

Proof of theorem 1.3. Lemmas 3.1 and 3.2 give the desired results of theorem 1.3 after a simple proof by contradiction. □

4. Global minimum

Given another smooth closed hypersurface $\tilde{\Sigma} \subset \mathcal{Q}$, similarly as the geometric ground in § 1, for some $\tilde{\delta} > 0$, we define

$$\tilde{S} = \{x \in \Omega : \mathrm{dist}(x, \tilde{\Sigma}) < 2\tilde{\delta}\}, \qquad \tilde{\Upsilon} = [-2\tilde{\delta}, 2\tilde{\delta}].$$

We parameterize elements $x \in \tilde{S}$ using their closest point z in $\tilde{\Sigma}$ and their distance t. Define the diffeomorphism $\tilde{\Gamma} : \tilde{\Sigma} \times \tilde{\Upsilon} \to \tilde{S}$ by

$$\tilde{\Gamma}(z,t) = z + t\tilde{\mathbf{n}}(z).$$

Letting $\tilde{\varphi}$ be the corresponding immersion into $\mathbb{R}^N$, we have

$$\begin{cases} \dfrac{\partial\tilde{\Gamma}}{\partial z_i}(z,t) = \dfrac{\partial\tilde{\varphi}}{\partial z_i}(z) + t\tilde{\kappa}_i^j(z)\dfrac{\partial\tilde{\varphi}}{\partial z_j}(z) \quad \text{for } i = 1,\dots,N-1,\\ \dfrac{\partial\tilde{\Gamma}}{\partial t}(z,t) = \tilde{\mathbf{n}}(z). \end{cases}$$

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Let also $(\bar{\tilde{g}}_{ij})_{ij}$ be the coefficients of the metric on $\tilde{\Sigma}$ in the above coordinates z. Then, letting $\tilde{g}$ denote the metric on Ω induced by $\mathbb{R}^N$, we have

$$\tilde{g}_{IJ} = \begin{pmatrix} \{\tilde{g}_{ij}\} & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\tilde{g}_{ij} = \bar{\tilde{g}}_{ij} + t(\tilde{\kappa}_i^m \bar{\tilde{g}}_{mj} + \tilde{\kappa}_j^n \bar{\tilde{g}}_{in}) + t^2 \tilde{\kappa}_i^m \tilde{\kappa}_j^n \bar{\tilde{g}}_{mn}.$$

We have also

$$\det \tilde{g} = \det \bar{\tilde{g}}[1 + 2t\tilde{\kappa}_i^i] + \mathrm{o}(t) =: \det \bar{\tilde{g}}[1 + 2t\tilde{\kappa}] + \mathrm{o}(t),$$

and

$$\mathrm{d}V_{\tilde{g}} = \sqrt{\det \tilde{g}}\mathrm{d}z\mathrm{d}t = (1 + t\tilde{\kappa} + \mathrm{o}(t))\sqrt{\det \bar{\tilde{g}}}\mathrm{d}z\mathrm{d}t = (1 + t\tilde{\kappa} + \mathrm{o}(t))\mathrm{d}V_{\bar{\tilde{g}}}\mathrm{d}t.$$

For h satisfying $\|h\|_{L^\infty(\tilde{\Sigma})} \leqslant 2\tilde{\delta}$, we define the perturbed closed hypersurface

$$\tilde{\Sigma}_h := \{\tilde{\Gamma}(z, h(z)) : z \in \tilde{\Sigma}\}.$$

LEMMA 4.1. *Assume that u_ϵ is a family of global minimizer of $\bar{J}_\epsilon$, we have*

$$\bar{J}_\epsilon(u_\epsilon) \leqslant \sqrt{2}\int_{\tilde{\Sigma}} \Lambda \mathrm{d}V_{\bar{\tilde{g}}} + \int_{\Omega_+\backslash\Omega_{\tilde{\Sigma}}} \frac{1}{\epsilon} F(x, b_2(x))\mathrm{d}x + o(1). \tag{4.1}$$

Proof. First, we also assume that $\mathcal{Q}$ is simply connected. Similar to that of § 3 we define $\tilde{\rho}_\epsilon : \Omega \to \mathbb{R}$ by

$$\tilde{\rho}_\epsilon(x) = \begin{cases} b_2(x), & x \in \Omega\backslash\Omega_{\tilde{\Sigma}_{\tilde{\delta}}}, \\ b_\epsilon(z, t; t), & x = \tilde{\varphi}(z) + t\tilde{\mathbf{n}}(z) \in \Omega_{\tilde{\Sigma}_{\tilde{\delta}}}\backslash\Omega_{\tilde{\Sigma}_{-\tilde{\delta}}}, \\ b_1(x), & x \in \Omega_{\tilde{\Sigma}_{-\tilde{\delta}}}. \end{cases}$$

Decompose

$$\begin{aligned} \bar{J}_\epsilon(\tilde{\rho}_\epsilon) &= \bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega\backslash\Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}}) + \bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}}\backslash\Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}) \\ &\quad + \bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}\backslash\Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}}) + \bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}}\backslash\Omega_{\tilde{\Sigma}_{-2\sqrt{\epsilon}}}) \\ &\quad + \bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{-2\sqrt{\epsilon}}}). \end{aligned}$$

Similar to that of (3.5), we have

$$\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega\backslash\Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}}) = \frac{1}{\epsilon}\int_{\Omega_+} F(x, b_2(x))\mathrm{d}x + \mathrm{O}(\epsilon), \tag{4.2}$$

and

$$\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{-2\sqrt{\epsilon}}}) = \mathrm{O}(\epsilon). \tag{4.3}$$

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We have

$$\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}) = \frac{1}{\epsilon}\int_{\Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}} F(x, \tilde{\rho}_\epsilon(x))\mathrm{d}x + \mathrm{o}(\epsilon).$$

Using that $F(x,\, b_2(x)) = 0 = F_u(x,\, b_2(x))$ and $F_{uu}(x,\, b_2(x)) > 0$ again, we have

$$\begin{aligned}
&\frac{1}{\epsilon}\int_{\Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}} F(x, \tilde{\rho}_\epsilon(x))\mathrm{d}x \\
&\quad = \frac{1}{\epsilon}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \int_{\tilde{\Sigma}_t} F\left(z, t, \left[b_2(z,t) - W\left(z, t; \frac{1}{\sqrt{\epsilon}}\right)\right] \frac{t-2\sqrt{\epsilon}}{\sqrt{\epsilon}}\right. \\
&\qquad \left. + b_2(z,t)\right)\mathrm{d}V_{\bar{\tilde{g}}}\mathrm{d}t \\
&\quad \leqslant \frac{1}{\epsilon}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \int_{\tilde{\Sigma}_t} F\left(z, t, q_1\mathrm{e}^{-\alpha/\sqrt{\epsilon}} + b_2(z,t)\right)\mathrm{d}V_{\bar{\tilde{g}}}\mathrm{d}t \\
&\quad \leqslant \frac{1}{\epsilon}\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} q_1\mathrm{e}^{-\alpha/\sqrt{\epsilon}}\tilde{\gamma}_\epsilon \mathrm{d}t \\
&\quad = \mathrm{o}(\epsilon),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\gamma}_\epsilon := \sup\Bigg\{\int_{\tilde{\Sigma}_\mu} F_u(z, \mu, \tau)\mathrm{d}V_{\bar{\tilde{g}}} : \sqrt{\epsilon} < \mu < 2\sqrt{\epsilon}, \\
b_2(z,\mu) < \tau < b_2(z,\mu) + q_1\mathrm{e}^{-\alpha/\sqrt{\epsilon}}\Bigg\}.
\end{aligned}$$

Therefore, we have

$$\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}) = \mathrm{o}(\epsilon). \tag{4.4}$$

Similarly we have

$$\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{-2\sqrt{\epsilon}}}) = \mathrm{o}(\epsilon). \tag{4.5}$$

For the integral $\bar{J}_\epsilon(\tilde{\rho}_\epsilon,\, \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}})$, we have

$$\begin{aligned}
&\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}}) \\
&\quad = \int_{\Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}}} \frac{\epsilon}{2} a \left|\nabla_{\tilde{g}} W\left(z, t, \frac{t}{\epsilon}\right)\right|^2 + \frac{1}{\epsilon} F\left(z, t, W\left(z, t, \frac{t}{\epsilon}\right)\right) \\
&\quad = \int_{\Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}} \backslash \Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}}} \frac{\epsilon}{2} a \left\{\left|\nabla_{\bar{\tilde{g}}} W\left(z, t, \frac{t}{\epsilon}\right)\right|^2 (1 + \mathrm{O}(t))\right. \\
&\qquad \left. + \left[\partial_2 W + \frac{1}{\epsilon}\partial_3 W\right]^2\right\} + \frac{1}{\epsilon} F\left(z, t, W\left(z, t, \frac{t}{\epsilon}\right)\right)\mathrm{d}V_{\bar{\tilde{g}}}\mathrm{d}t
\end{aligned}$$

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$$= \int_{\Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}\backslash\Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}}} \frac{\epsilon}{2} a \left\{ \left| \nabla_{\bar{\tilde{g}}} W\left(z,t,\frac{t}{\epsilon}\right)\right|^2 (1+\mathrm{O}(t)) \right.$$

$$\left. + \left[\partial_2 W + \frac{1}{\epsilon}\sqrt{\frac{2F}{a}}\right]^2 \right\} + \frac{1}{\epsilon} F\left(z,t,W\left(z,t,\frac{t}{\epsilon}\right)\right)$$

$$= \int_{\Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}\backslash\Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}}} \frac{1}{2} \left\{ \epsilon a \left[\left| \nabla_{\bar{\tilde{g}}} W\left(z,t,\frac{t}{\epsilon}\right)\right|^2 (1+\mathrm{O}(t)) + (\partial_2 W)^2 \right] \right.$$

Note that

$$\epsilon a \left[\left| \nabla_{\bar{\tilde{g}}} W\left(\tilde{z},\tilde{t},\frac{\tilde{t}}{\epsilon}\right)\right|^2 (1+\mathrm{O}(t)) + (\partial_2 W)^2 \right]$$

is bounded in $\Omega_{\Sigma_{h+\sqrt{\epsilon}}}\backslash\Omega_{\Sigma_{h-\sqrt{\epsilon}}}$ in virtue of the properties of the solution W of (2.2). Hence, letting $\mu = \frac{t}{\epsilon}$ and so $t = t(\mu) = \epsilon\mu$, we have

$$\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}\backslash\Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}})$$
$$= \int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \int_{\tilde{\Sigma}} 2F\left(z,\epsilon\mu,W(z,\epsilon\mu,\mu)\right)(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\mathrm{d}V_{\bar{\tilde{g}}}\mathrm{d}\mu + \mathrm{O}(\sqrt{\epsilon}). \tag{4.6}$$

One has

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\int_{\tilde{\Sigma}}\int_{b_1(z,\epsilon\mu)}^{W(z,\epsilon\mu;\mu)} \sqrt{\frac{1}{2}a(z,\epsilon\mu)F\left(z,\epsilon\mu,\tau\right)}(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\mathrm{d}\tau\mathrm{d}V_{\bar{\tilde{g}}}$$
$$= \int_{\tilde{\Sigma}}\int_{b_1(z,\epsilon\mu)}^{W(z,\epsilon\mu;\mu)} \frac{\mathrm{d}}{\mathrm{d}\mu}\left[\sqrt{\frac{1}{2}a(z,\epsilon\mu)F\left(z,\epsilon\mu,\tau\right)}(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\right]\mathrm{d}\tau\mathrm{d}V_{\bar{\tilde{g}}}$$
$$+ \int_{\tilde{\Sigma}} \sqrt{\frac{1}{2}a(z,\epsilon\mu)F\left(z,\epsilon\mu,W(z,\epsilon\mu;\mu)\right)}$$
$$\times (\epsilon\partial_2 W + \partial_3 W)(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\mathrm{d}V_{\bar{\tilde{g}}} \tag{4.7}$$

Note that

$$\sqrt{\frac{1}{2}a(z,\epsilon\mu)F\left(z,\epsilon\mu,W(z,\epsilon\mu;\mu)\right)}\partial_3 W(z,h+\epsilon\mu;\mu)$$
$$= F\left(z,\epsilon\mu,W(z,\epsilon\mu;\mu)\right). \tag{4.8}$$

By (4.7) and (4.8) we have

$$\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \int_{\tilde{\Sigma}} 2F\left(z,\epsilon\mu,W(z,\epsilon\mu,\mu)\right)(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\mathrm{d}V_{\bar{\tilde{g}}}\mathrm{d}\mu$$
$$= \int_{\tilde{\Sigma}}\int_{b_1(z,\sqrt{\epsilon})}^{W(z,\sqrt{\epsilon};1/\sqrt{\epsilon})} \sqrt{2a(z,\sqrt{\epsilon})F\left(z,\sqrt{\epsilon},\tau\right)}(1+\sqrt{\epsilon}\tilde{\kappa}+\mathrm{o}(\sqrt{\epsilon}))\mathrm{d}\tau\mathrm{d}V_{\bar{\tilde{g}}}$$

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$$- \int_{\tilde{\Sigma}} \int_{b_1(z,-\sqrt{\epsilon})}^{W(z,-\sqrt{\epsilon};-1/\sqrt{\epsilon})} \sqrt{2a(z,-\sqrt{\epsilon})F\left(z,-\sqrt{\epsilon},\tau\right)}(1-\sqrt{\epsilon}\tilde{\kappa}+\mathrm{o}(\sqrt{\epsilon}))\mathrm{d}\tau\mathrm{d}V_{\tilde{\bar{g}}}$$
$$- 2\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \tilde{I}_{1,\epsilon}\mathrm{d}\mu - 2\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \tilde{I}_{2,\epsilon}\mathrm{d}\mu, \tag{4.9}$$

where

$$\tilde{I}_{1,\epsilon} = \int_{\tilde{\Sigma}} \int_{b_1(z,\epsilon\mu)}^{W(z,\epsilon\mu;\mu)} \frac{\mathrm{d}}{\mathrm{d}\mu}\left[\sqrt{\frac{1}{2}a(z,\epsilon\mu)F\left(z,\epsilon\mu,\tau\right)}(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\right]\mathrm{d}\tau\mathrm{d}V_{\tilde{\bar{g}}},$$
$$\tilde{I}_{2,\epsilon} = \epsilon\int_{\tilde{\Sigma}} \sqrt{\frac{1}{2}a(z,\epsilon\mu)F\left(z,\epsilon\mu,W(z,\epsilon\mu;\mu)\right)}\partial_2 W(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\mathrm{d}V_{\tilde{\bar{g}}}.$$

Plainly

$$\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \tilde{I}_{2,\epsilon}\mathrm{d}\mu = \mathrm{O}(\sqrt{\epsilon}). \tag{4.10}$$

Recalling $t = t(\mu) = \epsilon\mu$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\sqrt{\frac{1}{2}a(z,\epsilon\mu)F\left(z,\epsilon\mu,\tau\right)}(1+\epsilon\mu\tilde{\kappa}+\mathrm{o}(\epsilon\mu))\right]$$
$$= \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{\frac{1}{2}a(\tilde{z},\tilde{t})F\left(z,t,\tau\right)}(1+t\tilde{\kappa}+\mathrm{o}(t))\right],$$

which yields

$$\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \tilde{I}_{1,\epsilon}\mathrm{d}\mu = \mathrm{O}(\sqrt{\epsilon}). \tag{4.11}$$

From (4.6), (4.9), (4.10) and (4.11) we obtain

$$\lim_{\epsilon\to 0} \bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega_{\tilde{\Sigma}_{\sqrt{\epsilon}}}\backslash\Omega_{\tilde{\Sigma}_{-\sqrt{\epsilon}}})$$
$$= \int_{\tilde{\Sigma}} \int_{b_1(z,0)}^{b_2(z,0)} \sqrt{2a(z,0)F\left(z,0,\tau\right)}\mathrm{d}\tau\mathrm{d}V_{\tilde{\bar{g}}}$$
$$= \sqrt{2}\int_{\tilde{\Sigma}} \Lambda(z,0)\mathrm{d}V_{\tilde{\bar{g}}}. \tag{4.12}$$

The upper bound estimate (4.1) follows from (4.2), (4.3), (4.4), (4.5), (4.12) and the assumption that $\bar{J}_\epsilon(u_\epsilon) \leqslant \bar{J}_\epsilon(\tilde{\rho}_\epsilon)$, where the relation $\Omega_+\backslash\Omega_{\tilde{\Sigma}} = \Omega_+$ is used, since $\mathcal{Q}$ is simply connected.

For the case that $\mathcal{Q}$ is multiply connected, (4.2) becomes

$$\bar{J}_\epsilon(\tilde{\rho}_\epsilon, \Omega\backslash\Omega_{\tilde{\Sigma}_{2\sqrt{\epsilon}}}) = \frac{1}{\epsilon}\int_{\Omega_+\backslash\Omega_{\tilde{\Sigma}}} F(x, b_2(x))\mathrm{d}x + \mathrm{O}(\epsilon),$$

and the same argument as that of the simply connected domain case gives the desired result. □

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On the other hand, from lemma (3.1) we have

$$\bar{J}_\epsilon(u_\epsilon) \geqslant \sqrt{2}\int_{\Sigma_0} \Lambda \mathrm{d}V_{\bar{g}} + \int_{\Omega_+\backslash\Omega_{\Sigma_0}} \frac{1}{\epsilon}F(x, b_2(x))\mathrm{d}x + \mathrm{o}(1). \tag{4.13}$$

Proof of theorem 1.4. Recall the assumption that $\Omega_+\backslash\Omega_\Sigma = \Omega_+\backslash\Omega_{\Sigma_0}$ for any closed smooth $(N-1)$-dimensional nontrivial surface $\Sigma \subset \mathcal{Q}$. Combining this, lemma 4.1 and (4.13) we obtain the desired results of theorem 1.4. □

To find the locations of the interfaces of interior layers to L^1-local and global maximizers of the associated energy functional, or even to general layer solutions, seems to be an interesting question. What about H^1-local and global minimizers or maximizers is also deserved to be studied.

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