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ON THE NUMBER OF p-BLOCKS OF A p-SOLUBLE GROUP

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Abstract

A technique is described for calculating the number of block ideals of FG, where F is an algebraically closed field of characteristic p, and where G is a p-soluble finite group. Among its consequences are the following: if U is a G-invariant irreducible $FO_{p'}(G)$-module, then there is a unique block ideal of FG whose restriction to $O_{p'}(G)$ has all its composition factors isomorphic to U; and if G has p'-length 1, the number of block ideals of FG is the number of G-conjugacy classes of $O_{p'}(G)$.

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Suppose that G is a finite group, p a prime, and F an algebraically closed field of characteristic p. We can write the group algebra FG as a direct sum of indecomposable two-sided ideals: say

$$FG = B_1 \oplus \cdots \oplus B_t,$$

where B_i is an indecomposable two-sided ideal of FG, $i = 1, \ldots, t$. Following Huppert and Blackburn [3], we call B_i a *block ideal* of FG; and if f_i is the indecomposable central idempotent of B_i, we call the set of irreducible FG-modules V for which $Vf_i = V$ a *p-block* of G. Huppert and Blackburn [3, page 178] remark that a "description of t is unfortunately completely lacking in the case where $\operatorname{char} F$ divides $|G|$". Our aim here is to take a small step towards such a description for p-soluble groups.

For p-soluble groups, a description of t can be given in two extreme cases: for those p-soluble groups G with $O_{p'}(G) = 1$ (Huppert and Blackburn [3, Theorem 7.13.4]), and for the class of p-nilpotent groups (an immediate corollary to Theorem 7.16.10 of Huppert and Blackburn [3]). In both these cases, the number of p-blocks is the number of G-conjugacy classes of $O_{p'}(G)$. Unfortunately, this is not true for all p-soluble groups: for example, if $G = S_4$, and F has characteristic 3, then G has 3 3-blocks, but $O_{3'}(G)$ has only 2 G-conjugacy classes. However, the ideas used in the proofs of these two results can be used to obtain more information about t: we will be able to show that the number of p-blocks of G is just the number of G-conjugacy classes of $O_{p'}(G)$ if G has p'-length 1, thereby extending the result for p-nilpotent groups; moreover, the techniques will give a method of caluclating t.

We start by establishing some notation. In general, we shall follow the notation and conventions of Huppert and Blackburn [3]. Put $N = O_{p'}(G)$, and let $U_1, \ldots, U_n$ be a complete set of distinct irreducible FN-modules, with $U_i \leqslant FN$. Let H_i be the inertia subgroup of U_i in G. Finally, suppose that $U_1, \ldots, U_k$ is a complete set of distinct non G-conjugate irreducible FN-modules. An easy calculation gives that k is just the number of distinct G-conjugacy classes contained in N.

That the number of p-blocks of G is at least the number of G-conjugacy classes of irreducible FN-modules is the content of part (1) of the proof of Theorem 7.16.10 of [3]. Putting these last two comments together gives the following result.

LEMMA 1. *The number of p-blocks of G is at least the number of G-conjugacy classes of* $O_{p'}(G)$.

Each block ideal of FG has the property that its restriction to FN has every composition factor isomorphic to a G-conjugate of some fixed U_j $(1 \leqslant j \leqslant k)$. This is an easy consequence of Lemma 1.5 of Cliff [1], and we shall follow his notation by saying that such a block ideal is of type U_j. Theorem 1.6 of Cliff [1] then tells us that the number of block ideals of FG of type U_j is the same as the number of block ideals of FH_j of type U_j. We have only been able to calculate this number in the following special case.

LEMMA 2. *Suppose that G is p-soluble, and that* $O_{p'}(H_j) = O_{p'}(G)$. *Then there is a unique block ideal of* FH_j *of type* U_j.

We shall defer the proof of Lemma 2 and first derive some consequences of it (and we assume for the rest of this paper that G is p-soluble).

COROLLARY 1. *If* U_j *is G-invariant, then there is a unique block ideal of FG of type* U_j.

This follows immediately from the fact that $H_j = G$. Note that Theorem 7.13.5 of [3] is a special case of this result.

We also obtain a method for calculating the number of block ideals of G. Each G-invariant irreducible FN-module contributes exactly one block ideal to the count, while for each irreducible FN-module which is not G-invariant, we get the appropriate number as the number of block ideals of the same type of a smaller group (the inertia subgroup).

In general, we need not have $O_{p'}(H_j) = N$. However, there are some cases where we can ensure that this condition is met. If G is of p'-length 1, then $H_j \geqslant O_p(G)$, and $O_{p'}(H_j) \leqslant O_{pp'}(G)$. Put $Q = O_{pp'}(G)$. We have $O_{p'}(H_j) \leqslant C_Q(O_{p'p}(Q)/O_{p'}(Q)) \leqslant O_{p'p}(Q)$ (by Theorem 6.3.2 of Gorenstein [2]). Thus $O_{p'}(H_j) \leqslant O_{p'}(Q) = O_{p'}(G)$. Now Lemma 2 and Theorem 1.6 of Cliff [1] give us that there is exactly one block ideal of G for each G-conjugacy class of irreducible FN-modules, giving the following result.

COROLLARY 2. *If G has p'-length* 1, *then the number of p-blocks of G is the number of G-conjugacy classes of $O_{p'}(G)$.*

We now turn to the proof of Lemma 2. We start by putting $U = U_j$ and $H = H_j$.

Suppose that B and C are blocks ideals of FH of type U with $B \neq C$, and with indecomposable central idempotents f_1 and f_2, respectively. Then B_N and C_N have all composition factors isomorphic to U.

Put $M = O_{p'p}(H)$. Then U has a unique extension to an irreducible FM-module W (the extension by Theorem 4 of Isaacs [4], and the uniqueness from Corollary 7.9.13 of [3]). Moreover, if V is an irreducible FM-module of type U, then it follows from Lemma 7.9.19 of [13] that $V \cong W$. We then have

$$\mathrm{Hom}_{FM}(W, V) = \mathrm{Hom}_{FN}(U, V_N).$$

Also, if V is a direct sum of copies of W, then

$$\mathrm{End}_{FM}(V) = \mathrm{End}_{FN}(V_N).$$

Let $x \in H \setminus M$, and set $Z = C_H(x)$. Since H is p-soluble, we have $M \nleqslant NZ$ (by Theorem 6.3.2 of Gorenstein [2]), and so if S is a transversal for $N(M \cap Z)$ in M, we have $|S| > 1$ and is a power of p. We can then find transversals R for MZ in H, and T for $N \cap Z$ in N such that $\{rst\colon r \in R, s \in S, t \in T\}$ is a transversal for Z in H. Note also that for $r \in R$ and $s \in S$, the set $\{t^{-1}(s^{-1}f^{-1}xr^s)t\colon t \in T\}$ is the set of N-conjugates of $s^{-1}r^{-1}xrs$. It then follows easily that, for $n \in N$, the set $\{n^{-1}t^{-1}(s^{-1}r^{-1}xrs)tn\colon t \in T\}$ is also the set of N-conjugates of $s^{-1}r^{-1}xrs$, and that $\{s^{-1}t^{-1}s(r^{-1}xr)s^{-1}ts\colon t \in T\}$ is the set of N-conjugates of $r^{-1}xr$.

Let V be an irreducible FH-module such that V_N is a direct sum of copies of U. Fix $r \in R$ and $s \in S$, and put $\tau_{s,r} = \sum_{t \in T} t^{-1}s^{-1}r^{-1}xrst$. Then $\tau_{s,r} \in \mathrm{End}_{FN}(V_N)$, since $n^{-1}\tau_{s,r}n = \tau_{s,r}$, and so $\tau_{s,r} \in \mathrm{End}_{FM}(V_M)$. Thus, if $v \in V$, we

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have

$$v\tau_{s,r} = vs\tau_{s,r}s^{-1} = v\tau_{1,r}.$$

This then gives

$$v\Big(\sum_{s\in S}\tau_{s,r}\Big) = v\big(|S|\tau_{1,r}\big) = 0,$$

since $|S|$ is divisible by p. If we now put $\sigma_x = \sum_{R,S,T} t^{-1}s^{-1}r^{-1}xrst$, then we have, for all $v \in V$, that $v\sigma_x = 0$.

We can write $f_i = \sum \alpha^i_x \sigma_x$, where $\alpha^i_x \in F$, and where the sum is taken over the class sums of conjugacy classes of p'-elements of H ([3], Theorem 7.12.8). Put $f_i^* = \sum \alpha^i_x \sigma_x$, where the α^i_x are as for f_i, and where the sum is taken over the class sums of H-conjugacy classes contained in N.

If V is a composition factor of B, then we have, for $v \in V$, that $v = vf_1 = vf_1^*$ (since $v\sigma_x = 0$ for $x \notin N$). Since V_N is a direct sum of copies of U, and $f_1^* \in FN$, we have $uf_1^* = u$ for $u \in U$. If W is a composition factor of C, then W_N is a direct sum of copies of U, and so for $w \in W$, we obtain $w = wf_1^* = wf_1$ (since $w\sigma_x = 0$ for $x \notin N$). However, $wf_2 = w$, and $f_1 f_2 = 0$, so that $w = wf_1 = (wf_1)f_2 = 0$ for all $w \in W$. This contradiction completes the proof.

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