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ON THE SCHERTZ CONJECTURE

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Abstract Schertz conjectured that every finite abelian extension of imaginary quadratic fields can be generated by the norm of the Siegel–Ramachandra invariants. We present a conditional proof of his conjecture by means of the characters on class groups and the second Kronecker limit formula.

Keywords: class field theory; modular units

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1. Introduction

Let K be an imaginary quadratic field, $\mathfrak{f}$ a non-zero integral ideal of K and $\mathrm{Cl}(\mathfrak{f})$ the ray class group of K modulo $\mathfrak{f}$. Then there exists a unique abelian extension $K_\mathfrak{f}$ of K whose Galois group is isomorphic to $\mathrm{Cl}(\mathfrak{f})$ via the Artin map

$$\sigma_\mathfrak{f} : \mathrm{Cl}(\mathfrak{f}) \xrightarrow{\sim} \mathrm{Gal}(K_\mathfrak{f}/K), \tag{1.1}$$

which is called the *ray class field* of K modulo $\mathfrak{f}$. By class field theory, any abelian extension of K is contained in some ray class field $K_\mathfrak{f}$; hence, it is important to construct the ray class fields of K to determine the maximal abelian extension of K.

In 1964, Ramachandra [**6**, Theorem 10] constructed a primitive generator of $K_\mathfrak{f}$ over K in terms of a certain elliptic unit and showed that arbitrary finite abelian extension of K could be generated by the norm of this unit, which settled Kronecker's Jugendtraum over an imaginary quadratic field. However, his unit involves products of singular values of the Klein forms and the discriminant Δ-function that are too complicated to use in practice. On the other hand, Schertz [**7**, Theorem 6.8.4] presented a relatively simple ray class invariant over K by means of the singular value of a certain Siegel function, namely, the Siegel–Ramachandra invariant. He further conjectured that every finite abelian extension of K could be generated by the norm of the Siegel–Ramachandra invariant [**7**, Conjecture 6.8.3] as follows.

Conjecture 1.1. *Let $\mathfrak{f}$ be a non-zero proper integral ideal of K and let L be a finite abelian extension of K such that $K \subset L \subset K_{\mathfrak{f}}$. Then for every non-zero integer n and $C \in \mathrm{Cl}(\mathfrak{f})$,*

$$L = K(N_{K_{\mathfrak{f}}/L}(g_{\mathfrak{f}}(C)^n)),$$

where $g_{\mathfrak{f}}(C)$ is the Siegel–Ramachandra invariant of conductor $\mathfrak{f}$ at C defined in (2.1).

Recently, Koo–Yoon generated ray class fields $K_{\mathfrak{f}}$ over K via Siegel–Ramachandra invariants by making use of the characters on class groups and the second Kronecker limit formula [**3**, Theorem 4.6]. In this paper, by improving their idea, we give a conditional proof of the conjecture with a certain assumption depending only on the extension degree $[K_{\mathfrak{f}} : LH_K]$, where H_K denotes the Hilbert class field of K (Theorem 2.6 and Example 2.10).

Notation 1.2. For $z \in \mathbb{C}$, we denote by $\overline{z}$ the complex conjugate of z. If G is a group and $g_1, g_2, \ldots, g_r$ are elements of G, let $\langle g_1, g_2, \ldots, g_r\rangle$ be the subgroup of G generated by $g_1, g_2, \ldots, g_r$. Moreover, if H is a subgroup of G and $g \in G$, by $[g]$ we mean the coset gH of H in G. For a number field K, let $\mathcal{O}_K$ be the ring of integers of K. If $a \in \mathcal{O}_K$, we denote by (a) the principal ideal of K generated by a.

2. Main theorem

For a rational vector $\mathbf{r} = \left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we define the *Siegel function* $g_{\mathbf{r}}(\tau)$ on the complex upper half plane $\mathbb{H}$ by the following infinite product:

$$g_{\mathbf{r}}(\tau) = -q^{1/2\mathbf{B}_2(r_1)}\mathrm{e}^{\pi \mathrm{i} r_2(r_1-1)}(1 - q^{r_1}\mathrm{e}^{2\pi \mathrm{i} r_2})\prod_{n=1}^{\infty}(1 - q^{n+r_1}\mathrm{e}^{2\pi \mathrm{i} r_2})(1 - q^{n-r_1}\mathrm{e}^{-2\pi \mathrm{i} r_2}),$$

where $\mathbf{B}_2(X) = X^2 - X + 1/6$ is the second Bernoulli polynomial and $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$. Then, a Siegel function is a modular unit, namely, it is a modular function whose zeros and poles are supported only at the cusps [**9**] or [**4**, p. 36]. In particular, if $\mathbf{r} \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2$ with an integer $N \geq 2$ then the function $g_{\mathbf{r}}(\tau)^{12N}$ belongs to $\mathcal{F}_N$ [**2**, Proposition 1.1], where $\mathcal{F}_N$ is the field of meromorphic modular functions for the principal congruence subgroup $\Gamma(N)$ whose Fourier coefficients lie in the Nth cyclotomic field $\mathbb{Q}(\mathrm{e}^{2\pi \mathrm{i}/N})$.

Let K be an imaginary quadratic field of discriminant d_K, $\mathfrak{f}$ a non-zero proper integral ideal of K and N the smallest positive integer in $\mathfrak{f}$. For $C \in \mathrm{Cl}(\mathfrak{f})$, we take any integral ideal $\mathfrak{c}$ in C and choose a basis $[\omega_1, \omega_2]$ of $\mathfrak{f}\mathfrak{c}^{-1}$ such that $\omega_1/\omega_2 \in \mathbb{H}$. Then one can write

$$N = r_1\omega_1 + r_2\omega_2$$

for some $r_1, r_2 \in \mathbb{Z}$. We define the *Siegel–Ramachandra invariant* of conductor $\mathfrak{f}$ at C by

$$g_{\mathfrak{f}}(C) = g_{\left[\begin{smallmatrix} r_1/N \\ r_2/N \end{smallmatrix}\right]}(\omega_1/\omega_2)^{12N}. \tag{2.1}$$

This value depends only on the class C and $\mathfrak{f}$, not on the choice of $\mathfrak{c}$.

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Proposition 2.1. *Let $C, C' \in \mathrm{Cl}(\mathfrak{f})$ with $\mathfrak{f} \neq \mathcal{O}_K$.*

(i) *$g_\mathfrak{f}(C)$ belongs to $K_\mathfrak{f}$ as an algebraic integer. If N is composite, $g_\mathfrak{f}(C)$ is a unit in $K_\mathfrak{f}$.*

(ii) *We have the transformation formula*

$$g_\mathfrak{f}(C)^{\sigma_\mathfrak{f}(C')} = g_\mathfrak{f}(CC'),$$

where $\sigma_\mathfrak{f}$ is the Artin map stated in (1.1).

Proof. [5, Chapter 19, Theorem 3] and [4, Chapter 11, Theorem 1.2]. □

Let χ be a non-trivial character of $\mathrm{Cl}(\mathfrak{f})$ with $\mathfrak{f} \neq \mathcal{O}_K$, $\mathfrak{f}_\chi$ a conductor of χ and χ_0 the primitive character of $\mathrm{Cl}(\mathfrak{f}_\chi)$ corresponding to χ. The *Stickelberger element* and the *L-function* for χ are defined by

$$S_\mathfrak{f}(\chi) = \sum_{C \in \mathrm{Cl}(\mathfrak{f})} \chi(C) \log |g_\mathfrak{f}(C)|,$$

$$L_\mathfrak{f}(s, \chi) = \sum_{\substack{(0) \neq \mathfrak{a} \subset \mathcal{O}_K \\ \gcd(\mathfrak{a}, \mathfrak{f}) = 1}} \frac{\chi(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s} \quad (s \in \mathbb{C}),$$

respectively, where $\mathcal{N}(\mathfrak{a})$ is the absolute norm of an ideal $\mathfrak{a}$. The second Kronecker limit formula describes the relation between the Stickelberger element and the L-function as follows.

Proposition 2.2. *Let χ be a non-trivial character of $\mathrm{Cl}(\mathfrak{f})$ with $\mathfrak{f}_\chi \neq \mathcal{O}_K$. Then we have*

$$L_{\mathfrak{f}_\chi}(1, \chi_0) \prod_{\substack{\mathfrak{p} \mid \mathfrak{f} \\ \mathfrak{p} \nmid \mathfrak{f}_\chi}} (1 - \overline{\chi_0}([\mathfrak{p}])) = -\frac{2\pi \chi_0([\gamma \mathfrak{d}_K \mathfrak{f}_\chi])}{6N(\mathfrak{f}_\chi)\omega(\mathfrak{f}_\chi) T_\gamma(\overline{\chi_0})\sqrt{-d_K}} \cdot S_\mathfrak{f}(\overline{\chi}),$$

where $\mathfrak{d}_K$ is the different ideal of $K/\mathbb{Q}$, γ is an element of K such that $\gamma \mathfrak{d}_K \mathfrak{f}_\chi$ is an integral ideal of K prime to $\mathfrak{f}_\chi$, $N(\mathfrak{f}_\chi)$ is the smallest positive integer in $\mathfrak{f}_\chi$, $\omega(\mathfrak{f}_\chi)$ is the number of roots of unity in K which are congruent to 1 modulo $\mathfrak{f}_\chi$ and

$$T_\gamma(\overline{\chi_0}) = \sum_{x + \mathfrak{f}_\chi \in (\mathcal{O}_K/\mathfrak{f}_\chi)^\times} \overline{\chi_0}([x\mathcal{O}_K]) \mathrm{e}^{2\pi \mathrm{i} \mathrm{Tr}_{K/\mathbb{Q}}(\gamma x)}.$$

Proof. See [4, Chapter 11 §2, LF 2]. □

Remark 2.3. Since χ_0 is a non-trivial primitive character of $\mathrm{Cl}(\mathfrak{f}_\chi)$, both $L_{\mathfrak{f}_\chi}(1, \chi_0)$ and the Gauss sum $T_\gamma(\overline{\chi}_0)$ are non-zero [1, Chapter V, Theorem 10.2], [5, Chapter 22 §1, G 3]. If every prime ideal factor of $\mathfrak{f}$ divides $\mathfrak{f}_\chi$ then we understand the Euler factor $\prod_{\mathfrak{p} \mid \mathfrak{f},\ \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \overline{\chi_0}([\psi]))$ to be 1, and hence we conclude that $S_\mathfrak{f}(\overline{\chi}) \neq 0$.

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For an intermediate field L of the extension $K_\mathfrak{f}/K$, we denote by $\mathrm{Cl}(K_\mathfrak{f}/L)$ the subgroup of $\mathrm{Cl}(\mathfrak{f})$ corresponding to $\mathrm{Gal}(K_\mathfrak{f}/L)$ via the Artin map (1.1). Then one can identify $\mathrm{Cl}(K_\mathfrak{f}/H_K)$ with the quotient group

$$(\mathcal{O}_K/\mathfrak{f})^\times/\{\alpha+\mathfrak{f}\in(\mathcal{O}_K/\mathfrak{f})^\times \mid \alpha\in\mathcal{O}_K^\times\}$$

via the natural homomorphism

$$\begin{aligned}(\mathcal{O}_K/\mathfrak{f})^\times &\longrightarrow \mathrm{Cl}(K_\mathfrak{f}/H_K)\\ \alpha+\mathfrak{f} &\longmapsto [(\alpha)].\end{aligned}$$

Let $\mathfrak{f}=\prod_\mathfrak{p}\mathfrak{p}^{e_\mathfrak{p}}$ be a prime ideal factorization of $\mathfrak{f}$. For each prime ideal $\mathfrak{p}$, we set

$$\mathbf{G}_\mathfrak{p}=(\mathcal{O}_K/\mathfrak{p}^{e_\mathfrak{p}})^\times/\{\alpha+\mathfrak{p}^{e_\mathfrak{p}}\in(\mathcal{O}_K/\mathfrak{p}^{e_\mathfrak{p}})^\times \mid \alpha\in\mathcal{O}_K^\times\}$$

so that $\mathbf{G}_\mathfrak{p}\cong\mathrm{Cl}(K_{\mathfrak{p}^{e_\mathfrak{p}}}/H_K)\subset\mathrm{Cl}(\mathfrak{p}^{e_\mathfrak{p}})$. Then we have

$$|\mathbf{G}_\mathfrak{p}|=\phi(\mathfrak{p}^{e_\mathfrak{p}})\frac{\omega(\mathfrak{p}^{e_\mathfrak{p}})}{\omega_K}$$

where $\phi(\mathfrak{p}^{e_\mathfrak{p}})=|(\mathcal{O}_K/\mathfrak{p}^{e_\mathfrak{p}})^\times|$, ω_K is the number of roots of unity in K and $\omega(\mathfrak{p}^{e_\mathfrak{p}})$ is the number of roots of unity in K which are congruent to 1 modulo $\mathfrak{p}^{e_\mathfrak{p}}$.

Lemma 2.4. *Let $H\subset G$ be two finite abelian groups, $g\in G\setminus H$, and let n be the order of the coset $[g]$ in G/H. Then for any character χ of H, we can extend it to a character ψ of G in such a way that $\psi(g)$ is any fixed nth root of $\chi(g^n)$.*

Proof. See [8, Chapter VI, Proposition 1]. □

Let L be a finite abelian extension of K such that $K\subsetneq L\subset K_\mathfrak{f}$ and $L\not\subset H_K$. Replacing $\mathfrak{f}$ by $\mathfrak{f}\mathfrak{p}^{-e_\mathfrak{p}}$ if necessary, we may assume that $L\not\subset K_{\mathfrak{f}\mathfrak{p}^{-e_\mathfrak{p}}}$ for every prime ideal factor $\mathfrak{p}$ of $\mathfrak{f}$.

Lemma 2.5. *Assume that for each prime ideal factor $\mathfrak{p}$ of $\mathfrak{f}$ there is a rational prime $\nu_\mathfrak{p}$ satisfying $\mathrm{ord}_{\nu_\mathfrak{p}}(|\mathbf{G}_\mathfrak{p}|)>\mathrm{ord}_{\nu_\mathfrak{p}}([K_\mathfrak{f}:LH_K])+i_\mathfrak{p}$ where*

$$i_\mathfrak{p}=\begin{cases}0 & \text{if } \nu_\mathfrak{p}\neq 2,\\ 1 & \text{if } \nu_\mathfrak{p}=2.\end{cases}$$

Then, for any class $D\in\mathrm{Cl}(\mathfrak{f})\setminus\mathrm{Cl}(K_\mathfrak{f}/L)$, there exists a character χ of $\mathrm{Cl}(\mathfrak{f})$ such that $\chi|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1$, $\chi(D)\neq 1$ and $\mathfrak{p}\mid\mathfrak{f}_\chi$ for every prime ideal factor $\mathfrak{p}$ of $\mathfrak{f}$.

Proof. By Lemma 2.4, there exists a character χ of $\mathrm{Cl}(\mathfrak{f})$ satisfying $\chi|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1$ and $\chi(D)\neq 1$. For each $\mathfrak{p}$, we define a homomorphism $\varphi_\mathfrak{p}$ by

$$\begin{aligned}\varphi_\mathfrak{p}:\mathrm{Cl}(K_\mathfrak{f}/H_K) &\rightarrow \mathbf{G}_\mathfrak{p}\\ [\alpha+\mathfrak{f}] &\longmapsto [\alpha+\mathfrak{p}^{e_\mathfrak{p}}].\end{aligned}$$

Suppose that $\mathfrak{p}\nmid\mathfrak{f}_\chi$ for some $\mathfrak{p}$. Let n be the order of the class D in the quotient group $\mathrm{Cl}(\mathfrak{f})/\mathrm{Cl}(K_\mathfrak{f}/H_K)$. Then $D^n=[(\beta)]$ for some $\beta\in\mathcal{O}_K$ which is prime to $\mathfrak{f}$. Note that

$$\mathrm{Cl}(K_\mathfrak{f}/L)\cap\mathrm{Cl}(K_\mathfrak{f}/H_K)=\mathrm{Cl}(K_\mathfrak{f}/LH_K).$$

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Case 1. First, suppose that $\mathbf{G}_\mathfrak{p}/\mathrm{Im}(\varphi_\mathfrak{p}|_{\mathrm{Cl}(K_\mathfrak{f}/LH_K)}) \neq \langle[\beta+\mathfrak{p}^{e_\mathfrak{p}}]\rangle$. Then there exists a non-trivial character ψ of $\mathbf{G}_\mathfrak{p}$ in such a way that ψ is trivial on $\mathrm{Im}(\varphi_\mathfrak{p}|_{\mathrm{Cl}(K_\mathfrak{f}/LH_K)})$ and $\psi([\beta+\mathfrak{p}^{e_\mathfrak{p}}])=1$. Let $\psi'=\psi\circ\varphi_\mathfrak{p}$ be a character of $\mathrm{Cl}(K_\mathfrak{f}/H_K)$. Then it is possible for us to extend ψ' to a character $\psi_\mathfrak{p}$ of $\mathrm{Cl}(\mathfrak{f})$ such that $\psi_\mathfrak{p}|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1$ and $\psi_\mathfrak{p}(D)=1$ by Lemma 2.4.

Case 2. Now, assume that $\mathbf{G}_\mathfrak{p}/\mathrm{Im}(\varphi_\mathfrak{p}|_{\mathrm{Cl}(K_\mathfrak{f}/LH_K)}) = \langle[\beta+\mathfrak{p}^{e_\mathfrak{p}}]\rangle$. By the hypothesis, there is a non-trivial character ψ of $\mathbf{G}_\mathfrak{p}$ such that ψ is trivial on $\mathrm{Im}(\varphi_\mathfrak{p}|_{\mathrm{Cl}(K_\mathfrak{f}/LH_K)})$ and $\psi([\beta+\mathfrak{p}^{e_\mathfrak{p}}])\neq 1,\chi(D^n)^{-1}$. Similar to Case 1, one can extend ψ to a character $\psi_\mathfrak{p}$ of $\mathrm{Cl}(\mathfrak{f})$ for which $\psi_\mathfrak{p}|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1$ and $\psi_\mathfrak{p}(D)\neq\chi(D)^{-1}$.

Here we observe that $\psi_\mathfrak{p}$ is a non-trivial character whose conductor is solely divisible by $\mathfrak{p}$ in both cases. Hence the character $\chi\psi_\mathfrak{p}$ of $\mathrm{Cl}(\mathfrak{f})$ satisfies $\chi\psi_\mathfrak{p}|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1$, $\chi\psi_\mathfrak{p}(D)\neq 1$, $\mathfrak{p}\,|\,\mathfrak{f}_{\chi\psi_\mathfrak{p}}$ and $\mathfrak{f}_\chi\,|\,\mathfrak{f}_{\chi\psi_\mathfrak{p}}$. Thus, we replace χ by $\chi\psi_\mathfrak{p}$. By continuing this process for every $\mathfrak{p}$, we get the lemma. □

Let $\mathbf{h}_{L,\mathfrak{f}}$ be the set of prime ideal factors $\mathfrak{p}$ of $\mathfrak{f}$ such that there is no rational prime $\nu_\mathfrak{p}$ satisfying $\mathrm{ord}_{\nu_\mathfrak{p}}(|\mathbf{G}_\mathfrak{p}|) > \mathrm{ord}_{\nu_\mathfrak{p}}([K_\mathfrak{f}:LH_K])+i_\mathfrak{p}$.

Theorem 2.6. *Let $\mathfrak{f}=\prod_\mathfrak{p}\mathfrak{p}^{e_\mathfrak{p}}$ be a non-zero proper integral ideal of K, and let L be a finite abelian extension of K such that $K\subset L\subset K_\mathfrak{f}$. Assume that $L\not\subset H_K$ and $L\not\subset K_{\mathfrak{f}\mathfrak{p}^{-e_\mathfrak{p}}}$ for every prime ideal factor $\mathfrak{p}$ of $\mathfrak{f}$, and*

$$\sum_{\mathfrak{p}\in\mathbf{h}_{L,\mathfrak{f}}}\frac{1}{[L:L\cap K_{\mathfrak{f}\mathfrak{p}^{-e_\mathfrak{p}}}]}\leq\frac{1}{2}. \tag{2.2}$$

Then, for any non-zero integer n and $C\in\mathrm{Cl}(\mathfrak{f})$, the singular value

$$N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C)^n)$$

generates L over K. In particular, if $|\mathbf{h}_{L,\mathfrak{f}}|=0$ or 1, then the assumption (2.2) is always true and so we have the desired result.

Proof. The proof is clear when $L=K$, and so we may assume that $K\subsetneq L$. Let

$$L'=K(N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C_0)^n)),$$

where C_0 is the unit class in $\mathrm{Cl}(\mathfrak{f})$. On the contrary, suppose $L'\subsetneq L$. Then we claim that there exists a character χ of $\mathrm{Cl}(\mathfrak{f})$ satisfying $\chi|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1$, $\chi|_{\mathrm{Cl}(K_\mathfrak{f}/L')}\neq 1$ and $\mathfrak{p}\,|\,\mathfrak{f}_\chi$ for every $\mathfrak{p}\in\mathbf{h}_{L,\mathfrak{f}}$. Indeed, if $|\mathbf{h}_{L,\mathfrak{f}}|=0$ then the proof is clear by Lemma 2.4. Suppose $|\mathbf{h}_{L,\mathfrak{f}}|\geq 1$. Let

$$G_1=\{\text{characters }\chi\text{ of }\mathrm{Cl}(\mathfrak{f})\mid \chi|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1,\chi|_{\mathrm{Cl}(K_\mathfrak{f}/L')}\neq 1\},$$

$$G_2=\{\text{non-trivial characters }\chi\text{ of }\mathrm{Cl}(\mathfrak{f})\mid \chi|_{\mathrm{Cl}(K_\mathfrak{f}/L)}=1\text{ and }\mathfrak{p}\nmid\mathfrak{f}_\chi\text{ for some }\mathfrak{p}\in\mathbf{h}_{L,\mathfrak{f}}\}.$$

Observe that all characters in G_1 are non-trivial. Then we have

$$
\begin{aligned}
|G_1| &= |\{\text{characters } \chi \text{ of } \mathrm{Gal}(L/K) \mid \chi|_{\mathrm{Gal}(L/L')} \neq 1\}| \\
&\quad \text{since } \mathrm{Cl}(\mathfrak{f})/\mathrm{Cl}(K_{\mathfrak{f}}/L) \cong \mathrm{Gal}(L/K) \\
&= |\{\text{characters } \chi \text{ of } \mathrm{Gal}(L/K)\}| - |\{\text{characters } \chi \text{ of } \mathrm{Gal}(L/K) \mid \chi|_{\mathrm{Gal}(L/L')} = 1\}| \\
&= [L:K] - [L':K] \\
&= [L:K]\left(1 - \frac{1}{[L:L']}\right) \\
&\geq \tfrac{1}{2}[L:K].
\end{aligned}
$$

On the other hand, we deduce

$$
\begin{aligned}
|G_2| &= |\{\text{characters } \chi \text{ of } \mathrm{Cl}(\mathfrak{f}) \mid \chi|_{\mathrm{Cl}(K_{\mathfrak{f}}/L)} = 1 \text{ and } \mathfrak{f}_\chi \mid \mathfrak{f}\mathfrak{p}^{-e_{\mathfrak{p}}} \text{ for some } \mathfrak{p} \in \mathbf{h}_{L,\mathfrak{f}}\}| - 1 \\
&\leq \sum_{\mathfrak{p} \in \mathbf{h}_{L,\mathfrak{f}}} |\{\text{characters } \chi \text{ of } \mathrm{Cl}(\mathfrak{f}\mathfrak{p}^{-e_{\mathfrak{p}}}) \mid \chi|_{\mathrm{Cl}(K_{\mathfrak{f}\mathfrak{p}^{-e_{\mathfrak{p}}}}/L \cap K_{\mathfrak{f}\mathfrak{p}^{-e_{\mathfrak{p}}}})} = 1\}| - 1 \\
&= \sum_{\mathfrak{p} \in \mathbf{h}_{L,\mathfrak{f}}} [L \cap K_{\mathfrak{f}\mathfrak{p}^{-e_{\mathfrak{p}}}} : K] - 1 \\
&= [L:K]\left(\sum_{\mathfrak{p} \in \mathbf{h}_{L,\mathfrak{f}}} \frac{1}{[L : L \cap K_{\mathfrak{f}\mathfrak{p}^{-e_{\mathfrak{p}}}}]}\right) - 1 \\
&\leq \tfrac{1}{2}[L:K] - 1 \quad \text{by (2.2)}.
\end{aligned}
$$

Hence $|G_1| > |G_2|$ and so the claim is proved.

Choose a class $D \in \mathrm{Cl}(K_{\mathfrak{f}}/L') \setminus \mathrm{Cl}(K_{\mathfrak{f}}/L)$ such that $\chi(D) \neq 1$. We then see from the proof of Lemma 2.5 that there is a character ψ of $\mathrm{Cl}(\mathfrak{f})$ satisfying $\chi\psi|_{\mathrm{Cl}(K_{\mathfrak{f}}/L)} = 1$, $\chi\psi(D) \neq 1$, $\mathfrak{f}_\chi \mid \mathfrak{f}_{\chi\psi}$ and $\mathfrak{p} \mid \mathfrak{f}_{\chi\psi}$ for every prime ideal factor $\mathfrak{p}$ of $\mathfrak{f}$. We replace χ by $\chi\psi$.

Since χ is non-trivial and $\mathfrak{f}_\chi \neq \mathcal{O}_K$, we obtain $S_{\mathfrak{f}}(\overline{\chi}) \neq 0$ by Proposition 2.2. On the other hand, we derive that

$$
\begin{aligned}
S_{\mathfrak{f}}(\overline{\chi}) &= \frac{1}{n} \sum_{E \in \mathrm{Cl}(\mathfrak{f})} \overline{\chi}(E) \log |g_{\mathfrak{f}}(E)^n| \\
&= \frac{1}{n} \sum_{E \in \mathrm{Cl}(\mathfrak{f})} \overline{\chi}(E) \log |(g_{\mathfrak{f}}(C_0)^n)^{\sigma_{\mathfrak{f}}(E)}| \quad \text{(by Proposition 2.1)} \\
&= \frac{1}{n} \sum_{\substack{E_1 \in \mathrm{Cl}(\mathfrak{f}) \\ E_1 \bmod \mathrm{Cl}(K_{\mathfrak{f}}/L')}} \sum_{\substack{E_2 \in \mathrm{Cl}(K_{\mathfrak{f}}/L') \\ E_2 \bmod \mathrm{Cl}(K_{\mathfrak{f}}/L)}} \\
&\quad \times \sum_{E_3 \in \mathrm{Cl}(K_{\mathfrak{f}}/L)} \overline{\chi}(E_1E_2E_3) \log |(g_{\mathfrak{f}}(C_0)^n)^{\sigma_{\mathfrak{f}}(E_1E_2E_3)}|
\end{aligned}
$$

$$= \frac{1}{n}\sum_{E_1}\overline{\chi}(E_1)\sum_{E_2}\overline{\chi}(E_2)\log|N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C_0)^n)^{\sigma_\mathfrak{f}(E_1)\sigma_\mathfrak{f}(E_2)}| \quad \text{since } \chi|_{\mathrm{Cl}(K_\mathfrak{f}/L)} = 1$$
$$= \frac{1}{n}\sum_{E_1}\overline{\chi}(E_1)\log|N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C_0)^n)^{\sigma_\mathfrak{f}(E_1)}|\left(\sum_{E_2}\overline{\chi}(E_2)\right)$$
$$= 0,$$

because $N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C_0)^n) \in L'$ and $\chi|_{\mathrm{Cl}(K_\mathfrak{f}/L')} \neq 1$. This is a contradiction, and so $L' = L$.

Since L' is an abelian extension of K and

$$N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C_0)^n)^{\sigma_\mathfrak{f}(C)} = N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C)^n) \quad \text{for } C \in \mathrm{Cl}(\mathfrak{f}),$$

we conclude that $L = L' = K(N_{K_\mathfrak{f}/L}(g_\mathfrak{f}(C)^n))$ as desired. □

Remark 2.7. If $\mathfrak{f} = \mathfrak{p}^n$ is a power of a prime ideal $\mathfrak{p}$ of K, then the assumption (2.2) is always satisfied since $|\mathbf{h}_{L,\mathfrak{f}}| \leq 1$.

Now, consider the case where $L = K_\mathfrak{f}$. One can readily show that

$$\mathbf{h}_{K_\mathfrak{f},\mathfrak{f}} = \{\text{a prime ideal factor } \mathfrak{p} \text{ of } \mathfrak{f} \mid |\mathbf{G}_\mathfrak{p}| = 1 \text{ or } 2\},$$

and hence [3, Theorem 4.6] is a special case of Theorem 2.6 for $L = K_\mathfrak{f}$ as follows.

Corollary 2.8. *Let $\mathfrak{f} = \prod_\mathfrak{p} \mathfrak{p}^{e_\mathfrak{p}}$ be a non-zero proper integral ideal of K. Assume that $K_\mathfrak{f} \neq K_{\mathfrak{f}\mathfrak{p}^{-e_\mathfrak{p}}}$ for every prime ideal factor $\mathfrak{p}$ of $\mathfrak{f}$, and*

$$\sum_{\mathfrak{p}\in\mathbf{h}_{K_\mathfrak{f},\mathfrak{f}}} \frac{1}{\phi(\mathfrak{p}^{e_\mathfrak{p}})} \leq \frac{1}{2}.$$

Then, for any non-zero integer n and $C \in \mathrm{Cl}(\mathfrak{f})$, we have

$$K_\mathfrak{f} = K(g_\mathfrak{f}(C)^n).$$

Proof. See [3, Theorem 4.6]. □

Remark 2.9. We see from [3, Lemma 4.4] that $|\mathbf{G}_\mathfrak{p}| = 1$ or 2 if and only if $\mathfrak{p}^{e_\mathfrak{p}}$ satisfies one of the following conditions.

Case 1. $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$.

- 2 is not inert in K, $\mathfrak{p}$ is lying over 2 and $e_\mathfrak{p} = 1, 2$ or 3.
- 3 is not inert in K, $\mathfrak{p}$ is lying over 3 and $e_\mathfrak{p} = 1$.
- 5 is not inert in K, $\mathfrak{p}$ is lying over 5 and $e_\mathfrak{p} = 1$.

Case 2. $K = \mathbb{Q}(\sqrt{-1})$.

- $\mathfrak{p}$ is lying over 2 and $e_\mathfrak{p} = 1, 2, 3$ or 4.
- $\mathfrak{p}$ is lying over 3 and $e_\mathfrak{p} = 1$.
- $\mathfrak{p}$ is lying over 5 and $e_\mathfrak{p} = 1$.

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Case 3. $K = \mathbb{Q}(\sqrt{-3})$.

- $\mathfrak{p}$ is lying over 2 and $e_{\mathfrak{p}} = 1$ or 2.
- $\mathfrak{p}$ is lying over 3 and $e_{\mathfrak{p}} = 1$ or 2.
- $\mathfrak{p}$ is lying over 7 and $e_{\mathfrak{p}} = 1$.
- $\mathfrak{p}$ is lying over 13 and $e_{\mathfrak{p}} = 1$.

Example 2.10. Let $K = \mathbb{Q}(\sqrt{-11})$ and let L be a finite abelian extension of K such that $K \subsetneq L \subset K_{\mathfrak{f}}$ for some non-zero proper integral ideal $\mathfrak{f}$ of K. Then $H_K = K$.

(i) Let $\mathfrak{f} = 22\mathcal{O}_K$. Then $\mathfrak{f} = \mathfrak{p}_1\mathfrak{p}_2^2$ with prime ideals $\mathfrak{p}_1 = 2\mathcal{O}_K$ and $\mathfrak{p}_2 = \sqrt{-11}\mathcal{O}_K$. Observe that $|\mathbf{G}_{\mathfrak{p}_1}| = 3$, $|\mathbf{G}_{\mathfrak{p}_2}| = 55$ and $[K_{\mathfrak{f}} : K] = 165$. Hence $[K_{\mathfrak{f}} : L] = 1, 3, 5, 11, 15, 33$ or 55. Since

$$\begin{aligned} \operatorname{ord}_3(|\mathbf{G}_{\mathfrak{p}_1}|) &> \operatorname{ord}_3([K_{\mathfrak{f}} : L]) \quad \text{if } [K_{\mathfrak{f}} : L] = 1, 5, 11, 55, \\ \operatorname{ord}_5(|\mathbf{G}_{\mathfrak{p}_2}|) &> \operatorname{ord}_5([K_{\mathfrak{f}} : L]) \quad \text{if } [K_{\mathfrak{f}} : L] = 3, 33, \\ \operatorname{ord}_{11}(|\mathbf{G}_{\mathfrak{p}_2}|) &> \operatorname{ord}_{11}([K_{\mathfrak{f}} : L]) \quad \text{if } [K_{\mathfrak{f}} : L] = 15, \end{aligned}$$

we get $|\mathbf{h}_{L,\mathfrak{f}}| = 0$ or 1 for any case. Therefore, it follows from Theorem 2.6 that

$$L = K(N_{K_{\mathfrak{f}}/L}(g_{\mathfrak{f}}(C)^n))$$

for any non-zero integer n and $C \in \mathrm{Cl}(\mathfrak{f})$.

(ii) Let $\mathfrak{f} = 5\mathcal{O}_K$. Then $\mathfrak{f} = \mathfrak{p}\overline{\mathfrak{p}}$ with the prime ideal $\mathfrak{p} = (5, 2 + \sqrt{-11})$. Consider the case where $L = K_{\mathfrak{f}}$. Note that we cannot apply [7, Theorem 6.8.4] to this case because the exponents of $(\mathcal{O}_K/\mathfrak{p})^\times$ and $(\mathcal{O}_K/\overline{\mathfrak{p}})^\times$ are 4. On the other hand, one can easily show that $|\mathbf{G}_{\mathfrak{p}}| = |\mathbf{G}_{\overline{\mathfrak{p}}}| = 2$ and so $\mathbf{h}_{K_{\mathfrak{f}},\mathfrak{f}} = \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Since

$$\frac{1}{\phi(\mathfrak{p})} + \frac{1}{\phi(\overline{\mathfrak{p}})} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

we deduce by Corollary 2.8 that

$$K_{\mathfrak{f}} = K(g_{\mathfrak{f}}(C)^n)$$

for any non-zero integer n and $C \in \mathrm{Cl}(\mathfrak{f})$.

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