Archimedes playing with a computer

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1. *Introduction*

It was known before Archimedes (287-212 BC) that the circumference of a circle was proportional to its diameter and that the area was proportional to the square of its radius. It was Archimedes who first supplied a rigorous proof that these two proportionality constants were the same, now called π [1]. He started with inscribed and circumscribed hexagons and increased the number of sides from 6 up to 96 by successively doubling it. His result was not a single value. In fact he generated five intervals each of which contained π. He calculated a lower bound from the inscribed polygon and an upper bound from the circumscribed polygon of 96 sides. This gave him the interval $\left(3\frac{10}{71},\ 3\frac{1}{7}\right)$ or $\left(\mathbf{3.14}0845,\ \mathbf{3.14}2857\right)$, which is less accurate than the interval bounded by half-perimeters of the inscribed and circumscribed 96-gons, which is $\left(\mathbf{3.141}031,\ \mathbf{3.14}2714\right)$.

Archimedes' method is a nice example for introducing undergraduate and graduate students in mathematics to numerical analysis methods in order to get information from sequences. This is the main purpose of this Article. In this short paper we introduce such techniques and try to imagine what Archimedes could have done with the help of a computer. What information he would have been able to obtain from numerical sequences generated by his method to approximate π. Maybe he would have had the idea of Richardson (1881-1953) to improve his sequences and obtain better approximations of π [3]. Finally, as described in [4], using trigonometric relations related to the polygons, and with the help of Taylor (1685-1731) and series expansions for functions, it is possible to verify all the information on the sequences obtained from the computer.

2. *The setting with Thales* (c.625-547 BC) *and Pythagoras* (c.580-495 BC)}

We consider inscribed and circumscribed regular τ_n-gons with respect to the unit circle decomposed into τ_n equal triangles. The sequence of integer numbers $\{\tau_n\}_{n=0}^{\infty}$ is such that $\tau_0 = 6$ (we start with an hexagon) and $\tau_{n+1} = 2\tau_n$.

For triangles associated to inscribed τ_n-gons, let their base, height, and side be b_n, h_n, and $s_n = 1$ (see Figure 1). From Pythagoras' Theorem we have

$$\begin{cases} h_n^2 + \left(\frac{b_n}{2}\right)^2 = 1, \\ (1 - h_n)^2 + \left(\frac{b_n}{2}\right)^2 = b_{n+1}^2, \end{cases}$$

from which we deduce that

$$\begin{cases} b_{n+1} = \sqrt{2(1 - h_n)}, \\ h_{n+1} = \sqrt{\frac{1 + h_n}{2}}. \end{cases}$$

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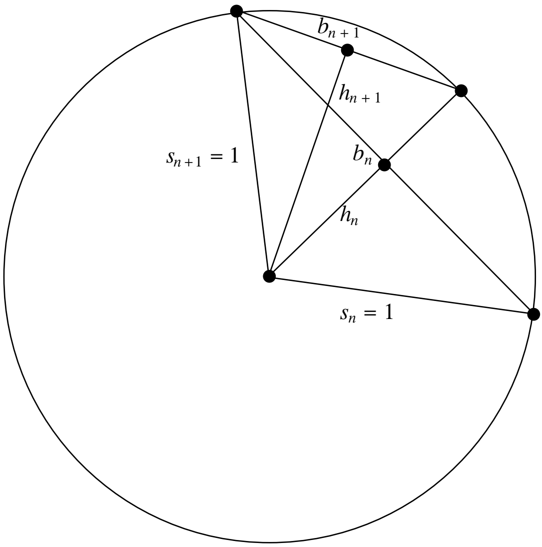

FIGURE 1: Triangles related to inscribed polygons

The half-perimeter p_n and area a_n of the τ_n-gons are

$$\begin{cases} p_n = \tau_n \frac{b_n}{2}, \\ a_n = \tau_n \frac{b_n h_n}{2} = p_n h_n, \end{cases}$$

so $a_n < p_n$ because $h_n < 1$. Similarly, for triangles associated with circumscribed τ_n-gons, let their base, height, and side be B_n, $H_n = 1$ and S_n (see Figure 2). From Pythagoras' Theorem we have

$$\begin{cases} 1 + \left(\frac{B_n}{2}\right)^2 = S_n^2, \\ (S_n - 1)^2 + \left(\frac{B_{n+1}}{2}\right)^2 = \left(\frac{B_n - B_{n+1}}{2}\right)^2. \end{cases}$$

The half-perimeter P_n and area A_n of the τ_n-gons are

$$\begin{cases} P_n = \tau_n \frac{B_n}{2}, \\ A_n = \tau_n \frac{B_n H_n}{2}, \end{cases}$$

so $A_n = P_n$ because $H_n = 1$. Moreover, from Thales' intercept theorem we get

$$\frac{B_n}{b_n} = \frac{S_n}{s_n} = \frac{H_n}{h_n},$$

where $H_n = 1$ and $s_n = 1$ (see Figure 3). From these relations we get $a_n < p_n < \frac{p_n}{h_n} = P_n = A_n$, and

$$a_{n+1} = \tau_{n+1} \frac{b_{n+1} h_{n+1}}{2} = \tau_n \sqrt{1 - h_n^2} = \tau_n \frac{b_n}{2} = p_n.$$

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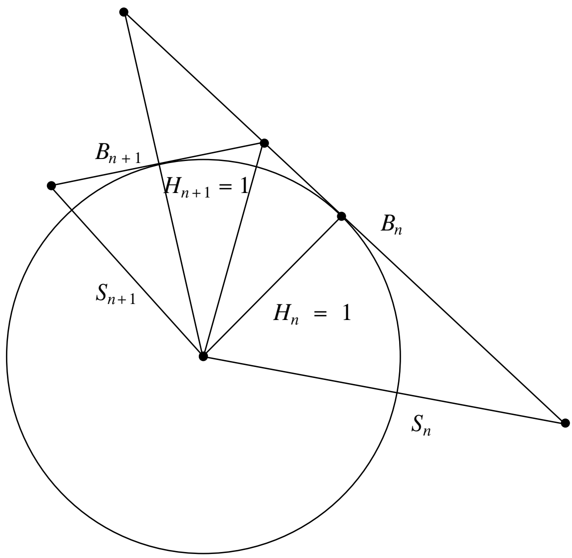

FIGURE 2: Triangles related to circumscribed polygons

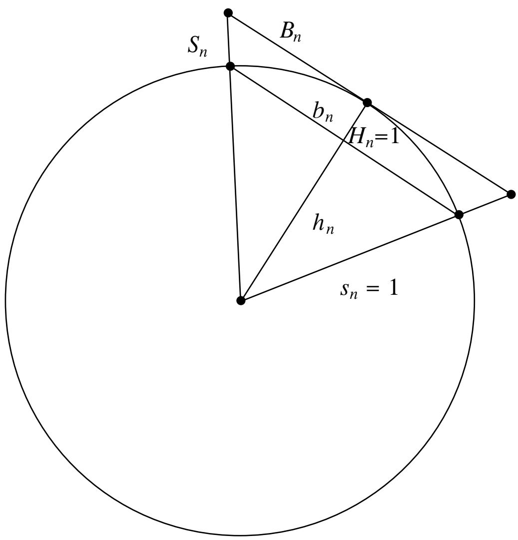

FIGURE 3: Similar inscribed and circumscribed triangles associated with a τ_n-gon

3. *Archimedes' method*

Since the inscribed τ_n-gon is included into the unit circle and the unit circle is itself included into the circumscribed τ_n-gon, it is normal to accept that

$$a_n \leqslant \text{ area of the unit circle } \leqslant A_n.$$

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For the perimeter of the unit circle, which is the length of a curve, it is less easy. We use the fact that inscribed τ_n-gon, unit circle, and circumscribed τ_n-gon are convex bodies nested within each other and accept that

$$2p_n \leqslant \text{ circumference of the unit circle } \leqslant 2P_n.$$

Theorem 1

The increasing sequence $\{p_n\}_{n=0}^{\infty}$ and the decreasing sequence $\{P_n\}_{n=0}^{\infty}$ both converge to half the circumference of the unit circle, which will be denoted by π.

Proof

We have $\frac{1}{2}b_n < b_{n+1}$ and $B_{n+1} < \frac{1}{2}B_n$, hence

$$\begin{cases} p_{n+1} = \tau_{n+1}\frac{b_{n+1}}{2} = \tau_n b_{n+1} > \tau_n \frac{b_n}{2} = p_n, \\ P_{n+1} = \tau_{n+1}\frac{B_{n+1}}{2} = \tau_n B_{n+1} > \tau_n \frac{B_n}{2} = P_n, \end{cases}$$

so

$$p_n < p_{n+1} < \ \dots \ < P_{n+1} < P_n.$$

Since the sequence $\{p_n\}_{n=0}^{\infty}$ is increasing and upper bounded, it converges say to p. In the same way, since the sequence $\{P_n\}_{n=0}^{\infty}$ is decreasing and lower bounded, it converges say to P. We also have $p \leqslant P$. Moreover,

$$0 < h_n < h_{n+1} = \sqrt{\frac{1 + h_n}{2}} < 1.$$

so the increasing sequence $\{h_n\}_{n=0}^{\infty}$ is upper bounded by 1. It converges to h such that $0 < h \leqslant 1$. Using $2h_{n+1}^2 = h_n + 1$ and taking the limit, we get $2h^2 = h + 1$. So $h = 1$ or $h = -\frac{1}{2}$, but $h > 0$, and it follows that $h = 1$. Since we have

$$P_n - p_n = p_n \frac{1 - h_n}{h_n},$$

taking the limit on both sides, we get $P - p = 0$, so the two series both have the same limit, which will be denoted by π.

Corollary

The increasing sequence $\{a_n\}_{n=0}^{\infty}$ and the decreasing sequence $\{A_n\}_{n=0}^{\infty}$ both converge to area of the unit circle, π.

Archimedes' method is based on a way to compute recursively the two sequences of half-perimeters. The next theorem presents the formula for this recursive method.

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Theorem 3

The successive terms of the two sequences $\{p_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$ are related by the following relations

$$\begin{cases} P_{n+1} = 2\frac{p_n P_n}{p_n + P_n}, \\ p_{n+1} = \sqrt{p_n P_{n+1}}. \end{cases}$$

Proof

For the first identity we have

$$P_{n+1} = \tau_{n+1}\frac{b_{n+1}}{2h_{n+1}} = 2\tau_n\sqrt{\frac{1 - h_n}{1 + h_n}} = \tau_n\frac{1}{1 + h_n}\sqrt{1 - h_n^2} = p_n\frac{2}{1 + h_n}.$$

Also

$$p_n P_n = p_n^2\frac{1}{h_n},$$

and

$$p_n + P_n = p_n\left(1 + \frac{1}{h_n}\right) = p_n\left(\frac{1 + h_n}{h_n}\right).$$

Therefore

$$2\frac{p_n P_n}{p_n + P_n} = 2\frac{p_n^2 / h}{p_n(1 + h_n)/h_n} = p_n\frac{2}{1 + h_n} = P_{n+1}.$$

For the second identity we have

$$p_{n+1} = \tau_{n+1}\frac{b_{n+1}}{2} = \tau_n\sqrt{2(1 - h_n)} = \tau_n\frac{b_n}{2}\sqrt{\frac{2(1 - h_n)}{1 - h_n^2}} = p_n\sqrt{\frac{2}{1 + h_n}},$$

and

$$p_n P_{n+1} = p_n^2\frac{2}{1 + h_n}.$$

Hence

$$p_{n+1} = \sqrt{p_n P_{n+1}}.$$

Corollary

The successive terms of the two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{A_n\}_{n=0}^{\infty}$ are related by the following relations

$$\begin{cases} a_{n+1} = \sqrt{a_n A_n}, \\ A_{n+1} = 2\frac{a_{n+1}A_n}{a_{n+1}+A_n}. \end{cases}$$

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We now have the following algorithm to generate the two sequences $\{p_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$ which will both converge to π.

STEP 1 Initialisation. Start with an hexagon.

$$\begin{aligned} \tau_0 &= 6; \\ b_0 &= 1,\ h_0 = \sqrt{3}/2; \\ p_0 &= \tfrac{\tau_0}{2} b_0 = 3; \\ P_0 &= \tfrac{\tau_0}{2} B_0 = \tfrac{\tau_0}{2}\tfrac{b_0}{h_0} = 2\sqrt{3}. \end{aligned}$$

STEP 2 Repeat for $n = 0,\ 1,\ 2,\ \ldots$

$$\begin{aligned} \tau_{n+1} &:= 2\tau_n; \\ P_{n+1} &:= 2\frac{p_n P_n}{p_n + P_n}; \\ p_{n+1} &:= \sqrt{p_n P_{n+1}}. \end{aligned}$$

The finite sequences $\{p_n\}_{n=0}^{20}$ and $\{P_n\}_{n=0}^{20}$ generated by this method are presented in Tables 1 and 2. The exact digits are indicated in boldface characters.

Archimedes' method					
n	τ_n	p_n	$p_{n+1} - p_n$	$K = \frac{p_{n+1}-p_{n+2}}{p_n-p_{n+1}}$	C_n
0	6	**3.**000000000000	0.105828541230	0.2532	-0.1417
1	12	**3.1**05828541230	0.026800072051	0.2508	-0.1426
2	24	**3.1**32628613281	0.006721589766	0.2502	-0.1432
3	48	**3.1**39350203047	0.001681747843	0.2501	-0.1434
4	96	**3.141**031950890	0.000420521395	0.2500	-0.1435
:					
9	3072	**3.141592**105999	0.000000410693	0.2500	-0.1436
10	6144	**3.141592**516692	0.000000102673	0.2500	-0.1436
11	12288	**3.141592**619365	0.000000025668	0.2500	-0.1436
:					
17	786432	**3.14159265358**1	0.000000000006	0.2500	-0.1434
18	1572864	**3.14159265358**8	0.000000000002	0.2497	-0.1465

TABLE 1: Results for Archimedes' method for p_n

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Archimedes' method					
n	τ_n	P_n	$P_{n+1} - P_n$	$K = \frac{P_{n+1}-P_{n+2}}{P_n-P_{n+1}}$	C_n
0	6	**3**.464101615138	0.248711305964	0.2241	0.3205
1	12	**3**.215390309173	0.055730367076	0.2436	0.3025
2	24	**3.1**59659942098	0.013573726966	0.2484	0.2927
3	48	**3.14**6086215131	0.003371615486	0.2496	0.2890
4	96	**3.14**2714599645	0.000841549666	0.2499	0.2877
:					
9	3072	**3.14159**3748771	0.000000821386	0.2500	0.2871
10	6144	**3.141592**927385	0.000000205346	0.2500	0.2871
11	12288	**3.141592**722039	0.000000051337	0.2500	0.2871
:					
17	786432	**3.141592653**607	0.000000000013	0.2500	0.2876
18	1572864	**3.1415926535**94	0.000000000003	0.2502	0.2835

TABLE 2: Results for Archimedes' method for P_n

4. *Analysis of the sequences*

4.1 *Convergence of the sequences*

The first question is: do we observe convergence? In Tables 1 and 2 errors are estimated at each iteration by computing $p_{n+1} - p_n$ instead $\pi - p_n$, and $P_{n+1} - P_n$ instead of $P_n - \pi$, because the value of π is unknown. We observe that the estimated errors decrease, so the two sequences seem to converge, both to the same value $\pi \approx 3.14159265359$ (which has 11 correct decimal places, 12 correct digits).

The second question is how fast do the sequences converge? Suppose a sequence $\{x_n\}_{n=0}^{\infty}$ converges to α. Let us consider the following limit if it exists:

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{x_n - \alpha} = K,$$

with $|K| < 1$ (because the errors decrease). Since α is not known, we replace α by x_{n+1}, which means that x_{n+1} looks like α to x_n. Then we consider the limit of the resulting sequence of ratios

$$\left\{K_n = \frac{x_{n+1} - x_{n+2}}{x_n - x_{n+1}}\right\}_{n=0}^{\infty},$$

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which is also K. Indeed we have

$$K_n = \frac{x_{n+1} - x_{n+2}}{x_n - x_{n+1}} = \left(\frac{x_{n+1} - \alpha}{x_n - \alpha}\right)\frac{\left(1 - \frac{x_{n+2} - \alpha}{x_{n+1} - \alpha}\right)}{\left(1 - \frac{x_{n+1} - \alpha}{x_n - \alpha}\right)}$$

and, if K exists,

$$\lim_{n \to \infty} K_n = K\frac{(1 - K)}{(1 - K)} = K.$$

So it is justified to compute K_n to approximate K. From Tables 1 and 2, we get $K = 0.2500 = 1/4$ for both sequences $\{p_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$.

If we continue, using the approximations

$$x_{n+k+1} - x_{n+k+2} \cong K\left(x_{n+k} - x_{n+k+1}\right) \cong K^{k+1}\left(x_n - x_{n+1}\right),$$

and $x_{n+k+2} \cong \alpha$, we have

$$x_{n+k} \cong \alpha + K^k\left(x_n - x_{n+1}\right),$$

which means, if we set $n = 0$ and then replace k by n, that

$$x_n = \alpha + O\left(K^n\right).$$

Hence we get

$$\begin{cases} p_n = \pi + O\left(\frac{1}{4^n}\right), \\ P_n = \pi + O\left(\frac{1}{4^n}\right). \end{cases}$$

4.2 *Towards a better approximation*

We can go further by trying to find the relative location of π within each interval $\left(p_n, P_n\right)$. We use

$$\begin{cases} p_{n+k} \cong \pi + K^k(p_n - p_{n+1}), \\ P_{n+k} \cong \pi + K^k(P_n - P_{n+1}). \end{cases}$$

Let us consider the ratios

$$\begin{aligned} r_n &= \frac{\pi - p_{n+k}}{P_{n+k} - p_{n+k}} \\ &\cong \frac{-K^k(p_n - p_{n+1})}{K^k\left[\left(P_n - P_{n+1}\right) - \left(p_n - p_{n+1}\right)\right]} \\ &= \frac{(p_{n+1} - p_n)}{\left[\left(P_n - P_{n+1}\right) + \left(p_{n+1} - p_n\right)\right]}, \end{aligned}$$

and

$$R_n = \frac{P_{n+k} - \pi}{P_{n+k} - p_{n+k}}$$

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$$\cong \frac{K^k(P_n - P_{n+1})}{K^k[(P_n - P_{n+1}) - (p_n - p_{n+1})]}$$

$$= \frac{P_n - P_{n+1}}{[(P_n - P_{n+1}) + (p_{n+1} - p_n)]}.$$

From Table 3, we have $\lim_{n\to\infty} r_n = \frac{1}{3}$, and $\lim_{n\to\infty} R_n = \frac{2}{3}$. So if we consider the new approximation $\hat{\pi} = \frac{2}{3}p_n + \frac{1}{3}P_n$, as expected, Table 3 indicates that this sequence converges faster; $K_{\hat{\pi}} = 0.0625 = 1/16 = 1/4^2$ or $K_{\hat{\pi}} = K^2$, and

$$\hat{\pi} = \pi + O\left(\frac{1}{4^{2n}}\right).$$

Better approximation							
n	τ_n	r_n	R_n	$\hat{\pi} = \frac{1}{3}(2p_n + P_n)$	$\hat{\pi}_n - \hat{\pi}_{n+1}$	$K=\frac{\hat{\pi}_{n+1}-\hat{\pi}_{n+2}}{\hat{\pi}_n-\hat{\pi}_{n+1}}$	C_n
0	6	0.2985	0.7015	**3.1**54700538379	0.012351407835	0.0575	0.0131
1	12	0.3247	0.6753	**3.14**2349130545	0.000710074325	0.0613	0.0131
2	24	0.3312	0.6688	**3.141**639056220	0.000043515812	0.0622	0.0120
3	48	0.3328	0.6672	**3.141595**540408	0.000002706600	0.0624	0.0119
4	96	0.3332	0.6668	**3.141592**833809	0.000000168959	0.0625	0.0118
5	192	0.3333	0.6667	**3.1415926**64850	0.000000010557	0.0625	0.0118
6	384	0.3333	0.6667	**3.14159265**4294	0.000000000660	0.0625	0.0118
7	768	0.3333	0.6667	**3.141592653**634	0.000000000041	0.0625	0.0118
8	1536	0.3333	0.6667	**3.1415926535**93	0.000000000003	0.0626	0.0117
9	3072	0.3333	0.6667	**3.141592653590**	0.000000000000	0.0634	0.0105

TABLE 3: Results for the better approximation $\hat{\pi}$

4.3 *Finding the first terms of series expansions*

Now if we suppose that

$$x_n = \alpha + CK^n + O(K^{2n}),$$

where

$$\frac{O(K^{2n})}{K^{2n}} \underset{n\to\infty}{\rightarrow} L,$$

we can try to estimate C from the data. Indeed

$$\frac{x_n - \alpha}{K^n} = C + K^n\frac{O(K^{2n})}{K^{2n}},$$

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and we have

$$\lim_{n \to \infty} \frac{x_n - \alpha}{K^n} = C.$$

We consider as before $x_n - x_{n+1}$ instead of $x_n = \alpha$, and we have

$$\frac{x_n - x_{n+1}}{K^n} = C(1 - K) + K^n \frac{O(K^{2n})}{K^{2n}} \underset{n \to \infty}{\to} C(1 - K).$$

Now we also have

$$K_n = K\left(1 + O\left(K^n\right)\right),$$

from which we deduce that

$$\lim_{n \to \infty} \left(\frac{K_n}{K}\right)^n = 1.$$

So, using K_n instead of K, we consider the limit

$$C_n = \frac{x_n - x_{n+1}}{\left(1 - K_n\right)K_n^n} \underset{n \to \infty}{\to} C.$$

Constants C_p, C_P and $C_{\hat{\pi}}$ are computed in Tables 1, 2 and 3 for the sequences $\{p_n\}_{n=0}^{\infty}$, $\{P_n\}_{n=0}^{\infty}$ and $\{\hat{\pi}_n\}_{n=0}^{\infty}$. In Table 4 we compare the computed numerical approximations of these constants and their theoretical values (presented in Section 6).

Approximate and true value of C		
C	Numerical approximation	True value
C_p	-0.1436	$-\frac{\pi^3}{3!.6^2}$ = -0.143547577
C_P	0.2871	$\frac{\pi^3}{3.6^2}$ = 0.287095154
$C_{\hat{\pi}}$	0.0118	$\frac{\pi^3}{20.6^4}$ = 0.011806315

TABLE 4: Approximate and true values of C

In fact at this point we have used p_n and P_n to cancel out the constant C of $O\left(\frac{1}{4^n}\right)$ from expressions for p_n and P_n

$$p_n = \pi + C_p\left(\frac{1}{4^n}\right) + \ldots$$

and

$$P_n = \pi + C_P\left(\frac{1}{4^n}\right) + \ldots,$$

to get

$$\hat{\pi}_n = \pi + C_{\hat{\pi}}\left(\frac{1}{4^{2n}}\right) + \ldots,$$

because

$$2C_p + C_P = 0.$$

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Remark

There is a price to pay for using x_{n+1} instead of α in numerical computations. An expression like $x_n - x_{n+1}$ produces catastrophic cancellations so it introduces non-significant digits or errors in the computations. We can see this effect in Tables 1, 2 and 3 for columns K_n and C_n. In these columns the computations are not precise enough in the first lines, and in the last lines the catastrophic cancellations introduce errors, so the choice of the constants K and C coincides with values on lines around the middle of the tables. Catastrophic cancellations are related to the machine epsilon of the computer and software used; it gives a lower bound on the relative errors due to rounding in floating point arithmetic [2]. For the computations we used MATLAB with machine epsilon equal to 2×10^{-16}, which means that we can get 15 to 16 significant digits.

5. *Richardson's process*

To get the approximation $\hat{\pi}$ we have eliminated a term of the form $O\left(\frac{1}{4^n}\right) = O\left(K^n\right)$, with $K = \frac{1}{4}$, in the series of p_n and P_n. So let us consider an expression of the form

$$x_n = \alpha + C_1K^n + C_2K^{2n} + \ldots = \alpha + \sum_{\ell \geqslant 1} C_\ell K^{\ell n},$$

which we rewrite as

$$\alpha = x_n - \sum_{\ell \geqslant 1} C_\ell K^{\ell n}.$$

There are systematic methods to perform the above-mentioned elimination.

One of these methods is Richardson's extrapolation process which successively, for increasing ℓ, removes the term $O\left(K^{\ell n}\right)$ in a series [3]. The principle is quite simple. Suppose, for a fixed k, we have a sequence of formulas $\{Q_{k,n}\}_{n+1}^{+\infty}$ to find α such that

$$\begin{aligned}\alpha &= Q_{k,n} + G_{k,k}K^{kn} + G_{k,k+1}K^{(k+1)n} + G_{k,k+2}K^{(k+2)n} + \ldots \\ &= Q_{k,n} + \sum_{\ell \geqslant k} G_{k,\ell}K^{\ell n},\end{aligned}$$

for each integer $n \geqslant 0$. To eliminate the term $G_{k,k}K^{kn}$, we consider

$$\begin{aligned}\alpha &= Q_{k,n+1} + \sum_{\ell \geqslant k} G_{k,\ell}K^{\ell(n+1)} \\ &= Q_{k,n+1} + \sum_{\ell \geqslant k} G_{k,\ell}K^{\ell n + \ell},\end{aligned}$$

and

$$K^k\alpha = K^kQ_{k,n} = \sum_{\ell \geqslant k} G_{k,\ell}K^{\ell n + k}.$$

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Then we subtract these expressions to get

$$\left(1 - K^k\right)\alpha = \left(Q_{k,n+1} - K^k Q_{k,n}\right) = \sum_{\ell \geqslant k} G_{k,\ell}\left(K^{\ell n + \ell} - K^{\ell n + k}\right)$$

$$= \left(Q_{k,n+1} - K^k Q_{k,n}\right) + \sum_{\ell \geqslant k} G_{k,\ell}\left(K^{\ell} - K^k\right)K^{\ell n}.$$

So we write

$$\alpha = Q_{k+1,n} + \sum_{\ell \geqslant k+1} G_{k+1,\ell} K^{\ell n},$$

where the new formula $Q_{k+1,n}$ is

$$Q_{k+1,n} = \frac{Q_{k,n+1} - K^k Q_{k,n}}{1 - K^k} = Q_{k,n+1} + K^k \frac{Q_{k,n+1} - Q_{k,n}}{1 - K^k},$$

and

$$G_{k+1,\ell} = G_{k,\ell}\frac{\left(K^{\ell} - K^k\right)}{\left(1 - K^k\right)}.$$

We can organise the computation as illustrated in Table 5 below.

	k										
n	1	…	2	…	3	…	$k - 1$	…	k	…	$k + 1$
		⋰		⋰		⋰		⋰		⋰	
0	$Q_{1,0}$	…	$Q_{2,0}$	…	$Q_{3,0}$	…	$Q_{k-1,0}$	…	$Q_{k,0}$	…	$Q_{k+1,0}$
		⋰		⋰		⋰		⋰		⋰	
1	$Q_{1,1}$	…	$Q_{2,1}$	…	$Q_{3,1}$	…	$Q_{k-1,1}$	…	$Q_{k,1}$		
		⋰		⋰		⋰		⋰			
2	$Q_{1,2}$	…	$Q_{2,2}$	…	$Q_{3,2}$	…	$Q_{k-1,2}$				
⋮	⋮		⋮		⋮	⋰					
$k - 2$	$Q_{1,k-2}$	…	$Q_{2,k-2}$	…	$Q_{3,k-2}$						
		⋰		⋰							
$k - 1$	$Q_{1,k-1}$	…	$Q_{2,k-1}$								
		⋰									
k	$Q_{1,k}$										
⋮											
n	$O\left(\frac{1}{4^n}\right)$		$O\left(\frac{1}{4^{2n}}\right)$		$O\left(\frac{1}{4^{3n}}\right)$	…	$O\left(\frac{1}{4^{(k-1)n}}\right)$		$O\left(\frac{1}{4^{kn}}\right)$		$O\left(\frac{1}{4^{(k+1)n}}\right)$

TABLE 5: Richardson's extrapolation table

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For our problem, let us use the series for $\alpha = \pi$ for each case: p_n, P_n and $\hat{\pi}_n$. Using the half-circumference of inscribed τ_n-gons and

$$\pi = p_n - C_{p,1}\left(\frac{1}{4^n}\right) - C_{p,2}\left(\frac{1}{4^{2n}}\right) + \ \dots ,$$

we set $p_n = Q_{1,n}$ and the results are given in Table 6. It is remarkable that $Q_{5,0}$ is a very accurate value of π. It only needs five values of the sequences of p_n, already generated by Archimedes, to get a very good results for $Q_{5,0}$. For the half-circumference of circumscribed τ_n-gons for

$$\pi = P_n - C_{P,1}\left(\frac{1}{4^n}\right) - C_{P,2}\left(\frac{1}{4^{2n}}\right) + \ \dots ,$$

we set $P_n = Q_{1,n}$ and the results are given in Table 7. In this case $Q_{5,0}$ is less accurate. For $\hat{\pi}_n$, since we have

$$\pi = \hat{\pi} - C_{\hat{\pi},1}\left(\frac{1}{4^{2n}}\right) + \ \dots ,$$

we skip the first column because, if we set $\hat{\pi}_n = Q_{2,n}$, and we only need five columns to get the result given in Table 8. For the same computational complexity, we can go up to $Q_{6,0}$ which is very good.

So if Archimedes had known this method he would have been able to find a very precise value of π.

		k				
n	m	$Q_{1,n}$	$Q_{2,n}$	$Q_{3,n}$	$Q_{4,n}$	$Q_{5,n}$
0	6	**3**.000000000000	**3.141**104721640	**3.141592**453898	**3.1415926535**78	**3.141592653590**
1	12	**3.1**05828541230	**3.1415**61970632	**3.14159265**0458	**3.141592653590**	
2	24	**3.1**32628613281	**3.14159**0732969	**3.1415926535**41		
3	48	**3.1**39350203047	**3.141592**533505			
4	96	**3.141**031950891				

TABLE 6: Richardson's extrapolation table for p_n

		k				
n	m	$Q_{1,n}$	$Q_{2,n}$	$Q_{3,n}$	$Q_{4,n}$	$Q_{5,n}$
0	6	**3**.464101615138	**3.1**32486540519	**3.141**656260576	**3.141592**542982	**3.141592653**637
1	12	**3**.215390309173	**3.141**083153072	**3.14159**3538570	**3.141592653**206	
2	24	**3.1**59659942098	**3.1415**61639476	**3.1415926**67039		
3	48	**3.14**6086215131	**3.14159**0727817			
4	96	**3.14**2714599645				

TABLE 7: Richardson's extrapolation table for P_n

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		k				
n	m	$Q_{2,n}$	$Q_{3,n}$	$Q_{4,n}$	$Q_{5,n}$	$Q_{6,n}$
0	6	**3.1**54700538379	**3.1415**25703356	**3.141592**765782	**3.1415926535**42	**3.141592653590**
1	12	**3.14**2349130545	**3.141591**717932	**3.141592653**980	**3.141592653590**	
2	24	**3.141**639056220	**3.1415926**39354	**3.141592653**591		
3	48	**3.14159**5540408	**3.141592653**369			
4	96	**3.141592**833809				

TABLE 8: Richardson's extrapolation table for $\hat{\pi}_n$

6. *Taylor and trigonometry*

As suggested in [4], if Archimedes had met Taylor, or in other words if we assume that he knew the trigonometric functions and their Taylor's expansions, the observations of the preceding sections on Archimedes' sequences can be verified. Indeed, let the central angle be

$$\theta_n = \frac{\text{length of the circumference}}{\text{number of sides of the } \tau_n\text{-gon}} = \frac{2\pi}{\tau_n},$$

so $\theta_{n+1} = \theta_n / 2$. We have

$$\begin{cases} p_n = \tau_n \frac{b_n}{2} = \tau_n \sin\frac{\theta_n}{2} = \tau_n \sin\frac{\pi}{\tau_n}, \\ P_n = \tau_n \frac{B_n}{2} = \tau_n \sin\frac{\theta_n}{2} = \tau_n \sin\frac{\pi}{\tau_n}, \end{cases}$$

and we consider Taylor's expansions

$$\begin{cases} \sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + O(\theta^7), \\ \tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + O(\theta^7). \end{cases}$$

Since $\theta = \frac{\pi}{\tau_n}$ and $t_n = 6 \cdot 2^n$, we get

$$\begin{cases} p_n = \pi - \frac{\pi^3}{3!}\left(\frac{1}{\tau_n^2}\right) + \frac{\pi^5}{5!}\left(\frac{1}{\tau_n^4}\right) + O\left(\frac{1}{\tau_n^6}\right), \\ \quad\; = \pi - \frac{\pi^3}{3! \cdot 6^2}\left(\frac{1}{4^n}\right) + \frac{\pi^5}{5! \cdot 6^4}\left(\frac{1}{4^{2n}}\right) + O\left(\frac{1}{4^{3n}}\right), \\ P_n = \pi + \frac{\pi^3}{3}\left(\frac{1}{\tau_n^2}\right) + \frac{2\pi^5}{15}\left(\frac{1}{\tau_n^4}\right) + O\left(\frac{1}{\tau_n^6}\right), \\ \quad\; = \pi + \frac{\pi^3}{3 \cdot 6^2}\left(\frac{1}{4^n}\right) + \frac{2\pi^5}{15 \cdot 6^4}\left(\frac{1}{4^{2n}}\right) + O\left(\frac{1}{4^{3n}}\right). \end{cases}$$

For the asymptotic constant K, we have for each sequence

$$K_p = \lim_{n \to \infty} \frac{p_{n+1} - \pi}{p_n - \pi} = \lim_{n \to \infty} \frac{1/4^{n+1}}{1/4^n} = \frac{1}{4},$$

and

$$K_P = \lim_{n \to \infty} \frac{P_{n+1} - \pi}{P_n - \pi} = \lim_{n \to \infty} \frac{1/4^{n+1}}{1/4^n} = \frac{1}{4}.$$

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For the relative location of π in the intervals we consider

$$r_n = \frac{\pi - p_n}{P_n - p_n} = \frac{\frac{\pi^3}{3!\cdot 6^2}\left(\frac{1}{4^n}\right) + O\left(\frac{1}{4^{2n}}\right)}{\frac{\pi^3}{2\cdot 6^2}\left(\frac{1}{4^n}\right) + O\left(\frac{1}{4^{2n}}\right)} \underset{n\to\infty}{\to} \frac{1}{3},$$

and

$$R_n = \frac{P_n - \pi}{P_n - p_n} = \frac{\frac{\pi^3}{3!\cdot 6^2}\left(\frac{1}{4^n}\right) + O\left(\frac{1}{4^{2n}}\right)}{\frac{\pi^3}{2\cdot 6^2}\left(\frac{1}{4^n}\right) + O\left(\frac{1}{4^{2n}}\right)} \underset{n\to\infty}{\to} \frac{2}{3}.$$

For $\hat{\pi}$ we have

$$\hat{\pi} = \frac{2p_n + P_n}{3} = \pi + \frac{\pi^5}{20\cdot 6^4}\left(\frac{1}{4^{2n}}\right) + O\left(\frac{1}{4^{3n}}\right),$$

which means we have eliminated the term in $O\left(\frac{1}{4^n}\right)$ in the expansions for p_n and P_n. It follows that

$$K_{\hat{\pi}} = \lim_{n\to\infty} \frac{\hat{\pi}_{n+1} - \pi}{\hat{\pi} - \pi} = \lim_{n\to\infty} \frac{1/4^{2(n+1)}}{1/4^{2n}} = \frac{1}{16}.$$

From these expansions, we get the true values of the constants C_p, C_P and $C_{\hat{\pi}}$ reported in Table 4.

7. *Conclusion*

Using the sequences generated by the Archimedes' method to approximate π from the half-perimeter of inscribed and circumscribed regular polygons with respect to the unit circle, we have seen how to extract information from a sequence about its convergence. Moreover we were able to modify a sequence to get faster convergence. So, with the simple acceleration method we used, Richardson's extrapolation method, Archimedes would have obtained a more accurate approximation of π. Finally we have considered the analysis of this problem with trigonometric functions and their Taylor's expansions as suggested for example in [4], and verified our results with a computer.

As an exercise, the interested reader may repeat the preceding analysis for the sequences generated by

$$\begin{cases} X_n = \left(1 + \frac{1}{2^n}\right)^{2^n + 1} \\ x_n = \left(1 + \frac{1}{2^n}\right)^{2^n} \end{cases}$$

for $n = 0, 1, 2, \ldots$, to get the limit and also propose sequences which converge faster.

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Quotations for *Nemo* (continued from page 399)

2. It was a sound ill suited to the place, and reminded Sir Kenneth how necessary it was he should be upon his guard. He started from his knee, and laid his hand upon his poniard. A creaking sound, as of a screw or pulleys, succeeded, and a light streaming upwards, as from an opening in the floor, showed that a trap-door had been raised or depressed.

3. Look where the pulley's tied above!
   Great God! (said I) what have I seen!
   On what poor engines move
   The thoughts of monarchs and designs of states!
   What petty motives rule their fates!

4. She had not contented herself with opening the door from above by the usual arrangement of a creaking pulley, though she had looked down on me first from an upper window, dropping the curious challenge which in Italy precedes the act of admission.

5. The kennel, to make amends, ran down the middle of the street—when it ran at all: which was only after heavy rains, and then it ran, by many eccentric fits, into the houses. Across the streets, at wide intervals, one clumsy lamp was slung by a rope and pulley; at night, when the lamplighter had let these down, and lighted, and hoisted them again, a feeble grove of dim wicks swung in a sickly manner overhead, as if they were at sea. Indeed they were at sea, and the ship and crew were in peril of tempest.

6. A distinguished psychologist, who is well acquainted with physiology, has told me that parts of himself are certainly levers, while other parts are probably pulleys, but that after feeling himself carefully all over, he cannot find a wheel anywhere.