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# Ion acoustic waves in expanding strongly coupled plasmas

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**Abstract.** We consider the excitation and dispersion of ion acoustic waves in expanding ultracold plasmas, taking into account the influence of boundary conditions. A cylindrical plasma geometry is assumed. We show that temporal changes in the medium lead to a wave frequency shift, associated with an evolving radial and standing wave mode structure, and to the temporal change of the background plasma parameters. A non-collisional model for the cylindrical geometry is also proposed.

## 1. Introduction

In recent years, an increasing attention has been given to ultracold neutral plasmas, or Rydberg plasmas (Killian et al. 2007; Rolston 2008). This plasma medium contrasts with the traditional views of a plasma as a very hot gas, due to its low electron temperatures of a few Kelvin, and ion temperatures in the milli-Kelvin domain. They also display novel properties, such as self-ionization of Rydberg atoms (Robinson et al. 2000), strongly correlated ions (Shukla and Avinash 2011), and new wave dispersive properties (Mendonça et al. 2009, 2010). Several waves and oscillations have been identified, such as electron plasma waves (Kulin et al. 2000), Tonks-Dattner modes (Fletcher et al. 2006), and electron drift instabilities (Zhang et al. 2008).

In ultracold plasmas, the thermal energy of the charged particles (mostly ions) can be much less than the Coulomb interaction energy between nearest neighbors, making them strongly coupled systems, similar to dusty plasmas with strongly correlated highly charged dust grains (Fortov et al. 2005). In strongly coupled ultracold plasmas, we have the possibility of Coulomb crystallization of the positive ions. It was recently shown that the dispersion relation of ion acoustic waves can be significantly modified in a strongly coupled plasma (Shukla 2010). Therefore, the study of non-stationary plasmas can eventually reveal the existence of strong ion coupling, and be used as a diagnostic technique to estimate the ion coupling parameter.

Here, we consider another aspect of ultracold plasmas, by addressing the problem of ion acoustic waves propagating in a non-stationary and strongly coupled plasma. This is related to recent experiments (Castro et al. 2010), as discussed theoretically by (Mendonça and Shukla 2011), where however strong coupling and boundary conditions were ignored. As we will show, these two features are determinant for the understanding of wave excitation in ultracold plasmas.

In this work, we deal with the properties of global plasma wave modes in a time-varying well-defined spatial structure. In particular, we focus on the case of an expanding plasma with cylindrical shape. This allows us to describe some of the main features that have been observed in the experiments by (Castro et al. 2010), by an explicitly evaluation of the various contributions to wave dispersion. These are (i) the boundary conditions, (ii) the time-varying plasma parameters and (iii) the viscosity effects associated with strong coupling. We use modified ion fluid equations, where a source term associated with the ionization and/or recombination processes and a non-local viscosity term associated with ion–ion coupling are included. We assume that the electrons are in Boltzmann equilibrium in the wave potential. We consider an arbitrary temporal variation of the plasma density, associated not only with plasma expansion but also with ionization and/or recombination processes. We determine the temporal evolution of the dispersion properties of ion acoustic modes, showing that these modes satisfy a time-varying dispersion relation. This will characterize the spectrum of ion acoustic waves, which can eventually be excited in ultracold plasma experiments.

## 2. Basic description

The excitation of ion acoustic waves in a strongly coupled and time-varying plasma can be described by the ion fluid equations for the ion mean density $n_i$ and mean velocity $\mathbf{v}_i$. We use the ion continuity equation with a source term $S_i$ which accounts for ionization and/or recombination processes

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}) = S_i, \tag{2.1}$$

and the modified momentum equation with a viscosity term associated with the ion–ion coupling, as given by

(Kaw and Sen 1998),

$$\frac{d\mathbf{v}_i}{dt} = -\frac{Ze}{m_i}\nabla\phi + \frac{\nabla P_i}{m_i n_i} + \int_{-\infty}^{t} dt' \int_{vol.} d\mathbf{r}' \eta_i(\mathbf{r}-\mathbf{r}', t-t')\mathbf{v}_i(\mathbf{r}', t'), \quad (2.2)$$

where $P_i$ is the ion pressure, and $\eta_i$ is a non-local viscoelastic operator which accounts for the non-local and memory effects, to be specified below. The electrostatic potential $\phi$ is determined by the Poisson equation

$$\nabla^2\phi = \frac{e}{\epsilon_0}(n_e - Zn_i), \quad n_e = n_{0e}\exp(eV/T_e), \quad (2.3)$$

where $Z$ is the degree of ionization, $n_{0e} \equiv n_0(\mathbf{r})f(t)$ is the quasi-equilibrium plasma density, where $f(t)$ is a temporal form function, and the electron density $n_e$ is assumed in the Boltzmann equilibrium at a temperature $T_e \neq T_i$. A plasma quasi-equilibrium is established on a very short time scale as compared with the long ion time scales to be considered here. This allows the equilibrium plasma density and temperature to vary on such a long time scale.

In order to establish the viscoelastic operator, we notice that the memory effects are generically characterized by a relaxation time $\tau_m$. The space and time Fourier transform of $\eta_i$ can be written as (Kaw and Sen 1998)

$$\eta_i(k,\omega) = \frac{1}{(1-i\omega\tau_m)m_i n_{0i}}\left[\eta k^2 + \left(\frac{\eta}{3}+\zeta\right)\mathbf{k}(\mathbf{k}\cdot\,)\right], \quad (2.4)$$

where $\eta$ and $\zeta$ are considered here as phenomenological parameters. We assume that a plasma quasi-equilibrium can been achieved, such that $n_{0i} = n_{0e}f(t)/Z$ and $\mathbf{v}_i = 0$. We then consider $n_i = n_{0i}(t) + \tilde{n}$, where $\tilde{n}$ describes the ion wave perturbation. We take the form function $f(t)$ as independent from the ionization rate $S$, because the temporal changes in the plasma can be due to expansion, and not just to ionization. In the absence of expansion, we simply have $S = n_0(df/dt)$. Linearizing the above fluid equations, we get the equation

$$D_\tau\left[\frac{\partial^2}{\partial t^2}\tilde{n} - \beta\nabla^2\phi\right] = \frac{1}{m_i n_{0i}}\left[\left(\frac{\eta}{3}+\zeta\right)\nabla(\nabla\cdot\tilde{n}) + \eta\nabla^2\tilde{n}\right] - v_{thi}^2\nabla^2\tilde{n} + F(t), \quad (2.5)$$

with $D_\tau = (1+\tau_m\partial/\partial t)$, $\beta = Zen_{i0}(1-R)/m_i$, and $R = e^2/4T_e\lambda_{De}$ . The source term associated with the temporal plasma variation is defined as

$$F(t) = \nu(t)\left[\frac{\partial\tilde{n}}{\partial t} - S_i\right] + \frac{dS_i}{dt}, \quad (2.6)$$

and where the quantity $\nu(t)$ and the ion plasma frequency $\omega_{pi}(t)$ are defined by

$$\nu(t) = d\ln f(t)/dt, \quad \omega_{pi}^2(t) = \frac{Ze^2 n_0 f(t)}{\epsilon_0 M}. \quad (2.7)$$

Equation (2.6) has to be coupled with the linearized Poisson's equation, which can be written as

$$\nabla^2 V = \omega_{pi}^2(t)\frac{MV}{ZT_e} - Z\frac{e\tilde{n}}{\epsilon_0}. \quad (2.8)$$

## 3. Cylindrical plasma

Let us assume a cylindrical plasma shape, with radius $a \equiv a(t)$ and length $L \equiv L(t)$, assumed as slowly varying functions of time. To solve (2.6)–(2.8), we write the Laplacian in cylindrical coordinates $\mathbf{r} = (r,\theta,z)$ and seek for a general solution of the form

$$V(\mathbf{r},t) = \sum_{lm} V_{lm}(z,t)J_m(k_{lm}r)\exp(im\theta), \quad (3.1)$$

with $m$ integer, and $k_{lm} = \alpha_{lm}/a$, where $\alpha_{lm}$ are the $l$th zeros of the Bessel functions $J_m$. This satisfies the (moving) boundary conditions, $V(r = a(t)) = 0$. In this solution, we also use

$$V_{lm} = \sum_j a_{jlm}(t)\sin(k_j z), \quad k_j = \frac{2\pi j}{L(t)}. \quad (3.2)$$

This also satisfies the additional boundary conditions $V(z = 0) = 0$ and $V(z = L(t)) = 0$. In order to derive the dispersion relation, we can therefore use perturbations of the form $(V,\tilde{n})_{jlm} \propto J_m(k_{lm}r)\exp(im\theta + ik_j z)$. Replacing this in the Poisson's equation (2.8), we get

$$-\left[K^2 + \omega_{pi}^2\frac{m_i}{ZT_e}\right]V = -\frac{Ze}{\epsilon_0}\tilde{n}, \quad K^2 = (k_{lm}^2 + k_j^2). \quad (3.3)$$

On the other hand, (2.6) becomes

$$D_\tau\left[\left(\frac{\partial^2}{\partial t^2} + v_{thi}^2K^2\right)\tilde{n} + \beta K^2 V\right] = -\frac{\eta_*}{m_i n_{0i}}\frac{\partial}{\partial t}\left(K^2\tilde{n}\right), \quad (3.4)$$

where $\eta_* = (\eta/3+\zeta)$. Here, we have neglected the source term in (2.6) $F(t) \sim 0$. At this point, we should notice that $K^2$, $\beta$ and $v_{thi}$ can be slowly time-varying quantities. Assuming that $(V,\tilde{n})$ behave as $\exp -i\int^t \omega(t')dt'$, and that the frequency $\omega(t)$ satisfies at any time $t$ the linear dispersion relation, we obtain

$$(\omega^2 - K_{jlm}^2 v_{thi}^2) - \frac{v_{ac}^2 K_{lmn}^2(1-R)}{1+K_{klm}^2\lambda_{De}^2} + i\frac{\omega K_{jlm}^2\eta_*}{\rho_i(1-i\omega\tau_m)} = 0, \quad (3.5)$$

with $v_{ac}^2 = \omega_{pi}^2\lambda_{De}^2 = ZT_e/m_i$. This is the modified ion acoustic wave, similar to that recently discussed by (Shukla 2010) for strongly correlated plasmas, but with time-dependent parameters corresponding to an expanding plasma cylinder. In particular, we have introduced the quantity

$$K_{jlm}^2(t) = \left[\frac{\alpha_{lm}}{a(t)}\right]^2 + \left[\frac{2\pi j}{L(t)}\right]^2. \quad (3.6)$$

This leads to a time-dependent frequency, $\omega = \omega(K_{jlm})$. Such a time dependence is due not only to the slow temporal evolution of the plasma parameters, such as the background density and temperature, but also due to the expansion of the boundaries, which imply the temporal variation of the wavenumber of the plasma eigenmodes. This frequency shift is a characteristic feature of *time refraction*, a concept first introduced by (Mendonça 2001), in the case of transverse photons, and later extended to longitudinal photons or plasmons (Mendonça 2009).

## 4. Ionization regime

Let us now consider the opposite case of a very slow expansion, such that we can neglect mode reflection, but in the presence of a significant ionization rate $S \equiv S(\mathbf{r}, t)$ which dominates the process. In this case, we can start with a spatial Fourier transformation of (2.6) and (2.8), leading to

$$\left[\frac{d^2}{dt^2} + \omega_{jlm}^2(t)\right] n_{jlm} = \frac{d}{dt} S_{jlm}, \tag{4.1}$$

where $n_{jlm}(t)$ and $S_{jlm}$ are the components of $\tilde{n}$ and $S$ on the basis of the cylindrical eigenmodes, and the mode frequency $\omega_{jlm}$ is determined by the dispersion relation (3.5). For simplicity, we neglect the first term in the source function $F(t)$ of (2.6), which corresponds to the case where ionization overtakes expansion. Equation (4.1) can easily be solved, leading to

$$n_{jlm}(t) = N_{jlm}(t) \exp\left[-i \int^t \omega_{jlm}(t')dt'\right], \tag{4.2}$$

where the mode amplitude is determined by

$$N_{jlm}(t) = \frac{1}{2} \int^t S_{jlm}(t') \exp\left[i \int^{t'} \omega_{jlm}(t'')dt''\right] dt'. \tag{4.3}$$

This generalizes the results previously obtained by us (Mendonça and Shukla 2011), for an unbounded plasma, to the case of an expanding cylindrical plasma. In the case of photoionization, as induced by a laser beam with a periodic modulation mask, we can assume that the laser beam propagates in some direction defined by $\theta =$ const., and the ionization mask is located in a plane $Oyz$, perpendicular to that direction. We can then write for the relevant mask component

$$S_{jlm} = \frac{1}{4\pi} \int_{-a}^{a} y dy J_m(k_{lm}|y|) S(y, z) e^{-i2\pi jz/L} dz \tag{4.4}$$

in the axial direction $Oz$, we can write the ionization rate as

$$S(\mathbf{r}, t) = S_0(t) \cos(k_0 z). \tag{4.5}$$

In this case, an expanding (but standing) ion acoustic mode will be excited in the plasma, such that $k_j(0) = k_0$, where the initial wavelength is imposed by the ionization mask, and a spectrum of transverse cylindrical modes $(lm)$ is determined by the $y$ dependence of the mask. This is valid for very short laser pulses, as used in current experiments (Castro et al. 2010), with duration much sorter than the period of the ion acoustic wave $2\pi/\omega_k$, where it is appropriate to use $S_0(t) = N_0 \delta(t = 0)$. The resulting space and time-varying ion acoustic perturbation will then be given by

$$\tilde{n}(\mathbf{r}, t) = \frac{1}{4} N_0 \sum_{lm} \exp\left[ik_j t - i \int^t \omega_{jlm}(t')dt'\right] + c.c., \tag{4.6}$$

where $\omega_{jlm}(0) \equiv \omega_{jlm}(k_j(0) = k_0)$.

## 5. Expansion process

Let us now consider the expansion process. It is currently assumed that the expansion of ultracold plasmas can be described by a non-collisional model, where the kinetic expansion dominates over the collision-induced ambipolar diffusion. According to these non-collisional expansion models (Manfredi et al. 1993; Robicheaux and Hanson 2003), we expect the expansion to be governed by the ion acoustic velocity $v_{ac}$. A direct application of this assumption to the cylindrical geometry leads to the following expansion law for the plasma radius:

$$a(t) = a_0 \sqrt{1 + t^2/\tau_a^2}, \quad \tau_a = \frac{a(0)}{v_{ac}(0)} = a_0 \sqrt{\frac{m_i}{T_e(0)}}. \tag{5.1}$$

Similarly, of the plasma length, we get

$$L(t) = L_0 \sqrt{1 + t^2/\tau_L^2}, \quad \tau_L = \left(\frac{L_0}{a_0}\right) \tau_a. \tag{5.2}$$

The total number of particles being assumed constant during plasma expansion, the mean plasma density will evolve as

$$n_0(t) = n_0(0) f(t), \quad f(t) = \frac{1}{(1 + t^2/\tau_a^2)(1 + t^2/\tau_L^2)^{1/2}}. \tag{5.3}$$

As for the temperatures, we can use the law

$$T_j(t) = T_j(0) f^{2/3}(t), \quad (j = e, i). \tag{5.4}$$

For the usual spherical model, where $\tau_a = \tau_L = \tau_{exp}$, this reduces to the well-known expression (Killian et al. 2007),

$$T_j(t) = \frac{T_j(0)}{(1 + t^2/\tau_{exp}^2)}. \tag{5.5}$$

If we now replace these results in (3.6), the quantity $K_{jlm}^2$ becomes

$$K_{jlm}^2(t) = \frac{k_{lm}^2(0)}{(1 + t^2/\tau_a^2)} + \frac{k_j^2(0)}{(1 + t^2/\tau_L^2)}. \tag{5.6}$$

Finally, the electron Debye length will evolve as

$$\lambda_{De}^2 = \frac{\lambda_{De}^2(0)}{f^{1/3}(t)}. \tag{5.7}$$

Replacing all these assumptions in the dispersion relation (3.5), and making assumptions on the temporal evolution of the phenomenological parameters $\eta_*(t), \tau_m(t)$ and $R(t)$, we will be able to determine the temporal evolution of the frequency $\omega$, associated with the modified ion acoustic modes in a strongly coupled expanding plasma.

## 6. Conclusions

In conclusion, in this work we have considered the temporal evolution of modified ion acoustic modes, which can be excited in expanding plasma cylinders. This is valid for long time scales, such that the typical expansion time is much larger than the wave period, $\omega\tau_a \gg 1$. We have shown that the resulting frequency shifts are due to two distinct factors. First, the temporal changes of the background plasma parameters, such as density and electron and ion temperatures. Second, the temporal changes of the mode wavenumbers, which are imposed by the moving boundary conditions. A comparison of these features with experiments can eventually give us access to the internal properties of the expanding plasmas, and in particular to the importance of the ion coupling. The present analysis is in good qualitative agreement with the experiments in ultracold plasmas (Castro et al. 2010). Our model could also be easily extended to the case of dust acoustic waves in strongly coupled plasmas (Shukla et al. 2003).

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