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FROM HERMITE POLYNOMIALS TO MULTIFRACTIONAL PROCESSES

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Abstract

We consider a class of multifractional processes related to Hermite polynomials. We show that these processes satisfy an invariance principle. To prove the main result of this paper, we use properties of the Hermite polynomials and the multiple Wiener integrals. Because of the multifractionality, we also need to deal with variations of the Hurst index by means of some uniform estimates.

Keywords: Multifractional process; Hermite polynomial; limit theorem; sample path properties

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1. Introduction

Hermite processes have attracted a lot of attention for many years because they have nice properties and generalize fractional Brownian motion [7], [14], [17]. Let $m \in \mathbb{N}^*$ and $H \in (\frac{1}{2}, 1)$. The Hermite process $W_{m,H}$ of order m and Hurst index H can be defined in terms of Wiener–Itô–Dobrushin integrals [6] by

$$W_{m,H}\colon t \mapsto W_{m,H}(t) = \int_{\mathbb{R}^m} f_{m,H}(x_1, \ldots, x_m, t)\,\mathrm{d}\hat{B}_{x_1} \cdots \mathrm{d}\hat{B}_{x_m} \tag{1.1}$$

with the function $f_{m,H}$ given for every t in $\mathbb{R}$ and almost every $(x_1, \ldots, x_m)$ in $\mathbb{R}^m$ by

$$f_{m,H}(x_1, \ldots, x_m, t) = C(m, H)\frac{\exp(\mathrm{i}t(x_1 + \cdots + x_m)) - 1}{\mathrm{i}(x_1 + \cdots + x_m)|x_1 \cdots x_m|^{(2H-2+m)/2m}},$$

where $C(m, H)$ is a normalizing constant and $\mathrm{d}\hat{B}$ is a complex Gaussian measure such that (1.1) defines a real process. Note that, for $m = 1$, the Hermite process $W_{1,H}$ is the fractional Brownian motion with Hurst index H. Moreover, for every integer $m \geq 2$, the Hermite process $W_{m,H}$ and fractional Brownian motion share many properties. For instance, $W_{m,H}$ is self-similar with parameter H: for every $a > 0$, the process $\{W_{m,H}(at)\}_{t\geq 0}$ is equal in distribution to $\{a^H W_{m,H}(t)\}_{t\geq 0}$. It is also a basic model for long-range dependence: the sequence of its increments $\delta W_{m,H} = \{\delta W_{m,H}(j) = W_{m,H}(j + 1) - W_{m,H}(j)\}_{j\in\mathbb{N}}$ is stationary and satisfies the long-range property

$$\mathbb{E}[\delta W_{m,H}(0)\delta W_{m,H}(j)] \sim \frac{c_{m,H}}{j^{2-2H}} \quad \text{as } j \to \infty,$$

where $c_{m,H}$ is a positive constant.

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The invariance principle [5], [7], [16], [17] is one of the most important properties of Hermite processes for applications. It can be stated as follows. Let $X = \{X_j\}_{j\in\mathbb{N}}$ be a Gaussian stationary sequence of centered random variables with variance 1. We assume that there exists a positive constant c such that X satisfies the long-range property

$$\mathbb{E}[X_0 X_j] \sim \frac{c}{j^{2(1-H)/m}} \quad \text{as } j \to \infty. \tag{1.2}$$

We consider a real-valued function ϕ satisfying $\int_{\mathbb{R}} \phi(x)^2 \mathrm{e}^{-x^2/2}\,\mathrm{d}x < \infty$ and with Hermite rank equal to m. Roughly speaking, this means that there exists a sequence of real numbers $\{\phi_j\}_{j\geq m}$ such that $\phi_m \neq 0$ and $\phi = \sum_{j=m}^{\infty} \phi_j P_j$, where, for each $j \geq m$, P_j is the jth Hermite polynomial. We define the sequence of processes $\{S^N_{\phi,H}\}_{N\in\mathbb{N}}$ by

$$S^N_{\phi,H}(t) = \frac{1}{N^H} \sum_{j=1}^{\lfloor Nt \rfloor} \phi(X_j) \tag{1.3}$$

for every $t \geq 0$ and $N \in \mathbb{N}$. The invariance principle states that, as N goes to ∞, the finite-dimensional distributions of $S^N_{\phi,H}$ converge to those of a Hermite process $W_{m,H}$ defined by (1.1) with a suitable constant $C(m, H)$. This is a remarkable property because it means that Hermite processes can be universal self-similar models in many applications of probability when long-range dependence arises. For instance, they have recently been used, in [10], to describe random media with long-range correlations for the study of wave propagation.

A limitation of Hermite processes and other fractional processes is the strong homogeneity of their properties as self-similarity, which are governed by the Hurst index H. In order to generalize fractional processes to less homogeneous processes, multifractional processes, as the class of multifractional Brownian motions [2], [13] for instance, have been introduced. Multifractional processes have locally, but not globally, the same properties as fractional processes. These properties are governed by a function h substituting the constant H in a suitable sense. For instance, multifractional processes satisfy the so-called local self-similarity property [2], [13].

As fractional Brownian motion and other Hermite processes, some nontrivial multifractional Gaussian processes satisfy invariance principles. From [4], we have the following result (see also [9] for a multidimensional version). Let $\{X_j(H)\}_{(j,H)\in\mathbb{N}\times(1/2,1)}$ be a Gaussian field satisfying some long-range assumptions related to (1.2): roughly speaking, there exists a continuous and symmetric function $R\colon (H_1, H_2) \to R(H_1, H_2)$ such that

$$\mathbb{E}[X_j(H_1) X_k(H_2)] \sim R(H_1, H_2)|j-k|^{H_1+H_2-2} \quad \text{as } |j-k| \to \infty \tag{1.4}$$

uniformly in (H_1, H_2). Let h be a continuous function taking its values in $(\frac{1}{2}, 1)$. We define the sequence of processes $\{S^N_h\}_{N\in\mathbb{N}}$ by

$$S^N_h(t) = \sum_{j=1}^{\lfloor Nt \rfloor} \frac{X_j(h(j/N))}{N^{h(j/N)}} \tag{1.5}$$

for all $t \geq 0$ and $N \in \mathbb{N}$. Then, as N goes to ∞, the finite-dimensional distributions of S^N_h converge to those of a centered Gaussian process S_h whose covariance is given for all t and $s \geq 0$ by

$$\mathbb{E}[S_h(t) S_h(s)] = \int_0^t \mathrm{d}\theta \int_0^s \mathrm{d}\sigma\, R(h(\theta), h(\sigma))|\theta - \sigma|^{h(\theta)+h(\sigma)-2}, \tag{1.6}$$

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where R is the continuous function derived from (1.4). The process S_h is locally self-similar. If the function h is constant then the process S_h is a fractional Brownian motion. This result is then a generalization of the invariance principles presented above for $\phi = \mathrm{Id}\colon x \to x$. It also defines a class of multifractional Gaussian processes, which satisfies invariance principles.

In this work we generalize the invariance principles presented above. We study the asymptotic behavior of a sequence generalizing both (1.3) and (1.5). In particular, this sequence is defined by a Gaussian field $\{X_j(H)\}_{j,H}$ satisfying the long-range properties in (1.4), a function ϕ such that $\int_{\mathbb{R}} \phi(x)^2 \mathrm{e}^{-x^2/2}\,\mathrm{d}x < \infty$ with Hermite rank equal to $m \in \mathbb{N}^*$, and a Hurst function h taking its values in $(\frac{1}{2}, 1)$. We get as a limit a multifractional process $S_{m,h}$ that depends on the integer m and the function h. If the function is a constant H then the limit process is the Hermite process with Hurst index H and Hermite order m. If the integer m is equal to 1 then the limit process is a Gaussian multifractional process of the class obtained in [4]. Moreover, as Hermite processes, $S_{m,h}$ is Gaussian if and only if $m = 1$. Our result then defines a class of multifractional processes, which can be Gaussian or non-Gaussian. Because these processes satisfy invariance principles, they can be suitable models when local self-similarity arises.

In contrast to what occurs in [4], the processes we study can be non-Gaussian. Hence, our work cannot be based only on second-order moments as in [4]. To prove our result, we use the convergence of multiple Wiener–Itô integrals and some properties of the Hermite polynomials. Moreover, to deal with multifractionality, we prove some uniform estimates to control the fluctuations of the Hurst index.

The paper is organized as follows. In Section 2 we recall some definitions and preliminary results about Hermite polynomials and multiple stochastic integrals, which are used throughout the paper. In Section 3 we establish the main results of the paper. Section 4 is devoted to the proofs.

2. Preliminaries

In this section we make precise some definitions and recall some results about Hermite polynomials and multiple stochastic integrals we use throughout this paper.

2.1. Hermite polynomials

For each positive integer $m \in \mathbb{N}$, the mth Hermite polynomial P_m is defined by

$$P_m(x) = (-1)^m \mathrm{e}^{x^2/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \mathrm{e}^{-x^2/2}$$

for every $x \in \mathbb{R}$. The family of the Hermite polynomials $\{P_m,\ m \in \mathbb{N}\}$ is an orthogonal basis of the space $L^2(\mathrm{e}^{-x^2/2}\,\mathrm{d}x)$ defined by

$$L^2(\mathrm{e}^{-x^2/2}\,\mathrm{d}x) = \left\{\phi\colon \mathbb{R} \to \mathbb{R}, \phi \text{ measurable and } \int_{\mathbb{R}} (\phi(x))^2 \mathrm{e}^{-x^2/2}\,\mathrm{d}x < \infty\right\}$$

endowed with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \cdot, \cdot \rangle\colon (\phi_1, \phi_2) \mapsto \langle \phi_1, \phi_2 \rangle = \int_{\mathbb{R}} \phi_1(x)\phi_2(x)\mathrm{e}^{-x^2/2} \frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

The norm corresponding to $\langle \cdot, \cdot \rangle$ will be denoted by $\|\cdot\|$. For every (nonzero) function $\phi \in L^2(\mathrm{e}^{-x^2/2}\,\mathrm{d}x)$, there exists an integer m_ϕ such that $\langle \phi, P_{m_\phi} \rangle \neq 0$ and $\langle \phi, P_m \rangle = 0$ for every $m = 0, \ldots, m_\phi - 1$. The integer m_ϕ is called the Hermite index of the function ϕ. Hence, for every $\phi \in L^2(\mathrm{e}^{-x^2/2}\,\mathrm{d}x)$,

$$\phi = \sum_{m=0}^{\infty} \frac{\langle \phi, P_m \rangle}{m!} P_m = \sum_{m=m_\phi}^{\infty} \frac{\langle \phi, P_m \rangle}{m!} P_m, \tag{2.1}$$

where the convergence of the series holds for the norm $\|\cdot\|$. If X and Y are two Gaussian random variables with mean 0 and variance 1 then, for all j and k in $\mathbb{N}^*$,

$$\mathbb{E}[P_j(X)P_k(Y)] = \begin{cases} k!\,(\mathbb{E}[XY])^k & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases} \tag{2.2}$$

As a consequence, for every $\phi \in L^2(\mathrm{e}^{-x^2/2}\,\mathrm{d}x)$ and every Gaussian random variable X with mean 0 and variance 1,

$$\mathbb{E}[|\phi(X)|^2] = \sum_{m=0}^{\infty} \frac{\langle \phi, P_m \rangle^2}{m!} < \infty. \tag{2.3}$$

2.2. Multiple Wiener–Itô integrals

Many notions of multiple Wiener–Itô integrals [6], [8] with respect to Brownian motion have been introduced and have been used to define processes as Hermite processes [7], [17]. Here we have chosen to use the so-called multiple Wiener–Itô–Dobrushin integral defined in [6]. In this subsection we give a brief description of this integral, using its properties throughout the paper. We refer the reader to the seminal paper [6] for a complete construction and a detailed study.

For every $d \in \mathbb{N}^*$, we denote by $\hat{L}^2(\mathbb{R}^d)$ ($\hat{L}^2(\mathbb{R})$ when $d = 1$) the space of square-integrable functions $f\colon \mathbb{R}^d \to \mathbb{C}$ satisfying, for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$f(x_1, \ldots, x_d) = \overline{f(-x_1, \ldots, -x_d)},$$

and, for every permutation σ on $\{1, \ldots, d\}$,

$$f(x_1, \ldots, x_d) = f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}).$$

Let B be a real Brownian motion. We define the random measure $\hat{B}$ by

$$\hat{B}(\psi) := \int_{\mathbb{R}} \hat{\psi}(\xi)\,\mathrm{d}B_\xi$$

for every $\psi \in \hat{L}^2(\mathbb{R})$, where $\hat{\psi}$ is the Fourier transform of ψ and the integral of the right-hand side is the classical Wiener–Itô integral in dimension one. We also denote by $\int_{\mathbb{R}} \psi(x)\,\mathrm{d}\hat{B}_x$ the random variable $\hat{B}(\psi)$. Because $\psi \in \hat{L}^2(\mathbb{R})$, $\hat{B}(\psi)$ is a real Gaussian variable with mean 0 and variance $\int_{\mathbb{R}} |\psi(x)|^2\,\mathrm{d}x$. Let $d \in \mathbb{N}^*$, and consider a function $f \in \hat{L}^2(\mathbb{R}^d)$. A multiple random integral of f is defined in [6] from $\hat{B}$ by an approximation of f with step functions in $\hat{L}^2(\mathbb{R}^d)$. This is the so-called Wiener–Itô–Dobrushin integral of f, which we denote in this paper by $\int_{\mathbb{R}^d} f\,\mathrm{d}\hat{B}^{\otimes d}$ or

$$\int_{\mathbb{R}^d} f(x_1, \ldots, x_d)\,\mathrm{d}\hat{B}_{x_1} \cdots \mathrm{d}\hat{B}_{x_d}.$$

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It is a real random variable with mean 0 and variance

$$\mathbb{E}\bigg[\bigg(\int_{\mathbb{R}^d} f\,\mathrm{d}\hat{B}^{\otimes d}\bigg)^2\bigg] = d!\int_{\mathbb{R}^d} |f(x_1,\dots,x_d)|^2\,\mathrm{d}x_1\cdots\mathrm{d}x_d \tag{2.4}$$

and is Gaussian if and only if $d = 1$. The following Fubini-type formula is one of the most important properties of this integral [6].

Lemma 2.1. *Let P_d be the Hermite polynomial of rank d. For every $\psi \in \hat{L}^2(\mathbb{R})$ satisfying $\int_{\mathbb{R}} |\psi(\xi)|^2\,\mathrm{d}\xi = 1$,*

$$P_d\bigg(\int_{\mathbb{R}} \psi(x)\,\mathrm{d}\hat{B}_x\bigg) = \int_{\mathbb{R}^d} \psi(x_1)\cdots\psi(x_d)\,\mathrm{d}\hat{B}_{x_1}\cdots\mathrm{d}\hat{B}_{x_d}. \tag{2.5}$$

The following lemma states a change-of-variable formula. It is due to the self-similarity of Brownian motion [6].

Lemma 2.2. *For every $\varepsilon > 0$, we have the equality in distribution*

$$\int_{\mathbb{R}^d} f(x_1,\dots,x_d)\,\mathrm{d}\hat{B}_{x_1}\cdots\mathrm{d}\hat{B}_{x_d} \overset{\mathrm{D}}{=} \varepsilon^{m/2}\int_{\mathbb{R}^d} f(\varepsilon x_1,\dots,\varepsilon x_d)\,\mathrm{d}\hat{B}_{x_1}\cdots\mathrm{d}\hat{B}_{x_d}. \tag{2.6}$$

The following lemma gives another change-of-variable formula. It is a direct consequence of Proposition 4.2 of [6].

Lemma 2.3. *Let $z\colon \mathbb{R} \to \mathbb{C}$ be a bounded and measurable function satisfying $z(x) = \overline{z(-x)}$ and $|z(x)| = 1$ for every $x \in \mathbb{R}$. Then, we have the equality in distribution*

$$\int_{\mathbb{R}^d} f(x_1,\dots,x_d)\,\mathrm{d}\hat{B}_{x_1}\cdots\mathrm{d}\hat{B}_{x_d} \overset{\mathrm{D}}{=} \int_{\mathbb{R}^d} f(x_1,\dots,x_d)z(x_1)\cdots z(x_d)\,\mathrm{d}\hat{B}_{x_1}\cdots\mathrm{d}\hat{B}_{x_d}. \tag{2.7}$$

By using the linearity of the integral, (2.4), and the bounded convergence theorem, we can prove the following lemma.

Lemma 2.4. *Let $\{f_N\}_{N\in\mathbb{N}}$ be a sequence of functions in $\hat{L}^2(\mathbb{R}^d)$. We assume that there exist two functions f and f^* in $\hat{L}^2(\mathbb{R}^d)$ such that, for almost every $x \in \mathbb{R}^d$, $\lim_{N\to\infty} f_N(x) = f(x)$ and $\sup_N |f_N(x)| \le f^*(x)$. Then*

$$\lim_{N\to\infty} \mathbb{E}\bigg[\bigg(\int_{\mathbb{R}^d} f_N\,\mathrm{d}\hat{B}^{\otimes d} - \int_{\mathbb{R}^d} f\,\mathrm{d}\hat{B}^{\otimes d}\bigg)^2\bigg] = 0.$$

Finally, we can generalize (2.4) to every moment by using hypercontractivity arguments (see, for instance, [11] or [12]).

Lemma 2.5. *Let $p \in \mathbb{N}^*$. There exists a constant $C = C(d,p) > 0$ such that, for every $f \in \hat{L}^2(\mathbb{R}^d)$,*

$$\mathbb{E}\bigg[\bigg(\int_{\mathbb{R}^d} f\,\mathrm{d}\hat{B}^{\otimes d}\bigg)^{2p}\bigg] \le C(d,p)\bigg(\int_{\mathbb{R}^d} |f(x_1,\dots,x_d)|^2\,\mathrm{d}x_1\cdots\mathrm{d}x_d\bigg)^p.$$

3. Main results

We fix $m \in \mathbb{N}^*$, and define

$$b = 1 - \frac{1}{2m}.$$

We consider the Gaussian field $X = \{X_n(H)\}_{(n,H)\in\mathbb{N}\times(b,1)}$ defined by

$$X_n(H) = \int_{-a}^{a} \exp(\mathrm{i}nx) g(H, x)|x|^{1/2-H}\, \mathrm{d}\hat{B}_x \tag{3.1}$$

for all $n \in \mathbb{N}$ and $H \in (b, 1)$, where $a \in (0, 2\pi/m)$, $\hat{B}$ is the Fourier transform of the random Brownian measure, and $g\colon (b, 1) \times (-a, a) \to \mathbb{C}$ is a measurable function. The right-hand side of (3.1) is a stochastic integral defined as in Section 2.2. We assume that the function g satisfies the following properties.

- For every $(H, x) \in (b, 1) \times (0, a)$, $g(H, x) = \overline{g(H, -x)}$. This property ensures that the field X is real.
- For every $H \in (b, 1)$,
$$\int_{-a}^{a} |g(H, x)|^2 |x|^{1-2H}\, \mathrm{d}x = 1, \tag{3.2}$$
so that $\mathbb{E}[X_n(H)^2] = 1$.
- The function g is twice continuously differentiable on $(b, 1) \times (-a, a)$. Then, for every $(H, x) \in (b, 1) \times (-a, a)$, we define
$$g_0(H) = g(H, 0) \quad \text{and} \quad g_1(H, x) = \int_0^x \frac{\partial g}{\partial \xi}(H, \xi)\, \mathrm{d}\xi,$$
so that $g = g_0 + g_1$ and
$$\lim_{x\to 0} \sup_{H\in K} \biggl(|g_1(H, x)| + \biggl|\frac{\partial g_1}{\partial H}(H, x)\biggr|\biggr) = 0$$
for every compact set K of $(b, 1)$.

The assumptions above ensure that the covariance function satisfies the uniform long-range property of [4]. More precisely, Lemma A.1 (see Appendix A) states that, for every compact set $K \subset (b, 1)$,

$$\lim_{j-k\to\infty} \sup_{(H_1,H_2)\in K^2} |(j-k)^{2-H_1-H_2} \mathbb{E}[X_j(H_1)X_k(H_2)] - R(H_1, H_2)| = 0, \tag{3.3}$$

where

$$R(H_1, H_2) = g_0(H_1)g_0(H_2) \int_{\mathbb{R}} \exp(\mathrm{i}x)|x|^{1-H_1-H_2}\, \mathrm{d}x \tag{3.4}$$

for every $(H_1, H_2) \in (b, 1)^2$.

We consider a function $\phi \in L^2(\mathrm{e}^{-x^2/2}\, \mathrm{d}x)$ with Hermite rank equal to $m \in \mathbb{N}^*$. Note that ϕ then satisfies the centering condition

$$\int_{\mathbb{R}} \phi(x)\mathrm{e}^{-x^2/2}\, \mathrm{d}x = 0.$$

We consider a continuously differentiable function $h\colon [0,\infty) \to (\frac{1}{2}, 1)$ and set

$$\tilde{h} := 1 + \frac{h-1}{m}\colon [0,\infty) \to (b, 1).$$

For all $t \geq 0$ and $N \in \mathbb{N}^*$, we define

$$S^N_{\phi,h}(t) := \sum_{j=1}^{\lfloor Nt \rfloor} \frac{\phi(X_j(h^N_j))}{N^{h(j/N)}}, \tag{3.5}$$

where, for every $j \in \{1, \dots, \lfloor Nt \rfloor\}$,

$$h^N_j = 1 + \frac{h(j/N) - 1}{m} = \tilde{h}\biggl(\frac{j}{N}\biggr).$$

For every $(x_1, \dots, x_m, t) \in (\mathbb{R}^*)^m \times [0,\infty)$, we set

$$f_{m,h}(x_1, \dots, x_m, t) = \int_0^t \exp\biggl(\mathrm{i}\theta \sum_{l=1}^m x_l\biggr) \tilde{g}(\theta) |x_1 \cdots x_m|^{1/2 - \tilde{h}(\theta)} \,\mathrm{d}\theta,$$

where

$$\tilde{g} = \frac{\langle \phi, P_m \rangle}{m!} (g_0 \circ \tilde{h})^m.$$

For every $t \geq 0$, we define

$$S_{m,h}(t) = \int_{\mathbb{R}^m} f_{m,h}(x_1, \dots, x_m, t) \,\mathrm{d}\hat{B}_{x_1} \cdots \mathrm{d}\hat{B}_{x_m}.$$

For all $t, u \geq 0$ and $\varepsilon \in (0,1)$, we set

$$T^{\varepsilon}_{m,h,t}(u) = \frac{S_{m,h}(t + \varepsilon u) - S_{m,h}(t)}{\varepsilon^{h(t)}}$$

and

$$T_{m,h,t}(u) = \tilde{g}(t) \int_{\mathbb{R}^m} \frac{\exp(\mathrm{i}u \sum_{l=1}^m x_l) - 1}{\mathrm{i}(\sum_{l=1}^m x_l) |x_1 \cdots x_m|^{\tilde{h}(t) - 1/2}} \,\mathrm{d}\hat{B}_{x_1} \cdots \mathrm{d}\hat{B}_{x_m}.$$

For any real interval I, $\mathcal{D}(I)$ is the space of càdlàg functions on I with the Skorokhod topology (see [3, Chapter 3]) and $\mathcal{C}(I)$ is the space of continuous functions on I with the uniform topology on each compact set. For a real continuous function w and a point t in the domain of definition of w, the local Hölder exponent of w around t is denoted by $\alpha_w(t)$ and defined by

$$\alpha_w(t) = \sup\biggl\{\alpha \in (0,1]\colon \lim_{s \to t} \frac{|w(t) - w(s)|}{|t - s|^{\alpha}} < \infty\biggr\}.$$

The two main results of this paper can now be stated. The first result concerns an invariance principle.

Theorem 3.1. *The process $S_{m,h} = \{S_{m,h}(t)\}_{t \geq 0}$ is continuous (up to a modification) and $S^N_{\phi,h} = \{S^N_{\phi,h}(t)\}_{t \geq 0}$ converges in distribution to $S_{m,h}$ in $\mathcal{D}([0,\infty))$ as $N \to \infty$.*

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The second main result deals with sample path properties (local self-similarity and local Hölder exponent) of the limit process $S_{m,h}$.

Theorem 3.2. *Let $t \geq 0$. The process $T^{\varepsilon}_{m,h,t} = \{T^{\varepsilon}_{m,h,t}(u)\}_{u\geq 0}$ converges in distribution to $T_{m,h,t} = \{T_{m,h,t}(u)\}_{u\geq 0}$ in $\mathcal{C}([0,\infty))$ as $\varepsilon \to 0$. Moreover, the local Hölder exponent of $S_{m,h}$ around t is equal to $h(t)$.*

Theorems 3.1 and 3.2 establish that sequences of processes defined as in (3.5), in particular from a function ϕ in $L^2(\mathrm{e}^{-x^2/2}\,\mathrm{d}x)$ of Hermite rank m and a Hurst function h, converge to a multifractional process $S_{m,h}$ with Hurst function h and represented as a multiple integral of order m. Therefore, because the process $S_{m,h}$ is defined as the limit of an invariance principle, it can be a universal model when local self-similarity and long-range dependence arise in a Gaussian or non-Gaussian framework.

Theorem 3.1 generalizes the results of [7] and [17] to a multifractional setting. Indeed, if we assume that $h \equiv H \in (\frac{1}{2}, 1)$ then Theorem 3.1 is the main result of [7] and [17]. In particular, the limit process $S_{m,H}$ can be written as $W_{m,H}$ in (1.1) with the constant

$$C(m,H) = \frac{\langle \phi, P_m\rangle}{m!}(g_0(\tilde{H}))^m = \frac{\langle \phi, P_m\rangle}{m!}\left(\frac{R(\tilde{H},\tilde{H})}{\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi}|\xi|^{1-2\tilde{H}}\,\mathrm{d}\xi}\right)^{m/2},$$

where

$$\tilde{H} := 1 + \frac{H-1}{m} \in (b, 1).$$

Theorem 3.1 is also an extension of the main result of [4] (Theorem 2) to a non-Gaussian framework. Indeed, if we assume that $\phi = \mathrm{Id}\colon x \to x$ then $m = 1$ and the limit process is $S_{1,h}$, which is a centered Gaussian process of covariance

$$(t,s) \mapsto \mathbb{E}[S_{1,h}(t)S_{1,h}(s)] = \int_0^t \mathrm{d}\theta \int_0^s \mathrm{d}\sigma\, R(h(\theta), h(\sigma))|\theta - \sigma|^{h(\theta)+h(\sigma)-2}$$

with R defined by (3.4).

To conclude this section, let us note the connection between our work and [15]. Let $Y_h = \{Y_h(t)\}_{t\geq 0}$ be the process defined for every $t \geq 0$ by

$$Y_h(t) = \mathcal{R}\int_{\mathbb{R}}\left(\int_0^t \frac{\exp(\mathrm{i}y\theta)}{|y|^{h(\theta)-1/2}}\,\mathrm{d}\theta\right)\mathrm{d}\tilde{B}(y), \tag{3.6}$$

where $\mathrm{d}\tilde{B}$ is a complex Gaussian measure and $\mathcal{R}$ stands for the real part. This process is called integrated fractional white noise and has been introduced in [15] as an alternative to multifractional Brownian motion. An advantage is the fact that it is a multifractional Gaussian process without undesirable oscillations that multifractional Brownian motion has (see [15]). Note that if we let $g_0 \equiv (m!/\langle \phi, P_m\rangle)^{1/m}$ then, for $m = 1$, the process $S_{1,h}$ has the same distributions as Y_h. Therefore, Theorem 3.1 states that Y_h is the limit of an invariance principle. This is another interest of Y_h. Moreover, for every $m \geq 2$, the process $S_{m,h}$ is a natural generalization of Y_h to a non-Gaussian framework.

4. Proofs

In this section we prove Theorems 3.1 and 3.2 and we proceed as follows. In Subsection 4.1 we establish a technical lemma. We prove the convergence of the finite-dimensional distributions of $S^N_{\phi,h}$ in Subsection 4.2. The regularity properties of $S_{m,h}$ are established in Subsection 4.3. Finally, we deal with the tightness for the Skorokhod topology in Subsection 4.4. Throughout this section, for every set E and every subset $A \subset E$, we denote by 1_A the function defined on E such that $1_A(a) = 1$ if $a \in A$ and $1_A(a) = 0$ if $a \in E - A$.

4.1. Technical lemma

In the following lemma we prove, for every $T > 0$, the existence of a function $\tilde{f}_T$ that is useful in the sequel to establish uniform bounds.

Lemma 4.1. *For every $T > 0$, there exists a function $\tilde{f}_T \in L^2(\mathbb{R}^m, \mathbb{R})$ such that, for every $t \in [0, T]$, every $H \in [\min \tilde{h}, \max \tilde{h}]$, and almost every $(x_1, \ldots, x_m) \in \mathbb{R}^m$,*

$$\left|\frac{\exp(\mathrm{i}t \sum_{l=1}^m x_l) - 1}{|x_1 \cdots x_m|^{H-1/2} \sum_{l=1}^m x_l}\right|(1 + |\ln|x_1 \cdots x_m||) \le \tilde{f}_T(x).$$

Proof. We fix $T > 0$. We define $L(x_1, \ldots, x_m) = (1 + |\ln|x_1 \cdots x_m||)^2$ and

$$\tilde{f}_T(x_1, \ldots, x_m) = \sqrt{\sum_{H=\min\tilde{h},\max\tilde{h}} \frac{T^2 1_{\{|\sum_{l=1}^m x_l| \le 1\}} + 4|\sum_{l=1}^m x_l|^{-2} 1_{\{|\sum_{l=1}^m x_l| > 1\}}}{|x_1 \cdots x_m|^{2H-1}} L(x_1, \ldots, x_m)}.$$

We have

$$\max_{H \in [\min\tilde{h}, \max\tilde{h}]} \frac{1}{|x_1 \cdots x_m|^{2H-1}} \le \frac{1}{|x_1 \cdots x_m|^{2\min\tilde{h}-1}} + \frac{1}{|x_1 \cdots x_m|^{2\max\tilde{h}-1}}.$$

Then, for every $t \in [0, T]$,

$$\left|\frac{\exp(\mathrm{i}t \sum_{l=1}^m x_l) - 1}{\sum_{l=1}^m x_l}\right|^2 \max_{H \in [\min\tilde{h}, \max\tilde{h}]} \frac{L(x_1, \ldots, x_m)}{|x_1 \cdots x_m|^{2H-1}} \le \tilde{f}_T(x_1, \ldots, x_m)^2.$$

It is then enough to prove that, for $H \in \{\min\tilde{h}, \max\tilde{h}\}$, the function

$$(x_1, \ldots, x_m) \mapsto \frac{T^2 1_{\{|\sum_{l=1}^m x_l| \le 1\}} + 4|\sum_{l=1}^m x_l|^{-2} 1_{\{|\sum_{l=1}^m x_l| > 1\}}}{|x_1 \cdots x_m|^{2H-1}} L(x_1, \ldots, x_m)$$

is integrable. We successively make the substitutions $y_j = x_1 + \cdots + x_j$ for every $j \in \{1, \ldots, m\}$, $z_k = y_k / y_{k+1}$ for every $k \in \{1, \ldots, m-1\}$, and $z_m = y_m$ to get

$$\begin{aligned}
&\int_{\mathbb{R}^m} \frac{T^2 1_{\{|\sum_{l=1}^m x_l| \le 1\}} + 4|\sum_{l=1}^m x_l|^{-2} 1_{\{|\sum_{l=1}^m x_l| > 1\}}}{|x_1 \cdots x_m|^{2H-1}} L(x_1, \ldots, x_m)\, \mathrm{d}x_1 \cdots \mathrm{d}x_m \\
&= \int_{\mathbb{R}^m} \frac{T^2 1_{\{|y_m| \le 1\}} + 4|y_m|^{-2} 1_{\{|y_m| > 1\}}}{|y_1(y_2 - y_1) \cdots (y_m - y_{m-1})|^{2H-1}} L(y_1, y_2 - y_1, \ldots, y_m - y_{m-1})\, \mathrm{d}y_1 \cdots \mathrm{d}y_m \\
&= \int_{\mathbb{R}} \frac{T^2 1_{\{|z_m| \le 1\}} + 4|z_m|^{-2} 1_{\{|z_m| > 1\}}}{|z_m|^{2m(H-1)+1}}\, \mathrm{d}z_m \int_{\mathbb{R}} \frac{\mathrm{d}z_{m-1}}{|1 - z_{m-1}|^{2H-1} |z_{m-1}|^{2(m-1)(H-1)+1}} \times \cdots \\
&\quad \times \int_{\mathbb{R}} \frac{\mathrm{d}z_1}{|1 - z_1|^{2H-1} |z_1|^{2H-1}} L\left(\prod_{k=1}^m z_k, (1 - z_1) \prod_{k=2}^m z_k, \ldots, (1 - z_{m-1}) z_m\right).
\end{aligned}$$

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The right-hand side above can be bounded by a finite sum of terms of the form

$$\int_{\mathbb{R}} \frac{T^2 1_{\{|z_m|\le 1\}} + 4|z_m|^{-2} 1_{\{|z_m|>1\}}}{|z_m|^{2m(H-1)+1}}\, \mathrm{d}z_m \int_{\mathbb{R}} \frac{\mathrm{d}z_{m-1}}{|1-z_{m-1}|^{2H-1}|z_{m-1}|^{2(m-1)(H-1)+1}} \times \cdots$$
$$\times \int_{\mathbb{R}} \frac{\mathrm{d}z_1}{|1-z_1|^{2H-1}|z_1|^{2H-1}} |\ln|z_k||^{\mu} |\ln|1-z_j||^{\nu}, \tag{4.1}$$

where k, j, μ, and ν are integers. The terms of the form (4.1) are finite since $H \in (1-1/(2m), 1)$ and by Bertrand's test. This concludes the proof.

4.2. Convergence of the finite-dimensional distributions

First, we deal with the sequence $\{S^N_{P_m,h}\}_{N\in\mathbb{N}}$ defined by

$$S^N_{P_m,h}(t) = \sum_{j=1}^{\lfloor Nt\rfloor} \frac{P_m(X_j(h^N_j))}{N^{h(j/N)}}$$

for all $t \ge 0$ and $N \in \mathbb{N}$. From now on, we denote by $\mathrm{d}\hat{B}^{\otimes m}_x$ the product $\prod_{l=1}^m \mathrm{d}\hat{B}_{x_l}$ when $x = (x_1, \dots, x_d)$.

Lemma 4.2. *For every $N \in \mathbb{N}$, the process $S^N_{P_m,h}$ is equal in distribution to the process $\tilde{S}^N_{m,h}$ defined by*

$$\tilde{S}^N_{m,h}(t) = \int_{(-Na,Na)^m} \mathrm{d}\hat{B}^{\otimes m}_x \frac{1}{N} \sum_{j=1}^{\lfloor Nt\rfloor} \prod_{l=1}^m \exp\Big(\frac{\mathrm{i} j x_l}{N}\Big) g\Big(h^N_j, \frac{x_l}{N}\Big) |x_l|^{1/2-h^N_j}$$

for every $t \ge 0$.

Proof. Using (2.5), we obtain

$$P_m(X_j(h^N_j)) = \int_{(-a,a)^m} \prod_{l=1}^m \exp(\mathrm{i} j x_l) g(h^N_j, x_l)|x_l|^{1/2-h^N_j}\, \mathrm{d}\hat{B}_{x_l}$$

almost surely. We then have

$$S^N_{P_m,h}(t) = \sum_{j=1}^{\lfloor Nt\rfloor} \frac{1}{N^{1-m/2}} \int_{(-a,a)^m} \mathrm{d}\hat{B}^{\otimes m}_x \prod_{l=1}^m \exp(\mathrm{i} j x_l) g(h^N_j, x_l)|N x_l|^{1/2-h^N_j}.$$

Making the substitution $x \to x/N$ and using (2.6), we obtain

$$S^N_{P_m,h} \overset{\mathrm{D}}{=} t \mapsto \sum_{j=1}^{\lfloor Nt\rfloor} \frac{1}{N} \int_{(-Na,Na)^m} \mathrm{d}\hat{B}^{\otimes m}_x \prod_{l=1}^m \exp\Big(\frac{\mathrm{i} j x_l}{N}\Big) g\Big(h^N_j, \frac{x_l}{N}\Big) |x_l|^{1/2-h^N_j}.$$

This concludes the proof by the linearity of the multiple integral.

We now aim to prove the convergence of $\{\tilde{S}^N_{m,h}(t)\}_{N\in\mathbb{N}}$ in $L^2(\Omega, \mathbb{R})$ for every t. We introduce the sequence of functions $\{f^N\}_{N\in\mathbb{N}}$ defined by

$$f^N\colon [0,\infty) \times \mathbb{R}^m \to \mathbb{C},$$
$$(t, x) \mapsto 1_{(-Na,Na)^m}(x) \frac{1}{N} \sum_{j=1}^{\lfloor Nt\rfloor} \prod_{l=1}^m \exp\Big(\frac{\mathrm{i} j x_l}{N}\Big) \frac{g(h^N_j, x_l/N)}{|x_l|^{h^N_j - 1/2}},$$

and we establish the following lemma.

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Lemma 4.3. *For every $t \geq 0$, there exists a function $f_t^* \in L^2(\mathbb{R}^m, \mathbb{R})$ such that, for all $x \in \mathbb{R}^m$ and $N \in \mathbb{N}$,*

$$|f^N(t, x)| \leq f_t^*(x).$$

Proof. We have

$$f^N(t, x) = 1_{(-Na,Na)}(x) \frac{\mathrm{i} \sum_{l=1}^m x_l/N}{1 - \exp(-\mathrm{i} \sum_{l=1}^m x_l/N)} \times \sum_{j=1}^{\lfloor Nt \rfloor} \frac{\exp(\mathrm{i} j \sum_{l=1}^m x_l/N) - \exp(\mathrm{i}(j-1) \sum_{l=1}^m x_l/N)}{\mathrm{i} \sum_{l=1}^m x_l} G_j^N(x),$$

where

$$G_j^N(x) = \prod_{l=1}^m \frac{g(h_j^N, x_l/N)}{|x_l|^{h_j^N - 1/2}}.$$

We write

$$f^N(t, x) = f^{N,1}(t, x) - f^{N,2}(t, x)$$

with

$$\begin{aligned} f^{N,1}(t, x) &= 1_{(-Na,Na)}(x) \frac{\mathrm{i} \sum_{l=1}^m x_l/N}{1 - \exp(-\mathrm{i} \sum_{l=1}^m x_l/N)} \\ &\quad \times \sum_{j=1}^{\lfloor Nt \rfloor} \frac{1}{\mathrm{i} \sum_{l=1}^m x_l} \bigg(G_j^N(x) \bigg(\exp\bigg(\mathrm{i} j \sum_{l=1}^m \frac{x_l}{N} \bigg) - 1 \bigg) \\ &\qquad - G_{j-1}^N(x) \bigg(\exp\bigg(\mathrm{i}(j-1) \sum_{l=1}^m \frac{x_l}{N} \bigg) - 1 \bigg) \bigg) \\ &= 1_{(-Na,Na)}(x) \frac{\mathrm{i} \sum_{l=1}^m x_l/N}{1 - \exp(-\mathrm{i} \sum_{l=1}^m x_l/N)} G_{\lfloor Nt \rfloor}^N(x) \frac{\exp(\mathrm{i} \lfloor Nt \rfloor \sum_{l=1}^m x_l/N) - 1}{\mathrm{i} \sum_{l=1}^m x_l} \end{aligned}$$

and

$$f^{N,2}(t, x) = 1_{(-Na,Na)}(x) \frac{\mathrm{i} \sum_{l=1}^m x_l/N}{1 - \exp(-\mathrm{i} \sum_{l=1}^m x_l/N)} \times \sum_{j=1}^{\lfloor Nt \rfloor} \frac{\exp(\mathrm{i}(j-1) \sum_{l=1}^m x_l/N) - 1}{\mathrm{i} \sum_{l=1}^m x_l} (G_j^N(x) - G_{j-1}^N(x)).$$

First, we deal with $f^{N,1}$. Because g is bounded, there exists $M_1 > 0$ such that, for every N and almost every x,

$$|f^{N,1}(t, x)| \leq M_1 \left| \frac{\exp(\mathrm{i} \lfloor Nt \rfloor \sum_{l=1}^m x_l/N) - 1}{|x_1 \cdots x_m|^{h_{\lfloor Nt \rfloor}^N - 1/2} \sum_{l=1}^m x_l} \right|.$$

Then, by Lemma 4.1, there exists a function $\tilde{f}_{t,1} \in L^2(\mathbb{R}^m, \mathbb{R})$ such that

$$\left| \frac{\exp(\mathrm{i} \lfloor Nt \rfloor \sum_{l=1}^m x_l/N) - 1}{|x_1 \cdots x_m|^{h_{\lfloor Nt \rfloor}^N - 1/2} \sum_{l=1}^m x_l} \right| \leq \tilde{f}_{t,1}(x),$$

so that we get

$$|f^{N,1}(t,x)| \leq M_1 \tilde{f}_{t,1}(x). \tag{4.2}$$

We now deal with $f^{N,2}$. By Taylor's formula we obtain

$$\begin{aligned}
&|G_j^N(x) - G_{j-1}^N(x)| \\
&\quad \leq \frac{\max|\tilde{h}'|}{N} \max_{H\in[\min\tilde{h},\max\tilde{h}]} \biggl| \frac{-\ln|x_1\cdots x_m|}{|x_1\cdots x_m|^{H-1/2}} \prod_{l=1}^m g\Bigl(H,\frac{x_l}{N}\Bigr) \\
&\qquad\qquad + \frac{1}{|x_1\cdots x_m|^{H-1/2}} \sum_{k=1}^m \Bigl(\frac{\partial g}{\partial H}\Bigl(H,\frac{x_k}{N}\Bigr)\Bigr) \prod_{l=1,\,l\neq k}^m g\Bigl(H,\frac{x_l}{N}\Bigr)\biggr|.
\end{aligned}$$

Since g and $\partial g/\partial H$ are bounded, there exists a constant $M_2 > 0$, which depends only on h and g, such that, for almost every x and every N,

$$|f^{N,2}(t,x)| \leq \frac{M_2}{N} \sum_{j=1}^{\lfloor Nt\rfloor} \frac{|\exp(\mathrm{i}(j-1)\sum_{l=1}^m x_l/N) - 1|}{|\sum_{l=1}^m x_l|} \max_{H\in[\min\tilde{h},\max\tilde{h}]} \frac{1+|\ln|x_1\cdots x_m||}{|x_1\cdots x_m|^{H-1/2}}.$$

As in the case of $f^{N,1}$, by Lemma 4.1, there exists a function $\tilde{f}_{t,2} \in L^2(\mathbb{R}^m,\mathbb{R})$ such that, for all N and j and almost every x,

$$\biggl|\frac{\exp(\mathrm{i}(j-1)\sum_{l=1}^m x_l/N) - 1}{\sum_{l=1}^m x_l}\biggr| \max_{H\in[\min\tilde{h},\max\tilde{h}]} \frac{1+|\ln|x_1\cdots x_m||}{|x_1\cdots x_m|^{H-1/2}} \leq \tilde{f}_{t,2}(x).$$

We then get

$$|f^{N,2}(t,x)| \leq M_2 \tilde{f}_{t,2}(x). \tag{4.3}$$

Hence, taking $\tilde{f}_t = M_1\tilde{f}_{t,1} + M_2\tilde{f}_{t,2}$ and combining (4.2) and (4.3) completes the proof.

The convergence of $\{\tilde{S}_{m,h}^N\}_{n\in\mathbb{N}}$ can now be established.

Lemma 4.4. *For every $t \geq 0$, as $N \to \infty$, $\tilde{S}_{m,h}^N(t)$ converges in $L^2(\Omega,\mathbb{R})$ to $\tilde{S}_{m,h}^\infty(t)$ given by*

$$\tilde{S}_{m,h}^\infty(t) = \int_{\mathbb{R}^m} \mathrm{d}\hat{B}_x^{\otimes m} \int_0^t \exp\Bigl(\mathrm{i}\theta\sum_{l=1}^m x_l\Bigr) g_0(\tilde{h}(\theta))^m |x_1\cdots x_m|^{1/2-\tilde{h}(\theta)}\,\mathrm{d}\theta.$$

Proof. We fix $t \geq 0$. By Lemmas 2.4 and 4.3, it suffices to prove that, for almost every x, $f^N(t,x)$ converges to $f^\infty(t,x)$ defined by

$$f^\infty(t,x) := \int_0^t \exp\Bigl(\mathrm{i}\theta\sum_{l=1}^m x_l\Bigr) g_0(\tilde{h}(\theta))^m |x_1\cdots x_m|^{1/2-\tilde{h}(\theta)}\,\mathrm{d}\theta.$$

We let

$$G_{j,0}^N(x) = g_0(h_j^N)^m |x_1\cdots x_m|^{1/2-h_j^N} \quad\text{and}\quad G_{j,1}^N(x) = G_j^N(x) - G_{j,0}^N(x),$$

where $G_j^N(x)$ is defined as in the proof of Lemma 4.3. We also consider the same decomposition $f^N = f^{N,1} - f^{N,2}$ as in the proof of Lemma 4.3 and we let

$$f^{N,1} = f^{N,1,0} - f^{N,1,1} \quad\text{and}\quad f^{N,2} = f^{N,2,0} - f^{N,2,1},$$

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where, for $\kappa \in \{0, 1\}$,

$$f^{N,1,\kappa}(t,x) = 1_{(-Na,Na)}(x) \frac{\mathrm{i} \sum_{l=1}^m x_l/N}{1-\exp(-\mathrm{i}\sum_{l=1}^m x_l/N)} G^N_{\lfloor Nt \rfloor,\kappa}(x) \frac{\exp(\mathrm{i}\lfloor Nt \rfloor \sum_{l=1}^m x_l/N)-1}{\mathrm{i}\sum_{l=1}^m x_l}$$

and

$$\begin{aligned} f^{N,2,\kappa}(t,x) &= 1_{(-Na,Na)}(x) \frac{\mathrm{i} \sum_{l=1}^m x_l/N}{1-\exp(-\mathrm{i}\sum_{l=1}^m x_l/N)} \\ &\quad \times \sum_{j=1}^{\lfloor Nt \rfloor} \frac{\exp(\mathrm{i}(j-1)\sum_{l=1}^m x_l/N)-1}{\mathrm{i}\sum_{l=1}^m x_l} (G^N_{j,\kappa}(x) - G^N_{j-1,\kappa}(x)). \end{aligned}$$

Because h and g_0 are continuously differentiable, we get, for almost every x,

$$\lim_{N\to\infty} f^{N,1,0}(t,x) = g_0(\tilde{h}(t))^m \frac{\exp(\mathrm{i}t\sum_{l=1}^m x_l)-1}{\mathrm{i}|x_1\cdots x_m|^{\tilde{h}(t)-1/2}\sum_{l=1}^m x_l}$$

and

$$\lim_{N\to\infty} f^{N,2,0}(t,x) = \int_0^t \frac{\exp(\mathrm{i}\theta\sum_{l=1}^m x_l)-1}{\mathrm{i}\sum_{l=1}^m x_l} \tilde{h}'(\theta) \frac{\partial}{\partial H}\left(\frac{g_0(H)^m}{|x_1\cdots x_m|^{H-1/2}} \right)\bigg|_{H=\tilde{h}(\theta)} \mathrm{d}\theta,$$

so that

$$\lim_{N\to\infty} (f^{N,1,0}(t,x) - f^{N,2,0}(t,x)) = f^{\infty}(t,x).$$

We now deal with $f^{N,1,1}$ and $f^{N,2,1}$. We write $G^N_{j,1}(x)$ as

$$G^N_{j,1}(x) = |x_1\cdots x_m|^{1/2-h^N_j} \sum_{k=1}^m g_1\left(h^N_j, \frac{x_k}{N}\right) g_0(h^N_j)^{k-1} \prod_{l=k+1}^m g\left(h^N_j, \frac{x_l}{N}\right).$$

Then, by Lemma 4.1 and because g_0 and g are bounded, there exist a constant $M_3 > 0$ and a function $\tilde{f}_{t,3} \in L^2(\mathbb{R}^m, \mathbb{R})$ such that, for almost every x,

$$|f^{N,1,1}(t,x)| \le M_3 \tilde{f}_{t,3}(x) \sum_{k=1}^m \sup_{H\in[\min\tilde{h},\max\tilde{h}]} \left| g_1\left(H, \frac{x_k}{N}\right)\right|,$$

so that $\lim_{N\to\infty} f^{N,1,1}(t,x) = 0$. Similarly, by Lemma 4.1, there exist a constant $M_4 > 0$ and a function $\tilde{f}_{t,4} \in L^2(\mathbb{R}^m, \mathbb{R})$ such that, for almost every x,

$$|f^{N,2,1}(t,x)| \le M_4 \tilde{f}_{t,4}(x) \sum_{k=1}^m \sup_{H\in[\min\tilde{h},\max\tilde{h}]} \left(\left| g_1\left(H, \frac{x_k}{N}\right)\right| + \left| \frac{\partial g_1}{\partial H}\left(H, \frac{x_k}{N}\right)\right| \right).$$

Therefore, $\lim_{N\to\infty} f^{N,2,1}(t,x) = 0$ and

$$\lim_{N\to\infty} (f^{N,1,1}(t,x) - f^{N,2,1}(t,x)) = 0,$$

which concludes the proof.

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The following lemma states that the convergence of $\{S^N_{\phi,h}\}_{N\in\mathbb{N}}$ can be reduced to the convergence of $\{S^N_{P_m,h}\}_{N\in\mathbb{N}}$ and, as a consequence of Lemma 4.2, to the convergence of $\{\tilde{S}^N_{m,h}\}_{N\in\mathbb{N}}$.

Lemma 4.5. *For every $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{E}\bigg[\bigg(S^N_{\phi,h}(t) - \frac{\langle \phi, P_m\rangle}{m!} S^N_{P_m,h}(t)\bigg)^2\bigg] = 0.$$

Proof. By (2.1) we have

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(S^N_{\phi,h}(t) - \frac{\langle \phi, P_m\rangle}{m!} S^N_{P_m,h}(t)\bigg)^2\bigg] \\
&\quad = \mathbb{E}\bigg[\bigg(\sum_{j=1}^{\lfloor Nt\rfloor} \frac{1}{N^{h(j/N)}} \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n\rangle}{n!} P_n(X_j(h^N_j))\bigg)^2\bigg] \\
&\quad = \sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor} \frac{1}{N^{h(j/N)+h(k/N)}} \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n\rangle^2}{(n!)^2} \mathbb{E}[P_n(X_j(h^N_j))P_n(X_k(h^N_k))].
\end{aligned}$$

By (2.2) and (3.2), we obtain, for every $n \geq m+1$,

$$\mathbb{E}[P_n(X_j(h^N_j))P_n(X_k(h^N_k))] = n!\,\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^n \leq n!\,|\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]|^{m+1},$$

so that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(S^N_{\phi,h}(t) - \frac{\langle \phi, P_m\rangle}{m!} S^N_{P_m,h}(t)\bigg)^2\bigg] \\
&\quad \leq \bigg(\sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n\rangle^2}{n!}\bigg) \sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor} \frac{|\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]|^{m+1}}{N^{h(j/N)+h(k/N)}}.
\end{aligned}$$

Let $\eta > 0$. By (3.1) and (3.3), there exists $N_\eta \in \mathbb{N}^*$ such that, for $|j-k| > N_\eta$ and $N \in \mathbb{N}^*$, $|\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]| \leq \eta$. Therefore,

$$\begin{aligned}
&\sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor} \frac{|\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]|^{m+1}}{N^{h(j/N)+h(k/N)}} \\
&\quad \leq \sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor} \frac{1_{\{|j-k|\leq N_\eta\}}}{N^{h(j/N)+h(k/N)}} + \eta \sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor} 1_{\{|j-k|\geq 1\}} \frac{|\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]|^{m}}{N^{h(j/N)+h(k/N)}}.
\end{aligned}$$

There exists $C_1(\eta) > 0$ such that

$$\sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor} \frac{1_{\{|j-k|\leq N_\eta\}}}{N^{h(j/N)+h(k/N)}} \leq \frac{C_1(\eta)}{N^{2\min h - 1}}.$$

Moreover, by (3.1) and (3.3), there exists a constant $C_2 > 0$, which is independent on η, such that, for all j, k, and N,

$$|\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]| \leq C_2 |j-k|^{h^N_j + h^N_k - 2}. \tag{4.4}$$

We then obtain

$$\sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor}\frac{|\mathbb{E}[X_j(h_j^N)X_k(h_k^N)]|^{m+1}}{N^{h(j/N)+h(k/N)}}$$
$$\leq \frac{C_1(\eta)}{N^{2\min h-1}}+\frac{\eta C_2}{N^2}\sum_{j=1}^{\lfloor Nt\rfloor}\sum_{k=1}^{\lfloor Nt\rfloor}1_{\{|j-k|\geq 1\}}\left|\frac{j-k}{N}\right|^{m(h_j^N+h_k^N-2)}.$$

Hence, for every $\eta>0$,

$$\limsup_{N\to\infty}\mathbb{E}\bigg[\bigg(S_{\phi,h}^N(t)-\frac{\langle\phi,P_m\rangle}{m!}S_{P_m,h}^N(t)\bigg)^2\bigg]$$
$$\leq \eta C_2\bigg(\sum_{n=m+1}^{\infty}\frac{\langle\phi,P_n\rangle^2}{n!}\bigg)\int_0^t\int_0^t|\theta-\sigma|^{h(\theta)+h(\sigma)-2}\,\mathrm{d}\theta\,\mathrm{d}\sigma.$$

The constants $\sum_{n=m+1}^{\infty}\langle\phi,P_n\rangle^2/n!$ and $\int_0^t\int_0^t|\theta-\sigma|^{h(\theta)+h(\sigma)-2}\,\mathrm{d}\theta\,\mathrm{d}\sigma$ are finite by (2.3) and because $\min h>\frac{1}{2}$, respectively. This concludes the proof.

We conclude this subsection with the following lemma.

Lemma 4.6. *As $N\to\infty$, the finite-dimensional distributions of $S_{\phi,h}^N$ converge to those of $S_{m,h}$, which can be defined by*

$$S_{m,h}(t):=\frac{\langle\phi,P_m\rangle}{m!}\int_{\mathbb{R}^m}f^{\infty}(t,x_1,\dots,x_m)\,\mathrm{d}\hat{B}_{x_1}\cdots\mathrm{d}\hat{B}_{x_m}$$

for every $t\geq 0$.

Proof. We fix $n\in\mathbb{N}$, $(t_1,\dots,t_n)\in[0,\infty)^n$, and a Lipschitz bounded function $\Psi\colon\mathbb{R}^n\to\mathbb{R}$. We define $\phi_m=\langle\phi,P_m\rangle/m!$. We have

$$|\mathbb{E}[\Psi(S_{\phi,h}^N(t_1),\dots,S_{\phi,h}^N(t_n))]-\mathbb{E}[\Psi(S_{m,h}(t_1),\dots,S_{m,h}(t_n))]|\leq E_1^N+E_2^N, \tag{4.5}$$

where

$$E_1^N=|\mathbb{E}[\Psi(S_{\phi,h}^N(t_1),\dots,S_{\phi,h}^N(t_n))-\Psi(\phi_m S_{P_m,h}^N(t_1),\dots,\phi_m S_{P_m,h}^N(t_n))]|$$

and

$$E_2^N=|\mathbb{E}[\Psi(\phi_m S_{P_m,h}^N(t_1),\dots,\phi_m S_{P_m,h}^N(t_n))]-\mathbb{E}[\Psi(S_{m,h}(t_1),\dots,S_{m,h}(t_n))]|.$$

Because Ψ is Lipschitz and by the Cauchy–Schwarz inequality, there exists $C_1>0$ such that, for every N,

$$E_1^N\leq C_1\sum_{j=1}^{n}\sqrt{\mathbb{E}[(S_{\phi,h}^N(t_j)-\phi_m S_{P_m,h}^N(t_j))^2]}.$$

Then, by Lemma 4.5,

$$\lim_{N\to\infty}E_1^N=0. \tag{4.6}$$

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By Lemma 4.2 we have

$$E_2^N = |\mathbb{E}[\Psi(\phi_m \tilde{S}_{m,h}^N(t_1), \dots, \phi_m \tilde{S}_{m,h}^N(t_n)) - \Psi(S_{m,h}(t_1), \dots, S_{m,h}(t_n))]|.$$

Thus, as in the E_1^N case, there exists $C_2 > 0$ such that, for every N,

$$E_2^N \leq C_2 \sum_{j=1}^{n} \sqrt{\mathbb{E}[(\phi_m \tilde{S}_{m,h}^N(t_j) - S_{m,h}(t_j))^2]}.$$

As a consequence, by Lemma 4.4,

$$\lim_{N\to\infty} E_2^N = 0. \tag{4.7}$$

Combining (4.5), (4.6), and (4.7) completes the proof.

4.3. Continuity, local self-similarity, and local Hölder exponent of the limit

In this subsection we first prove the convergence of the finite-dimensional distributions of $\{T_{m,h,t}^{\varepsilon}\}_{\varepsilon>0}$ for every t. Then, we prove the continuity of $S_{m,h}$ and the tightness of $\{T_{m,h,t}^{\varepsilon}\}_{\varepsilon>0}$ in $\mathcal{C}([0,\infty))$ to deduce the local self-similarity property. Finally, we deal with the local Hölder exponent of $S_{m,h}$.

Making the substitution $\theta \to \varepsilon\theta + t$, we obtain, for every $u \geq 0$,

$$T_{m,h,t}^{\varepsilon}(u) = \int_{\mathbb{R}^m} \exp\biggl(\mathrm{i}t \sum_{l=1}^{m} x_l\biggr) \psi_1(t,u,x,\varepsilon)\,\mathrm{d}\hat{B}_{x_1}\cdots \mathrm{d}\hat{B}_{x_m},$$

where

$$\psi_1(t,u,x,\varepsilon) = \varepsilon^{1-h(t)} \int_0^u \exp\biggl(\mathrm{i}\varepsilon\theta \sum_{l=1}^{m} x_l\biggr) \tilde{g}(\varepsilon\theta + t)|x_1\cdots x_m|^{1/2-\tilde{h}(\varepsilon\theta+t)}\,\mathrm{d}\theta.$$

By (2.6) and (2.7), we have

$$\{T_{m,h,t}^{\varepsilon}(u)\}_{u\geq 0} \overset{\mathrm{D}}{=} \biggl\{\int_{\mathbb{R}^m} \psi_2(t,u,x,\varepsilon)\,\mathrm{d}\hat{B}_{x_1}\cdots \mathrm{d}\hat{B}_{x_m}\biggr\}_{u\geq 0}$$

with

$$\psi_2(t,u,x,\varepsilon) = \int_0^u \varepsilon^{m(\tilde{h}(\varepsilon\theta+t)-\tilde{h}(t))} \exp\biggl(\mathrm{i}\theta \sum_{l=1}^{m} x_l\biggr) \tilde{g}(\varepsilon\theta + t)|x_1\cdots x_m|^{1/2-\tilde{h}(\varepsilon\theta+t)}\,\mathrm{d}\theta.$$

For all u and t and almost every x,

$$\lim_{\varepsilon\to 0} \psi_2(t,u,x,\varepsilon) = \tilde{g}(t) \frac{\exp(\mathrm{i}u \sum_{l=1}^{m} x_l) - 1}{\mathrm{i}(\sum_{l=1}^{m} x_l)|x_1\cdots x_m|^{\tilde{h}(t)-1/2}}.$$

By an integration by parts,

$$\begin{aligned}\psi_2(t,u,x,\varepsilon) &= \varepsilon^{m(\tilde{h}(\varepsilon u+t)-\tilde{h}(t))} \tilde{g}(\varepsilon u + t) \frac{\exp(\mathrm{i}u \sum_{l=1}^{m} x_l) - 1}{\mathrm{i}(\sum_{l=1}^{m} x_l)|x_1\cdots x_m|^{\tilde{h}(\varepsilon u+t)-1/2}} \\ &\quad + \varepsilon \int_0^u \varepsilon^{m(\tilde{h}(\varepsilon\theta+t)-\tilde{h}(t))} \frac{\exp(\mathrm{i}\theta \sum_{l=1}^{m} x_l) - 1}{\mathrm{i}(\sum_{l=1}^{m} x_l)|x_1\cdots x_m|^{\tilde{h}(\varepsilon\theta+t)-1/2}} \\ &\qquad \times \{\tilde{g}'(\varepsilon\theta+t) + \tilde{h}'(\varepsilon\theta+t)\tilde{g}(\varepsilon\theta+t)(\ln(\varepsilon^m) - \ln|x_1\cdots x_m|)\}\,\mathrm{d}\theta.\end{aligned}$$

As a consequence of the identity above and because of Lemma 4.1, there exists a function $\psi_{3,t,u} \in L^2(\mathbb{R}^m, \mathbb{R}_+)$ such that, for every $x \in \mathbb{R}^m$,

$$|\psi_2(t, u, x, \varepsilon)| \le \psi_{3,t,u}(x).$$

Because $\psi_{3,t,u}$ is square integrable and by Lemma 2.4, this proves the convergence of the finite-dimensional distributions of $T^{\varepsilon}_{m,h,t}$.

To prove the continuity of $S_{m,h}$, we use the Kolmogorov lemma. Let $T > 0$. As previously, by Lemma 4.1 again, there exists $C > 0$ such that, for all $0 \le s < t \le T$,

$$\begin{aligned}\mathbb{E}[(S_{m,h}(t) - S_{m,h}(s))^2] &= m!\,(t-s)^{2h(s)} \int_{\mathbb{R}^m} (\psi_2(s, 1, x, t-s))^2 \,\mathrm{d}x_1 \cdots \mathrm{d}x_m \\ &\le C(t-s)^{2h(s)}, \end{aligned} \tag{4.8}$$

which concludes the proof of the continuity of $S_{m,h}$.

Moreover, in a similar way, we prove that, for every compact set $U \subset [0, \infty)$, there exists a constant $C_U > 0$ such that, for all u and v in U such that $|u - v| < 1$, we have

$$\mathbb{E}\bigg[\bigg(\frac{S_{m,h}(t+\varepsilon u) - S_{m,h}(t+\varepsilon v)}{\varepsilon^{h(t)}}\bigg)^2\bigg] \le C_U |u-v|^{2h(t)}.$$

This proves the tightness of the family $\{T^{\varepsilon}_{m,h,t}\}_{\varepsilon>0}$ in $\mathcal{C}([0,\infty))$ by the Kolmogorov lemma [3], and then the local self-similarity property of $S_{m,h}$.

Finally, for each $t_0 \ge 0$, we deal with the local Hölder exponent of $S_{m,h}$ around t_0, which is denoted by $\alpha_{S_{m,h}}(t_0)$. By (4.8) and Lemma 2.5, for every $p \in \mathbb{N}^*$, there exists $C_p > 0$ such that, for all $s \le t$ in a neighborhood of t_0,

$$\mathbb{E}[(S_{m,h}(t) - S_{m,h}(s))^{2p}] \le C_p (t-s)^{2p \inf_{[s,t]} h}.$$

By the Kolmogorov lemma, this implies that $\alpha_{S_{m,h}}(t_0) \ge h(t_0)$. Using the local self-similarity of $S_{m,h}$ and proceeding as in [1], we prove that $\alpha_{S_{m,h}}(t_0) \le h(t_0)$, which gives $\alpha_{S_{m,h}}(t_0) = h(t_0)$ and concludes this subsection.

4.4. Tightness

Because of Theorem 15.6 of [3], it is enough to show that there exist $C > 0$ and $\gamma > 1$ such that, for all $t_1, t_2, t_3 \in [0, T]$ satisfying $t_1 < t_2 < t_3$ and $t_3 - t_1 < 1$, we have

$$\mathbb{E}[|S^N_{\phi,h}(t_3) - S^N_{\phi,h}(t_2)||S^N_{\phi,h}(t_2) - S^N_{\phi,h}(t_1)|] \le C(t_3 - t_1)^{\gamma}. \tag{4.9}$$

First, we assume that $t_3 - t_1 < 1/N$. Hence, $\lfloor Nt_1 \rfloor = \lfloor Nt_2 \rfloor$ or $\lfloor Nt_2 \rfloor = \lfloor Nt_3 \rfloor$, which gives

$$\mathbb{E}[|S^N_{\phi,h}(t_3) - S^N_{\phi,h}(t_2)||S^N_{\phi,h}(t_2) - S^N_{\phi,h}(t_1)|] = 0. \tag{4.10}$$

Then, we assume that $1/N \le t_3 - t_1 < 1$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\mathbb{E}[|S^N_{\phi,h}(t_3) - S^N_{\phi,h}(t_2)||S^N_{\phi,h}(t_2) - S^N_{\phi,h}(t_1)|] \\ &\quad \le \sqrt{\mathbb{E}[|S^N_{\phi,h}(t_3) - S^N_{\phi,h}(t_2)|^2]}\sqrt{\mathbb{E}[|S^N_{\phi,h}(t_2) - S^N_{\phi,h}(t_1)|^2]}. \end{aligned} \tag{4.11}$$

Consequently, it is enough to prove that there exist $C > 0$ and $\gamma > 1$ such that, for $(s,t) = (t_1, t_2)$ or (t_2, t_3),

$$\mathbb{E}[|S^N_{\phi,h}(t) - S^N_{\phi,h}(s)|^2] \le C(t_3 - t_1)^{\gamma}. \tag{4.12}$$

Proceeding as in the proof of Lemma 4.5, we obtain

$$\mathbb{E}[|S^N_{\phi,h}(t) - S^N_{\phi,h}(s)|^2] \leq \left(\sum_{n=m}^{\infty} \frac{\langle \phi, P_n \rangle^2}{n!}\right) \sum_{j=\lfloor Ns \rfloor+1}^{\lfloor Nt \rfloor} \sum_{k=\lfloor Ns \rfloor+1}^{\lfloor Nt \rfloor} \frac{\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^m}{N^{h(j/N)+h(k/N)}}. \tag{4.13}$$

Because of (4.4), there exists $C_1 > 0$ such that

$$\begin{aligned}
&\sum_{j=\lfloor Ns \rfloor+1}^{\lfloor Nt \rfloor} \sum_{k=\lfloor Ns \rfloor+1}^{\lfloor Nt \rfloor} \frac{\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^m}{N^{h(j/N)+h(k/N)}} \\
&\quad \leq \sum_{j=\lfloor Nt_1 \rfloor+1}^{\lfloor Nt_3 \rfloor} \frac{|\mathbb{E}[X_0(h^N_j)^2]|^m}{N^{h(j/N)+h(k/N)}} + C_1 \sum_{j,k=\lfloor Nt_1 \rfloor+1,\ j\neq k}^{\lfloor Nt_3 \rfloor} \frac{|j-k|^{m(h^N_j+h^N_k-2)}}{N^{h(j/N)+h(k/N)}} \\
&\quad \leq C_2 \frac{\lfloor Nt_3 \rfloor - \lfloor Nt_1 \rfloor}{N^{2\min h}} + \frac{C_1}{N^2} \sum_{j,k=\lfloor Nt_1 \rfloor+1,\ j\neq k}^{\lfloor Nt_3 \rfloor} \left|\frac{j-k}{N}\right|^{2\min h - 2},
\end{aligned} \tag{4.14}$$

where $C_2 = \max_{H\in[\min h, \max h]} |\mathbb{E}[X_0(H)^2]|^m$. We have

$$\frac{\lfloor Nt_3 \rfloor - \lfloor Nt_1 \rfloor}{N^{2\min h}} \leq \frac{t_3 - t_1}{N^{2\min h - 1}} + \frac{1}{N^{2\min h}} \leq 2(t_3 - t_1)^{2\min h} \tag{4.15}$$

because $1/N \leq t_3 - t_1$. Moreover,

$$\begin{aligned}
\frac{1}{N^2} \sum_{j,k=\lfloor Nt_1 \rfloor+1,\ j\neq k}^{\lfloor Nt_3 \rfloor} \left|\frac{j-k}{N}\right|^{2\min h - 2} &\leq C_3 \int_s^t \mathrm{d}\theta \int_{t_1}^{t_3} \mathrm{d}\sigma \, |\theta - \sigma|^{2\min h - 2} \\
&\leq \frac{C_3 (t_3 - t_1)^{2\min h}}{(2\min h - 1)\min h},
\end{aligned} \tag{4.16}$$

where $C_3 > 0$ does not depend on (N, t_1, t_3). Combining (4.16), (4.15), (4.14), and (4.13), we get (4.12). Because of (4.11) and (4.10), this concludes the proof of (4.9).

Appendix A. Technical lemma

This section is devoted to the proof of a lemma that deals with the asymptotic behavior of a covariance function.

Lemma A.1. *Let $X = \{X_n(H)\}_{(n,H)\in\mathbb{N}\times(b,1)}$ be a random field as defined by (3.1). For every compact set $K \subset (b, 1)$,*

$$\lim_{n\to\infty} \sup_{(H_1,H_2)\in K^2} |n^{2-H_1-H_2}\mathbb{E}[X_n(H_1)X_0(H_2)] - R(H_1, H_2)| = 0,$$

where, for every $(H_1, H_2) \in (b, 1)^2$,

$$R(H_1, H_2) = g_0(H_1)g_0(H_2) \int_{\mathbb{R}} \exp(\mathrm{i}x)|x|^{1-H_1-H_2} \, \mathrm{d}x.$$

Proof. Let K be a compact set in $(b, 1)$. We fix $(H_1, H_2) \in K^2$ and $n \in \mathbb{N}^*$. We have

$$\begin{aligned}\mathbb{E}[X_n(H_1)X_0(H_2)] &= \int_{-a}^{a} \exp(\mathrm{i}nx)g(H_1, x)\overline{g(H_2, x)}|x|^{1-H_1-H_2}\,\mathrm{d}x \\ &= n^{H_1+H_2-2}\int_{-na}^{na} \exp(\mathrm{i}nx)g\left(H_1, \frac{x}{n}\right)\overline{g\left(H_2, \frac{x}{n}\right)}|x|^{1-H_1-H_2}\,\mathrm{d}x.\end{aligned}$$

Setting

$$\psi_{H_1,H_2}(y) = g(H_1, y)\overline{g(H_2, y)} - g_0(H_1)g_0(H_2)$$

for every $y \in \mathbb{R}$, we prove hereafter that

$$\lim_{n\to\infty} \sup_{(H_1,H_2)\in K^2} \left| g_0(H_1)g_0(H_2)\int_{-na}^{na} \exp(\mathrm{i}x)|x|^{1-H_1-H_2}\,\mathrm{d}x - R(H_1, H_2)\right| = 0 \tag{A.1}$$

and

$$\lim_{n\to\infty} \sup_{(H_1,H_2)\in K^2} \left|\int_{-na}^{na} \exp(\mathrm{i}x)\psi_{H_1,H_2}\left(\frac{x}{n}\right)|x|^{1-H_1-H_2}\,\mathrm{d}x\right| = 0. \tag{A.2}$$

By an integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\left(\int_{-na}^{na} \exp(\mathrm{i}x)|x|^{1-H_1-H_2}\,\mathrm{d}x - \int_{-\infty}^{\infty} \exp(\mathrm{i}x)|x|^{1-H_1-H_2}\,\mathrm{d}x\right) \\ &\qquad = -\int_{na}^{\infty} \cos(x)x^{1-H_1-H_2}\,\mathrm{d}x \\ &\qquad = \sin(na)(na)^{1-H_1-H_2} + (1 - H_1 - H_2)\int_{na}^{\infty} \sin(x)x^{-H_1-H_2}\,\mathrm{d}x.\end{aligned}$$

Hence, we get

$$\begin{aligned}&\sup_{(H_1,H_2)\in K^2} \left| g_0(H_1)g_0(H_2)\int_{-na}^{na} \exp(\mathrm{i}x)|x|^{1-H_1-H_2}\,\mathrm{d}x - R(H_1, H_2)\right| \\ &\qquad \le 2 \sup_{(H_1,H_2)\in K^2} \left| g_0(H_1)g_0(H_2)\left(|na|^{1-H_1-H_2} + \int_{na}^{\infty} |x|^{-H_1-H_2}\,\mathrm{d}x\right)\right|,\end{aligned}$$

which implies (A.1) because $K \subset (\frac{1}{2}, 1)$.

Now we prove (A.2). Again by integration by parts, we obtain

$$\begin{aligned}&\int_0^{na} \exp(\mathrm{i}x)\psi_{H_1,H_2}\left(\frac{x}{n}\right)x^{1-H_1-H_2}\,\mathrm{d}x \\ &\quad = \mathrm{i}(1 - \mathrm{e}^{\mathrm{i}na})\psi_{H_1,H_2}(a)(na)^{1-H_1-H_2} - \frac{\mathrm{i}}{n}\int_0^{na} (1 - \mathrm{e}^{\mathrm{i}x})\psi'_{H_1,H_2}\left(\frac{x}{n}\right)x^{1-H_1-H_2}\,\mathrm{d}x \\ &\qquad + \mathrm{i}(H_1 + H_2 - 1)\int_0^{na} (1 - \mathrm{e}^{\mathrm{i}x})\psi_{H_1,H_2}\left(\frac{x}{n}\right)x^{-H_1-H_2}\,\mathrm{d}x.\end{aligned} \tag{A.3}$$

Obviously, we have

$$\begin{aligned}&\sup_{(H_1,H_2)\in K^2} |\mathrm{i}(1 - \mathrm{e}^{\mathrm{i}na})\psi_{H_1,H_2}(a)(na)^{1-H_1-H_2}| \\ &\qquad \le 2 \sup_{(H_1,H_2)\in K^2} |\psi_{H_1,H_2}(a)| \sup_{(H_1,H_2)\in K^2} |(na)^{1-H_1-H_2}| \\ &\qquad \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{A.4}$$

By a change of variable we get

$$\begin{aligned}
&\sup_{(H_1,H_2)\in K^2}\left|\frac{\mathrm{i}}{n}\int_0^{na}(1-\mathrm{e}^{\mathrm{i}x})\psi'_{H_1,H_2}\left(\frac{x}{n}\right)x^{1-H_1-H_2}\,\mathrm{d}x\right| \\
&\quad\le \sup_{(H_1,H_2)\in K^2}\left|\int_0^a \psi'_{H_1,H_2}(x)x^{1-H_1-H_2}\,\mathrm{d}x\right| \sup_{(H_1,H_2)\in K^2}|n^{1-H_1-H_2}| \\
&\quad\to 0 \quad \text{as } n\to\infty. \qquad\qquad (A.5)
\end{aligned}$$

It remains to prove that

$$\lim_{n\to\infty}\sup_{(H_1,H_2)\in K^2}\left|(H_1+H_2-1)\int_0^{na}(1-\mathrm{e}^{\mathrm{i}x})\psi_{H_1,H_2}\left(\frac{x}{n}\right)x^{-H_1-H_2}\,\mathrm{d}x\right| = 0. \qquad (A.6)$$

By the assumptions on g_1, for almost every $x\in\mathbb{R}$,

$$\lim_{n\to\infty} 1_{(0,na)}(x)(1-\mathrm{e}^{\mathrm{i}x})\sup_{(H_1,H_2)\in K^2}\left|(H_1+H_2-1)\psi_{H_1,H_2}\left(\frac{x}{n}\right)x^{-H_1-H_2}\right| = 0.$$

Moreover,

$$\begin{aligned}
&1_{(0,na)}(x)(1-\mathrm{e}^{\mathrm{i}x})\sup_{(H_1,H_2)\in K^2}\left|(H_1+H_2-1)\psi_{H_1,H_2}\left(\frac{x}{n}\right)x^{-H_1-H_2}\right| \\
&\quad\le (1-\mathrm{e}^{\mathrm{i}x})\sup_{(H_1,H_2)\in K^2}|(H_1+H_2-1)x^{-H_1-H_2}| \sup_{(H_1,H_2,y)\in K^2\times(0,a)}|\psi_{H_1,H_2}(y)|.
\end{aligned}$$

The function $x\mapsto(1-\mathrm{e}^{\mathrm{i}x})\sup_{(H_1,H_2)\in K^2}|(H_1+H_2-1)x^{-H_1-H_2}|$ is integrable. Then, by the dominated convergence theorem we obtain (A.6). Combining (A.3), (A.4), (A.5), and (A.6), we obtain

$$\lim_{n\to\infty}\sup_{(H_1,H_2)\in K^2}\left|\int_0^{na}\mathrm{e}^{\mathrm{i}x}\psi_{H_1,H_2}\left(\frac{x}{n}\right)x^{1-H_1-H_2}\,\mathrm{d}x\right| = 0.$$

Similarly, we obtain

$$\lim_{n\to\infty}\sup_{(H_1,H_2)\in K^2}\left|\int_{-na}^{0}\mathrm{e}^{\mathrm{i}x}\psi_{H_1,H_2}\left(\frac{x}{n}\right)x^{1-H_1-H_2}\,\mathrm{d}x\right| = 0,$$

which proves (A.2).

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