Ergod. Th. & Dynam. Sys. (1998), **18**, 343–356
Printed in the United Kingdom

Spaces of cohomologies associated with linear functional equations

GENRICH BELITSKII†§ and NIKOLAI BYKOV‡¶

† *Department of Mathematics and Computer Sciences, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel*

‡ *Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel*

(*Received* 4 *November* 1995 *and accepted in revised form* 4 *January* 1997)

Abstract. Let $F : X \to X$ be a $C^k(X)$, $k = [0, \infty]$, map on a topological space (smooth manifold) X, $A : X \to \mathrm{End}(C^m)$ and let $\{U_\alpha\}$ be an F-invariant covering of X. We introduce spaces of cohomologies associated with $\{U_\alpha\}$ and an operator $T = I - R$, where $(R\phi)(x) = A(x)\phi(F(x))$ is a weighted substitution operator in $C^k(X)$. This yields a correspondence between $\mathrm{Im}\, T$ and $\mathrm{Im}\, T|U_\alpha$ and the description of $\mathrm{Im}\, T$ in cohomological terms. In particular, it is proven that for any structurally stable diffeomorphism on a circle and for large enough k, the operator T is semi-Fredholm, and a similar result holds for the substitution operators generated by simple multidimensional maps. On the other hand, we show that, in general, the closures of $\mathrm{Im}\, T$ and $\mathrm{Im}\, T|U_\alpha$ are independent.

1. *Introduction*

1.1. *Operator equation.* Let $F : X \to X$ be a $C^k(X)$, $k = [0, \infty]$, map on a locally compact, countable at infinity topological space (smooth manifold). Given $C^k(X)$ functions $A : X \to \mathrm{End}(C^m)$ and $\gamma : X \to \mathbb{C}^m$, the map F generates a linear functional equation in operator form:

$$(T_{F,A}\phi)(x) = ((I - R_{F,A})\phi)(x) = \phi(x) - A(x)\phi(F(x)) = \gamma(x). \qquad (1)$$

If $m = 1$ and $A(x) \equiv 1$, equation (1) is called the *homology equation*.

Operators R and T act in the space $C^k(X)$ of continuous (smooth) functions on X, endowed with a standard topology of convergence with all the derivatives up to order k on each compact subset of X, and are, obviously, continuous.

We note that equation (1) is typical in normal forms theory; for instance, it naturally appears in the problems of conjugacy of local diffeomorphisms [**1**], just as the homology equation appears in the corresponding problem for transitive diffeomorphisms of the circle.

§ Research partially supported by the Guastello foundation.
¶ Research partially supported by the Foundation 'Promathematica'.

1.2. *Algebraic obstacles to solvability.* Due to the well-known equality $\overline{\operatorname{Im} T} = (\operatorname{Ker} T^*)_\perp$, involving the conjugate operator T^* in the dual space $(C^k(X))^*$, we may describe algebraic obstacles to the solvability of equation (1) as functionals $h \in \operatorname{Ker} T^*$. It is also clear from this, that the sufficiency of 'algebraic' solvability (by this we mean the inclusion $\gamma(x) \in (\operatorname{Ker} T^*)_\perp$) in continuous (smooth) functions is equivalent to the closure of the subspace $\operatorname{Im} T$. On the other hand, the closure of $\operatorname{Im} T$ may be interpreted as the *normal solvability* of our equation, i.e. the case when given a convergent sequence of functions $\gamma_n(x)$ on the right-hand side of (1), we can guarantee that their limit $\gamma(x)$ belongs to $\operatorname{Im} T$ and that the corresponding solutions converge.

1.3. *Spectral data.* If X is a compact manifold and $k < \infty$, then the operator R acts in the Banach space $C^k(X)$, its spectrum having been studied in [**5**] (the Kamowitz–Sheinberg theorem for scalar substitution operators in $C^0(X)$) and [**7**] (the general scalar case). It was shown that Σ_R^k, the residual spectrum of R, is an annulus, so the operator $T_z = I - (1/z)R$ cannot be semi-Fredholm if z lies on the boundary circles (non-isolated boundary point of spectrum [**6**]).

Thus, we may reformulate our goal as the study, for a class of maps F, of the fine structure of Σ_R^k at its interior points.

1.4. *Invariant covering.* Suppose map F has a number of invariant domains U_α forming a covering of X. Though closure of the image is, in general, not a hereditary property, it often turns out much easier to investigate images of restrictions $T|U_\alpha$, so we can pose the natural question of how the properties of $\operatorname{Im} T$ and the $\operatorname{Im} T|U_\alpha$ are related. In particular, we are interested in finding the conditions which can provide the closure of $\operatorname{Im} T$, provided the $\operatorname{Im} T|U_\alpha$ are closed.

Before stating the general scheme in §3, in §2 we study in detail the leading example of a structurally stable C^k-diffeomorphism of the circle S^1, which, obviously, allows the covering by invariant arcs. In the case $k = \infty$, $A \in C^\infty(S^1)$, the operator T acts in $C^k(X)$ for any k and the following theorem holds.

THEOREM 1.1. *Let F be a structurally stable diffeomorphism of the circle and let the matrix $A(x)$ be non-degenerate for all $x \in S^1$. Then there exists $k_0 = k_0(F, A)$ such that for $k \geq k_0$ the operator T is semi-Fredholm with closed* $\operatorname{Im} T$ *and* $\dim \operatorname{Ker} T < \infty$.

The explicit value of k_0, and the description of $\operatorname{Im} T$, as well as comments on $\operatorname{Im} T$ for $k < k_0$, are presented in this case. Actually, we sketch the proof of the following.

THEOREM 1.2. *For any structurally stable diffeomorphism of the circle and non-degenerate matrix A, the operator R is semi-Fredholm in all interior points of Σ_R^k, with the possible exception of a finite number of resonant circles.*

In §3 we introduce spaces of cohomologies $H^p(T, \mathcal{U})$ associated with the operator T and covering $\{U_\alpha\}$ and the homomorphism θ, which is a chain of an exact sequence, such that the following holds.

THEOREM 1.3. $\operatorname{Im} T = \operatorname{Ker} \theta$ *and* $\operatorname{Im} T$ *is closed iff θ is continuous.*

Note that the spaces $H^p(T, U)$ and the homomorphism θ for the homology equation were built in [**1**]. The above theorem provides us with some tools for the investigation of $\operatorname{Im} T$ as corollaries.

In §4 we consider a collection of multidimensional examples illustrating the interaction of F and $F|U_\alpha$. These examples show that, in general, closure of $\operatorname{Im} T$ does not imply closure of $F|U_\alpha$, nor does closure of $F|U_\alpha$ imply closure of $\operatorname{Im} T$.

Nevertheless, another result is as follows.

THEOREM 1.4. *Let $k = \infty$ and let F be a gradient-like diffeomorphism of a compact manifold. Then the homology equation is normally solvable.*

2. $\operatorname{Im} T$ *for a structurally stable diffeomorphism on* S^1

2.1. Let us recall that any structurally stable diffeomorphism H of the circle S^1 has an even number of non-degenerate periodic points $x_1, \ldots, x_{2n} (n > 0)$ of the same period N, which are in turn attractors and repellers. Without loss of generality we may consider orientation-preserving maps so that all periodic points of the diffeomorphism $F = H^N$ are fixed and the corresponding derivatives at the fixed points are $\lambda_1, \ldots, \lambda_{2n}$, $0 < \lambda_i < 1 < \lambda_{i+1}$. Let $Q(x) = A(x)A(Hx)\cdots A(H^{N-1}x)$, and let $\{q_1^i, \ldots, q_m^i, q_l^i \neq 0\}$ be different eigenvalues of the matrices $Q(x_i)$. Set

$$k_0 = \max_{l,i}\{-\ln q_l^i / \ln \lambda_i\}.$$

THEOREM 2.1. *Let F be a structurally stable C^k-diffeomorphism and let $A(x)$ be a non-degenerate C^k-operator function, $k \geq k_0$. Then the subspace* $\operatorname{Im} T$ *is closed in* $C^k(S^1)$ *and* $\dim \operatorname{Ker} T < \infty$. *If $F, A \in C^\infty(S^1)$, then* $\operatorname{Im} T$ *is closed in* $C^\infty(S^1)$.

If $k < k_0$, then it may happen that $\operatorname{Im} T$ is not closed. For example, consider the homology equation for $k = 0$. It was proved in [**1**] that this equation on a compact space is normally solvable iff F is a periodic map. Since a structurally stable diffeomorphism is not periodic, $\operatorname{Im} T$ is not closed.

The proof of Theorem 2.1 will follow from lemmas we prove below. Our first step is to pass from H to $F = H^N$.

LEMMA 2.1. *Let R be a continuous linear operator in a linear topological space E and let the subspace* $\operatorname{Im}(I - R^N)$ *be closed. Then* $\operatorname{Im}(I - R)$ *is closed as well.*

Proof. Evidently,

$$\operatorname{Im}(I - R^N) = \left\{ \sum_{i=0}^{N-1} R^i e \mid e \in \operatorname{Im}(I - R) \right\} \subset \operatorname{Im}(I - R).$$

Now let $e_m \in \operatorname{Im}(I - R)$; $e_m \to e$. Then $\sum_{i=0}^{N-1} R^i e_m \in \operatorname{Im}(I - R^N)$. Since $\operatorname{Im}(I - R^N)$ is by assumption closed and R is continuous, $\sum_{i=0}^{N-1} R^i e_m \to w$, where $w \in \operatorname{Im}(I - R^N)$. Thus, there exists an element $v \in \operatorname{Im}(I - R)$ such that $w = \sum_{i=0}^{N-1} R^i v$. Defining $u = e - v$, we obtain

$$\sum_{i=0}^{N-1} R^i (e_m - v) \to 0, \quad \text{i.e. } \sum_{i=0}^{N-1} R^i u = 0. \tag{2}$$

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Finally, let $y = -1/N \sum_{i=1}^{N-1} iR^{N-i-1}u$. Immediate calculations show that (2) implies $(I - R)y = u$, or $u \in \mathrm{Im}(I - R)$, and $e = u - v \in \mathrm{Im}(I - R)$. □

Note, that it follows from Lemma 2.1 that for equation (1) with periodic map H the subspace $\mathrm{Im}\, T$ is always closed. Actually, $H^N = I$, and the space

$$\mathrm{Im}(I - R^N) = \{\gamma(x) \in C^k(X), \gamma(x) \in \mathrm{Im}(I - R_{I,Q}), x \in X\}$$

is closed.

In this way (and since $\mathrm{Ker}(I - R) \subset \mathrm{Ker}(I - R^N)$), we only need to prove the theorem for the operator

$$T\phi = (I - R^N_{H,A})\phi = (I - R_{F,Q})\phi. \tag{3}$$

2.2. Open arcs $U_i = (x_{i-1}, x_{i+1}), i = 0, \dots, 2n$, form an open F-invariant covering of S^1, i.e. $\cup U_i = S^1$, $F(U_i) = U_i$. Denote by $T|U_i$ a restriction of the operator (3) on the arc U_i. In the presence of *resonances* $\lambda_i^j \cdot q_l^i = 1$ for some j, l, we arrive at the finite number of conditions necessary for $\gamma \in C^k(S^1)$ to belong to $\mathrm{Im}\, T|U_i$. Namely, if $\gamma(x) = (\gamma_1(x), \dots, \gamma_m(x))$, then for each resonant triple i, j, l, where $j \leq k$, we should have *formal resonance restrictions:*

$$\left\{ \begin{array}{ll} \gamma_l^{(j)}(x_i) = 0, & \text{for simple eigenvalues } q_l^i, \\ \gamma_{l+d(l)}^{(j)}(x_i) = 0, & \text{for jordan blocks of order } d(l). \end{array} \right\}$$

LEMMA 2.2. *If $k \geq k_0$, and $\gamma \in C^k(S^1)$ satisfies formal resonance restrictions, then γ belongs to* $\mathrm{Im}\, T|U_i$ *for every $i = 1, \dots, 2n$. In other words* $\mathrm{Im}(T|U_i) = (\mathrm{Ker}(T|U_i)^*)_\perp$, *and, therefore,* $\mathrm{Im}(T|U_i)$ *is closed for each arc.*

Proof. Given γ satisfying these formal restrictions, we may find $\hat{\phi} \in C^k(U_i)$ such that

$$((T\hat{\phi}) - \gamma)^{(j)}(x_i) = 0, \quad j \leq k. \tag{4}$$

Then the substitution $\phi(x) \to \phi(x) + \hat{\phi}(x)$ leads to (1) with the new function $\hat{\gamma} \equiv T\hat{\phi} - \gamma$.

First, suppose $\lambda_i < 1$. Set

$$\phi(x) = \sum_{j=0}^{\infty} Q(x)Q(Fx)\cdots Q(F^{j-1}x)\hat{\gamma}(F^j x). \tag{5}$$

If $k \geq k_0$ then $|q_l^i \cdot \lambda_i^k| < 1, l = 1, \dots, m$, and we conclude from (4) that series (5) converges uniformly in every compact set in U_i with all its derivatives up to the kth order. Obviously, $\phi(x)$ satisfies (3). If $\lambda_i > 1$, making use of the non-degeneracy of matrix $Q(x)$, we consider the series

$$\phi(x) = \sum_{j=1}^{\infty} Q^{-1}(x)Q^{-1}(F^{-1}x)\cdots Q^{-1}(F^{-j}x)\hat{\gamma}(F^{-j}x). \tag{6}$$

Like the previous series, it yields a C^k-solution if $k \geq k_0$, $|q_l^i \cdot \lambda_i^k| < 1$ for all $l = 1, \dots, m$. □

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2.3. Assume now that $\gamma \in \operatorname{Im} T|U_i$ for every $i = 1, \ldots, 2n$, i.e. $\gamma|U_i = T\phi_i$ for some $\phi_i \in C^k(U_i)$. Let us define

$$[\gamma](x) \equiv \{\phi_{i+1}(x) - \phi_i(x)\}, \quad \text{for } x \in U_{i+1} \cap U_i,\ i = 1, \ldots, 2n.$$

This set of functions depends on the choice of solutions $\phi_i(x)$, but in any case its entries are elements of $\operatorname{Ker} T|U_{i+1} \cap U_i$.

LEMMA 2.3. *The function γ belonging to $\operatorname{Im} T|U_i$ for every $i = 1, \ldots, 2n$, lies in $\operatorname{Im} T$ iff there exist functions $c_i \in \operatorname{Ker} T|U_i$ such that*

$$[\gamma](x) = \{c_{i+1}(x) - c_i(x)\}, \quad i = 1, \ldots, 2n. \tag{7}$$

Proof. Let $\gamma \in \operatorname{Im} S^1$, $\gamma = T\phi$. Then for every $i = 1, \ldots, 2n$ we have $\gamma|U_i \in \operatorname{Im}(T|U_i)$, i.e. $\gamma|U_i = T\phi_i$. So $c_i(x) = \phi(x) - \phi_i(x) \in \operatorname{Ker}(T|U_i)$, and

$$\phi(x) = c_i(x) + \phi_i(x) = c_{i+1} + \phi_{i+1}(x), \quad x \in U_{i+1} \cap U_i.$$

Thus we have established that $[\gamma](x) = \{c_{i+1}(x) - c_i(x)\}$, this property being independent of the choice of solutions $\phi_i(x)$.

On the other hand, if (7) holds, set $\tilde{\phi}_i = \phi_i + c_i$. Then, since $\tilde{\phi}_i(x) = \tilde{\phi}_{i+1}(x)$ for $x \in U_{i+1} \cap U_i$, the function

$$\phi(x) = \{\tilde{\phi}_i(x), x \in U_i\}$$

is C^k-smooth on S^1 and $\gamma = T\phi$. This proves Lemma 2.3. □

2.4. Let us now describe $\operatorname{Ker}(T|U_i)$ for $k \geq k_0$.

LEMMA 2.4. *Provided $k \geq k_0$, the space $\operatorname{Ker} T|U_i$ is finite-dimensional for each arc.*

Proof. Here we may think of U_i as $]-\infty, \infty[$, $x_i = 0$. Let j_1 be an *order of the first resonance*. This means that $j_1 = \min\{j : \exists l, \lambda_i^j \cdot q_l^i = 1\}$. We set $j_1 = 0$ in the absence of resonances. Considering a C^k function p satisfying the equation $p(x) = Q(x)p(Fx)$, $x \in U_i$, we note that $p^{(s)}(0) = 0, s < j_1$. Setting $p^{(j_1)}(0) = c_1 \in \mathbb{C}^m$, one can uniquely compute $p^{(s)}(0), s < j_2$, up to the next resonance order etc. Thus, the finite number (not more than the number of resonances) of derivatives $p^{(j)}(0) = c_j$ for resonance values $j \leq k$ uniquely defines the Taylor series of p at the origin.

To prove our assertion we only need to show that for any such vectors c_j there exists at most one function $p \in \operatorname{Ker}(T|U_i)$ with the prescribed Taylor series. Indeed, if we are given two such functions, their difference r is an element of $\operatorname{Ker}(T|U_i)$ with zero Taylor series. Then for any integer d one has

$$r(x) = Q(x)Q(Fx)\cdots Q(F^{d-1}x)r(F^d x).$$

Taking into account that $r(F^d x)/(F^d x)^k \to 0$ as $d \to \infty$, we see that $Q(x)Q(Fx)\cdots Q(F^{d-1}x)\cdot(F^d x)^k \to 0$, if $k \geq k_0$. Hence $r(x) \equiv 0$. □

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Proof of Theorem 2.1. Suppose $k \geq k_0$, and we are given a convergent sequence $\gamma_s \to \gamma$, $\gamma_s \in \operatorname{Im} T$. Then the restrictions $\gamma_s | U_i$ belong to $\operatorname{Im} T | U_i$, $i = 1, \ldots, 2n$. According to Lemma 2.1, there exist such functions $\phi_i^s \in C^k(U_i)$, $i = 1, \ldots, 2n$, such that $\gamma_s | U_i = T\phi_i^s$, $\phi_i^s \to \phi_i$, $\gamma | U_i = T\phi_i$.

Furthermore, according to Lemma 2.3,

$$[\gamma_s](x) = \{\phi_{i+1}^s(x) - \phi_i^s(x)\} = \{c_{i+1}^s(x) - c_i^s(x)\} \to \{\phi_{i+1}(x) - \phi_i(x)\}$$

for some $c_i^s \in \operatorname{Ker}(T|U_i)$. Thus, the sequences $\{c_{i+1}^s(x) - c_i^s(x)\}$ C^k-converge in $U_{i+1} \cap U_i$, $i = 1, \ldots, 2n$.

Now, since according to Lemma 2.4 the subspace $\operatorname{Ker}(T|U_{i+1}) \oplus \operatorname{Ker}(T|U_i) \subset \operatorname{Ker}(T|U_{i+1} \bigcap U_i)$ is finite-dimensional, we deduce that

$$[\gamma](x) = \phi_{i+1}(x) - \phi_i(x) = c_{i+1}(x) - c_i(x)$$

for some $c_i \in \operatorname{Ker}(T|U_i)$. Finally, applying the 'opposite arrow' of Lemma 2.3, we obtain the inclusion $\gamma \in \operatorname{Im} T|S^1$. □

2.5. Do global obstructions to solvability really exist?

PROPOSITION 1. (Realization of the obstruction cocycle.) *For each set of C^k-functions $\{t_i(x) \in \operatorname{Ker} T|U_{i+1} \cap U_i\}_{i=1}^{2n}$ one may choose a function $\gamma(x) \in C^k(S^1)$ such that*

$$[\gamma](x) = \{t_i(x)\}.$$

Proof. Choose C^k-functions $\phi_i(x)$ on U_i such that $\phi_{i+1}(x) - \phi_i(x) = t_i(x)$ for $x \in U_{i+1} \cap U_i$. Let

$$\gamma(x) \equiv (T\phi_i)(x), \quad x \in U_i, i = 1, \ldots, 2n.$$

Then $\gamma(x) \in C^k(S^1)$, and $[\gamma](x) = \{t_i(x)\}$. □

Thus, although the subspace $\operatorname{Im} T$ in this case is always infinite-dimensional, restriction (7) is very severe; in fact, $\dim \operatorname{Ker}(T|U_{i+1} \cap U_i) = \infty$, but $\dim \operatorname{Ker} T|U_i \leq m$.

2.6. Now we will give Theorem 2.1 a slightly different form, considering the homology equation for the map with fixed points. For the latter, the only resonances are those of zero order, $k_0 = 1$, and $\operatorname{Ker} T|U_i = \{c_i = \text{constant}\}$. Further, the condition $\gamma(x) \in \operatorname{Im} T$, for $k \geq 1$ is, as we saw when proving Lemma 2.3, equivalent to the conditions

$$\gamma(x_i) = 0, \quad [\gamma](x) = \{c_{i+1} - c_i\}. \tag{8}$$

Fix a number of arbitrary points $\mathcal{X} = \{z_i \in U_{i+1} \cap U_i\}_{i=1}^{2n}$. Taking into account (5) and (6) we obtain that

$$c_{i+1} - c_i = \phi_{i+1}(x) - \phi_i(x) = \sum_{-\infty}^{\infty} \gamma(F^j(z_i)).$$

Let $\lambda_1 < 1$. Define $h_{\mathcal{X}} \in \operatorname{Ker} T^*$ by

$$\begin{aligned} (h_{\mathcal{X}}, \gamma) \equiv & \sum_{i=1}^{n} \left\{ \sum_{j=0}^{\infty} [\gamma(F^j(z_{2i-1})) - \gamma(x_{2i-1})] + \sum_{j=1}^{\infty} [\gamma(F^{-j}(z_{2i-1})) - \gamma(x_{2i})] \right\} \\ & + \sum_{i=1}^{n} \left\{ \sum_{j=0}^{\infty} [\gamma(F^j(z_{2i})) - \gamma(x_{2i+1})] + \sum_{j=1}^{\infty} [\gamma(F^{-j}(z_{2i})) - \gamma(x_{2i})] \right\}. \end{aligned}$$

Thus, writing down conditions (8), we may reformulate Theorem 2.1 in greater detail as follows (see the next section for the $k = 0$ case).

COROLLARY 2.1. *Assume F is a structurally stable map on the circle with* $\operatorname{Per} F = \operatorname{Fix} F$ *and $k \geq 1$. Then the homology equation for the function $\gamma \in C^k(S^1)$ has a C^k-solution iff $\gamma(x_i) = 0$, $i = 1, \ldots, 2n$, and $(h_{\mathcal{X}}, \gamma) = 0$ for every set $\mathcal{X}$.*

Thus, $\operatorname{Im} T = (\operatorname{Ker}(T)^*)_\perp$, $\operatorname{Ker}(T)^* = \{\delta_{x_i}, h_{\mathcal{X}}\}$. A similar representation of $\operatorname{Im} T$ holds for an arbitrary non-singular matrix $A(x)$ and a structurally stable map $F : S^1 \to S^1$.

2.7. Concluding this section, we should comment on whether the assumptions of Theorem 2.1 are necessary.

(i) First consider equation (1) for $k = 0$. As was proved in [**1**], for the homology equation on a compact space, $\operatorname{Im} T$ is closed in $C^0(X)$ iff F is a periodic map. Since a structurally stable map is not periodic, $\operatorname{Im} T$ is not closed in $C^0(S^1)$.

(ii) On the other hand, $\operatorname{Im} T$ is closed for all *non-resonant values* of $k < k_0$. In order not to complicate the proof, we will briefly sketch the additional aspects to be dealt with. Lemma 2.2 still applies, although the solutions to (1) on the arcs are no longer given by convergent series, and one needs to reduce matrix $Q(x)$ to the block form (see [**3**]). In this case the space $\operatorname{Ker} T$ may be infinite-dimensional (then $\dim \operatorname{Ker} T^* < \infty$ and T is still semi-Fredholm), but the general scheme of the next section makes it possible to complete the proof. Note that in the case of more than two periodic points, resonant circles may lie inside the spectrum of R. This implies the following.

THEOREM 2.2. *For any structurally stable diffeomorphism of the circle and any non-degenerate matrix A, operator R is semi-Fredholm in all interior points of Σ_R^k, with the possible exception of a finite number of resonant circles.*

(iii) For a constant matrix A, the non-degeneracy condition is redundant (Lemma 2.2 still holds), but it is essential in the general case.

3. *Spaces of cocycles*

In this section we generalize the scheme developed above for the maps on the circle.

Let $\mathcal{U} = \{U_\alpha\}$ be an open countable F-invariant covering of $X : X = \cup U_\alpha$, $F(U_\alpha) = U_\alpha$. This yields the natural continuous injections

$$j_\alpha : C^k(X) \to C^k(U_\alpha); \quad i : \operatorname{Im} T \to \cap_\alpha j_\alpha^{-1}(\operatorname{Im}(T|U_\alpha)).$$

The latter inclusion is, as a rule, strong and as we will show, the *defect space*

$$\mathcal{L}(T; \mathcal{U}) = \cap_\alpha j_\alpha^{-1}(\operatorname{Im}(T|U_\alpha))/i(\operatorname{Im} T)$$

may be described in cohomological terms.

Consider the sheaf of C^k-function germs on X. The covering $\mathcal{U}$ generates a space of p-cochains $Ch^p(\mathcal{U})$ [**4**], which we endow with the topology of a direct product. Actually, $p = 0, 1, 2$ suffice for our purposes. As usual, $\delta : Ch^p(\mathcal{U}) \to Ch^{p+1}(\mathcal{U})$

is the coboundary operator, $B^p(\mathcal{U}) = \operatorname{Im}\delta$ and $Z^p(\mathcal{U}) = \operatorname{Ker}\delta$ are the corresponding spaces of coboundaries and cocycles.

Now we introduce linear spaces associated with the covering $\mathcal{U}$ and operator T:

$$Ch^p(T;\mathcal{U}) = \{c = (c_{\alpha_0\ldots\alpha_p}) \mid c_{\alpha_0\ldots\alpha_p} \in \operatorname{Ker}(T| \cap_{i=0}^{p} U_{\alpha_i})\},$$
$$Z^p(T;\mathcal{U}) = Z^p(\mathcal{U}) \cap Ch^p(T;\mathcal{U});\ B^p(T;\mathcal{U}) = \delta(Ch^{p-1}(T;\mathcal{U})).$$

In a standard way, the space of cohomologies generated by $\mathcal{U}$, T is

$$H^p(T;\mathcal{U}) = Z^p(T;\mathcal{U})/B^p(T;\mathcal{U}).$$

Let $\gamma \in \cap_\alpha j_\alpha^{-1}(\operatorname{Im} T|U_\alpha)$, i.e. $\gamma_\alpha = T\phi_\alpha$ for some $\phi_\alpha \in C^k(U_\alpha)$. Setting

$$[\gamma]_{\alpha\beta}(x) = \phi_\alpha(x) - \phi_\beta(x),$$

we obtain a cocycle $[\gamma] \in Z^1(T;\mathcal{U})$. Thus we have defined the homomorphism

$$\theta(\gamma) = \overline{[\gamma]}, \quad \theta : \cap_\alpha j_\alpha^{-1}(\operatorname{Im} T|U_\alpha) \to Z^1(T;\mathcal{U}).$$

We will see below, that continuity of θ is a characteristic feature of $\mathcal{U}$, T.

THEOREM 3.1. *The following sequence of homomorphisms*

$$0 \to \operatorname{Im} T \xrightarrow{i} \cap_\alpha j_\alpha^{-1}(\operatorname{Im} T|U_\alpha) \xrightarrow{\theta} H^1(T;\mathcal{U}) \to 0$$

is exact.

Thus we arrive at the following description of the 'defect' space and the image:

$$\mathcal{L}(T;\mathcal{U}) = H^1(T;\mathcal{U}); \quad i(\operatorname{Im} T) = \operatorname{Ker}\theta.$$

Proof. (i) $\operatorname{Im} i \subset \operatorname{Ker}\theta$. Namely, if $\gamma = T\phi$, we may set $\gamma|U_\alpha = T\phi_\alpha$ and $\phi_\alpha = \phi|U_\alpha$. Therefore, $[\gamma]_{\alpha\beta}(x) = \phi_\alpha(x) - \phi_\beta(x)$ for $x \in U_\alpha \cap U_\beta$, and $\theta(\gamma) = 0$.

(ii) $\operatorname{Ker}\theta \subset \operatorname{Im} i$. Actually, if $\gamma_\alpha = T\phi_\alpha$, and $\theta(\gamma) = 0$, then

$$[\gamma]_{\alpha\beta}(x) = c_\alpha(x) - c_\beta(x), \quad c_\alpha \in \operatorname{Ker}(T|U_\alpha).$$

Defining $\phi(x) = \phi_\alpha(x) - \phi_\beta(x)$ $(x \in U_\alpha)$, we obtain a C^k-function on X such that $\gamma = T\phi$.

(iii) $\operatorname{Im}\theta = H^1(T;\mathcal{U})$. For every $c = \{c_{\alpha\beta}\} \in Z^1(T;\mathcal{U})$ one has

$$c_{\alpha\beta} \in (\operatorname{Ker}(T|U_\alpha \cap U_\beta)), \quad \delta(c) = 0.$$

Since our sheaf is fine **[2]**, there exist functions $\phi_\alpha \in C^k(U_\alpha)$ such that $c_{\alpha\beta} = \phi_\alpha(x) - \phi_\beta(x)$. The function $\gamma(x) = (T\phi)(x)$ $(x \in U_\alpha)$ lies in the intersection $\cap_\alpha j_\alpha^{-1}(\operatorname{Im}(T|U_\alpha))$. Obviously, $\theta(\gamma) = c$. □

Let us draw consequences from the above theorem.

COROLLARY 3.1. *Assume that all the subspaces* $\operatorname{Im}(T|U_\alpha)$ *are closed. Then the following conditions are equivalent:*

(a) $\operatorname{Im} T$ *is closed;*

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(b) *$B^1(T;\mathcal{U})$ is closed;*
(c) *$H^1(T;\mathcal{U})$ is Hausdorff;*
(d) *θ is a continuous map.*

Proof. (b)⇔(c). This is a well-known fact from the theory of linear topological spaces.

(a)⇒(b). Let $\{c^n\}$ be a sequence in $B^1(T;\mathcal{U})$ converging to an element $c \in Z^1(T;\mathcal{U})$. We need to show that $c \in B^1(T;\mathcal{U})$. Choose γ so that $[\gamma] = c$ and again set

$$\gamma|U_\alpha = T\phi_\alpha; \quad c_{\alpha\beta} = \phi_\alpha(x) - \phi_\beta(x).$$

Since $Z^1(\mathcal{U}) = B^1(\mathcal{U}) = \delta(Ch^0(\mathcal{U}))$ (the sheaf is fine), $\delta : Ch^0(\mathcal{U}) \to Z^1(\mathcal{U})$ is a normally solvable operator. Thus one can find sequences $\{\phi^n_\lambda\}^\infty_{n=1}$, $\phi^n_\lambda \in C^k(U_\alpha)$, such that $\phi^n_\lambda \to \phi_\lambda$ as $n \to \infty$ and $c^n_{\alpha\beta} = \phi^n_\alpha - \phi^n_\beta$. Defining $\gamma^n|U_\alpha = T\phi^n_\alpha$, we obtain that $\gamma^n \in \operatorname{Im} T$ and $\gamma^n \to \gamma$. Closure of $\operatorname{Im} T$ implies that $\gamma \in \operatorname{Im} T$, and hence $c \in B^1(T;\mathcal{U})$.

(b)⇒(d). Given a sequence $\gamma^n \in \cap_\alpha j^{-1}_\alpha(\operatorname{Im}(T|U_\alpha))$ converging to zero, we will prove that $\theta(\gamma^n) \to 0$. Since all the $\operatorname{Im}(T|U_\alpha)$ are closed, choose $\phi^n_\lambda \in C^k(U_\alpha)$ satisfying the conditions $\gamma^n = T\phi^n_\alpha$, $\phi^n_\lambda \to 0$ $(n \to \infty)$. Then, since $B^1(T;\mathcal{U})$ is closed,

$$[\gamma^n]_{\alpha\beta} = \phi^n_\alpha - \phi^n_\beta \to 0; \quad \theta(\gamma^n) = \overline{[\gamma^n]} \to 0.$$

(d)⇒(a) follows from the equality $\operatorname{Im} T = \operatorname{Ker} \theta$. □

COROLLARY 3.2. *Assume that the subspaces* $\operatorname{Im}(T|U_\alpha)$ *are closed and* $\dim \operatorname{Ker} T|U_\alpha < \infty$. *Then* $\operatorname{Im} T$ *is closed.*

Proof. Indeed, in this case the space $(Ch^0(T;\mathcal{U}))$ is also finite-dimensional, so recalling the definition $B^1(T;\mathcal{U}) = \delta(Ch^0(T;\mathcal{U}))$ and the fact that δ is a continuous map, we see that item (b) of Corollary 3.1 is applicable.

4. *Examples and applications*

4.1. Corollary 3.1 provides us with a straightforward tool to study $\operatorname{Im} T$ for maps of Morse–Smale type [**8**] which have a finite number of hyperbolic periodic points, their basins of attraction covering the whole manifold X. Corollary 3.2 yields another result for a natural generalization of maps on S^1 from §2 as follows.

Example 4.1. Consider a C^k-diffeomorphism F on the sphere S^n, having only two fixed points: an attractor x_1 with the eigenvalues of its linear part $\{\lambda_1, \ldots, \lambda_n\}$, $0 < |\lambda_i| < 1$, and a repeller x_2 with the eigenvalues $\{\mu_1, \ldots, \mu_n\}$, $1 < |\mu_i|$. Let $\{q^i_1, \ldots, q^i_m, q^i_l \neq 0\}$ be the eigenvalues of non-degenerate matrices $A(x_i)$, $i = 1, 2$, and let k_0 be defined as in the case of S^1. Let us show that the subspace $\operatorname{Im} T$ is closed if $k \geq k_0$. Indeed, the covering $U_1 = S^n \backslash \{x_2\}$, $U_2 = S^n \backslash \{x_1\}$ is finite and F-invariant. Arguments identical to those for Lemma 2.4 show that spaces $\operatorname{Ker} T|U_1$ and $\operatorname{Ker} T|U_2$ are finite-dimensional. In order to apply Corollary 3.2, we only need to convince ourselves that for $k \geq k_0$ both spaces $\operatorname{Im} T|U_1$ and $\operatorname{Im} T|U_2$ are closed. But due to the choice of k_0, we may repeat the arguments from Lemma 2.2 showing that the formal series (5) and (6) yield smooth solutions. Therefore, $\operatorname{Im} T$ is closed.

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4.2. We further show that, in general, closure of $\operatorname{Im} T|U_\alpha$ does not imply closure of $\operatorname{Im} T$.

Example 4.2. Let the space X be a punctured quadrant on the plane:

$$X = \{(\xi, \eta) \in R^2 \backslash \{0\}; \xi, \eta \geq 0\};$$

and let the map F be a linear saddle: $F(\xi, \eta) = (\lambda\xi, \mu\eta), 0 < \lambda < 1 < \mu; m = 1$, $A \equiv \mathrm{Id}$. Take the covering $\mathcal{U} = \{U_1 = X\backslash\{\xi = 0\}; U_2 = X\backslash\{\eta = 0\}\}$.

First we show that $\operatorname{Im} T|U_1$, and $\operatorname{Im} T|U_2$ are closed in $C^k(X)$ for any $k = 0, \ldots, \infty$. Actually, both operators $T|U_1$ and $T|U_2$ are surjective, and to show this for, say, $T|U_1$ we introduce two closed non-intersecting strips in U_1:

$$P_+ = \{(\xi, \eta), 2 \leq \xi < \infty, 0 \leq \eta \leq 1\}, \quad P_- = \{(\xi, \eta), 0 < \xi \leq 1, 0 \leq \eta < \infty\}.$$

Every C^k-function $\gamma(\xi, \eta)$ on U_1 may be written as a sum $\gamma = \gamma_+ + \gamma_-$ of C^k-functions such that $\gamma_+ \equiv 0$ in P_+ and $\gamma_- \equiv 0$ in P_-. Thus, if $T\phi_\pm = \gamma_\pm$, then $T(\phi_+ + \phi_-) = \gamma$. For the first equations set

$$\phi_+(\xi, \eta) = \sum_{j=0}^{\infty} \gamma_+(F^j(\xi, \eta)), \quad \phi_-(\xi, \eta) = \sum_{j=1}^{\infty} \gamma_-(F^{-j}(\xi, \eta)).$$

Since each point (ξ, η) is mapped uniformly with its small neighborhood into P_+ by F and into P_- by F^{-1}, these are actually C^k-solutions. The case of $T|U_2$ does not differ.

Now let us convince ourselves that $B^1(T; \mathcal{U})$ is not closed. If $\phi(\xi, \eta) \in \operatorname{Ker}(T|U_1)$, i.e. $\phi(\lambda\xi, \mu\eta) = \phi(\xi, \eta)$ for $\eta \geq 0, \xi > 0$, set $\hat{\phi}(x, y) = \phi(\exp(x \ln \lambda), \exp(y \ln \mu))$. Then we obtain the equivalent equation

$$\hat{\phi}(x + 1, y + 1) = \hat{\phi}(x, y), \quad -\infty < x < \infty, -\infty < y < \infty. \tag{9}$$

Any solution to (9) is defined by its values on the strip D bounded by the lines $y = -x$ and $y = -x + 2$. Dividing it into squares with sides of length $2^{1/2}$, we may shift D to the vertical strip $1 < x < 3$. Since the function $\phi(\xi, \eta)$ is bounded on any compact set separated from the $\{\xi = 0\}$-line, the function $\hat{\phi}(x, y)$ is bounded on the half-strip $1 < x < 3, y < 0$, or, equivalently, as we have seen, in the half-strip $\{D, x > 0, y < 0\}$. This means that each function $\phi(\xi, \eta) \in \operatorname{Ker} T|U_1$ is bounded at zero. Identical arguments show that every function $\delta(\xi, \eta) \in \operatorname{Ker} T|U_2$ has the same property. Hence, each function $c(\xi, \eta) \in B^1(T; \mathcal{U})$ is bounded at zero as well.

Finally, define $\nu = \ln \mu / \ln \lambda$ and let $\tau; \mathbb{R}^1 \to (0, 1)$ be a C^∞-function such that $\tau(t) = t$, $t \in (0, \frac{1}{4})$. Set

$$c^n(\xi, \eta) = \sum_{j=0}^{n} (1 - \tau(\xi^\nu \eta))^j, \quad x = (\xi, \eta) \in U_1.$$

Evidently, $c^n \in \operatorname{Ker}(T|U_1) \subset B^1(T; \mathcal{U})$ and this sequence converges in the space $C^\infty(U_1 \bigcap U_2)$ to the function $1/\tau(\xi^\nu \eta)$, which, as just established, cannot belong to $B^1(T; \mathcal{U})$. Hence, $\operatorname{Im} T$ is not closed.

Another way to see that $\operatorname{Im} T$ is not closed is to note that F has no invariant compact subsets and as a consequence has no invariant measures. It follows from this that

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$\operatorname{Ker} T^* = \{0\}$, hence $\overline{\operatorname{Im} T} = C^k(X)$. However, T is not surjective. For instance, the *Abelian equation*

$$\varphi(Fx) = \varphi(x) + 1$$

has no continuous solution. Indeed, let φ be some solution, then

$$\varphi(\lambda^n \xi, \eta) = \varphi(\xi, \mu^{-n} \eta) + n$$

and, letting $n \to \infty$, we see that φ cannot be continuous.

In order to demonstrate the converse (i.e. that closure of $\operatorname{Im} T$ is not a hereditary property), we only have to add zero to the quadrant, $\hat{X} = X \cup \{0\}$, and consider another covering $\hat{\mathcal{U}} = \{U_1 = X, U_2 = \hat{X}\}$. As we have just seen, $\operatorname{Im} T|U_1$ is not closed. On the other hand, at a hyperbolic saddle point every formal C^∞-solution can be restored to a smooth one [**3**], so $\operatorname{Im} T$ is closed in $C^\infty(\hat{X})$.

4.3. Let us now show that $\operatorname{Im} T$ is closed in $C^\infty(X)$ for the following model example:

Example 4.3. Let X be a closed half-strip on the plane:

$$X = \{(\xi, \eta) \in R^2; 0 \le \xi \le 1, \eta \ge 0\},$$

and let $F(\xi, \eta)$ now have two fixed points: a saddle at the origin with eigenvalues $0 < \lambda < 1 < \mu$ and a repeller at the point $(1, 0)$; let $A \equiv \mathrm{Id}$. For simplicity we assume that the saddle point is non-resonant.

Further let O_ϵ be an open ϵ-ball in X centered at zero. Let $U_1 = \cup_{-\infty<n<\infty} F^n(O_\epsilon)$, which makes U_1 an ϵ-hyperbolic neighborhood of the global stable manifold $W_1^+ = \{\xi = 0\}$ and the global unstable manifold $W_1^- = \{0 \le \xi < 1, \eta = 0\}$ of the saddle. Also take $U_2 = X \backslash \{\xi = 0\} = W_2^-$, where W_2^- stands for the global unstable manifold of the repeller. Obviously, $\mathcal{U} = \{U_1, U_2\}$ is an open finite F-invariant covering.

As we have just mentioned, every formal solution to (1) at zero can be restored to a C^∞-solution in O_ϵ and by definition to the whole domain U_1, and any C^∞-function which is zero at the point $(1, 0)$ belongs to $\operatorname{Im}(T|U_2)$. Thus, both $\operatorname{Im}(T|U_1)$ and $\operatorname{Im}(T|U_2)$ are closed, and due to the absence of higher-order resonances, $\operatorname{Ker}(T|U_2) = \{D = \text{constant}\}$.

We further show that the space $B^1(T; \mathcal{U})$ is closed and, hence, according to Corollary 3.1, $\operatorname{Im} T$ itself is closed. Namely, first let the C^∞-diffeomorphism $G(\xi, \eta) = (g_1(\xi, \eta), g_2(\xi, \eta))$ linearize F in O_ϵ (hence in U_1) [**9**]. Then we obtain the following equation equivalent to $T\phi = 0$:

$$\psi(x, y) = \psi(\lambda x, \mu y), (x, y) \in G(U_1); \quad \phi(\xi, \eta) = \psi(g_1(\xi, \eta), g_2(\xi, \eta)). \tag{10}$$

Recall that the elements of $B^1(T; \mathcal{U})$ in this case are functions $c(\xi, \eta)$ defined in $U_1 \cap U_2$ which may be represented as

$$c(\xi, \eta) = \phi(\xi, \eta) - \delta(\xi, \eta), \quad \phi \in \operatorname{Ker}(T|U_1), \quad \delta \in \operatorname{Ker}(T|U_2). \tag{11}$$

Now let the sequence

$$c_n(\xi, \eta) = \varphi_n(\xi, \eta) - D_n \in B^1(T; \mathcal{U})$$

converge in $C^\infty(U_1 \bigcap U_2)$ to $c \in \operatorname{Ker}(T|U_1 \bigcap U_2)$. We need to show that $c \in B^1(T;\mathcal{U})$. To this end let us set

$$\varphi_n(\xi,\eta) = \psi_n(g_1(\xi,\eta), g_2(\xi,\eta)), \quad \hat{c}_n(x,y) = \psi_n(x,y) - D_n.$$

Then $\hat{c}_n \to \hat{c}$ in the domain $G(U_1 \bigcap U_2)$.

Due to the absence of resonances, $\psi_n(x,y) = \omega_n(x,y) + \psi_n(0)$, where $\omega_n(x,y)$ is equal to zero with all its derivatives *flat*, i.e. it is *flat* on the coordinate cross $W_1^+ \bigcup W_1^-$. Thus

$$\hat{c}_n = \omega_n - k_n \to \hat{c}, \quad k_n = D_n - \psi_n(0). \tag{12}$$

Since $\hat{c} \in \operatorname{Ker}(T|G(U_1 \bigcap U_2))$, we have a representation

$$\hat{c}(x,y) = \omega(x,y) + \hat{c}(x,0), \quad 0 < x < 1$$

where $\omega(x,y)$ is flat on the interval $\{0 < x < 1,\ y = 0\}$. Setting $y = 0$ in (12) we obtain $k_n \to \hat{c}(x,0)$, hence $\hat{c}(x,0) = k =$ constant, and $\omega_n(x,y) \to \omega(x,y)$ in the domain $G(U_1 \bigcap U_2)$. Setting $\omega(0,y) = 0$ by definition, let us show that $\omega \in C^\infty(G(U_1))$. This means that $\hat{c} \in B^1(T;\mathcal{U})$.

Let $z_n = (x_n, y_n) \to (0, y_0)$. Since $\omega(x_n, y_n) = \omega(\lambda x_n, \mu y_n)$ we can choose l_n such that all points $\{(\lambda^{-l_n} x_n, \mu^{-l_n} y_n)\}$ lie in the compact set separated from the W_1^+-line (Example 4.2). Since, additionally, $\mu^{-l_n} y_n \to 0$, we obtain $\omega(z_n) \to 0$. Applying identical arguments to all the derivatives of ω we see that $\omega(x,y)$ is a C^∞-function in $G(U_1)$. Hence, $\operatorname{Im} T$ is closed.

4.4. Next we prove that if map F has the trivial sort of dynamics, glued together from the models of Example 4.3, then the homology equation is normally solvable. Consider the generalizing as follows.

Example 4.4. Let X be a compact d-dimensional C^∞-manifold and let F be a Morse–Smale C^∞-diffeomorphism on X. We want to exclude heteroclinic structures from our considerations, so we will assume that F is a gradient-like map [**8**] and, as above, all the saddles are non-resonant. Under this assumption we obtain the following.

THEOREM 4.1. *The homology equation is normally solvable in C^∞.*

Proof. Naturally, we substitute for the initial map the map F^N, whose only periodic points are fixed. So we have

$$\operatorname{Fix} F = \{x_1^{\mathrm{r}}, \dots, x_k^{\mathrm{r}}; x_1^{\mathrm{a}}, \dots, x_l^{\mathrm{a}}; x_1^{\mathrm{s}}, \dots, x_m^{\mathrm{s}}\},$$

where subscripts r, a, s stand for repeller, attractor and saddle respectively, and W^+ and W^- for the global stable and unstable manifolds, respectively, of a fixed point. For each saddle set $V_i = \cup_{-\infty<n<\infty} F^n(O_\epsilon)$, where O_ϵ is a full neighborhood of the saddle point so small that it does not contain any other elements of $\operatorname{Fix} F$. Consider the covering

$$\mathcal{U} = \{W^+_{x_i^{\mathrm{a}}}, i = 1, \dots, k; W^-_{x_j^{\mathrm{r}}}, j = 1, \dots, l; V_{x_p^{\mathrm{s}}}, p = 1, \dots, m\}.$$

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In the same way as in Example 4.4, $\mathcal{U}$ is an open finite F-invariant covering such that $\gamma \in \operatorname{Im} T|U_\alpha \Leftrightarrow \gamma(x_\alpha) = 0$ for every $\gamma \in C^\infty(U_\alpha)$. Therefore $\operatorname{Im} T|U_\alpha$ is closed for each α.

Again, let $\{c_n(x)\} \to \{c(x)\}$ be in $B^1(T; \mathcal{U})$. Now we take into account simple facts about our covering:

(i) for $i \neq j$, one always has

$$W^+_{x^{\mathrm{a}}_i} \cap W^+_{x^{\mathrm{a}}_j} = \emptyset, \quad W^-_{x^{\mathrm{r}}_i} \cap W^-_{x^{\mathrm{r}}_j} = \emptyset,$$

and all elements of $\operatorname{Ker} T|W^+_{x^{\mathrm{a}}_i}, W^-_{x^{\mathrm{r}}_j}$ are constants;

(ii) we can choose O_ϵ so small that $V_{x^{\mathrm{s}}_p} \cap V_{x^{\mathrm{s}}_q} = \emptyset$ for $p \neq q$;

(iii) with the exception of the circle with two fixed points, all the domains

$$W^+_{x^{\mathrm{a}}_i} \cap V_{x^{\mathrm{s}}_p}, \quad W^-_{x^{\mathrm{r}}_j} \cap V_{x^{\mathrm{s}}_p}, \quad W^+_{x^{\mathrm{a}}_i} \cap W^-_{x^{\mathrm{r}}_j}$$

are connected;

(iv)

$$V_{x^{\mathrm{s}}_p} \backslash \{x^{\mathrm{s}}_p\} = \{(\cup_i V_{x^{\mathrm{s}}_p} \cap W^+_{x^{\mathrm{a}}_i}) \cup (\cup_j V_{x^{\mathrm{s}}_p} \cap W^-_{x^{\mathrm{r}}_j})\}.$$

Thus $\{c_n(x)\}$ is of the form

$$\{c_n(x)\} \equiv \left\{ \begin{array}{lll} C^n_i - B^n_p - \phi^n_p(x), & x \in W^+_{x^{\mathrm{a}}_i} \cap V_{x^{\mathrm{s}}_p}; & C^n_i, D^n_j, B^n_p = \text{constant}, \\ D^n_j - B^n_p - \phi^n_p(x), & x \in W^-_{x^{\mathrm{r}}_j} \cap V_{x^{\mathrm{s}}_p}; & \phi^n_p(x) \in \operatorname{Ker} T|V_{x^{\mathrm{s}}_p}, \phi^n_p(x^{\mathrm{s}}_p) = B^n_p \\ C^n_i - D^n_j, & x \in W^+_{x^{\mathrm{a}}_i} \cap W^-_{x^{\mathrm{r}}_j}; & \phi^n_p(x) - B^n_p \text{ is flat on } W^+_{x^{\mathrm{s}}_p} \cup W^-_{x^{\mathrm{s}}_p} \end{array} \right\}.$$

This means that sequences $\phi^n_p(x)$ converge everywhere in the punctured domains $V_{x^{\mathrm{s}}_p} \backslash \{x^{\mathrm{s}}_p\}$ (item (iv)), and carrying over the arguments from Example 4.4, we see that

$$\phi^n_p(x) \to \phi_p(x) \text{ for all } x \in V_{x^{\mathrm{s}}_p}, \ p = 1, \dots, m, \text{ where } \phi_p(x) \in \operatorname{Ker} T|V_{x^{\mathrm{s}}_p}.$$

Then we have

$$C^n_i - B^n_p \to R_{ip}, \quad D^n_j - B^n_p \to S_{jp}, \quad C^n_i - D^n_j \to T_{ij}.$$

In this way, we have reduced our case to the finite-dimensional case, and by Corollary 3.2 we can choose constants C_i, D_j, B_p such that

$$\{c(x)\} \equiv \left\{ \begin{array}{lll} C_i - \phi_p(x), & x \in W^+_{x^{\mathrm{a}}_i} \cap V_{x^{\mathrm{s}}_p}; & \phi_p(x) \in C^\infty(V_{x^{\mathrm{s}}_p}) \\ D_j - \phi_p(x), & x \in W^-_{x^{\mathrm{r}}_j} \cap V_{x^{\mathrm{s}}_p}; & \phi_p(x) \in \operatorname{Ker} T|V_{x^{\mathrm{s}}_p} \\ C_i - D_j, & x \in W^+_{x^{\mathrm{a}}_i} \cap W^-_{x^{\mathrm{r}}_j}; & C_i, D_j = \text{constant} \end{array} \right\}.$$

Thus $\{c(x)\} \in B^1(T; \mathcal{U})$ and $\operatorname{Im} T|X$ is C^∞-closed. The last step is to return to the initial map using Lemma 2.1.

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