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LINEARIZATION OF THE PRODUCT OF JACOBI POLYNOMIALS. II

GEORGE GASPER

1. Introduction. Let [**3**, p. 170, (16)]

$$(1.1)\qquad P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} F(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2)$$

denote the Jacobi polynomial of order (α, β), $\alpha, \beta > -1$, and let $g(k, m, n; \alpha, \beta)$ be defined by

$$(1.2)\qquad R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} g(k, m, n; \alpha, \beta) R_k^{(\alpha,\beta)}(x),$$

where $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. It is well known [**1**; **2**; **4**; **5**; **6**] that the harmonic analysis of Jacobi polynomials depends, at crucial points, on the answers to the following two questions.

Question 1. *For which* (α, β) *do we have*

$$(1.3)\qquad g(k, m, n; \alpha, \beta) \geqq 0, \qquad k, m, n = 0, 1, \ldots ?$$

Question 2. *For which* (α, β) *do we have*

$$(1.4)\qquad \sum_k |g(k, m, n; \alpha, \beta)| \leqq G,$$

where G depends only on (α, β)?

Notice that (1.3) implies (1.4); in fact, since $R_n^{(\alpha,\beta)}(1) = 1$, (1.2) and (1.3) yield

$$\sum_k |g(k, m, n; \alpha, \beta)| = 1.$$

In [**4**] we mentioned several applications of (1.3) and (1.4), and we proved that if $\alpha \geqq \beta$ and $\alpha + \beta + 1 \geqq 0$, then (1.3) holds. Our aim in this paper is to give the answer (Theorem 1) to Question 1 and a partial answer (Theorem 2) to Question 2.

THEOREM 1. *Let* $\alpha > -1$, $\beta > -1$, $a = \alpha + \beta + 1$, $b = \alpha - \beta$, *and*

$$V = \{(\alpha, \beta): \alpha \geqq \beta, a(a+5)(a+3)^2 \geqq (a^2 - 7a - 24)b^2\}.$$

If $(\alpha, \beta) \in V$, *then* (1.3) *holds. However, if* $(\alpha, \beta) \notin V$, *then there exist positive integers* k, m, *and* n *such that* $g(k, m, n; \alpha, \beta) < 0$. *In particular:*

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(i) *If* $\alpha \geqq \beta$ *and* $(\alpha, \beta) \notin V$, *then* $g(2, 2, 2; \alpha, \beta) < 0$;
(ii) *If* $\beta > \alpha$, *then* $g(n - m + 1, m, n; \alpha, \beta) < 0, n \geqq m \geqq 1$.

THEOREM 2. *Let a and b be defined as in Theorem* 1 *and let*

$$W = \{(\alpha, \beta): \alpha \geqq \beta, 2b^2 > -a(a + 3)\} \cup \{(-\tfrac{1}{2}, -\tfrac{1}{2})\}.$$

If $(\alpha, \beta) \in W$, *then* (1.4) *holds. However, if* $-1 < \alpha < -\frac{1}{2}$, *then* $g(0, n, n; \alpha, \beta)$ *is not bounded and so* (1.4) *does not hold.*

Observe that

$$\{(\alpha, \beta): \alpha \geqq \beta > -1, \alpha + \beta + 1 \geqq 0\} \subset V \subset W.$$

For $-1 < \beta \leqq -\frac{1}{2}$, the set W is bounded on the left by the curve

$$b = w(a) = [-a(a + 3)/2]^{\frac{1}{2}}, \qquad -\tfrac{1}{3} \leqq a \leqq 0. \tag{1.5}$$

By considering $w'(a)$ we find that (1.5) determines a path in the (α, β)-plane which starts at $(-\frac{1}{3}, -1)$ and approaches the line $\alpha + \beta + 1 = 0$ tangentially from the left, meeting it at $(-\frac{1}{2}, -\frac{1}{2})$.

Similarly, for $-1 < \beta \leqq -\frac{1}{2}$, the set V is bounded on the left by a curve which starts at $((-11 + (73)^{\frac{1}{2}})/8, -1)$ and approaches the line $\alpha + \beta + 1 = 0$ tangentially, meeting it at $(-\frac{1}{2}, -\frac{1}{2})$. Therefore

$$\begin{aligned} &V \subset \{(\alpha, \beta): a > \tfrac{1}{8}(-11 + (73)^{\frac{1}{2}}) = -0.3069\ldots\}, \\ &W \subset \{(\alpha, \beta): a > -\tfrac{1}{3}\}. \end{aligned} \tag{1.6}$$

Before proving Theorems 1 and 2 we present some applications, the most important of which is the following convolution structure.

Consider (α, β) fixed and let $g(k, m, n) = g(k, m, n; \alpha, \beta)$,

$$\gamma(k, m, n) = \int_{-1}^{1} R_k^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x)(1 - x)^\alpha (1 + x)^\beta \, dx,$$

$$h(n) = \left(\int_{-1}^{1} [R_n^{(\alpha,\beta)}(x)]^2 (1 - x)^\alpha (1 + x)^\beta \, dx\right)^{-1}$$

$$= \frac{(2n + \alpha + \beta + 1)\Gamma(n + \alpha + 1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha+\beta+1}\,\Gamma(n + 1)\Gamma(\alpha + 1)\Gamma(\alpha + 1)\Gamma(n + \beta + 1)}.$$

Then $g(k, m, n) = \gamma(k, m, n)h(k)$. If $F(n)$ is defined for $n = 0, 1, \ldots,$ then we say that $F(n)$ belongs to the class $b^{(\alpha,\beta)}$ whenever its norm

$$||F|| = \sum_{n=0}^{\infty} |F(n)|h(n)$$

is finite. For $F_1(n)$, $F_2(n) \in b^{(\alpha,\beta)}$ we define their convolution $F_1 * F_2$ by

$$(F_1 * F_2)(n) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} F_1(k)F_2(m)\gamma(k, m, n)h(k)h(m).$$

For $F(n) \in b^{(\alpha,\beta)}$ we define its transform $F^\wedge(x)$ by

$$F^\wedge(x) = \sum_{n=0}^{\infty} F(n) R_n^{(\alpha,\beta)}(x) h(n), \qquad -1 \leqq x \leqq 1,$$

and so the inversion formula is

$$F(n) = \int_{-1}^{1} F^\wedge(x) R_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \, dx.$$

Then, as in [**5**] for the ultraspherical case $\alpha = \beta$, Theorems 1 and 2 yield Corollary 1 and the usual Banach algebra proof of the Wiener-Lévy theorem yields Corollary 2.

COROLLARY 1. *If* $(\alpha, \beta) \in W$ *and* $F_j(n) \in b^{(\alpha,\beta)}, j = 1, 2, 3,$ *then*

$$(F_1 * F_2)(n) \in b^{(\alpha,\beta)}$$

and

(i) $$\|F_1 * F_2\| \leqq G\|F_1\| \, \|F_2\|,$$

(ii) $$F_1 * F_2 = F_2 * F_1,$$

(iii) $$F_1 * (F_2 * F_3) = (F_1 * F_2) * F_3,$$

(iv) $$(F_1 * F_2)^\wedge(x) = F_1^\wedge(x) F_2^\wedge(x),$$

where G *depends only on* (α, β). *If* (α, β) *also belongs to* V, *then* (i) *holds with* $G = 1$.

COROLLARY 2. *Suppose that* $(\alpha, \beta) \in W$,

$$f(x) = \sum_{n=0}^{\infty} a(n) R_n^{(\alpha,\beta)}(x), \qquad \sum_{n=0}^{\infty} |a(n)| < \infty,$$

and ϕ *is a function holomorphic on an open set containing the range of* f. *Then*

$$\phi(f(x)) = \sum_{n=0}^{\infty} b(n) R_n^{(\alpha,\beta)}(x) \quad \textit{with} \sum_{n=0}^{\infty} |b(n)| < \infty.$$

Closely connected with the above convolution structure is the generalized translation operator for which we now have the following result.

COROLLARY 3. *Suppose that* $f(x)$ *is integrable on* $(-1, 1)$ *with respect to* $(1-x)^\alpha(1+x)^\beta$ *and let*

$$F(n) = \int_{-1}^{1} f(x) R_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \, dx,$$

$$F(n, m) = \int_{-1}^{1} f(x) R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \, dx.$$

If $(\alpha, \beta) \in V$, *then the operator which takes* $F(n)$ *into* $F(n, m)$, *the generalized* *translate of $F(n)$, is a positive operator in the sense that if $F(n) \geqq 0, n = 0, 1, \ldots,$ then $F(n, m) \geqq 0, n, m = 0, 1, \ldots$.*

By setting

$$S_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)},$$
$$V^* = \{(\alpha, \beta): (\beta, \alpha) \in V\},$$
$$W^* = \{(\alpha, \beta): (\beta, \alpha) \in W\},$$

and using $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$, one obtains analogous results with $R_n^{(\alpha,\beta)}(x)$, V, and, W replaced by $S_n^{(\alpha,\beta)}(x)$, V^*, and W^*, respectively.

2. Proof of Theorem 1. Our main tool in [**4**] was a recurrence formula for a positive multiple of $g_k = g(k, m, n) = g(k, m, n; \alpha, \beta)$. In order to work directly with g_k, we first obtain its recurrence formula.

In [**6**] Hylleraas let

$$y_n(z) = F(-n, n + p; q; z), \qquad p + 1 > q > 0,$$

and derived a recurrence formula for $c_k = c(k, m, n)$, where c_k is defined by

$$y_n y_m = \sum_{k=n-m}^{n+m} c_k y_k$$

and it is assumed that $n \geqq m$. Setting $p = \alpha + \beta + 1$, $q = \alpha + 1$, and $z = (1 - x)/2$, we find from (1.1) and (1.2) that $y_n(z) = R_n^{(\alpha,\beta)}(x)$ and $c_k = g_k = g(k, m, n)$. We also set $a = \alpha + \beta + 1$, $b = \alpha - \beta$, $s = n - m$, and $k = s + j$. Observe that $2(\alpha + 1) = a + b + 1 > 0$, $2(\beta + 1) = a - b + 1 > 0$, $a > -1$, $s \geqq 0$, and that $b \geqq 0$ if and only if $\alpha \geqq \beta$. The recurrence formula [**6**, (4.13)] for c_k yields

$$(2.1) \quad \frac{(j+1)(2s+j+1)(2n+j+a+1)}{(2s+2j+a+1)} \times \frac{(2m-j+a-1)(2s+2j+a-b+1)}{(2s+2j+a+2)} g_{s+j+1}$$
$$= b\left[\frac{(j+1)(2s+j+1)(2m-j)(2n+j+2a)}{(2s+2j+a+1)} - \frac{j(2s+j)(2m-j+1)(2n+j+2a-1)}{(2s+2j+a-1)}\right] g_{s+j}$$
$$+ \frac{(2m-j+1)(j+a-1)(2s+j+a-1)}{(2s+2j+a-2)} \times \frac{(2n+j+2a-1)(2s+2j+a+b-1)}{(2s+2j+a-1)} g_{s+j-1}$$

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and the formulas [**6**, (3.3) and (3.8)] for c_{n+m} and c_{n-m} yield

$$(2.2)\qquad g_{n+m} = \frac{\binom{2n+\alpha+\beta}{n}\binom{2m+\alpha+\beta}{m}\binom{n+m+\alpha}{n+m}}{\binom{2n+2m+\alpha+\beta}{n+m}\binom{n+\alpha}{n}\binom{m+\alpha}{m}},$$

$$(2.3)\qquad g_{n-m} = \frac{\binom{n}{m}\binom{2m+\alpha+\beta}{m}\binom{n+\beta}{m}}{\binom{2m}{m}\binom{2n+\alpha+\beta+1}{2m}\binom{m+\alpha}{m}}.$$

Clearly $g_{n+m} > 0$ and $g_{n-m} > 0$. Setting $j = 0$ and then $j = 1$ in (2.1) and using $g_{s-1} = 0$, we obtain:

$$(2.4)\qquad g_{s+1} = \frac{4bm(n+a)(2s+a+2)}{(2n+a+1)(2m+a-1)(2s+a-b+1)} g_s$$

and

$$(2.5)\qquad \frac{(s+1)(2n+a+2)(2m+a-2)(2s+a-b+3)}{(2s+a+3)(2s+a+4)} g_{s+2} = Cm(n+a)(aA+b^2B)g_s,$$

where

$$A = A(m,n,a) = (2n+a+1)(2m+a-1)(2s+a+3)(2s+a+1)^2,$$
$$B = B(m,n,a) = 4(2s+a+2)[(s+1)(2m-1)(2n+2a+1)(2s+a+1) - m(n+a)(2s+1)(2s+a+3)] - a(2n+a+1)(2m+a-1)(2s+a+3),$$
$$C = C(m,n,a,b) = [(2n+a+1)(2m+a-1)(2s+a+1)(2s+a+3) \times (2s+a-b+1)]^{-1}.$$

Note that $A > 0$ and $C > 0$ when $n \geqq m \geqq 1$. Since $g_s > 0$, it follows from (2.4) that if $\beta > \alpha$, then $g_{s+1} = g(n-m+1, m, n) < 0$, $n \geqq m \geqq 1$, while if $\alpha \geqq \beta$, then $g_{s+1} = g(n-m+1,m,n) \geqq 0$, $n \geqq m \geqq 1$. Hence, because $g_{n-m} > 0$ and $g_{n+m} > 0$, we have $g(k, m, n) \geqq 0$ when $\alpha \geqq \beta$ and $n \geqq m = 1$. When $n = m = 2$ we have

$$aA + b^2B = (a+1)^2[a(a+5)(a+3)^2 - (a^2 - 7a - 24)b^2],$$

so that by (2.5), $g_{s+2} = g(2,2,2) \geqq 0$ if and only if

$$a(a+5)(a+3)^2 \geqq (a^2 - 7a - 24)b^2.$$

Consequently, in view of the definition of V, we have reduced the proof to showing that if $(\alpha, \beta) \in V$, $n \geqq m \geqq 2$, and $n \geqq 3$, then

$$(2.6)\qquad g_{s+j+1} = g(s+j+1, m, n) \geqq 0, \qquad j = 1, 2, \ldots, 2m-2.$$

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In proving this we may assume that $a < 0$, for we have already considered the case $a \geqq 0$ in [**4**]. Set $J = j - 1$ and write the coefficient of g_{s+j} in (2.1) in the form

$$\text{coef}(g_{s+j}) = \frac{bF(J)}{(2s + 2J + a + 1)(2s + 2J + a + 3)}, \tag{2.7}$$

where

$$\begin{aligned} F(J) = {} & (J + 2)(2s + J + 2)(2m - J - 1)(2s + 2m + J + 2a + 1) \\ & \times (2s + 2J + a + 1) - (J + 1)(2s + J + 1)(2m - J) \\ & \times (2s + 2m + J + 2a)(2s + 2J + a + 3) \\ = {} & -6J^4 - 12[2s + a + 2]J^3 + 2[-16s^2 + 4(m - 4a - 9)s \\ & +4m(m + a) - 3a^2 - 19a - 17]J^2 + 2[-8s^3 + 4(2m - 3a - 8)s^2 \\ & +2\{2m(2m + 3a + 2) - 2a^2 - 17a - 17\}s + 4m(m + a)(a + 2) \\ & -7a^2 - 19a - 10]J + [16(m - 1)s^3 + 8\{2m + (3a + 1) \\ & +3\}(m - 1)s^2 + 4\{2m(a + 2) + 2a^2 + 3(3a + 1) + 2\}(m - 1)s \\ & +(3m - 2)(4a^2 + 11a + 3) + \{2(3a + 1)(2m - 1) + a + 1\}(m - 2)] \\ = {} & a_4J^4 + a_3J^3 + a_2J^2 + a_1J + a_0. \end{aligned}$$

Since, from (1.6), $3a + 1 > 0$ and $4a^2 + 11a + 3 > 0$, it is clear that $a_4 < 0$, $a_3 < 0$, $a_0 > 0$, and

$$\begin{aligned} a_1 - 2sa_2 = {} & 2[24s^3 + 20(a + 2)s^2 \\ & +2\{2(m - 1) + 2(a + 1)(m + 1) + a^2\}s + a^2(4m - 7) \\ & +\{4(m^2 - 4) + 8(m - 2) + 13\}(a + 1) + 4m(m - 2) + 9] > 0. \end{aligned}$$

Hence, $F(J)$ has only one variation of sign, and so by Descartes' rule there exists a positive integer $J_0 = J_0(m, n, a)$ such that

$$F(J) \geqq 0, \qquad J = 0, 1, \ldots, J_0 - 1,$$

and $F(J) < 0$, $J = J_0, J_0 + 1, \ldots$. Thus by (2.7),

$$\text{coef}(g_{s+j}) \geqq 0, \qquad j = 1, 2, \ldots, J_0,$$

and $\text{coef}(g_{s+j}) \leqq 0$, $j = J_0 + 1, J_0 + 2, \ldots$. In (2.1) we have

$$\text{coef}(g_{s+j+1}) > 0, \qquad j = 1, 2, \ldots, 2m - 2,$$

and $\text{coef}(g_{s+j-1}) > 0$, $j = 2, 3, \ldots, 2m$. But $\text{coef}(g_{s+j+1}) < 0$ for $j = 2m - 1$ and $\text{coef}(g_{s+j-1}) < 0$ for $j = 1$ since $a < 0$. This presents difficulties not encountered in the case $a \geqq 0$. Nevertheless, if we could prove (2.6) for $j = 1, 2m - 3, 2m - 2$, then the general case would easily follow. For, with (2.6) for $j = 1, 2m - 3, 2m - 2$ and our previous observations, we would have

$$g_{s+j} \geqq 0, \qquad j = 0, 1, 2, 2m - 2, 2m - 1, 2m,$$

and so by successive applications of (2.1) with $j = 2, 3 \ldots, \min(J_0, 2m - 4)$ and (if $J_0 < 2m - 4$) $j = 2m - 2, 2m - 3, \ldots, J_0 + 1$ we would obtain (2.6). Consequently, it suffices to prove (2.6) for $j = 1, 2m - 3, 2m - 2$.

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Let us first consider the case $j = 1$; i.e., $g_{s+2} \geqq 0$. Put

$$D(a) = (a^2 - 7a - 24)A + (a + 5)(a + 3)^2 B.$$

If $D(a) \geqq 0$, then by the definition of V, we have

$$(a + 5)(a + 3)^2[aA + b^2 B] \geqq b^2 D(a) \geqq 0,$$

which implies that $g_{s+2} \geqq 0$. We obtain $D(a) \geqq 0$, $-\frac{1}{3} < a < 0$, by demonstrating that

$$(2.8) \quad D(-\tfrac{1}{3}) \geqq 0, \qquad D'(-\tfrac{1}{3}) \geqq 0, \qquad D''(a) \geqq 0, \qquad -\tfrac{1}{3} \leqq a \leqq 0,$$

where the primes indicate differentiations with respect to a. A long computation yields

$$\begin{aligned} D(a) = 4\{&[(2m - 1)s + 3m - 6]a^6 + [(10m - 5)s^2 + (2m^2 + 43m - 58)s \\ &+ 3m^2 + 24m - 60]a^5 + [(16m - 8)s^3 + (8m^2 + 156m - 188)s^2 \\ &+ (40m^2 + 222m - 436)s + 24m^2 + 72m - 240]a^4 + [(8m - 4)s^4 \\ &+ (8m^2 + 212m - 252)s^3 + (116m^2 + 622m - 1096)s^2 \\ &+ (198m^2 + 470m - 1342)s + 72m^2 + 102m - 492]a^3 \\ &+ [(96m - 120)s^4 + (96m^2 + 680m - 1120)s^3 \\ &+ (424m^2 + 954m - 2482)s^2 + (398m^2 + 454m - 2023)s \\ &+ 102m^2 + 69m - 546]a^2 + [(256m - 380)s^4 \\ &+ (256m^2 + 724m - 1668)s^3 + (556m^2 + 592m - 2451)s^2 \\ &+ (352m^2 + 183m - 1480)s + 69m^2 + 18m - 312]a \\ &+ [(168m - 264)s^4 + (168m^2 + 240m - 792)s^3 \\ &+ (240m^2 + 114m - 882)s^2 + (114m^2 + 18m - 420)s + 18m^2 - 72]\} \\ = 4\{&d_6 a^6 + d_5 a^5 + \ldots + d_1 a + d_0\}. \end{aligned}$$

Each d_k is positive since $m \geqq 2$. Therefore

$$\begin{aligned} D\left(-\frac{1}{3}\right) &\geqq \frac{4}{27}\{-d_5 - d_3 + 3d_2 - 9d_1 + 27d_0\} \\ &= \frac{4}{27}\{[2512(m - 2) + 960]s^4 + [2512(m^2 - 4) \\ &\quad + 1792m + 568]s^3 + [2632(m - 2)^2 + 10508(m - 2) \\ &\quad + 2388]s^2 + [904(m - 2)^2 + 3304(m - 2) + 303]s \\ &\quad + 96(m - 2)^2 + 303(m - 2)\} \geqq 0, \end{aligned}$$

$$\begin{aligned} D'\left(-\frac{1}{3}\right) &\geqq \frac{4}{27}\{-d_6 - 4d_4 + 9d_3 - 18d_2 + 27d_1\} \\ &= \frac{4}{27}\{[5256(m - 2) + 2376]s^4 + [5256(m^2 - 4) \\ &\quad + 9152(m - 2) + 12216]s^3 + [8392(m^2 - 4) + 3786m \\ &\quad + 2955]s^2 + [3962(m^2 - 4) + 109m + 1969]s \\ &\quad + 579(m - 2)^2 + 2187(m - 2)\} \geqq 0, \end{aligned}$$

and, for $-\frac{1}{3} \leqq a \leqq 0$,

$$\begin{aligned} D''(a) \geqq 4\{-d_5 - 2d_3 + 2d_2\} \\ = 4\{[176(m-2)+120]s^4 + [176m^2 + 936(m-2) \\ +136]s^3 + [616(m-2)^2 + 3118(m-2) + 1005]s^2 \\ +[398(m-2)^2 + 1517(m-2) + 138]s + 57(m-2)^2 \\ +138(m-2)\} \geqq 0. \end{aligned}$$

This yields (2.8) and hence (2.6) for $j = 1$.

Now we consider the cases $j = 2m - 2$ and $j = 2m - 3$ of (2.6). Setting $j = 2m$ and then $j = 2m - 1$ in (2.1) and using $g_{s+2m+1} = 0$, we obtain

$$(2.9) \quad g_{n+m-1} = \frac{4bnm(2n+2m+a-2)}{(2n+2m+a+b-1)(2n+a-1)(2m+a-1)} g_{n+m}$$

and

$$(2.10) \quad \frac{(2n+2m+a+b-3)(n+m+a-1)}{(2n+2m+a-4)} \times \frac{(2n+a-2)(2m+a-2)}{(2n+2m+a-3)} g_{n+m-2} = Mnm(aK + b^2L)g_{n+m},$$

where

$$\begin{aligned} K = K(m, n, a) &= (2n+2m+a-3)(2n+a-1) \\ &\quad \times (2m+a-1)(2n+2m+a-1)^2, \\ L = L(m, n, a) &= 4(2n+2m+a-2)[(2n-1)(2m-1) \\ &\quad \times (n+m+a-1)(2n+2m+a-1) \\ &\quad - nm(2n+2m+2a-1)(2n+2m+a-3)] \\ &\quad - a(2n+a-1)(2m+a-1)(2n+2m+a-3), \\ M = M(m, n, a, b)& \\ = [(2n+a-1)&(2m+a-1)(2n+2m+a+b-1) \\ &\quad \times (2n+2m+a-3)(2n+2m+a-1)]^{-1}. \end{aligned}$$

Note that $K > 0$ and $M > 0$ for $n \geqq m \geqq 1$. From (2.9), $g_{n+m-1} \geqq 0$ which is (2.6) for $j = 2m - 2$. For the remaining case $j = 2m - 3$ of (2.6), we observe by an argument similar to the one which precedes (2.8) that it is enough to prove

$$(2.11) \quad E(-\tfrac{1}{3}) > 0, \qquad E'(-\tfrac{1}{3}) > 0, \qquad E''(a) > 0, \qquad -\tfrac{1}{3} \leqq a \leqq 0,$$

where

$$E(a) = (a^2 - 7a - 24)K + (a+5)(a+3)^2 L,$$

$n \geqq m \geqq 2$, and $n \geqq 3$. Let $t = n + m - 5$. Then $t \geqq 0$ and

$$\begin{aligned}
E(a) = 4\{&[(2m-1)t - 2m^2 + 10m - 9]a^6 + [(10m-5)t^2 \\
&+(-10m^2 + 117m - 92)t - 67m^2 + 335m - 315]a^5 \\
&+[(16m-8)t^3 + (-16m^2 + 336m - 272)t^2 \\
&+(-256m^2 + 2166m - 2036)t - 886m^2 + 4430m - 4356]a^4 \\
&+[(8m-4)t^4 + (-8m^2 + 332m - 308)t^3 \\
&+(-292m^2 + 3898m - 4036)t^2 + (-2438m^2 + 18018m - 18934)t \\
&-5828m^2 + 29140m - 29898]a^3 + [(96m - 120)t^4 \\
&+(-96m^2 + 2264m - 2800)t^3 + (-1784m^2 + 19350m - 23062)t^2 \\
&+(-10430m^2 + 71710m - 81123)t - 19560m^2 \\
&+97800m - 104013]a^2 + [(256m - 380)t^4 \\
&+(-256m^2 + 5068m - 6988)t^3 + (-3788m^2 + 37492m - 47895)t^2 \\
&+(-18552m^2 + 122865m - 145134)t - 30105m^2 \\
&+150525m - 164187]a + [(168m - 264)t^4 \\
&+(-168m^2 + 3120m - 4488)t^3 + (-2280m^2 + 21714m - 28602)t^2 \\
&+(-10314m^2 + 67122m - 81000)t - 15552m^2 + 77760m - 86022]\} \\
= 4\{&e_6a^6 + e_5a^5 + \ldots + e_1a + e_0\}.
\end{aligned}$$

Each e_k is positive when $n \geqq m \geqq 2$ and $n \geqq 3$. This can be seen by appropriately rewriting each e_k as a sum of positive terms of the form $n(m-2)t^3$, $m(n-3)t^3$, $m(n-m)t$, $n(m-2)$, etc. To illustrate one such arrangement we write

$$\begin{aligned}
e_1 = [&\{190n(m-2) + 66m(n-3)\}t^3 + \{2544n(m-2) \\
&+182m(n-m) + 880m(n-3)\}t^2 + \{11228n(m-2) + n \\
&+1538m(n-m) + 4248m(n-3)\}t + 16430n(m-2) + n \\
&+9633m(n-3) + 2021m(n-m) + 6145m + 108],
\end{aligned}$$

from which its positivity is obvious. Due to the positivity of each e_k we have

$$\begin{aligned}
E\left(-\frac{1}{3}\right) &\geqq \frac{4}{27}\{-e_5 - e_3 + 3e_2 - 9e_1 + 27e_0\} \\
&= \frac{4}{27}\{[2032n(m-2) + 480m(n-3)]t^3 + [23028n(m-2) \\
&\quad + 2624m(n-m) + 4252m(n-3)]t^2 + [88032n(m-2) \\
&\quad + 20180m(n-m) + 11960m(n-3)]t + 112409n(m-2) \\
&\quad + n + 39284m(n-m) + 10767m(n-3) + 3904(m-2) \\
&\quad + 5156\} > 0,
\end{aligned}$$

$$\begin{aligned}
E'\left(-\frac{1}{3}\right) &\geqq \frac{4}{27}\{-e_6 - 4e_4 + 9e_3 - 18e_2 + 27e_1\} \\
&= \frac{4}{27}\{[4068n(m-2) + 1188m(n-3)]t^3 + [50168n(m-2) \\
&\quad + 4572m(n-m) + 13416m(n-3)]t^2 + [205803n(m-2) \\
&\quad + n + 37228m(n-m) + 53823m(n-3)]t + 281320n(m-2) \\
&\quad + 63996m(n-m) + 100349m(n-3) + 58387m + 736\} > 0
\end{aligned}$$

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and, for $-\frac{1}{3} \leqq a \leqq 0$,

$$\begin{aligned} E''(a) \geqq 4\{-e_5 - 2e_3 + 2e_2\} &= 4\{[116n(m-2) + 60m(n-3)]t^3 + [1912n(m-2) \\ &+ 52m(n-m) + 968m(n-3)]t^2 + [9464n(m-2) \\ &+ n + 660m(n-m) + 5190m(n-3)]t + 14836n(m-2) \\ &+ n + 57m(n-m) + 12447m(n-3) + 7955m + 440\} > 0. \end{aligned}$$

This concludes the proof.

3. Proof of Theorem 2. In order to indicate the origin of the set W and to give the main idea behind our proof, we begin by mentioning that W is also a best possible set in the sense that it is the answer to the following question.

Question 3. *Find each* (α, β) *for which there exists a number* $N = N(\alpha, \beta)$ *such that*

$$g(k, m, n; \alpha, \beta) \geqq 0, \qquad n \geqq N, \quad n \geqq m. \tag{3.1}$$

Since (3.1) implies (1.4) and since the unboundedness of $g(0, n, n; \alpha, \beta)$ for $-1 < \alpha < -\frac{1}{2}$ follows immediately by applying Stirling's formula to (2.3), we may confine ourselves to proving that the set W answers Question 3.

Due to our observations in § 2 we may assume that $n \geqq m \geqq 2$, $\alpha \geqq \beta$, and $a < 0$. Then, from (2.5), $g(s+2, m, n) \geqq 0$ if and only if $aA + b^2B \geqq 0$. Since

$$B(m, n, -1) = 32(m-1)[s^2 + (m+1)s + m]s^2 \geqq 0,$$
$$B'(m, n, -1) = 4[16(m-1)s^2 + (8m^2 + 8m - 15)s + 8m(m-1)]s \geqq 0,$$

and, for $-1 \leqq a \leqq 0$,

$$\begin{aligned} B''(m, n, a) &= -12a^2 + 6a[(8m-12)s + 8m - 11] + (80m - 88)s^2 \\ &\quad + (16m^2 + 112m - 144)s + 16m^2 + 32m - 54 \\ &\geqq 8[(10m - 11)s^2 + (2m^2 + 8m - 9)s + 2m(m-1)] > 0, \end{aligned}$$

where primes indicate differentiations with respect to a, we have $B > 0$ for $-1 < a \leqq 0$. Hence $b^2 = -aA/B$ determines a curve which we denote by $\gamma(m, n)$. Since

$$\lim_{n\to\infty} \frac{-aA(n, n, a)}{B(n, n, a)} = -\frac{a(a+3)}{2},$$

the curves $\gamma(n, n)$ tend to the curve $b^2 = -a(a+3)/2$ as $n \to \infty$. In addition, if $b^2 \leqq -a(a+3)/2$, then by the positivity of B, we have

$$\begin{aligned} 2[aA(n, n, a) + b^2B(n, n, a)] &\leqq 2aA(n, n, a) - a(a+3)B(n, n, a) \\ &= 3a(a+1)^3(a+2)(a+3) < 0, \end{aligned}$$

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i.e., $g(2, n, n; \alpha, \beta) < 0$ when $b^2 \leqq -a(a+3)/2$ and $a < 0$. Consequently, (3.1) does not hold when $(\alpha, \beta) \notin W$.

To show that (3.1) holds when $(\alpha, \beta) \in W$, we first consider the function $F(J) = a_4 J^4 + \ldots + a_1 J + a_0$ defined in § 2. It is clear that we still have $a_4 < 0$, $a_3 < 0$, and $a_1 - 2sa_2 > 0$. Even though a_0 can now take on negative values, it is positive provided that n is sufficiently large (depending only on (α, β)). For it follows from

$$a_0 = 4[4(n+a+1)(m-1)s^2 + 2\{n(a+2) + a^2 + 3a + 2\}(m-1)s + (3a+1)(n+a+1)(m-1) + a^2 + a]$$

and $3a + 1 > 0$ that there exists a number $N_1 = N_1(a)$ such that $a_0 > 0$ whenever $n \geqq N_1$. Consequently, our remarks in § 2 concerning $F(J)$ are still valid when $n \geqq N_1$, and so it suffices to prove for each (α, β) under consideration that $aA + b^2B \geqq 0$ and $aK + b^2L \geqq 0$ whenever $n \geqq N = N(\alpha, \beta) \geqq N_1$.

Fix (α, β) and choose $\epsilon = \epsilon(\alpha, \beta) > 0$ so small that $2b^2 > \epsilon - a(a+3)$. This is possible by the definition of W. Since (2.11) implies that $E(a) > 0$, $-\frac{1}{3} < a < 0$, it follows that we also have $L > 0$, $-\frac{1}{3} < a < 0$. Thus, if

$$X = 2aA + (\epsilon - a^2 - 3a)B \geqq 0$$

and

$$Y = 2aK + (\epsilon - a^2 - 3a)L \geqq 0,$$

then $aA + b^2B > 0$ and $aK + b^2L > 0$. We shall now show that there is a number $N = N(\alpha, \beta) \geqq N_1$ such that $X \geqq 0$ and $Y \geqq 0$ for $n \geqq N$. To handle X we write

$$\begin{aligned} X = 32[\epsilon(m-1) - a(a+1)(m-2)]s^4 &+ 32[\epsilon m^2 - a(a+1)m^2 \\ &+ S(\epsilon, a, m)]s^3 + 32[\epsilon(a+2)m^2 - a(a+1)^2m^2 \\ &+ S(\epsilon, a, m)]s^2 + 8[\epsilon(a+1)(a+5)m^2 - a(a+1)^3m^2 \\ &+ S(\epsilon, a, m)]s + 8\epsilon(a+1)^2m^2 + S(\epsilon, a, m), \end{aligned}$$

where $S(\epsilon, a, m)$ denotes a polynomial in ϵ, a, and m, *not* necessarily the same at each occurrence, which contains m to at most the first power.

From this representation of X it is clear that there exists a number $N_2 = N_2(\alpha, \beta)$ such that when $m \geqq N_2$ the function X, as a polynomial in s, has positive coefficients and so is positive. Hence, since $s = n - m$ and the coefficient of s^4 is (strictly) positive, there also exists a number $N_3 = N_3(\alpha, \beta)$ such that $X \geqq 0$ when $m \leqq N_2$ and $n \geqq N_3$. Thus $X \geqq 0$ when $n \geqq N_3$. Next

$$\begin{aligned} Y = 32[\{-a(a+1)(1-2a) + 2(1-a)\epsilon\}(n+m)^2 & \\ + T(\epsilon, a, n+m; 1)](m-2)^2 &+ 32[\{-a(a+1)n \\ + \epsilon n + 2a\epsilon - 3\epsilon &- 2a^3 + a^2 + 3a\}(n+m)^3 \\ + T(\epsilon, a, n+m; 2)](m-2) &+ 32[\epsilon(n+2a-3)(n+m)^3 \\ &+ T(\epsilon, a, n+m; 2)], \end{aligned}$$

where $T(\epsilon, a, n+m; k)$ denotes a polynomial in ϵ, a and $n + m$, *not* necessarily the same at each occurrence, which contains $n + m$ to at most the kth power.

Since $n \geqq m \geqq 2$ and $-\frac{1}{3} < a < 0$, it follows from this representation of Y that there is a number $N_4 = N_4(\alpha, \beta)$ such that each term in brackets is positive when $n \geqq N_4$, and so $Y \geqq 0$ when $n \geqq N_4$. The proof is complete once we put $N = \max(N_1, N_3, N_4)$.

Appendix. We shall show here that if $\beta > \alpha > -1$, then (1.4) does *not* hold. Setting $x = -1$ in (1.2) and using

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n\binom{n+\beta}{n},$$

we obtain

$$u^2(n;\alpha,\beta) = \sum_{k=0}^{2n} (-1)^k g(k,n,n;\alpha,\beta)u(k;\alpha,\beta),$$

where

$$u(n;\alpha,\beta) = \binom{n+\beta}{n}\binom{n+\alpha}{n}^{-1}.$$

If (1.4) held for some (α, β) with $\beta > \alpha$, then, since $u(k;\alpha,\beta)$ is an increasing function of k when $\beta > \alpha$, we would have

$$u^2(n;\alpha,\beta) \leqq Gu(2n;\alpha,\beta).$$

But by Stirling's formula this inequality cannot be true for all n. This contradiction proves that (1.4) cannot hold whenever $\beta > \alpha$.

With this result and Theorem 2, we have answered Question 2 for all (α, β) except those belonging to the small set

$$Z = \{(\alpha,\beta): -\tfrac{1}{2} \leqq \alpha < -\tfrac{1}{3}, -1 < \beta < -\tfrac{1}{2}, (\alpha,\beta) \notin W\}.$$

Added in proof. In a joint paper with R. Askey (in preparation) it will be shown that (1.4) also holds for the set Z.

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University of Toronto,
Toronto, Ontario