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Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations

Teresa D'Aprile
Dipartimento Interuniversitario di Matematica,
via E. Orabona 4, 70125 Bari, Italy (daprile@dm.uniba.it)

Dimitri Mugnai
Dipartimento di Matematica e Informatica,
via Vanvitelli 1, 06123 Perugia, Italy (mugnai@dipmat.unipg.it)

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In this paper we study the existence of radially symmetric solitary waves for nonlinear Klein–Gordon equations and nonlinear Schrödinger equations coupled with Maxwell equations. The method relies on a variational approach and the solutions are obtained as mountain-pass critical points for the associated energy functional.

1. Introduction

This paper has been motivated by the search of non-trivial solutions for the following nonlinear equations of Klein–Gordon type,

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0, \quad x \in \mathbb{R}^3, \tag{1.1}$$

or of Schrödinger type,

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^3, \tag{1.2}$$

where $\hbar > 0$, $m > 0$, $p > 2$, $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$.

In recent years, many papers have been devoted to finding standing waves of (1.1) or (1.2), i.e. solutions of the form

$$\psi(x,t) = \mathrm{e}^{\mathrm{i}\omega t}u(x), \quad \omega \in \mathbb{R}.$$

With this ansatz, the nonlinear Klein–Gordon equation, as well as the nonlinear Schrödinger equation, is reduced to a semilinear elliptic equation, and existence theorems have been established to show whether u is radially symmetric and real (see [8, 9]) or non-radially symmetric and complex (see [13, 16]). In this paper, we investigate the existence of nonlinear Klein–Gordon or Schrödinger fields interacting with an electromagnetic field $\boldsymbol{E} - \boldsymbol{H}$. Such a problem has been extensively pursued in the case of assigned electromagnetic fields (see [2, 3, 12]). Following the ideas already introduced in [5–7,10,11,14,15], we do not assume that the electromagnetic field is assigned. Then we have to study a system of equations whose unknowns are the wave function $\psi = \psi(x,t)$ and the gauge potentials $\boldsymbol{A}$, Φ,

$$\boldsymbol{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \qquad \Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R},$$

which are related to $\boldsymbol{E} - \boldsymbol{H}$ by the Maxwell equations

$$\boldsymbol{E} = -\left(\nabla\Phi + \frac{\partial \boldsymbol{A}}{\partial t}\right), \qquad \boldsymbol{H} = \nabla \times \boldsymbol{A}.$$

Let us first consider equation (1.1). The Lagrangian density related to (1.1) is given by

$$\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}\left[\left|\frac{\partial\psi}{\partial t}\right|^2 - |\nabla\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

The interaction of ψ with the electromagnetic field is described by the minimal coupling rule, that is, the formal substitution

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \mathrm{i}e\Phi, \qquad \nabla \mapsto \nabla - \mathrm{i}e\boldsymbol{A},$$

where e is the electric charge. Then the Lagrangian density becomes

$$\mathcal{L}_{\mathrm{KGM}} = \frac{1}{2}\left[\left|\frac{\partial\psi}{\partial t} + \mathrm{i}e\psi\Phi\right|^2 - |\nabla\psi - \mathrm{i}e\boldsymbol{A}\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

If we set

$$\psi(x,t) = u(x,t)\mathrm{e}^{\mathrm{i}S(x,t)},$$

where $u, S : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, the Lagrangian density takes the form

$$\mathcal{L}_{\mathrm{KGM}} = \tfrac{1}{2}\{u_t^2 - |\nabla u|^2 - [|\nabla S - e\boldsymbol{A}|^2 - (S_t + e\Phi)^2 + m^2]u^2\} + \frac{1}{p}|u|^p.$$

Now consider the Lagrangian density of the electromagnetic field $\boldsymbol{E} - \boldsymbol{H}$,

$$\mathcal{L}_0 = \tfrac{1}{2}(|\boldsymbol{E}|^2 - |\boldsymbol{H}|^2) = \tfrac{1}{2}|\boldsymbol{A}_t + \nabla\Phi|^2 - \tfrac{1}{2}|\nabla \times \boldsymbol{A}|^2. \tag{1.3}$$

Therefore, the total action is given by

$$\mathcal{S} = \iint \mathcal{L}_{\mathrm{KGM}} + \mathcal{L}_0.$$

Making the variation of $\mathcal{S}$ with respect to u, S, Φ and $\boldsymbol{A}$, respectively, we get

$$u_{tt} - \Delta u + [|\nabla S - e\boldsymbol{A}|^2 - (S_t + e\Phi)^2 + m^2]u - |u|^{p-2}u = 0, \tag{1.4}$$

$$\frac{\partial}{\partial t}[(S_t + e\Phi)u^2] - \operatorname{div}[(\nabla S - e\boldsymbol{A})u^2] = 0, \tag{1.5}$$

$$\operatorname{div}(\boldsymbol{A}_t + \nabla\Phi) = e(S_t + e\Phi)u^2, \tag{1.6}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) + \frac{\partial}{\partial t}(\boldsymbol{A}_t + \nabla\Phi) = e(\nabla S - e\boldsymbol{A})u^2. \tag{1.7}$$

We are interested in finding standing (or *solitary*) waves of (1.4)–(1.7), that is, solutions having the form

$$u = u(x), \quad S = \omega t, \quad \boldsymbol{A} = 0, \quad \Phi = \Phi(x), \quad \omega \in \mathbb{R}.$$

Then equations (1.5) and (1.7) are identically satisfied, while (1.4) and (1.6) become

$$-\Delta u + [m^2 - (\omega + e\Phi)^2]u - |u|^{p-2}u = 0, \tag{1.8}$$

$$-\Delta\Phi + e^2u^2\Phi = -e\omega u^2. \tag{1.9}$$

In [6], the authors proved the existence of infinitely many symmetric solutions (u_n, Φ_n) of (1.8), (1.9), under the assumption $4 < p < 6$, by using an equivariant version of the mountain-pass theorem (see [1,4]).

The object of the first part of this paper is to extend this result as follows.

Theorem 1.1. *Assume that one of the following two hypotheses hold:*

(a) $m > \omega > 0$ *and* $4 \leqslant p < 6$*; or*

(b) $m\sqrt{p-2} > \sqrt{2}\omega > 0$ *and* $2 < p < 4$.

Then system (1.8), (1.9) has infinitely many radially symmetric solutions (u_n, Φ_n), $u_n \not\equiv 0$ *and* $\Phi_n \not\equiv 0$, *with* $u_n \in H^1(\mathbb{R}^3)$, $\Phi_n \in L^6(\mathbb{R}^3)$ *and* $|\nabla\Phi_n| \in L^2(\mathbb{R}^3)$.

In the second part of the paper, we study the Schrödinger equation for a particle in an electromagnetic field.

Consider the Lagrangian associated to (1.2),

$$\mathcal{L}_{\mathrm{S}} = \frac{1}{2}\left[\mathrm{i}\hbar\frac{\partial\psi}{\partial t}\bar{\psi} - \frac{\hbar^2}{2m}|\nabla\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

By using the formal substitution

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \mathrm{i}\frac{e}{\hbar}\Phi, \qquad \nabla \mapsto \nabla - \mathrm{i}\frac{e}{\hbar}\boldsymbol{A},$$

we obtain

$$\mathcal{L}_{\mathrm{SM}} = \frac{1}{2}\left[\mathrm{i}\hbar\frac{\partial\psi}{\partial t}\bar{\psi} - e\Phi|\psi|^2 - \frac{\hbar^2}{2m}|\nabla\psi - \mathrm{i}\frac{e}{\hbar}\boldsymbol{A}\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

Now take

$$\psi(x,t) = u(x,t)\mathrm{e}^{\mathrm{i}S(x,t)/\hbar}.$$

With this ansatz, the Lagrangian $\mathcal{L}_{\mathrm{SM}}$ becomes

$$\mathcal{L}_{\mathrm{SM}} = \frac{1}{2}\left[\mathrm{i}\hbar uu_t - \frac{\hbar^2}{2m}|\nabla u|^2 - \left(S_t + e\Phi + \frac{1}{2m}|\nabla S - e\boldsymbol{A}|^2\right)u^2\right] + \frac{1}{p}|\psi|^p.$$

Proceeding as in [5], we consider the total action

$$\mathcal{S} = \iint\left[\mathcal{L}_{\mathrm{SM}} + \frac{1}{8\pi}(|\boldsymbol{E}|^2 - |\boldsymbol{H}|^2)\right]$$

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of the system 'particle-electromagnetic field'. Then the Euler–Lagrange equations associated to the functional $\mathcal{S} = \mathcal{S}(u, S, \Phi, \boldsymbol{A})$ give rise to the following system of equations:

$$-\frac{\hbar^2}{2m}\Delta u + \left(S_t + e\Phi + \frac{1}{2m}|\nabla S - e\boldsymbol{A}|^2\right)u - |u|^{p-2}u = 0, \tag{1.10}$$

$$\frac{\partial}{\partial t}u^2 + \frac{1}{m}\operatorname{div}[(\nabla S - e\boldsymbol{A})u^2] = 0, \tag{1.11}$$

$$eu^2 = -\frac{1}{4\pi}\operatorname{div}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right), \tag{1.12}$$

$$\frac{e}{2m}(\nabla S - e\boldsymbol{A})u^2 = \frac{1}{4\pi}\left[\frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right) + \nabla \times (\nabla \times \boldsymbol{A})\right]. \tag{1.13}$$

If we look for solitary wave solutions in the electrostatic case, i.e.

$$u = u(x), \quad S = \omega t, \quad \Phi = \Phi(x), \quad \boldsymbol{A} = 0, \quad \omega \in \mathbb{R},$$

then (1.11) and (1.13) are identically satisfied, while (1.10) and (1.12) become

$$-\frac{\hbar^2}{2m}\Delta u + e\Phi u - |u|^{p-2}u + \omega u = 0, \tag{1.14}$$

$$-\Delta\Phi = 4\pi e u^2. \tag{1.15}$$

The existence of solutions of (1.14), (1.15) was already studied for $4 < p < 6$; in [5], existence of infinitely many radial solutions was proved, while in [13] existence of a non-radially symmetric solution was established. In the second part of the paper we prove the following result.

Theorem 1.2. *Let $\omega > 0$ and $4 \leqslant p < 6$. Then the system (1.14), (1.15) has at least a radially symmetric solution (u, Φ), $u \neq 0$ and $\Phi \neq 0$, with $u \in H^1(\mathbb{R}^3)$, $\Phi \in L^6(\mathbb{R}^3)$ and $|\nabla\Phi| \in L^2(\mathbb{R}^3)$.*

2. Nonlinear Klein–Gordon equations coupled with Maxwell equations

In this section we prove theorem 1.1. For the sake of simplicity, assume $e = 1$, so that (1.8), (1.9) give rise to the following system in $\mathbb{R}^3$:

$$-\Delta u + [m^2 - (\omega + \Phi)^2]u - |u|^{p-2}u = 0, \tag{2.1}$$

$$-\Delta\Phi + u^2\Phi = -\omega u^2. \tag{2.2}$$

Assume that one of the following hypotheses hold:

(a) $m > \omega > 0$, $4 \leqslant p < 6$; or

(b) $m\sqrt{p-2} > \sqrt{2}\omega > 0$, $2 < p < 4$.

We note that $q = 6$ is the critical exponent for the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$.

It is clear that (2.1), (2.2) are the Euler–Lagrange equations of the functional $F : H^1 \times D^{1,2} \to \mathbb{R}$ defined as

$$F(u,\Phi) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 - |\nabla\Phi|^2 + [m^2 - (\omega+\Phi)^2]u^2)\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,\mathrm{d}x.$$

Here, $H^1 \equiv H^1(\mathbb{R}^3)$ denotes the usual Sobolev space endowed with the norm

$$\|u\|_{H^1} \equiv \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)\,\mathrm{d}x\right)^{1/2} \tag{2.3}$$

and $D^{1,2} \equiv D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{R})$ with respect to the norm

$$\|u\|_{D^{1,2}} \equiv \left(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\right)^{1/2}. \tag{2.4}$$

The following two propositions hold.

PROPOSITION 2.1. *The functional F belongs to $C^1(H^1 \times D^{1,2},\mathbb{R})$ and its critical points are the solutions of (2.1), (2.2).*

Proof. We refer to [6]. □

PROPOSITION 2.2. *For every $u \in H^1$, there exists a unique $\Phi = \Phi[u] \in D^{1,2}$ that solves (2.2). Furthermore, the following hold.*

(i) $\Phi[u] \leqslant 0$.

(ii) $\Phi[u] \geqslant -\omega$ *in the set* $\{x \mid u(x) \neq 0\}$.

(iii) *If u is radially symmetric, then $\Phi[u]$ is radial too.*

Proof. For fixed $u \in H^1$, consider the following bilinear form on $D^{1,2}$:

$$a(\phi,\psi) = \int_{\mathbb{R}^3} (\nabla\psi\nabla\psi + u^2\phi\psi)\,\mathrm{d}x.$$

Obviously,

$$a(\phi,\phi) \geqslant \|\phi\|^2_{D^{1,2}}.$$

Observe that, since $H^1(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$, then $u^2 \in L^{3/2}(\mathbb{R}^3)$. On the other hand, $D^{1,2}$ is continuously embedded in $L^6(\mathbb{R}^3)$, and hence, by Hölder's inequality,

$$a(\phi,\psi) \leqslant \|\phi\|_{D^{1,2}}\|\psi\|_{D^{1,2}} + \|u^2\|_{L^{3/2}}\|\phi\|_{L^6}\|\psi\|_{L^6} \leqslant (1 + C\|u\|^2_{L^3})\|\phi\|_{D^{1,2}}\|\psi\|_{D^{1,2}}$$

for some positive constant C, given by the Sobolev inequality (see [20]). Therefore, a defines an inner product, equivalent to the standard inner product in $D^{1,2}$.

Moreover, $H^1(\mathbb{R}^3) \subset L^{12/5}(\mathbb{R}^3)$, and then

$$\left|\int_{\mathbb{R}^3} u^2\psi\,\mathrm{d}x\right| \leqslant \|u^2\|_{L^{6/5}}\|\psi\|_{L^6} \leqslant c\|u\|^2_{L^{12/5}}\|\psi\|_{D^{1,2}}. \tag{2.5}$$

Therefore, the linear map

$$\psi \in D^{1,2} \mapsto \int_{\mathbb{R}^3} u^2\psi\,\mathrm{d}x$$

is continuous. By the Lax–Milgram lemma, we get the existence of a unique $\Phi \in D^{1,2}$ such that

$$\int_{\mathbb{R}^3} (\nabla\Phi\nabla\psi + u^2\Phi\psi)\,\mathrm{d}x = -\omega \int_{\mathbb{R}^3} u^2\psi\,\mathrm{d}x \quad \forall\psi \in D^{1,2},$$

i.e. Φ is the unique solution of (2.2). Furthermore, Φ achieves the minimum

$$\inf_{\phi\in D^{1,2}} \int_{\mathbb{R}^3} (\tfrac{1}{2}(|\nabla\phi|^2 + u^2|\phi|^2) + \omega u^2\phi)\,\mathrm{d}x = \int_{\mathbb{R}^3} (\tfrac{1}{2}(|\nabla\Phi|^2 + u^2|\Phi|^2) + \omega u^2\Phi)\,\mathrm{d}x.$$

Note that $-|\Phi|$ also achieves such a minimum; then, by uniqueness, $\Phi = -|\Phi| \leqslant 0$. Now let $O(3)$ denote the group of rotations in $\mathbb{R}^3$. Then, for every $g \in O(3)$ and $f : \mathbb{R}^3 \to \mathbb{R}$, set $T_g(f)(x) = f(gx)$. Note that T_g does not change the norms in H^1, $D^{1,2}$ and L^p. In lemma 4.2 of [6], it was proved that $T_g\Phi[u] = \Phi[T_g u]$. In this way, if u is radial, we get $T_g\Phi[u] = \Phi[u]$.

Finally, following the same idea of [17], with fixed $u \in H^1$, if we multiply (2.2) by $(\omega + \Phi[u])^- \equiv -\min\{\omega + \Phi[u], 0\}$, which is an admissible test function, since $\omega > 0$, we get

$$-\int_{\Phi[u]<-\omega} |D\Phi[u]|^2\,\mathrm{d}x - \int_{\Phi[u]<-\omega} (\omega + \Phi[u])^2u^2\,\mathrm{d}x = 0,$$

so that $\Phi[u] \geqslant -\omega$, where $u \neq 0$. □

Remark 2.3. The result of proposition 2.2 (ii) can be strengthened in some cases. Indeed, take $\bar{u}$ in $H^1(\mathbb{R}^3) \cap C^\infty$ radially symmetric such that $\bar{u} > 0$ in $B(0, R)$ and $\bar{u} \equiv 0$ in $\mathbb{R}^3 \setminus B(0, R)$ for some $R > 0$. Then we get

$$-\omega \leqslant \Phi[\bar{u}](x) \leqslant 0 \quad \forall x \in \mathbb{R}^3.$$

In fact, since $\Phi[\bar{u}]$ solves (2.2), by standard regularity results for elliptic equations, $\bar{u} \in C^\infty$ implies $\Phi[\bar{u}] \in C^\infty$. By proposition 2.2, $\Phi[\bar{u}]$ is radial; moreover, $\Phi[\bar{u}]$ is harmonic outside $B(0, R)$. Since $\Phi[\bar{u}] \in D^{1,2}$, then

$$\Phi[\bar{u}](x) = -\frac{c}{|x|}, \quad |x| \geqslant R,$$

for some $c > 0$. Setting $\tilde{\Phi}(r) = \Phi[\bar{u}](x)$ for $|x| = r$, it follows that $\tilde{\Phi}'(R) > 0$ and $\tilde{\Phi}(r) > \tilde{\Phi}(R)$ for every $r > R$. Therefore, the minimum of $\Phi[\bar{u}]$ is achieved in $B(0, R)$. Let $\bar{x}$ be a minimum point for $\Phi[\bar{u}]$. Then (2.2) implies

$$\Phi[\bar{u}](\bar{x}) = \frac{-\omega\bar{u}^2(\bar{x}) + \Delta\Phi[\bar{u}](\bar{x})}{\bar{u}^2(\bar{x})} \geqslant -\omega.$$

In view of proposition 2.2, we can define the map

$$\Phi : H^1 \to D^{1,2},$$

which maps each $u \in H^1$ in the unique solution of (2.2). From standard arguments, we have $\Phi \in C^1(H^1, D^{1,2})$ and from the very definition of Φ we get that

$$F'_\phi(u, \Phi[u]) = 0 \quad \forall u \in H^1. \tag{2.6}$$

Now let us consider the functional

$$J : H^1 \to \mathbb{R}, \qquad J(u) := F(u, \Phi[u]).$$

By proposition 2.1, $J \in C^1(H^1, \mathbb{R})$ and, by (2.6), we have

$$J'(u) = F'_u(u, \Phi[u]).$$

By the definition of F, we obtain

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 - |\nabla \Phi[u]|^2 + [m^2 - \omega^2]u^2 - u^2\Phi[u]^2)\,\mathrm{d}x \\ - \omega \int_{\mathbb{R}^3} u^2\Phi[u] - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,\mathrm{d}x.$$

Multiplying both members of (2.2) by $\Phi[u]$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla \Phi[u]|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} |u|^2|\Phi[u]|^2\,\mathrm{d}x = -\omega \int_{\mathbb{R}^3} |u|^2\Phi[u]\,\mathrm{d}x. \tag{2.7}$$

Using (2.7), the functional J may be written as

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + [m^2 - \omega^2]u^2 - \omega u^2\Phi[u])\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,\mathrm{d}x. \tag{2.8}$$

The next lemma states a relationship between the critical points of the functionals F and J (the proof can be found in [6]).

LEMMA 2.4. *The following statements are equivalent.*

(i) $(u, \Phi) \in H^1 \times D^{1,2}$ *is a critical point of* F.

(ii) u *is a critical point of* J *and* $\Phi = \Phi[u]$.

Then, in order to get solutions of (2.1), (2.2), we look for critical points of J.

THEOREM 2.5. *Assume hypotheses (a) and (b). Then the functional* J *has infinitely many critical points* $u_n \in H^1$ *having a radial symmetry.*

Proof. Our aim is to apply the equivariant version of the mountain-pass theorem (see [1, theorem 2.13] or [18, theorem 9.12]). Since J is invariant under the group of translations, there is clearly a lack of compactness. In order to overcome this difficulty, we consider radially symmetric functions. More precisely, we introduce the subspace

$$H^1_r = \{u \in H^1 \mid u(x) = u(|x|)\}.$$

We divide the remaining part of the proof into three steps.

STEP 1. *Any critical point* $u \in H^1_r$ *of* $J_{|H^1_r}$ *is also a critical point of* J.

The proof can be found in [6].

STEP 2. *The functional $J_{|H^1_r}$ satisfies the Palais–Smale condition, i.e. any sequence $\{u_n\}_n \subset H^1_r$ such that $J(u_n)$ is bounded and $J'_{|H^1_r}(u_n) \to 0$ contains a convergent subsequence.*

For the sake of simplicity, from now on, we set $\Omega = m^2 - \omega^2 > 0$. Let $\{u_n\}_n \subset H^1_r$ be such that

$$|J(u_n)| \leqslant M, \qquad J'_{|H^1_r}(u_n) \to 0$$

for some constant $M > 0$. Then, using the form of J given in (2.8),

$$\begin{aligned} pJ(u_n) - J'(u_n)u_n &= (\tfrac{1}{2}p - 1)\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2)\,\mathrm{d}x \\ &\quad - \omega(\tfrac{1}{2}p - 2)\int_{\mathbb{R}^3} u_n^2 \Phi[u_n]\,\mathrm{d}x + \int_{\mathbb{R}^3} u_n^2(\Phi[u_n])^2\,\mathrm{d}x \\ &\geqslant (\tfrac{1}{2}p - 1)\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2)\,\mathrm{d}x - \omega(\tfrac{1}{2}p - 2)\int_{\mathbb{R}^3} u_n^2 \Phi[u_n]\,\mathrm{d}x. \end{aligned} \tag{2.9}$$

We distinguish two cases: either $p \geqslant 4$ or $2 < p < 4$.

If $p \geqslant 4$, by (2.9), using proposition 2.2, we immediately deduce that

$$pJ(u_n) - J'(u_n)u_n \geqslant (\tfrac{1}{2}p - 1)\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2)\,\mathrm{d}x. \tag{2.10}$$

Moreover, by hypothesis (a),

$$(\tfrac{1}{2}p - 1)\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2)\,\mathrm{d}x \geqslant c_1\|u_n\|^2, \tag{2.11}$$

and, by assumption,

$$pJ(u_n) - J'(u_n)u_n \leqslant pM + c_2\|u_n\| \tag{2.12}$$

for some positive constants c_1 and c_2.

Combining (2.10), (2.11) and (2.12), we deduce that $\{u_n\}_n$ is bounded in H^1_r.

If $2 < p < 4$, by proposition 2.2 and (2.9), we get

$$\begin{aligned} pJ(u_n) - J'(u_n)u_n &\geqslant (\tfrac{1}{2}p - 1)\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2)\,\mathrm{d}x - \omega^2(2 - \tfrac{1}{2}p)\int_{\mathbb{R}^3} u_n^2\,\mathrm{d}x \\ &= (\tfrac{1}{2}p - 1)\int_{\mathbb{R}^3} |\nabla u_n|^2 + \tfrac{1}{2}(m^2(p-2) - 2\omega^2)\int_{\mathbb{R}^3} |u_n|^2\,\mathrm{d}x. \end{aligned}$$

By hypothesis (b), $m^2(p-2) - 2\omega^2 > 0$, and we repeat the same argument as for $p \geqslant 4$ and obtain the boundedness of $\{u_n\}_n$ in H^1_r.

On the other hand, using equation (2.2) and proceeding as in (2.5), we get

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla\Phi[u_n]|^2\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^3} |\nabla\Phi[u_n]|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} |u_n|^2|\Phi[u_n]|^2\,\mathrm{d}x \\ &= -\omega\int_{\mathbb{R}^3} u_n^2\Phi[u_n]\,\mathrm{d}x \\ &\leqslant c\omega\|u_n\|^2_{L^{12/5}}\|\Phi[u_n]\|_{D^{1,2}}, \end{aligned}$$

which implies that $\{\Phi[u_n]\}_n$ is bounded in $D^{1,2}$.

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Then, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } H_r^1,$$
$$\Phi[u_n] \rightharpoonup \phi \quad \text{in } D^{1,2}.$$

If $L : H_r^1 \to (H_r^1)'$ is defined as

$$L(u) = -\Delta u + \Omega u,$$

then

$$L(u_n) = \omega u_n \Phi[u_n] + |u_n|^{p-2} u_n + \varepsilon_n,$$

where $\varepsilon_n \to 0$ in $(H_r^1)'$, that is,

$$u_n = L^{-1}(\omega u_n \Phi[u_n]) + L^{-1}(|u_n|^{p-2} u_n) + L^{-1}(\varepsilon_n).$$

Now note that $\{u_n \Phi[u_n]\}$ is bounded in $L_r^{3/2}$; in fact, by Hölder's inequality,

$$\|u_n \Phi[u_n]\|_{L_r^{3/2}} \leqslant \|u_n\|_{L_r^2} \|\Phi[u_n]\|_{L_r^6} \leqslant c \|u_n\|_{L_r^2} \|\Phi[u_n]\|_{D^{1,2}}.$$

Moreover, $\{|u_n|^{p-2} u_n\}$ is bounded in $L_r^{p'}$ (where $1/p + 1/p' = 1$). The immersions $H_r^1 \hookrightarrow L_r^3$ and $H_r^1 \hookrightarrow L_r^p$ are compact (see [8] or [19]), and thus, by duality, $L_r^{3/2}$ and $L_r^{p'}$ are compactly embedded in $(H_r^1)'$. Then, by standard arguments, $L^{-1}(\omega u_n \Phi[u_n])$ and $L^{-1}(|u_n|^{p-2} u_n)$ strongly converge in H_r^1. Then we conclude that

$$u_n \to u \quad \text{in } H_r^1.$$

STEP 3. *The functional $J_{|H_r^1}$ satisfies the geometrical hypothesis of the equivariant version of the mountain-pass theorem.*

First of all, we observe that $J(0) = 0$. Moreover, by proposition 2.2 and (2.8),

$$J(u) \geqslant \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x + \tfrac{1}{2} \Omega \int_{\mathbb{R}^3} |u|^2 \, \mathrm{d}x - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, \mathrm{d}x.$$

The hypothesis $2 < p < 6$ and the continuous embedding $H^1 \subset L^p$ imply that there exists $\rho > 0$ small enough such that

$$\inf_{\|u\|_{H^1} = \rho} J(u) > 0.$$

Since J is even, the thesis of step 3 will follow if we prove that, for every finite-dimensional subset V of H_r^1, we have

$$\lim_{\substack{u \in V, \\ \|u\|_{H^1} \to +\infty}} J(u) = -\infty. \tag{2.13}$$

Let V be an m-dimensional subspace of H_r^1 and let $u \in V$. By proposition 2.2, $\Phi[u] \geqslant -\omega$, where $u \neq 0$, so that

$$J(u) \leqslant \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \Omega |u|^2 + \omega^2 u^2) \, \mathrm{d}x - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \leqslant c \|u\|_{H^1}^2 - \frac{1}{p} \|u\|_{L^p}^p$$

and (2.13) follows, since all norms in V are equivalent. □

Proof of theorem 1.1. Lemma 2.4 and theorem 2.5. □

REMARK 2.6. In view of remark 2.3, the existence of one non-trivial critical point for the functional J follows from the classical mountain-pass theorem; more precisely, with $\bar{u} \in H^1_r \cap C^\infty$ as in remark 2.3, since $\|\Phi[\bar{u}]\|_\infty \leqslant \omega$, we have

$$J(t\bar{u}) \leqslant \tfrac{1}{2}t^2 \int_{\mathbb{R}^3} (|\nabla \bar{u}|^2 + \Omega|\bar{u}|^2 + \omega^2 \bar{u}^2)\,\mathrm{d}x - \frac{t^p}{p}\int_{\mathbb{R}^3} |\bar{u}|^p \to -\infty \quad \text{as } t \to +\infty.$$

3. Nonlinear Schrödinger equations coupled with Maxwell equations

For the sake of simplicity, we assume that $\hbar = m = e = 1$ in (1.14), (1.15). Then we are reduced to studying the following system in $\mathbb{R}^3$:

$$-\tfrac{1}{2}\Delta u + \Phi u + \omega u - |u|^{p-2}u = 0, \tag{3.1}$$

$$-\Delta \Phi = 4\pi u^2. \tag{3.2}$$

We will assume that

(a) $\omega > 0$, and

(b) $4 \leqslant p < 6$.

Of course, equalities (3.1), (3.2) are the Euler–Lagrange equations of the functional $\mathcal{F} : H^1 \times D^{1,2} \to \mathbb{R}$ defined as

$$\mathcal{F}(u,\Phi) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x - \frac{1}{16\pi}\int_{\mathbb{R}^3} |\nabla \Phi|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} (\Phi u^2 + \omega u^2)\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,\mathrm{d}x,$$

where H^1 and $D^{1,2}$ are defined as in the previous section.

It is easy to prove the analogous of proposition 2.1, i.e. that $\mathcal{F} \in C^1(H^1 \times D^{1,2}, \mathbb{R})$ and that its critical points are solutions of (3.1), (3.2).

Moreover, we have the following proposition.

PROPOSITION 3.1. *For every $u \in H^1$, there exists a unique solution $\Phi = \Phi[u] \in D^{1,2}$ of (3.2) such that the following hold.*

(i) *$\Phi[u] \geqslant 0$.*

(ii) *$\Phi[tu] = t^2\Phi[u]$ for every $u \in H^1$ and $t \in \mathbb{R}$.*

Proof. Let us consider the linear map

$$\phi \in D^{1,2} \mapsto \int_{\mathbb{R}^3} u^2 \phi\,\mathrm{d}x,$$

which is continuous by (2.5). By Lax–Milgram's lemma, we get the existence of a unique $\Phi \in D^{1,2}$ such that

$$\int_{\mathbb{R}^3} \nabla\Phi\nabla\phi\,\mathrm{d}x = 4\pi\int_{\mathbb{R}^3} u^2\phi\,\mathrm{d}x \quad \forall \phi \in D^{1,2},$$

i.e. Φ is the unique solution of (3.2). Furthermore, Φ achieves the minimum

$$\inf_{\phi \in D^{1,2}} \left\{ \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 - 4\pi\int_{\mathbb{R}^3} u^2\phi\,\mathrm{d}x \right\} = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\Phi|^2\,\mathrm{d}x - 4\pi\int_{\mathbb{R}^3} u^2\Phi\,\mathrm{d}x.$$

Note that $|\Phi|$ also achieves such a minimum. Then, by uniqueness, $\Phi = |\Phi| \geqslant 0$.

Finally,

$$-\Delta\Phi[tu] = 4\pi t^2 u^2 = -t^2\Delta\Phi[u] = -\Delta(t^2\Phi[u]).$$

Thus, by uniqueness, $\Phi[tu] = t^2\Phi[u]$. □

Proceeding as in the previous section, we can define the map

$$\Phi : H^1 \to D^{1,2},$$

which maps each $u \in H^1$ in the unique solution of equation (3.2). As before, $\Phi \in C^1(H^1, D^{1,2})$ and

$$\mathcal{F}'_\Phi(u, \Phi[u]) = 0 \quad \forall u \in H^1.$$

Now consider the functional $\mathcal{J} : H^1 \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \mathcal{F}(u, \Phi[u]).$$

$\mathcal{J}$ belongs to $C^1(H^1, \mathbb{R})$ and satisfies $\mathcal{J}'(u) = \mathcal{F}_u(u, \Phi[u])$. Using the definition of $\mathcal{F}$ and equation (3.2), we obtain

$$\mathcal{J}(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \tfrac{1}{2}\omega\int_{\mathbb{R}^3} |u|^2 \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} |u|^2\Phi[u] \,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x.$$

As before, one can prove the following lemma.

Lemma 3.2. *The following statements are equivalent.*

(i) $(u, \Phi) \in H^1 \times D^{1,2}$ *is a critical point of* $\mathcal{F}$*.*

(ii) u *is a critical point of* $\mathcal{J}$ *and* $\Phi = \Phi[u]$*.*

Now we are ready to prove the existence result for equations (3.1), (3.2).

Theorem 3.3. *Assume hypotheses (a) and (b). Then the functional* $\mathcal{J}$ *has a non-trivial critical point* $u \in H^1$ *having a radial symmetry.*

Proof. Let H_r^1 be defined as in theorem 2.5.

Step 1. *Any critical point* $u \in H_r^1$ *of* $\mathcal{J}_{|H_r^1}$ *is also a critical point of* $\mathcal{J}$*.*

The proof is as in theorem 2.5.

Step 2. *The functional* $\mathcal{J}_{|H_r^1}$ *satisfies the Palais–Smale condition.*

Let $\{u_n\}_n \subset H_r^1$ be such that

$$|\mathcal{J}(u_n)| \leqslant M, \qquad \mathcal{J}'_{|H_r^1}(u_n) \to 0$$

for some constant $M > 0$. Then

$$\begin{aligned} p\mathcal{J}(u_n) - \mathcal{J}'(u_n)u_n &\\ &= (\tfrac{1}{4}p - \tfrac{1}{2})\int_{\mathbb{R}^3} |\nabla u_n|^2 + (\tfrac{1}{4}p - 1)\int_{\mathbb{R}^3} \Phi[u_n]u_n^2 \,\mathrm{d}x + (\tfrac{1}{2}p - 1)\omega\int_{\mathbb{R}^3} |u_n|^2 \,\mathrm{d}x \\ &\geqslant (\tfrac{1}{4}p - \tfrac{1}{2})\int_{\mathbb{R}^3} |\nabla u_n|^2 + (\tfrac{1}{2}p - 1)\omega\int_{\mathbb{R}^3} |u_n|^2 \,\mathrm{d}x \end{aligned}$$

by proposition 3.1, since $p \geqslant 4$. Moreover,

$$(\tfrac{1}{4}p - \tfrac{1}{2}) \int_{\mathbb{R}^3} (|\nabla u|^2 + \omega |u|^2)\,\mathrm{d}x \geqslant c_1 \|u_n\|^2$$

and, by assumption,

$$p\mathcal{J}(u_n) - \mathcal{J}'(u_n)u_n \leqslant pM + c_2\|u_n\|_{H^1}$$

for some positive constants c_1 and c_2.

We have thus proved that $\{u_n\}_n$ is bounded in H^1_r.

On the other hand,

$$\|\Phi[u_n]\|^2_{D^{1,2}} = 4\pi \int_{\mathbb{R}^3} u^2 \Phi[u_n]\,\mathrm{d}x,$$

and then, using inequality (2.5), we easily deduce that $\{\Phi[u_n]\}_n$ is bounded in $D^{1,2}$.

The remaining part of the proof follows as in step 2 of theorem 2.5, after replacing L with $\mathcal{L} : H^1_r \to (H^1_r)'$ defined as $\mathcal{L}(u) = -\tfrac{1}{2}\Delta u + \omega u$.

STEP 3. *The functional $\mathcal{J}_{|H^1_r}$ satisfies the three geometrical hypothesis of the mountain-pass theorem.*

By proposition 3.1, we have

$$\mathcal{J}(u) \geqslant \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + \tfrac{1}{2}\omega \int_{\mathbb{R}^3} |u|^2\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,\mathrm{d}x.$$

Then, using the continuous embedding $H^1 \subset L^p$, we deduce that $\mathcal{J}$ has a strict local minimum in 0.

We introduce the following notation: if $u : \mathbb{R}^3 \to \mathbb{R}$, we set

$$u_{\lambda,\alpha,\beta}(x) = \lambda^{\beta} u(\lambda^{\alpha} x), \quad \lambda > 0, \quad \alpha, \beta \in \mathbb{R}.$$

Now fix $u \in H^1_r$. We want to show that

$$\Phi[u_{\lambda,\alpha,\beta}] = (\Phi[u])_{\lambda,\alpha,2(\beta-\alpha)}. \tag{3.3}$$

In fact,

$$\begin{aligned} -\Delta\Phi[u_{\lambda,\alpha,\beta}](x) &= 4\pi u^2_{\lambda,\alpha,\beta}(x) = 4\pi\lambda^{2\beta}u^2(\lambda^{\alpha}x) \\ &= -\lambda^{2\beta}(\Delta\Phi[u])(\lambda^{\alpha}x) = -\Delta((\Phi[u])_{\lambda,\alpha,2(\beta-\alpha)})(x). \end{aligned}$$

By uniqueness (see proposition 3.1), equation (3.3) follows.

Now take $u \not\equiv 0$ in H^1_r and evaluate

$$\begin{aligned} \mathcal{J}(u_{\lambda,\alpha,\beta}) = \tfrac{1}{4}(\lambda^{2\beta-\alpha}) \int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + \tfrac{1}{2}\omega\lambda^{2\beta-3\alpha}\int_{\mathbb{R}^3} u^2\,\mathrm{d}x \\ + \tfrac{1}{4}(\lambda^{4\beta-5\alpha})\int_{\mathbb{R}^3} u^2\Phi[u]\,\mathrm{d}x - \frac{\lambda^{\beta p-3\alpha}}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x. \end{aligned}$$

We want to prove that $\mathcal{J}(u_{\lambda,\alpha,\beta}) < \mathcal{J}(0)$ for some suitable choice of λ, α and β.

For example, assume that

$$\left.\begin{aligned} \beta p - 3\alpha &< 0,\\ \beta p - 3\alpha &< 2\beta - \alpha,\\ \beta p - 3\alpha &< 2\beta - 3\alpha,\\ \beta p - 3\alpha &< 4\beta - 5\alpha. \end{aligned}\right\} \tag{3.4}$$

Then it is clear that $\mathcal{J}(u_{\lambda,\alpha,\beta}) \to -\infty$ as $\lambda \to 0$.

So we look for a couple (α, β) which satisfies (3.4). From the third inequality, we get $\beta < 0$. Combining the second and the fourth ones, we derive

$$4 - p < \frac{2\alpha}{\beta} < p - 2. \tag{3.5}$$

Such an inequality is satisfied by taking $\beta = 2\alpha$, which also satisfies the first inequality in (3.4).

In a similar way, one can prove that if

$$\left.\begin{aligned} \beta p - 3\alpha &> 0,\\ \beta p - 3\alpha &> 2\beta - \alpha,\\ \beta p - 3\alpha &> 2\beta - 3\alpha,\\ \beta p - 3\alpha &> 4\beta - 5\alpha, \end{aligned}\right\} \tag{3.6}$$

then $\mathcal{J}(u_{\lambda,\alpha,\beta}) \to -\infty$ as $\lambda \to +\infty$, with the same choice $\beta = 2\alpha$. □

REMARK 3.4. Notice that the systems (3.4) or (3.6) have a solution for every $p > 3$. More precisely, for every $p > 3$, there is a couple (α, β) which satisfies the inequality (3.5) and, consequently, $\mathcal{J}(u_{\lambda,\alpha,\beta}) \to -\infty$. The restriction $p \geqslant 4$ appears in proving the Palais–Smale condition.

Proof of theorem 1.2. Lemma 3.2 and theorem 3.3. □

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