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A Remark on the Number of Complete and Empty Subgraphs

RICHARD H. SCHELP[1]† and ANDREW THOMASON[2]

[1,2] Department of Mathematical Sciences,
University of Memphis, Memphis TN 38152, USA
(e-mail: schelpr@msci.memphis.edu)

[2] Department of Pure Mathematics and Mathematical Statistics,
16 Mill Lane, Cambridge CB2 1SB, UK
(e-mail: A.G.Thomason@dpmms.cam.ac.uk)

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Let $k_p(G)$ denote the number of complete subgraphs of order p in the graph G. Bollobás proved that any real linear combination of the form $\sum a_p k_p(G)$ attains its maximum on a complete multipartite graph. We show that the same is true for a linear combination of the form $\sum a_p k_p(G) + b_p k_p(\overline{G})$, provided $b_p \geqslant 0$ for every p.

A theorem of Bollobás [2] (see the elegant proof on page 298 of [1]) states that if $k_p(G)$ is the number of complete subgraphs of order p in a graph G, then the real linear combination $bk_p(G) + ck_q(G)$ finds its maximum (among graphs of given order) on a complete multipartite graph; the same, therefore, also holds true for the minimum. The proof readily generalizes to linear combinations involving more than two terms. However, there can be no theorem like Bollobás's involving general linear combinations of *both* $k_p(\overline{G})$ and $k_p(G)$, where $\overline{G}$ is the complement of G. It is known that the minimum of $k_3(\overline{G}) + k_3(G)$ is not attained on a complete multipartite graph or its complement if n is odd (see Goodman [3] and Lorden [5]). Though finding the minimum of $k_p(\overline{G}) + k_p(G)$ remains open for $p \geqslant 4$, it is certain that the extremal graphs are neither Turán graphs nor their complements (see Jagger, Št'ovíček and Thomason [4] and Thomason [7]).

Nevertheless, Bollobás's theorem does extend to incorporate linear combinations involving the terms $k_p(\overline{G})$ provided the coefficients of these extra terms are *positive*.

In fact, the theorem can be extended still further. Given a graph F, let $i_F(G)$ be the number of induced subgraphs of G isomorphic to F. Thus $k_p(\overline{G}) = i_{\overline{K}_p}(G)$. Then it can

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be shown that any real linear combination $\sum_F \alpha_F i_F(G)$, the sum being over complete multipartite graphs F, finds its maximum on a complete multipartite graph G provided $\alpha_F \geqslant 0$ whenever F is not complete.

Theorem 1. *Let $i_F(G)$ denote the number of induced subgraphs of G isomorphic to F. Let $f(G) = \sum_F \alpha_F i_F(G)$ be a real linear combination where each F is a complete multipartite graph and where $\alpha_F \geqslant 0$ unless F is complete. Then, among graphs of given order, the function $f(G)$ finds its maximum on a complete multipartite graph.*

Moreover, if $\alpha_{\overline{K}_3} > 0$ then $f(G)$ has no other maxima.

Proof. We may suppose that $\alpha_{\overline{K}_3} > 0$, the case $\alpha_{\overline{K}_3} = 0$ following by a limiting argument. Choose a graph G of order n which maximizes $f(G)$. Suppose G is not a complete multipartite graph. Then G contains non-adjacent vertices x and y whose neighbourhoods X and Y differ.

The number $i_F(G)$ comprises contributions from four different kinds of induced subgraph isomorphic to F, namely those containing, respectively, x but not y, y but not x, both x and y, and neither x nor y. Note that the first contribution depends only on X and the second only on Y. Moreover, the third depends only on $X \cap Y$ and $V - (X \cup Y)$ where $V = V(G)$, since no copy of the complete multipartite graph F can contain vertices outside these two sets if it contains both x and y. This fourfold division of $i_F(G)$ gives rise to a corresponding partition of the sum $f(G)$; we denote this by $f(G) = g(X) + g(Y) + h(X \cap Y, V - (X \cup Y)) + C$ where C is independent of X and Y. Note that $h(A, B) \leqslant h(A', B')$ if $A \subset A'$ and $B \subset B'$ because $i_F(G)$ makes no contribution to $h(A, B)$ if F is complete and $\alpha_F \geqslant 0$ otherwise. Moreover, if $B \neq B'$ then $h(A, B) < h(A', B')$, because $\alpha_{\overline{K}_3} > 0$ and $i_{\overline{K}_3}(G)$ contributes exactly $\alpha_{\overline{K}_3}|B|$ to $h(A, B)$.

We may suppose that $g(X) \geqslant g(Y)$ and, if $g(X) = g(Y)$, that $|X| \leqslant |Y|$ and so $X \neq X \cup Y$. Therefore $g(X) + h(X, V - X) > g(Y) + h(X \cap Y, V - (X \cup Y))$. Form H from G by removing the edges between y and Y and inserting all edges between y and X. Then $f(H) = 2g(X) + h(X, V - X) + C > f(G)$, contradicting the choice of G. □

This proof is, of course, essentially that of Bollobás. Curiously enough, though, the addition of the extra terms makes the proof technically simpler; the need in the original proof to introduce weights as a tie-breaking device is obviated here.

We remark that $f(G)$ has no other maxima also in the case $\alpha_P > 0$ where P is the graph of order 3 with 2 edges. In this case $h(A, B)$ is strictly increasing in A and we choose $|X| \geqslant |Y|$ if $g(X) = g(Y)$.

Our interest in Theorem 1 was aroused by our need to find the minimum of the graph function $(|G|-2)e(G)+2k_3(\overline{G})-2k_3(G)$ which arose in our study of 4-cycles in the cube [6]. As an application of Theorem 1 we therefore offer the following.

Theorem 2. *For every graph G of order n, the inequalities*

$$\binom{n}{3} \leqslant (n-2)e(G) + 2k_3(\overline{G}) - 2k_3(G) \leqslant 2\binom{n}{3}$$

hold. The upper bound is attained only when G is the complete bipartite graph $K_{k,n-k}$, where $0 \leqslant k \leqslant n$. The lower bound is attained only when $\overline{G} = K_{k,n-k}$.

Proof. There are not really two things to prove here; if we let $f(G) = (n-2)e(G) + 2k_3(\overline{G}) - 2k_3(G)$ then $f(\overline{G}) + f(G) = 3\binom{n}{3}$. So we establish just the upper bound.

Choose G so that $f(G)$ is maximal; by Theorem 1, G is a complete q-partite graph for some $q \geqslant 1$. It is easily checked that if $q \leqslant 2$ then $f(G) = 2\binom{n}{3}$, as claimed. Suppose then that $q \geqslant 3$. Let two of the vertex classes have orders a and b and let G' be the complete $(q-1)$-partite graph obtained from G by removing the ab edges between these two classes. Then

$$f(G') - f(G) = -(n-2)ab + 2\binom{a}{2}b + 2a\binom{b}{2} + 2ab(n-a-b) = ab(n-a-b) > 0,$$

a contradiction to the choice of G. □

The parameter $i_F(G)$ equals zero if G is complete multipartite but F is not. It follows that Theorem 1 cannot be extended to cover such graphs F; even the simple function $f(G) = i_F(G)$ will find its maximum somewhere other than on a complete multipartite graph.

A somewhat more informative example is the following. Let Q be the path of length 3 and let R be the triangle with a pendant edge. Neither Q, R nor their complements appear as induced subgraphs of a complete multipartite graph. Therefore, if G or $\overline{G}$ is a complete multipartite graph then the function $f(G) = i_Q(G) - i_R(G)$ is zero. However, if G is a random graph of order n with edge probability p then $f(G)$ is likely to be approximately $12[p^3(1-p)^3 - p^4(1-p)^2]\binom{n}{4}$. So if p is small then $f(G) > 0$ and if p is near to 1 then $f(G) < 0$. Hence $f(G)$ is neither maximized nor minimized on a complete multipartite graph or on the complement of a complete multipartite graph.

Added in proof

The transformation used in the proof of the main theorem does not increase the clique size or the chromatic number. Therefore the theorem remains true even if the maximum is sought over all graphs of given order and bounded clique size and chromatic number.

References

[1] Bollobás, B. (1978) *Extremal Graph Theory*. Academic Press, London.

[2] Bollobás, B. (1976) On complete subgraphs of different orders. *Math. Proc. Cambridge Phil. Soc.* **79** 19–24.

[3] Goodman, A. W. (1959) On sets of acquaintances and strangers at any party. *Amer. Math. Monthly* **66** 778–783.

[4] Jagger, C. N., Šťovíček, P. and Thomason, A. (1996) Multiplicities of subgraphs. *Combinatorica* **16** 123–141.

[5] Lorden, G. (1962) Blue-empty chromatic graphs. *Amer. Math. Monthly* **69** 114–120.

[6] Schelp, R. and Thomason, A. G. On quadrilaterals in layers of the cube and extremal problems for directed and oriented graphs. Submitted to *J. Graph Theory*.

[7] Thomason, A. (1997) Graph products and monochromatic multiplicities. *Combinatorica* **17** 125–134.