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ERRATUM

Erratum to 'From Curves to Currents'

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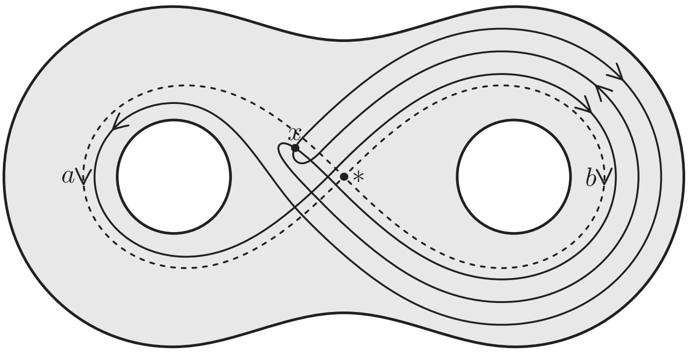
In the paper 'From Curves to Currents', the authors gave a definition of *essential crossing* [MGT21, Def. 2.6] that is too permissive. The main results remain true (with natural interpretation), and only one small part of the proof is affected.

For the issue with the definition, on the pair of pants P, consider the curve γ shown solid below, with marked crossing x.



If we take the natural generators a, b of $\pi_1(P, *)$ shown dashed, then γ is in the free homotopy class of $[ab]$. To see that the x fits [MGT21, Def. 2.6], note it divides γ into two subpaths, representing $p = bab$ and $q = B$ (adopting the convention that capital letters mean inverse, and connecting x to $*$ by the short path). Since $pq = ba$ and $qp = ab$, the corresponding lifts of γ to $\widetilde{P}$ cross at infinity since their axes $\overrightarrow{ab}$ and $\overrightarrow{ba}$ cross. But x is not an essential crossing in any reasonable sense: it can clearly be isotoped away, and if we take lengths with respect to any Riemannian metric, $\ell([pQ]) = \ell([ab^3]) > \ell([ab]) = \ell([pq])$, and not the other way around.

As a result, [MGT21, Lemma 2.8] is wrong. Instead, we essentially take the conclusion of that lemma as a definition.

Definition 1. A representative γ of a multi-curve C is *minimal* if it has only transverse self-intersections and a minimum number of them. A crossing that appears in some minimal representative is said to be *essential* and induces a reduction relation $C \searrow C_1$ and $C \searrow C_2$ for the two different ways of smoothing the crossing.

'Some' above can be replaced by 'every':

Lemma 2. *If there is a crossing x in some minimal representative γ of C, then for any other representative γ', there is a corresponding crossing x' giving the same smoothings.*

Proof. If γ is connected, by a result of Hass and Scott [HS94], γ' can be turned into the minimal form γ using Reidemeister I, II and III moves, with the Reidemeister I and II moves being used only in the forward (simplifying) direction. Since we know that γ has a crossing of the desired type, we can trace the crossings backwards through these moves: there is a bijection between the crossings before and after a Reidemeister III move that does not change the homotopy types of curves achievable by a single smoothing, and we can ignore the additional crossings created by backwards Reidemeister I and II moves.

For multi-curves, Hass and Scott show that we need to add one more move, swapping the position of two components that are homotopically powers of the same primitive curve δ [HS94, pp. 31–32]. Again, this operation induces a bijection between crossings that preserves the curves that result from smoothing. □

To give a more workable and algebraic criterion for essential crossings, we adopt a group-theoretic point of view. For $a, b \in \pi_1(S, *)$, write $[a]$ for the free homotopy class of curves represented by a; algebraically, this is the union of the conjugacy classes of a and of A. Write $[a][b]$ for the multi-curve $[a] \cup [b]$. Say that a factorization $[ab]$ *reduces* if some (equiv. every) minimal representative of $[ab]$ has a crossing so that one smoothing gives $[a][b]$ and the other smoothing gives $[aB]$. Similarly say that $[a][b]$ *reduces* if some/every minimal representative has a crossing whose smoothings give $[ab]$ and $[aB]$. (This is an abuse of notation since the reduction is not invariant under individual conjugation of a and b.)

For $a, b \in \pi_1(S, *)$, let $a * b$ be a 4-valent graph mapped into S obtained by taking the union of the paths a, b joined at the basepoint $*$, and let $[a * b]$ be its free homotopy class. Free homotopy means in particular invariance under simultaneous conjugation: $[a * b] = [caC * cbC]$. A *taut* representative of $[a * b]$ is one with a minimum number of transverse crossings.

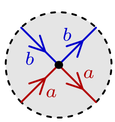
Finally, if we fix a hyperbolic structure on S, a nontrivial element $a \in \pi_1(S, *)$ determines a geodesic $\overrightarrow{a}$ in $\widetilde{S}$, the unique geodesic that stays a bounded distance from $a^k \cdot \widetilde{*}$, with endpoints a^+ and a^-. Depending on the cyclic order of the endpoints up to orientation reversal, we say that $\overrightarrow{a}$ and $\overrightarrow{b}$ are

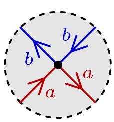
- *parallel* if they appear in order a^-, a^+, b^+, b^-;
- *anti-parallel* if they appear in order a^-, a^+, b^-, b^+; and
- *cross* if they appear in order a^-, b^-, a^+, b^+.

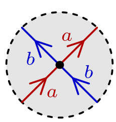
We can now reformulate essential crossings in the (interesting) first case of [MGT21, Def. 2.6], when a and b are not common powers.

Proposition 3. *Let $a, b \in \pi_1(S, *)$ be nontrivial elements that are not common powers. Then in the following three trialities, the same possibility holds in each case.*

1. *(Reduction) One of the following factorizations reduces to the other two:*
   (a) $[ab]$;   (b) $[aB]$; or   (c) $[a][b]$.
2. *(Geometric) For every taut embedding of $a * b$, around the 4-valent vertex we see in cyclic order, up to orientation reversal*

(a) 

; *(b)* 

; *or* *(c)* 

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3. *(Algebraic) The geodesics* $\overrightarrow{a}$ *and* $\overrightarrow{b}$ *are*
(a) parallel; (b) anti-parallel; or (c) cross.

Proof. Since a and b are not common powers, the endpoints of $\overrightarrow{a}$ and $\overrightarrow{b}$ are all distinct from each other, so the algebraic triality is mutually exclusive and exhaustive. For the geometric triality, for any given embedding of $a * b$, one of the three given possibilities holds, but it is not a priori clear that the same possibility holds for all taut embeddings. For the reduction triality, the possibilities are mutually exclusive (we cannot have circular reductions since at each reduction, length with respect to any Riemannian metric strictly decreases), but now it is not clear that any of the three possibilities holds.

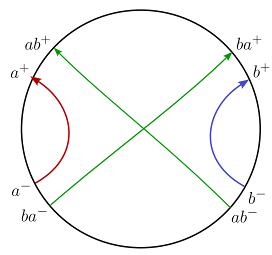
We start with the geometric triality. Consider any taut representative ρ of $[a*b]$ in one of the cases (a), (b) or (c). Consider the (concrete) multi-curve γ given by replacing the 4-valent vertex with a crossing. If we pick orientations on γ correctly, this is an embedding of $[ab]$, $[aB]$ or $[a][b]$, respectively, with a distinguished crossing x.

We claim that γ is minimal. Otherwise, there is some reducing isotopy using RIII moves and some RI/RII reduction [HS94]. Any RIII move on γ can be copied with an isotopy of ρ, as can an RI or RII reduction of γ that does not involve x. An RI reduction at x is only possible if a or b is the identity, and an RII reduction involving x also reduces ρ, reducing the number of intersections of ρ by one. In any case, this contradicts minimality of ρ or nontriviality of a, b.

We have thus shown that one of $[ab]$, $[aB]$ or $[a][b]$ has a minimal representative γ with a crossing x that can be smoothed to give the other two multi-curves. Since we cannot have a circular chain of essential smoothings (for instance. because we get strict inequalities of hyperbolic length), it follows that the cases are exclusive and thus that *every* taut representative of $[a * b]$ has the same cyclic order around the 4-valent vertex. Combined, this proves equivalence of the first two trialities.

For the remaining equivalence, suppose first we are in case (c) of the geometric triality. For the fixed hyperbolic structure, take γ to be the geodesic representative of the free homotopy class $[a][b]$, intersecting at the distinguished point $x \in S$, and let $\overrightarrow{a}$ and $\overrightarrow{b}$ be the respective lifts passing through some lift $\widetilde{x}$ of x. Then $\overrightarrow{a}$ and $\overrightarrow{b}$ intersect only once and thus cross at infinity, so we are also in case (c) of the algebraic triality.

If we are in case (a) of the geometric triality, again take the geodesic representative γ of ab, with distinguished crossing x. For concreteness, if necessary, reverse the orientation of S so the branches appear in the order shown below. If we perform the disconnected (oriented) smoothing at x, we get two components $\gamma_a \in [a]$ and $\gamma_b \in [b]$; these concrete curves are typically not minimal. However, γ_a and γ_b are regular isotopic to a +minimal curve [Thu14, Lemma 5.5] (i.e., there is no RI move required in making them minimal). This implies that their lifts $\widetilde{\gamma_a}$ and $\widetilde{\gamma_b}$ to $\widetilde{S}$ passing through a neighborhood of $\widetilde{x}$ are embedded in the disk (see, for example, [Thu14, Lemma 2.21]). Now consider the endpoints a^+, a^- of $\widetilde{\gamma_a}$. As an oriented curve, $\widetilde{\gamma_a}$ is geodesic outside of neighborhoods of lifts of x, and inside those neighborhoods turns to the left. It then follows that $(ab)^+, a^+, a^-, (ba)^-$ occur in that cyclic order on $\partial\widetilde{S}$, and similarly, $(ab)^-, b^-, b^+, (ba)^+$ occur in that cyclic order:



It therefore follows that $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel, so we are in case (a) of the algebraic triality. Case (b) follows in the same way, interchanging $b \leftrightarrow B$. □

The case when a and b are common powers is handled by the following result.

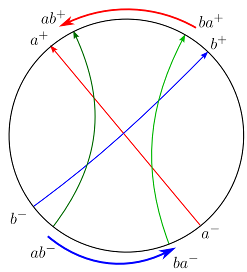
Proposition 4. *If $a = c^k$ and $b = c^\ell$ are common powers of a primitive element c with $k, \ell \geq 1$, then the factorization $[ab]$ reduces.*

Proof. See Hass and Scott [HS85, Lemma 1.12]. □

As a result of Proposition 3, [MGT21, Def. 2.6] is correct when looking at crossing between two different components of a multi-curve. There is only one place in the proof of our main results where we smoothed a crossing of one component with itself – namely, step (iii′) of the proof of [MGT21, Lemma 9.1(b)]. In the terminology there, we must verify that $\overrightarrow{\alpha_1}$ and $\overrightarrow{\beta_1}$ are parallel, whereas the written proof uses [MGT21, Lemma 12.3] to prove that $\overrightarrow{\gamma_1} = \overrightarrow{\alpha_1\beta_1}$ and $\overrightarrow{\gamma_2} = \overrightarrow{\beta_1\alpha_1}$ cross, which, as explained above, is not sufficient. However, by [MGT21, Lemma 12.2], $\overrightarrow{\alpha_1}$ and $\overrightarrow{\beta_1}$ both cross $\overrightarrow{\delta}$ in the same sense, so they cannot be anti-parallel. To see that $\overrightarrow{\alpha_1}$ and $\overrightarrow{\beta_1}$ cannot cross (and so must be parallel), use the lemma below and the established fact that $\overrightarrow{\gamma_1}$ and $\overrightarrow{\gamma_2}$ cross.

Lemma 5. *If a and b are two hyperbolic elements of $PSL(2, \mathbb{R})$ with crossing axes, then ab and ba are also hyperbolic with parallel axes.*

Proof. We will prove that the endpoints of the various geodesics must be laid out as follows, up to orientation reversal:



By the North-South dynamics of a and b on the circle, ab maps the interval (a^+, b^+) strictly inside itself, which implies that it is hyperbolic with attracting endpoint $(ab)^+$ in that interval; similarly, ab maps (a^-, b^-) to a strict superset of itself, so $(ab)^- \in (a^-, b^-)$. The actions of a and b interchange $(ab)^- \leftrightarrow (ba)^-$ and $(ab)^+ \leftrightarrow (ba)^+$, forcing the configuration shown. (The endpoints $(ab)^\pm, (ba)^\pm$ are necessarily on the portions of the circle where a and b move in opposite directions.) □

Remark 6. A very similar argument shows that if $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel, then $\overrightarrow{ab}$ and $\overrightarrow{ba}$ cross. So in the self-crossing case, the condition in [MGT21, Def. 2.6] is necessary but not sufficient.

Competing interest. The authors have no competing interest to declare.

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