Ergod. Th. & Dynam. Sys., (2023), **43**, 1492–1510
doi:10.1017/etds.2022.12

Topological full groups of minimal subshifts and quantifying local embeddings into finite groups

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(*Received* 25 *July* 2021 *and accepted in revised form* 16 *January* 2022)

Abstract. We investigate quantitative aspects of the locally embeddable into finite groups (LEF) property for subgroups of the topological full group $[\![\sigma]\!]$ of a two-sided minimal subshift over a finite alphabet, measured via the LEF growth function. We show that the LEF growth of $[\![\sigma]\!]'$ may be bounded from above and below in terms of the recurrence function and the complexity function of the subshift, respectively. As an application, we construct groups of previously unseen LEF growth types, and exhibit a continuum of finitely generated LEF groups which may be distinguished from one another by their LEF growth.

Key words: LEF, topological full group, minimal subshift
2020 Mathematics Subject Classification: 20E26 (Primary); 20F69, 37B10 (Secondary)

1. *Introduction*

Often in geometric group theory, one considers a *growth function* F_Γ, which describes some part of the asymptotic structure of a finitely generated group Γ. Examples include subgroup growth, word growth, conjugacy growth, the Dehn function, the Følner function and residual finiteness growth. Having introduced F_Γ, it is always natural to attempt to estimate F_Γ for some group Γ of interest, to relate the behaviour of F_Γ to structural features of Γ or its actions, and to explore the types of functions which can arise as F_Γ for some Γ. In this paper we make contributions to all three of these themes for the *LEF growth* function, by examining some groups arising in Cantor dynamics.

1.1. *Statement of results.*

A group Γ is *locally embeddable into finite groups* (LEF) if every finite subset of Γ admits an injective partial homomorphism (a local embedding) into a finite group. In other words, every finite subset of the multiplication table of Γ occurs in the multiplication table of some finite group. If Γ is finitely generated by S, the prototypical finite subsets are the balls $B_S(n)$ in the associated word metric. The *LEF growth function* $\mathcal{L}_\Gamma^S$ sends $n \in \mathbb{N}$ to the minimal order of a finite group into which $B_S(n)$ locally embeds. The dependence of the function on S is slight, so we suppress S from our notation for the rest of this introduction.

That the topological full group $[\![\varphi]\!]$ of a Cantor minimal system (X, φ) is an LEF group was proved by Grigorchuk and Medynets [8]. We prove an effective version of their result, in the case of a two-sided minimal subshift (X, σ) over a finite alphabet. Let $R_X : \mathbb{N} \to \mathbb{N}$ be the *recurrence function* of (X, σ) (see Definition 2.12 below).

THEOREM 1.1. *Let* Γ *be a finitely generated subgroup of* $[\![\sigma]\!]$. *Then*

$$\mathcal{L}_\Gamma(n) \preceq (2R_X(n))!\,. \tag{1.1}$$

In particular, this inequality holds for $\Gamma = [\![\sigma]\!]'$.

For non-decreasing unbounded functions F_1 and F_2 we write $F_1 \preceq F_2$ if, up to constant rescaling of the argument, F_2 bounds F_1 from above (see Definition 2.4 below). Note that the Cantor minimal system (X, φ) being isomorphic to a minimal subshift is a necessary and sufficient condition for $[\![\varphi]\!]'$ to be finitely generated [13]. In the other direction, we have the following lower bound on the LEF growth of $[\![\sigma]\!]'$. Let $p_X : \mathbb{N} \to \mathbb{N}$ be the *complexity function* of the minimal subshift (X, σ) (see Definition 2.10 below).

THEOREM 1.2. *There exists* $c > 0$ *such that* $\exp(cp_X(n^{1/2})) \preceq \mathcal{L}_{[\![\sigma]\!]'}(n)$.

Our proof yields $c = \log(60)/9 \approx 0.455$, but small modifications to the argument would enable us to make c arbitrarily large. The group $[\![\sigma]\!]'$ always has exponential word growth, which immediately implies that $\mathcal{L}_{[\![\sigma]\!]'}$ grows at least exponentially. One consequence of Theorem 1.1 is that when the subshift X is extremely ‘orderly’, this exponential lower bound is close to best possible.

Example 1.3. Let X be a linearly recurrent subshift. Then

$$\exp(n) \preceq \mathcal{L}_{[\![\sigma]\!]'}(n) \preceq n!. \tag{1.2}$$

On the other hand, if X is highly non-deterministic, then Theorem 1.2 gives a novel lower bound for the LEF growth of $[\![\sigma]\!]'$.

Example 1.4. Let X be a subshift of positive entropy. Then

$$\exp(\exp(n^{1/2})) \preceq \mathcal{L}_{[\![\sigma]\!]'}(n). \tag{1.3}$$

See §2.2 for definitions of linear recurrence and entropy. Although the upper and lower bounds proved in Theorems 1.1 and 1.2 are some distance apart, they are powerful enough to allow us to observe new phenomena in the kinds of functions which can arise as the LEF growth functions of groups. To achieve this, we adapt a construction of Jung, Lee and Park [10] to obtain a sufficient diversity of subshifts.

THEOREM 1.5. *For every $r \in [2, \infty)$, there are a minimal subshift (X_r, σ), constants $C_r, c_r > 0$, and an increasing sequence $(n_i^{(r)})$ of integers such that:*

(i) *for all $n \geq 2$, $R_{X_r}(n) \leq \exp(C_r(\log n)^r)$;*

(ii) *for all $i \in \mathbb{N}$, $p_{X_r}(n_i^{(r)}) \geq \exp(c_r(\log n_i^{(r)})^r)$.*

Applying Theorems 1.1 and 1.2 to the groups $[\![\sigma]\!]'$ arising from the subshifts (X_r, σ) of Theorem 1.5 yields that uncountably many inequivalent LEF growth functions occur among finitely generated LEF groups, answering a question posed in [5].

THEOREM 1.6. *For any $r \in [2, \infty)$, there exists a finitely generated LEF group $\Gamma^{(r)}$ such that:*

(i) *there exists $C_r > 0$ such that $\mathcal{L}_{\Gamma^{(r)}}(n) \preceq \exp(\exp(C_r(\log n)^r))$;*

(ii) *for any $2 \leq r' < r$, and for all $C > 0$, $\mathcal{L}_{\Gamma^{(r)}}(n) \npreceq \exp(\exp(C(\log n)^{r'}))$.*

Consequently, there is an uncountable family $\mathcal{F}$ of pairwise non-isomorphic finitely generated LEF groups such that, for $\Gamma_1, \Gamma_2 \in \mathcal{F}$, if $\Gamma_1 \neq \Gamma_2$ then $\mathcal{L}_{\Gamma_1} \not\approx \mathcal{L}_{\Gamma_2}$.

We write $F_1 \approx F_2$ if $F_1 \preceq F_2$ and $F_2 \preceq F_1$ (see Definition 2.4 below). Conclusions (i) and (ii) of Theorem 1.6 for the group $\Gamma^{(r)} = [\![\sigma]\!]'$ follow from Theorem 1.5(i) and (ii), by Theorems 1.1 and 1.2, respectively. Further examples of subshifts of 'intermediate' complexity could be a rich source of examples of new exotic behaviours in the LEF growth of groups, and this should be investigated further.

1.2. *Background.* The concept of an LEF group first appears in the work of Mal'cev, but was developed and popularized by Vershik and Gordon [16]. All residually finite groups are LEF, including all finitely generated nilpotent or linear groups, but local embeddability into finite groups enjoys some closure properties that residual finiteness does not: for instance, the (regular restricted) wreath product of LEF groups is LEF. Among finitely presented groups, the classes of LEF and residually finite groups coincide, and this observation provides a useful tool for proving that certain groups are not finitely presentable (see [8] for a proof along these lines for derived subgroups of topological full groups). LEF groups have also been studied in connection with weaker approximation properties of groups, such as soficity and hyperlinearity, since they provide a source of examples beyond those arising from residual finiteness or amenability. For instance, Elek and Szabó [7] used the LEF property to construct the first examples of sofic groups which are not residually amenable.

The LEF growth function was introduced independently in [1, 4] (in the latter under the name *geometric full residual finiteness growth*), and fits into the extensive literature on quantifying finite approximations of infinite groups which has developed over the last decade. This program started with the work of Bou-Rabee and collaborators on quantitative residual finiteness (see [3] and the references therein). Using results on quantitative residual finiteness, word growth, and finite presentability, the LEF growth function has been estimated for several natural classes of groups (see [5, §2.4]).

Example 1.7. Let Γ be a finitely generated group.

(i) If Γ is virtually $\mathbb{Z}^d$, then $\mathcal{L}_\Gamma(n) \approx n^d$.

(ii) $\mathcal{L}_\Gamma$ is bounded above by a polynomial function if and only if Γ is virtually nilpotent.

(iii) If $\Gamma \leq \mathrm{GL}_d(\mathbb{Z})$ is finitely generated, not virtually nilpotent, then $\mathcal{L}_\Gamma(n) \approx \exp(n)$.

Groups of larger LEF growth can be explicitly constructed using wreath products.

THEOREM 1.8. [5, Theorem 1.8] *If* Γ *is a finitely generated LEF group with word growth function* γ_Γ, *and* Δ *is a finite non-trivial centreless group, then* $\exp(\gamma_\Gamma(n)) \preceq \mathcal{L}_{\Delta\wr\Gamma}(n) \preceq \exp(\mathcal{L}_\Gamma(n))$.

In particular, using Example 1.7, Theorem 1.8 allows us to construct groups of LEF growth $\approx \exp(\exp(n))$ and $\approx \exp(n^d)$ (for any $d \in \mathbb{N}$). The only other functions which have been observed to arise as LEF growth functions of groups are inexplicit and very large (see [5, §5]). Our Theorem 1.6 therefore greatly extends the spectrum of known growth types.

The derived subgroup of the topological full group $[\![\sigma]\!]$ of a minimal subshift (X, σ) is a remarkable object in group theory. It is a finitely generated infinite simple group, which, as well as being LEF, is amenable [12] (and indeed was the first group discovered with this combination of properties). It is also a natural invariant from the point of view of topological dynamics. As shown in [2], for any Cantor minimal system (X, φ), $[\![\varphi]\!]'$ retains perfect information about the dynamics of (X, φ).

THEOREM 1.9. (Bezuglyi and Medynets) *Let* (X, φ) *and* (Y, ψ) *be Cantor minimal systems. Then* $[\![\varphi]\!]' \cong [\![\psi]\!]'$ *if and only if* (X, φ) *and* (Y, ψ) *are flip-conjugate.*

It is therefore reasonable to expect that group-theoretic asymptotic invariants of $[\![\varphi]\!]'$ should reflect asymptotic features of the dynamical system (X, φ). Our Theorems 1.1 and 1.2 are in this spirit: knowing the LEF growth function of $[\![\sigma]\!]'$ allows one to deduce some bounds on the recurrence or complexity functions of X.

1.3. *Methods and structure of the paper.* Our proof of Theorem 1.1 is based on Elek's streamlined proof of LEF for topological full groups [6]. Given $\Gamma \leq [\![\sigma]\!]$, finitely generated by S, and a two-sided sequence $\mathbf{x} \in X$, the σ-orbit O of $\mathbf{x}$ is dense in X, so Γ acts faithfully on O. Further, any short word in S moves some cylinder set $C \subseteq X$, defined by a short string in $\mathbf{x}$. Since only finitely many such cylinder sets C arise, each of which intersects O, there exists $M \in \mathbb{N}$ such that no non-identity element of $B_S(n)$ fixes $\{\sigma^i\mathbf{x} : 1 \leq i \leq M\}$ pointwise; moreover, we can take $M \leq R_X(Cn)$ for some $C > 0$. Carefully choosing the exact value of M to ensure consistency, we use this to construct a local embedding $B_S(n) \to \mathrm{Sym}(M)$.

For our lower bound, we observe that $[\![\sigma]\!]'$ contains many copies of the alternating group Alt(5), acting on disjoint subsets of X (hence generating their direct product). It follows that any finite group admitting a local embedding of a large ball in $[\![\sigma]\!]'$ also contains a direct product of many copies of Alt(5) as a subgroup, and so has large order. The supply of disjoint subsets on which to act, in this construction, is limited by the complexity function p_X, hence the appearance of p_X in Theorem 1.2.

This paper is structured as follows. In §§2.1–2.3 we collect necessary background results about LEF growth of groups, symbolic dynamics, and topological full groups, respectively. In §3 we construct the local embeddings required to prove Theorem 1.1. In §4 we prove

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Theorem 1.2. In §5 we describe the construction of the minimal subshifts arising in Theorem 1.5, and deduce Theorems 1.6.

2. *Preliminaries*

2.1. *LEF groups and Schreier graphs.*

Definition 2.1. For Γ, Δ groups and $F \subseteq \Gamma$, a *partial homomorphism* of F into Δ is a function $\phi : F \to \Delta$ such that, for all $g, h \in F$, if $gh \in F$, then $\phi(gh) = \phi(g)\phi(h)$. A partial homomorphism ϕ is called a *local embedding* if it is injective. Γ is *locally embeddable into finite groups* if, for all finite $F \subseteq \Gamma$, there exist a finite group Q and a local embedding of F into Q.

Henceforth suppose that Γ is LEF and generated by the finite set S. Let $B_S(n) \subseteq \Gamma$ denote those elements of length at most n, with respect to the word metric induced on Γ by S.

Definition 2.2. The *LEF growth* of Γ (with respect to S) is

$$\mathcal{L}_\Gamma^S(n) = \min\{|Q| : \text{ there exists } \phi : B_S(n) \to Q \text{ a local embedding}\},$$

and the *LEF action growth* is

$$\mathcal{LA}_\Gamma^S(n) = \min\{d : \text{ there exists } \phi : B_S(n) \to \mathrm{Sym}(d) \text{ a local embedding}\}.$$

Remark 2.3. It is clear that $\mathcal{LA}_\Gamma^S(n) \leq \mathcal{L}_\Gamma^S(n) \leq \mathcal{LA}_\Gamma^S(n)!$, though of course typically neither inequality will be sharp. Theorem 1.1 will be obtained from combining the second of these inequalities with an upper bound on the LEF action growth (see Theorem 3.7). We comment below (Remark 3.8) on whether our bound in Theorem 1.1 could be improved by avoiding the use of LEF action growth. Meanwhile the proof of Theorem 1.2 also yields a bound on $\mathcal{LA}$ which is slightly stronger than that which could be obtained by using the above inequalities and treating the conclusion of Theorem 1.2 as a black box (see Remark 4.9). In summary, for $\Gamma = [\![\sigma]\!]'$ as in Theorems 1.1 and 1.2, we have

$$p_X(n^{1/2}) \preceq \mathcal{LA}_\Gamma(n) \preceq 2R_X(n).$$

Definition 2.4. For $F_1, F_2 : \mathbb{N} \to \mathbb{N}$ non-decreasing functions, write $F_1 \preceq F_2$ if there exists $C > 0$ such that $F_1(n) \leq F_2(Cn)$ for all n. Write $F_1 \approx F_2$ if $F_1 \preceq F_2$ and $F_2 \preceq F_1$.

LEMMA 2.5. *Let $\mathcal{F} = \mathcal{L}$ or $\mathcal{LA}$. Let $\Delta \leq \Gamma$ be finitely generated by T. Then there exists $C > 0$ such that, for all n,*

$$\mathcal{F}_\Delta^T(n) \leq \mathcal{F}_\Gamma^S(Cn)$$

In particular, for T a second finite generating set for Γ, $\mathcal{F}_\Gamma^S \approx \mathcal{F}_\Gamma^T$.

Proof. This is proved for $\mathcal{F} = \mathcal{L}$ as Corollary 2.7 in [5]; the proof for $\mathcal{F} = \mathcal{LA}$ is identical. □

The next proposition is key to the proof of Theorem 1.2. It uses an idea already exploited in [5, Theorem 3.4] to control the LEF growth of wreath products.

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PROPOSITION 2.6. *Let $n \in \mathbb{N}$ and $m \geq 2$. Suppose $\Delta_1, \ldots, \Delta_m \leq \Gamma$ are finite centreless subgroups, generating their direct product, and that $\Delta_i \subseteq B_S(n)$. Suppose that Q is a finite group and that $\phi : B_S(2n) \to Q$ is a local embedding. Then the $\phi(\Delta_i) \leq Q$ generate their direct product, and $|Q| \geq \prod_i |\Delta_i|$.*

Proof. Since ϕ restricts to an injective homomorphism on each Δ_i, $\phi(\Delta_i)$ is a subgroup of Q, isomorphic to Δ_i. Certainly, for $i \neq j$ and $g_i \in \Delta_i$, $g_j \in \Delta_j$,

$$\phi(g_i)\phi(g_j) = \phi(g_i g_j) = \phi(g_j g_i) = \phi(g_j)\phi(g_i).$$

Therefore, if the $\phi(\Delta_i)$ fail to generate their direct product, there exist $1 \leq i \leq m$ and $1 \neq g \in \Delta_i$ such that $\phi(g) \in P$, where $P = \langle \phi(\Delta_j) : i \neq j \rangle \leq Q$. P centralizes $\phi(\Delta_i)$, since the $\phi(\Delta_j)$ do, so for $h \in \Delta_i$, $\phi(gh) = \phi(g)\phi(h) = \phi(h)\phi(g) = \phi(hg)$. By injectivity of ϕ restricted to Δ_j, g is central in Δ_i, a contradiction. □

A *based graph* is a pair (G, v), where G is a directed graph and $v \in V(G)$. A *morphism* of based graphs $(G_1, v_1) \to (G_2, v_2)$ is a graph morphism $\phi : G_1 \to G_2$ with $\phi(v_1) = v_2$. For C a set, an *edge colouring* of the graph G in C is a function $c : E(G) \to C$. A morphism $(G_1, v_1, c_1) \to (G_2, v_2, c_2)$ of based graphs with edge colourings in C is a morphism $\phi : (G_1, v_1) \to (G_2, v_2)$ of based graphs such that, for all $e \in E(G_1)$, $c_1(e) = c_2(\phi(e))$.

Definition 2.7. Let Γ be a group, Ω be a Γ-set, and $S \subseteq \Gamma$. The associated *Schreier graph* $\mathrm{Schr}(\Gamma, \Omega, \mathbf{S})$ is the graph with vertex set Ω and edge set $\Omega \times S$, with the edge (ω, s) running from ω to $s\omega$. Impose an ordering on the elements of S, to obtain an ordered $|S|$-tuple $\mathbf{S} \in \Gamma^{|S|}$ (equivalently, fix a bijection $c : S \to \{1, \ldots, |S|\}$). Then $\mathrm{Schr}(\Gamma, \Omega, \mathbf{S})$ is naturally an edge-coloured graph with colours in $\{1, \ldots, |S|\}$, via $c((\omega, s)) = c(s)$.

Definition 2.8. Let G_1, G_2 be directed edge-coloured graphs (with colours in C) and let $r \in \mathbb{N}$. We say that G_1 is *locally embedded inG_2 at radius r* if, for every $v \in V(G_1)$, there exists $w \in V(G_2)$ and an isomorphism of based coloured graphs $(B_v(r), v) \cong (B_w(r), w)$ (here $B_v(r) \subseteq G_1$ is the induced subgraph on the closed ball of radius r around v in the path metric on G_1, and likewise for $B_w(r) \subseteq G_2$). We say that G_1 and G_2 are *locally colour isomorphic at radius r* if each is locally embedded in the other at radius r, that is, G_1 and G_2 have the same set of isomorphism types of balls of radius r.

The next observation is our key tool for constructing the local embeddings needed in Theorem 1.1; it is proved as Lemma 4.2 in [5].

LEMMA 2.9. *For $i = 1, 2$, let Γ_i be a group acting faithfully on a set Ω_i, and let $\mathbf{S}_i$ be an ordered generating d-tuple in Γ_i. Suppose that the Schreier graphs $\mathrm{Schr}(\Gamma_i, \Omega_i, \mathbf{S}_i)$ are locally colour-isomorphic at some radius at least $\lceil 3r/2 \rceil$. Then there is a local embedding $B_{\mathbf{S}_1}(r) \to \Gamma_2$ extending $(\mathbf{S}_1)_j \mapsto (\mathbf{S}_2)_j$.*

2.2. *Words.* Throughout, the *alphabet* A will be a finite discrete set with $|A| \geq 2$. Let $A^* = \bigsqcup_n A^n$ be the set of *finite words* over A. An *infinite word* shall be an element of

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either $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$. We equip the latter sets with the product topology; note that both are thereby homeomorphic to the Cantor space.

Definition 2.10. Let $w \in A^{\mathbb{Z}}$. For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, the *kth n-factor* of w is $w_k w_{k+1} \cdots w_{k+n-1} \in A^n$. $v \in A^n$ is an n-factor of w if it is the kth n-factor for some k. n-factors of words in $A^{\mathbb{N}}$ and A^* are defined similarly, with the requirement that $k \in \mathbb{N}$ and, for $w \in A^*$, that $k \leq |w| - n + 1$.

In any case, the set of all n-factors of w is denoted by $F_n(w)$. For w an infinite word, the *complexity function* $p_w : \mathbb{N} \to \mathbb{N}$ of w is given by $p_w(n) = |F_n(w)|$.

It is immediate from the definitions that $p_w(n+m) \leq p_w(n)p_w(m)$ and $p_w(n) \leq |A|^n$ for all $n, m \in \mathbb{N}$.

Definition 2.11. The *entropy* of $w \in A^{\mathbb{Z}}$ is

$$h(w) = \lim_{n \to \infty} \frac{\log p_w(n)}{n}$$

(the limit is well defined, by the preceding remarks).

Definition 2.12. $\mathbf{x} \in A^{\mathbb{Z}}$ is *uniformly recurrent* if, for every $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that, for every $w \in F_n(\mathbf{x})$ and $v \in F_{M_n}(\mathbf{x})$, $w \in F_n(v)$. The smallest such M_n is denoted by $R_{\mathbf{x}}(n)$, and $R_{\mathbf{x}} : \mathbb{N} \to \mathbb{N}$ is the *recurrence function* of $\mathbf{x}$.

Henceforth assume $\mathbf{x} \in A^{\mathbb{Z}}$ to be uniformly recurrent non-periodic.

THEOREM 2.13. [14, Theorem 7.5] $R_{\mathbf{x}}(n) \geq p_{\mathbf{x}}(n) + n \geq 2n + 1$ *for all* $n \in \mathbb{N}$.

In particular, $R_{\mathbf{x}}$ grows at least linearly in n.

Definition 2.14. $\mathbf{x}$ is *linearly recurrent* if there exists $C > 0$ such that $R_{\mathbf{x}}(n) \leq Cn$ for all $n \in \mathbb{N}$.

Example 2.15. [15, Theorem 11.4] The Fibonacci word is linearly recurrent.

Definition 2.16. A *cylinder set* of $A^{\mathbb{Z}}$ is a set of the form

$$\ll u_k, \ldots, u_{-1}, \underline{u}_0, u_1, \ldots, u_l \gg = \{\mathbf{y} \in A^{\mathbb{Z}} : y_i = u_i \text{ for } k \leq i \leq l\},$$

where $k, l \in \mathbb{Z}$ with $k \leq 0 \leq l$ and $u_i \in A$ for $k \leq i \leq l$. For $u \in A^n$ and $1 \leq i \leq n$, we write $\ll u \gg_i$ for the cylinder set $\ll u_1, \ldots, u_{i-1}, \underline{u}_i, u_{i+1}, \ldots, u_n \gg$. Note that

$$\ll u_1, \ldots, u_{i-1}, \underline{u}_i, u_{i+1}, \ldots, u_n \gg = \ll v_k, \ldots, v_{-1}, \underline{v}_0, v_1, \ldots, v_l \gg$$

where $k = 1 - i$, $l = n - i$ and $v_j = u_{j+i}$ for $k \leq j \leq l$.

For $X \subseteq A^{\mathbb{Z}}$ clopen and $m \in \mathbb{N}$, an *m-cylinder of X* is a non-empty set of the form

$$X \cap \ll u_{-m}, \ldots, u_{-1}, \underline{u}_0, u_1, \ldots, u_m \gg .$$

The set of m-cylinders of X will be denoted by $\mathrm{Cyl}_X(m)$.

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Remark 2.17. Let $u \in A^n$ and $1 \leq i \leq n$.

(i) For any $k, l \in \mathbb{N}$,

$$\bigsqcup_{v \in A^k} \ll vu \gg_{k+i} \;=\; \ll u \gg_i = \bigsqcup_{w \in A^l} \ll uw \gg_i$$

In particular, for any $m \in \mathbb{N}$, the set of all m-cylinders of X forms a clopen partition of X, and for $1 \leq k \leq m$, any k-cylinder of X is the disjoint union of the m-cylinders of X which intersect it.

(ii) The cylinder sets form a basis for the topology on $A^{\mathbb{Z}}$. Hence, for any clopen subset Y of $A^{\mathbb{Z}}$, Y is the union of cylinder sets. By compactness, and by (i), there exists $C = C(Y) > 0$ such that, for all $m \geq C$, Y is the disjoint union of m-cylinders of $A^{\mathbb{Z}}$.

(iii) Since $\mathcal{O}(\mathbf{x}) = \{\sigma^i(\mathbf{x}) : i \in \mathbb{Z}\}$ is dense in X, for $u_i \in A$, $U = \ll u_{-m}, \ldots, u_{-1}, \underline{u}_0, u_1, \ldots, u_m \gg$ intersects X if and only if for some $i \in \mathbb{Z}$, $\sigma^i(\mathbf{x}) \in U$, which occurs if and only if $u_j = x_{i+j}$ for $-m \leq j \leq m$. That is, U intersects X if and only if $u_{-m} \cdots u_m$ is a factor of $\mathbf{x}$. Thus the map sending $u_{-m} \cdots u_m$ to $U \cap X$ is a bijection from $F_{2m+1}(\mathbf{x})$ to $\mathrm{Cyl}_X(m)$, hence $|\mathrm{Cyl}_X(m)| = p_X(2m+1)$.

2.3. *Topological full groups and minimal subshifts.* Let $\mathbf{C}$ be the Cantor space. A *Cantor dynamical system* is a pair (X, φ), where X is a space homeomorphic to $\mathbf{C}$, and $\varphi \in \mathrm{Homeo}(X)$ (we specify the space X, rather than always taking $X = \mathbf{C}$, in case the homeomorphism φ is described in terms of a particular model X of Cantor space; in particular, this will be the case when the system is a subshift). The system (X, φ) is called *minimal* if every orbit in X under the action of $\langle \varphi \rangle$ is dense in X.

Example 2.18. Let A be a finite discrete space with $|A| \geq 2$. Then $A^{\mathbb{Z}} \cong \mathbf{C}$. The *shift* over A is $\sigma \in \mathrm{Homeo}(A^{\mathbb{Z}})$ given, for $\mathbf{a} = (a_i)_{i \in \mathbb{Z}}$, by $\sigma(\mathbf{a})_i = a_{i+1}$. The Cantor dynamical system $(A^{\mathbb{Z}}, \sigma)$ is never minimal.

Definition 2.19. The *topological full group* $[\![\varphi]\!]$ of the system (X, φ) is the set of all homeomorphisms g of X such that there exists a continuous function $f_g : X \to \mathbb{Z}$ (called the *orbit cocycle* of g) such that for all $x \in X$, $g(x) = \varphi^{f_g(x)}(x)$ (here we assume $\mathbb{Z}$ equipped with the discrete topology).

Equivalently, $g \in \mathrm{Homeo}(X)$ lies in $[\![\varphi]\!]$ if there exist a finite clopen partition $C_1, \ldots, C_d$ of X and integers $a_1, \ldots, a_d$ such that for $1 \leq i \leq d$, $g|_{C_i} = \varphi^{a_i}|_{C_i}$ (taking $\{a_1, \ldots, a_d\} = \mathrm{im}(f_g)$, $C_i = f_g^{-1}(a_i)$). It is straightforward to check that $[\![\varphi]\!]$ is a subgroup of $\mathrm{Homeo}(X)$.

Remark 2.20. If (X, φ) is a minimal system, then the orbit cocycle f_g is uniquely determined by $g \in [\![\varphi]\!]$, since φ has no periodic points.

The next result gives the key source of minimal systems for our purposes.

PROPOSITION 2.21. *Suppose* $\mathbf{x} \in A^{\mathbb{Z}}$ *is uniformly recurrent non-periodic. Then* $\overline{\mathcal{O}(\mathbf{x})} \cong \mathbf{C}$, *and the system* $(\overline{\mathcal{O}(\mathbf{x})}, \sigma)$ *is minimal.*

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Proof. For minimality, see [11, Theorem 1.0.1]. To prove $\overline{O(\mathbf{x})} \cong \mathbf{C}$, note that the only property not automatically inherited by subspaces of $\mathbf{C}$ is being perfect. Then, if $Y = \overline{O(\mathbf{x})}$, a non-periodic $\mathbf{x}$ must have $|Y| = \infty$, which implies that Y has an accumulation point in $\mathbf{C}$. But Y contains its set of accumulation points $\mathrm{Acc}(Y)$ in $\mathbf{C}$ (by compactness), and $\mathrm{Acc}(Y)$ is closed and σ-invariant, so by minimality $Y = \mathrm{Acc}(Y)$. □

Definition 2.22. A subspace $X = \overline{O(\mathbf{x})} \subseteq A^{\mathbb{Z}}$ constructed as in the statement of Proposition 2.21 is called a *minimal subshift.*

Remark 2.23. Suppose $\mathbf{x} \in A^{\mathbb{Z}}$ is uniformly recurrent non-periodic. For any $\mathbf{y} \in \overline{O(\mathbf{x})}$ and $n \in \mathbb{N}$, $F_n(\mathbf{x}) = F_n(\mathbf{y})$. In particular, $p_{\mathbf{x}}$ and $R_{\mathbf{x}}$ depend only on X, and we may henceforth write p_X or R_X instead.

Let $[\![\varphi]\!]'$ denote the derived subgroup of $[\![\varphi]\!]$. The reason for our focus on minimal subshifts, among all minimal systems, is made clear by the next result.

THEOREM 2.24. [13, Theorem 5.4] *For (X, φ) a Cantor minimal system, $[\![\varphi]\!]'$ is a finitely generated group if and only if (X, φ) is isomorphic to a minimal subshift.*

THEOREM 2.25. [8, Proposition 2.4] *Let (X, σ) be a minimal subshift, and let S be a finite generating set for $[\![\sigma]\!]'$. Then $|B_S(n)| \succeq \exp(n)$.*

COROLLARY 2.26. *We have $\mathcal{L}_{[\![\sigma]\!]'}(n) \succeq \exp(n)$.*

Proof. This is immediate from the preceding theorem: if $\phi : B_S(n) \to Q$ is a local embedding, then ϕ is injective, so $|Q| \geq |B_S(n)|$. □

Thus our Theorem 1.2 is only new in the case $p_X(n) \not\preceq n^2$.

3. *Construction of local embeddings*

In this section we prove Theorem 1.1. Let (X, σ) be a minimal subshift over the alphabet A.

PROPOSITION 3.1. *Let $S \subseteq [\![\sigma]\!]$ be finite. Then there exists an integer $C_1 = C_1(S) \geq 1$ such that, for all $r \in \mathbb{N}$ and all $g \in B_S(r)$:*
(i) $\max\{|f_g(x)| : x \in X\} \leq C_1 r$;
(ii) *for all $m \geq C_1 r$, f_g is constant on every m-cylinder of X.*

Proof. For (i), for $g \in [\![\sigma]\!]$ write $\lambda(g) = \max\{|f_g(x)| : x \in X\}$. Since S is finite, we may choose $C_1 \geq \max\{\lambda(s) : s \in S\}$. For $g, h \in [\![\sigma]\!]$, $\lambda(gh) \leq \lambda(g) + \lambda(h)$, so by induction $\lambda(g) \leq C_1 r$ for all $g \in B_S(r)$.

For (ii), for every $s \in S \cup S^{-1}$ there is a finite clopen partition $\mathcal{C}_s$ of X, such that f_s is constant on the parts of $\mathcal{C}_s$. By Remark 2.17(ii), we may choose C_1 sufficiently large that, for all $s \in S \cup S^{-1}$ and $m \geq C_1$, f_s is constant on every m-cylinder of X. Let $r \geq 2$ and suppose by induction that the claim holds for smaller r. Let $m \geq C_1 r$, let U be an m-cylinder of X, and let $g \in B_S(r)$. Then there exist $h \in B_S(r-1)$ and $s \in S \cup S^{-1} \cup \{e\}$ such that $g = sh$. By inductive hypothesis, f_h is constant on U, say with value i. Observe that, for any $k, l \in \mathbb{N}$, if $x, y \in X$ lie in the same $(k+l)$-cylinder of X, then $\sigma^{\pm l}x, \sigma^{\pm l}y$

lie in the same k-cylinder of X. Thus, there is a unique $(m - |i|)$-cylinder containing $h(U)$. By (i), $i \in [-C_1(r-1), C_1(r-1)]$, so $m - |i| \geq C_1$. This implies that f_s is constant on $h(U)$, hence f_g is constant on U. □

Remark 3.2. By Proposition 3.1(i), for $k \in \mathbb{N}$ and $m \geq C_1 r$, if $x, y \in X$ lie in the same $(k+m)$-cylinder of X, then $g(x), g(y)$ lie in the same k-cylinder of X for any $g \in B_S(r)$.

LEMMA 3.3. *Let C_1 be as in Proposition 3.1. For all $r \in \mathbb{N}$, there exists $M = M(r) \in \mathbb{N}$ satisfying:*

(i) *$\{\sigma^i \mathbf{x} : 1 \leq i \leq M\}$ intersects every $C_1(r+1)$-cylinder of X;*
(ii) *$\mathbf{x}$ and $\sigma^M \mathbf{x}$ lie in the same $C_1(2r+1)$-cylinder of X;*
(iii) *$10C_1 r \leq M \leq 2R_X(10C_1 r)$.*

Proof. Let $u \in A^{2C_1(r+1)+1}$. Then $\ll u \gg_{C_1(r+1)+1} \cap X$ is a $C_1(r+1)$-cylinder of X if and only if $u \in F_{2C_1(r+1)+1}(\mathbf{x})$. Therefore (i) holds for any $M \geq R_X(2C_1(r+1)+1)$. For the same reason, a value of M satisfying (ii) occurs at least once in every interval of length $R_X(2C_1(2r+1)+1)$. Therefore we can choose $M \leq R_X(2C_1(2r+1)+1) + 10C_1 r$ satisfying (i), (ii) and the first inequality of (iii). Since $10C_1 r \leq R_X(5C_1 r)$ by Theorem 2.13, and since R_X is non-decreasing, the second inequality of (iii) also holds. □

COROLLARY 3.4. *Let M be as in Lemma 3.3. For any $|i| \leq C_1 r$, $\sigma^i \mathbf{x}$ and $\sigma^{M+i} \mathbf{x}$ lie in the same $C_1(r+1)$-cylinder of X.*

Proof. This is immediate from Lemma 3.3(ii), by Remark 3.2. □

Fix $\mathbf{x} \in X$ and let $O = O(\mathbf{x})$ be the orbit of $\mathbf{x}$ under σ. Then $[\![\sigma]\!]$ acts on O. Since X has no periodic points, this action induces a well-defined homomorphism $\phi : [\![\sigma]\!] \to \mathrm{Sym}(\mathbb{Z})$ by

$$g(\sigma^n \mathbf{x}) = \sigma^{\phi(g)[n]}(\mathbf{x}). \tag{3.1}$$

In other words, $\phi(g)[n] = n + f_g(\sigma^n \mathbf{x})$. Since O is dense in X, ϕ is injective.

Let $\Gamma \leq [\![\sigma]\!]$ be finitely generated, and let $\mathbf{S} = (s_1, \ldots, s_d)$ be an ordered generating d-tuple of elements of Γ. Then Γ acts faithfully on $\mathbb{Z}$ via ϕ. Let $G = \mathrm{Schr}(\Gamma, \mathbb{Z}, \mathbf{S})$ be the associated Schreier graph of the action of Γ on $\mathbb{Z}$ with respect to $\mathbf{S}$. Let $C_1 = C_1(\{s_1, \ldots, s_d\}) \geq 1$ be as in Proposition 3.1.

LEMMA 3.5. *For $n \in \mathbb{Z}$ and $r \in \mathbb{N}$, the isomorphism type of $B_G(n, r) \subseteq G$ (as a based, edge-coloured graph) depends only on the $C_1(r+1)$-cylinder of X in which $\sigma^n \mathbf{x}$ lies.*

Proof. By Proposition 3.1(i), $B_G(n, r) \subseteq [n - C_1 r, n + C_1 r] \cap \mathbb{Z}$. For $|i| \leq C_1 r$, $n + i \in B_G(n, r)$ if and only if, for some $g \in B_{\mathbf{S}}(r)$, $f_g(\sigma^n \mathbf{x}) = i$, and by Proposition 3.1(ii), this condition depends only on the $C_1 r$-cylinder of X containing $\sigma^n \mathbf{x}$.

For $n + i, n + j \in B_G(n, r)$, and $1 \leq c \leq d$, $(n + i, n + j)$ is a c-coloured edge in G if and only if $f_{s_c}(\sigma^{n+i} \mathbf{x}) = j - i$. This depends only on the C_1-cylinder of X containing $\sigma^{n+i} \mathbf{x}$, which in turn depends only on the $C_1(r+1)$-cylinder of X containing $\sigma^n \mathbf{x}$. □

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Let $r \geq 1$ and let $M = M(r)$ be as in Lemma 3.3. For $1 \leq c \leq d$ define $\overline{s}_c : \mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$ by

$$\overline{s}_c(n + M\mathbb{Z}) = \phi(s_c)[n] + M\mathbb{Z} \quad \text{for } 1 \leq n \leq M.$$

PROPOSITION 3.6. *The function* $\overline{s}_c$ *lies in* $\mathrm{Sym}(\mathbb{Z}/M\mathbb{Z})$.

Proof. We check that $\overline{s}_c$ is injective, and therefore is indeed a well-defined permutation. If $1 \leq m < n \leq M$ with $\phi(s_c)[n] \equiv \phi(s_c)[m] \bmod M$, then

$$n - m \equiv f_{s_c}(\sigma^m \mathbf{x}) - f_{s_c}(\sigma^n \mathbf{x}) \bmod M,$$

but by Proposition 3.1(i),

$$|f_{s_c}(\sigma^m \mathbf{x}) - f_{s_c}(\sigma^n \mathbf{x})| \leq 2C_1$$

while $1 \leq n - m \leq M - 1$, so either $1 \leq n - m \leq 2C_1$ or $M - 2C_1 \leq n - m \leq M - 1$. In the former case,

$$\phi(s_c)[n] \equiv \phi(s_c)[m] \bmod M \quad \text{but} \quad |\phi(s_c)[n] - \phi(s_c)[m]| \leq 4C_1,$$

and since $M \geq 10C_1$ by Lemma 3.3(iii), $\phi(s_c)[n] = \phi(s_c)[m]$, contradicting the injectivity of $\phi(s_c)$. The latter case is similar; we have

$$1 \leq m \leq 2C_1 \quad \text{and} \quad 1 \leq m + M - n \leq 2C_1.$$

By Corollary 3.4, $\sigma^{m+M}\mathbf{x}$ and $\sigma^m \mathbf{x}$ lie in the same $C_1(r+1)$-cylinder of X, so by Proposition 3.1, $f_{s_c}(\sigma^{m+M}\mathbf{x}) = f_{s_c}(\sigma^m \mathbf{x})$. We may therefore argue as in the former case, with $m + M$ replacing m. □

Theorem 1.1 is now immediate, by Remark 2.3, from the next result.

THEOREM 3.7. *We have* $\mathcal{LA}_\Gamma^S(\lfloor 2r/3 \rfloor) \leq 2R_X(10C_1 r)$.

Proof. Let M be as in Lemma 3.3. By the upper bound on M from Lemma 3.3(iii), it suffices to show that there is a local embedding $B_{\mathbf{S}}(\lfloor 2r/3 \rfloor) \to \mathrm{Sym}(\mathbb{Z}/M\mathbb{Z})$ sending s_c to $\overline{s}_c$ for $1 \leq c \leq d$. Let $\overline{\mathbf{S}} = \{\overline{s}_1, \ldots, \overline{s}_d\}$; let $\overline{\Gamma} = \langle \overline{\mathbf{S}} \rangle \leq \mathrm{Sym}(\mathbb{Z}/M\mathbb{Z})$, and let $\overline{G} = \mathrm{Schr}(\overline{\Gamma}, \mathbb{Z}/M\mathbb{Z}, \overline{\mathbf{S}})$. By Lemma 2.9, it suffices to show that G and $\overline{G}$ are locally colour-isomorphic at radius r.

By Lemmas 3.5 and 3.3(i), for every $m \in \mathbb{Z}$, $B_G(m, r) \subseteq G$ is isomorphic (as a based, edge-coloured graph) to $B_G(n, r)$, for some $1 \leq n \leq M$. Let $\pi_M : \mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$ be reduction modulo M. It suffices to check that for every $1 \leq n \leq M$ the restriction of π_M to $V(B_G(n, r)) \subseteq \mathbb{Z}$ induces an isomorphism of based edge-coloured graphs $B_G(n, r) \to B_{\overline{G}}(\pi_M(n), r)$.

Consider $I_n = [n - C_1 r, n + C_1 r] \cap \mathbb{Z}$. Since $2C_1 r < M$ (by Lemma 3.3), the restriction of π_M to I_n is injective. By Proposition 3.1(i), $V(B_G(n, r)) \subseteq I_n$, so the restriction of π_M to $V(B_G(n, r))$ is a bijection onto its image. Then it suffices to show that $\pi_M(\phi(s_c)[k]) = \overline{s}_c(\pi_M(k))$ for all $k \in I_n$ and all c.

Since $-C_1 r \leq k \leq M + C_1 r$, there exists $\epsilon \in \{0, \pm 1\}$ such that $k' = k + \epsilon M$ is the representative of $k + M\mathbb{Z}$ in $[1, M]$. By Corollary 3.4 and Proposition 3.1(ii) (or by $k = k'$)

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we have $f_{s_c}(\sigma^k \mathbf{x}) = f_{s_c}(\sigma^{k'} \mathbf{x})$, which means $\phi(s_c)[k] = \phi(s_c)[k'] + \epsilon M$. Then

$$\pi_M(\phi(s_c)[k]) = \pi_M(\phi(s_c)[k'] + \epsilon M) = \pi_M(\phi(s_c)[k']) = \overline{s}_c(k' + M\mathbb{Z}) = \overline{s}_c(\pi_M(k))$$

as claimed.

By the preceding paragraph, the restriction of π_M to $V(B_G(n, r))$ preserves edges and colours. In particular, the vertices of $B_{\overline{G}}(\pi_M(n), r)$, which are precisely the endpoints of (undirected) edge-paths of length no greater than r in $\overline{G}$ starting at $\pi_M(n)$, are exactly the image under π_M of the endpoints of edge paths of length no greater than r in G starting at n, namely the vertices of $B_G(n, r)$. Hence π_M does indeed induce the desired isomorphism of based edge-coloured graphs. □

Remark 3.8. In the proof of Theorem 3.7, it is unclear what can be said in general about the subgroup $\overline{\Gamma}$ of $\mathrm{Sym}(\mathbb{Z}/M\mathbb{Z})$ generated by $\overline{s}_1, \ldots, \overline{s}_d$. If $\overline{\Gamma}$ were to be much smaller than $\mathrm{Sym}(\mathbb{Z}/M\mathbb{Z})$, then we might deduce a better upper bound in Theorem 1.1 than that obtained by combining Theorem 1.1 with Remark 2.3. That said, it is known that a 'generic' subset of $\mathrm{Sym}(\mathbb{Z}/M\mathbb{Z})$ (in various senses) generates a subgroup containing $\mathrm{Alt}(\mathbb{Z}/M\mathbb{Z})$, so absent a specific reason for expecting the contrary, it is reasonable to suspect the index of $\overline{\Gamma}$ in $\mathrm{Sym}(\mathbb{Z}/M\mathbb{Z})$ to be small.

4. *Obstructions to small local embeddings*

Let (X, σ) be a minimal subshift over A, and fix $\mathbf{x} \in A^{\mathbb{Z}}$ with $X = \overline{O(\mathbf{x})}$. Let $S \subseteq [\![\sigma]\!]'$ be a finite generating set (such exists by Theorem 2.24), and let $|\cdot|_S$ be the word length function induced on $[\![\sigma]\!]'$ by S. Recall that $p_X = p_{\mathbf{x}} : \mathbb{N} \to \mathbb{N}$ is the *complexity function* of $\mathbf{x}$; $R_X = R_{\mathbf{x}} : \mathbb{N} \to \mathbb{N}$ is the *recurrence function* of $\mathbf{x}$, and $\mathrm{Cyl}_X(m)$ denotes the set of m-cylinders of X.

LEMMA 4.1. *Suppose that $m \geq (R_{\mathbf{x}}(4) - 1)/2$, and let $U \in \mathrm{Cyl}_X(m)$. Then the sets $\sigma^i(U)$, for $-2 \leq i \leq 2$, are pairwise disjoint.*

Proof. Suppose to the contrary that $\mathbf{y} \in \sigma^i(U) \cap \sigma^j(U)$, for some $-2 \leq i < j \leq 2$. Writing $k = j - i$, we have $y_l = y_{l+k}$ for $-m \leq l \leq m - k$. Letting $w = y_{-m} y_{-m+1} \cdots y_m \in F_{2m+1}(\mathbf{x})$, we have $|F_k(w)| \leq k$. On the other hand, $2m + 1 \geq R_{\mathbf{x}}(k)$, so by the definition of $R_{\mathbf{x}}$, $|F_k(w)| = |F_k(\mathbf{x})| = p_{\mathbf{x}}(k) \geq k + 1$ (the last inequality holding by Theorem 2.13), a contradiction. □

LEMMA 4.2. *Let $m \geq \max\{(R_{\mathbf{x}}(4) - 1)/2, 5\}$. There exists a set $\mathrm{DCyl}_X(m) \subseteq \mathrm{Cyl}_X(m)$ such that the sets $\sigma^i(U)$, for $U \in \mathrm{DCyl}_X(m)$ and $-2 \leq i \leq 2$, are pairwise disjoint, and*

$$|\mathrm{DCyl}_X(m)| \geq p_{\mathbf{x}}(2m - 7)/9. \tag{4.1}$$

Proof. We construct $\mathrm{DCyl}_X(m)$ via an iterative process as follows. At step 0, let U_0 be any m-cylinder. Then the $\sigma^i(U_0)$, for $-2 \leq i \leq 2$, are pairwise disjoint, by Lemma 4.1. Let A_0 be the set of $(m - 4)$-cylinders containing one of the $\sigma^j(U_0)$, for $-4 \leq j \leq 4$, so that $|A_0| \leq 9$, and let $B_0 = \{U_0\}$.

At each subsequent step $k + 1$, we start with B_k a family of m-cylinders and A_k a family of $(m - 4)$-cylinders such that, for any $U \in B_k$ and $-4 \leq j \leq 4$, there exists $V \in A_k$ such

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that $\sigma^j(U) \subseteq V$. If A_k is a cover of X, then set $\mathrm{DCyl}_X(m) = B_k$ and stop. Otherwise, choose $\mathbf{y}_k \in X \setminus (\bigcup_{V \in A_k} V)$ and let U_{k+1} be the m-cylinder of X containing $\mathbf{y}_k$. Then $\sigma^i(U_{k+1})$ $(-2 \leq i \leq 2)$ are pairwise disjoint (by Lemma 4.1). If there exist $0 \leq l \leq k$ and $-2 \leq i, j \leq 2$ such that $\sigma^i(U_{k+1}) \cap \sigma^j(U_l) \neq \emptyset$, then, letting $V \in A_k$ be such that $\sigma^{j-i}(U_l) \subseteq V$, $U_{k+1} \subseteq V$ (by Remark 2.17), contradicting the choice of $\mathbf{y}_k$.

We may therefore let $B_{k+1} = B_k \cup \{U_{k+1}\}$ and produce A_{k+1} by adding to A_k all $(m-4)$-cylinders containing one of the $\sigma^j(U_{k+1})$, $-4 \leq j \leq 4$. At every stage $|A_{k+1}| \leq |A_k| + 9$, $|B_{k+1}| \geq |B_k| + 1$, and by Remark 2.17(iii), the process terminates only when $|A_k| = p_{\mathbf{x}}(2m-7)$. □

Notation 4.3. Let $U \subseteq X$ be a non-empty clopen set, such that $\sigma^{-1}(U)$, U and $\sigma(U)$ are pairwise disjoint. We denote by f_U the element of $[\![\sigma]\!]$ given by

$$f_U(\mathbf{y}) = \begin{cases} \sigma(\mathbf{y}), & \mathbf{y} \in U \cup \sigma^{-1}(U), \\ \sigma^{-2}(\mathbf{y}), & \mathbf{y} \in \sigma(U), \\ \mathbf{y} & \text{otherwise}. \end{cases}$$

LEMMA 4.4. *All f_U lie in $[\![\sigma]\!]'$.*

Proof. Define $h_U \in [\![\sigma]\!]$ by

$$h_U(\mathbf{y}) = \begin{cases} \sigma(\mathbf{y}), & \mathbf{y} \in \sigma^{-1}(U), \\ \sigma^{-1}(\mathbf{y}), & \mathbf{y} \in U, \\ \mathbf{y} & \text{otherwise}. \end{cases}$$

Then $f_U = f_U^{-1} h_U^{-1} f_U h_U$. □

The following identities appear as [**13**, Lemma 5.3]; they are proved by direct calculation, some of which is explained in [**11**, Lemma 3.0.11].

LEMMA 4.5. *Let $U, V \subseteq X$ be non-empty clopen subsets.*

(i) *If $\sigma^i(V)$ are pairwise disjoint for $-2 \leq i \leq 2$, and $U \subseteq V$, then*

$$\tau_V f_U \tau_V^{-1} = f_{\sigma(U)},$$
$$\tau_V^{-1} f_U \tau_V = f_{\sigma^{-1}(U)},$$

where $\tau_V = f_{\sigma^{-1}(V)} f_{\sigma(V)}$.

(ii) *If $\sigma^{-1}(U)$, U, $\sigma(U) \cup \sigma^{-1}(V)$, V, $\sigma(V)$ are pairwise disjoint, then*

$$f_{\sigma(U) \cap \sigma^{-1}(V)} = f_V f_U^{-1} f_V^{-1} f_U.$$

PROPOSITION 4.6. *There exists $C_2 = C_2(X, S) > 0$ such that, for all $m \geq (R_{\mathbf{x}}(4) - 1)/2$, if $W \subseteq X$ is an m-cylinder of X, then $f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in B_S(C_2 m^2)$.*

Proof. Write $C_0 = \lceil (R_{\mathbf{x}}(4) - 1)/2 \rceil + 1$, and define $m_n = C_0(2^n + 1)$, so that $m_{n+1} = 2m_n - C_0$. We inductively construct a non-decreasing sequence (l_n) of positive integers, such that, if $W \subseteq X$ is an m-cylinder of X, with $C_0 \leq m \leq m_n$, then $f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in$

$B_S(l_n)$. We then analyse the growth of l_n. Since there are finitely many m-cylinders W of X with $C_0 \le m \le m_0 = 2C_0$, and, for each, $f_W \in [\![\sigma]\!]'$ by Lemma 4.4, there is a constant l_0 such that $f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in B_S(l_0)$ for all such W. For $n \ge 1$, suppose $m_{n-1} < m \le m_n$ and let

$$W = \ll w_{-m}, \ldots, w_{-1}, \underline{w_0}, w_1, \ldots, w_m \gg \cap X$$

be an m-cylinder of X. Set $C = C_0$ or $C_0 - 1$, such that $m + C$ is even, and let

$$U = \ll w_{-m}, \ldots, w_{-2}, \underline{w_{-1}}, w_0, \ldots, w_C \gg,$$
$$V = \ll w_{-C}, \ldots, w_0, \underline{w_1}, w_2, \ldots, w_m \gg$$

so that $\sigma(U) \cap \sigma^{-1}(V) = W$, and

$$\sigma(U) \cup \sigma^{-1}(V) \subseteq \ll w_{-C}, \ldots, w_{-1}, \underline{w_0}, w_1, \ldots, w_C \gg,$$

so by Lemma 4.1, U and V satisfy the conditions of Lemma 4.5(ii), and

$$|f_W|_S \le 2(|f_U|_S + |f_V|_S). \tag{4.2}$$

Now, $\sigma^{-(m-C-2)/2}(U)$ and $\sigma^{(m-C-2)/2}(V)$ are $(m+C)/2$-cylinders of X, and $(m+C)/2 \le m_{n-1}$, so by induction,

$$|f_{\sigma^{-(m-C-2)/2}(U)}|_S, |f_{\sigma^{(m-C-2)/2}(V)}|_S \le l_{n-1}. \tag{4.3}$$

Let U_i' be the C_0-cylinder of X containing $\sigma^{-i}(U)$, for $1 \le i \le (m - C - 2)/2$. Then by Lemma 4.5(i), f_U can be obtained from $f_{\sigma^{-(m-C-2)/2}(U)}$ by conjugating by $\tau_{U'_{(m-C-2)/2}}, \ldots, \tau_{U_1'}$ in sequence. By our base case, $|\tau_{U_i'}|_S \le 2l_0$ for all i, so

$$|f_U|_S \le |f_{\sigma^{-(m-C-2)/2}(U)}|_S + 2l_0(m - C - 2),$$

and arguing similarly for V,

$$|f_V|_S \le |f_{\sigma^{(m-C-2)/2}(V)}|_S + 2l_0(m - C - 2).$$

Combining with (4.2), (4.3), and our bound on m_n, we have

$$|f_W|_S \le 4l_{n-1} + C'2^n + C'' \tag{4.4}$$

for some constants $C', C'' > 0$. Finally, applying Lemma 4.5(i) a final time, we conjugate f_W by $\tau_{W'}$ or $\tau_{W'}^{-1}$ where W' is the C_0-cylinder of X containing W, and $|f_{\sigma(W)}|_S, |f_{\sigma^{-1}(W)}|_S$ also satisfy a bound as in (4.4) (for some larger C''). We may therefore take $l_n = 4l_{n-1} + C'2^n + C''$, so that $f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in B_S(l_n)$. Solving the recurrence for l_n, we obtain $l_n \le C_2' 4^n \le (4C_2'/C_0^2)m_{n-1}^2 \le (4C_2'/C_0^2)m^2$ for some $C_2' > 0$. □

Recall that for $g \in \mathrm{Homeo}(X)$, the *support* of g is

$$\mathrm{supp}(g) = \{x \in X : g(x) \ne x\}.$$

PROPOSITION 4.7. *Let m and $\mathrm{DCyl}_X(m)$ be as in Lemma 4.2. There exists $C_3 = C_3(X, S) > 0$ such that, for all $U \in \mathrm{DCyl}_X(m)$, there is a subgroup $\Delta_U \le [\![\sigma]\!]'$ satisfying:*

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(i) $\Delta_U \cong \mathrm{Alt}(5)$;
(ii) $\Delta_U \subseteq B_S(C_3 m^2)$;
(iii) *for all* $g \in \Delta_U$,

$$\mathrm{supp}(g) \subseteq \bigcup_{i=-2}^{2} \sigma^i(U). \tag{4.5}$$

Proof. For $U \in \mathrm{DCyl}_X(m)$, let $\Delta_U = \langle f_{\sigma^{-1}(U)}, f_U, f_{\sigma(U)} \rangle$. Then (iii) holds, since it holds for $g \in \{f_{\sigma^{-1}(U)}, f_U, f_{\sigma(U)}\}$. Now Δ_U acts, faithfully, on $\{\sigma^i(U) : |i| \leq 2\}$. Identifying this set in the obvious way with $\{-2, -1, 0, 1, 2\}$, $f_{\sigma^{-1}(U)}$, f_U and $f_{\sigma(U)}$ act, respectively, as the 3-cycles $(-2\ -1\ 0)$, $(-1\ 0\ 1)$ and $(0\ 1\ 2)$. As such, Δ_U acts as the alternating group on $\{-2, -1, 0, 1, 2\}$, and we have (i).

By Proposition 4.6, $f_{\sigma^{-1}(U)}, f_U, f_{\sigma(U)} \in B_S(C_2 m^2)$, and since $|\Delta_U| = 60$, $\Delta_U \subseteq B_S(60 C_2 m^2)$, whence (ii). □

Proof of Theorem 1.2. Let Q be a finite group and $\pi : B_S(r) \to Q$ be a local embedding. For any $m \leq (r/2C_3)^{1/2}$, we have $\Delta_U \leq B_S(r/2)$, where $C_3 > 0$ and Δ_U are as in Proposition 4.7. We apply Proposition 2.6 to the family $\{\Delta_U : U \in \mathrm{DCyl}(m)\}$ for $m \geq c' r^{1/2}$ ($c' > 0$ sufficiently small), and r larger than a constant such that $m \geq (R_{\mathbf{x}}(4) - 1)/2$. Since the Δ_U are disjointly supported (by Lemma 4.2 and (4.5)), they do indeed generate their direct product. From Proposition 2.6 we conclude that

$$\begin{aligned} |Q| &\geq \prod_{U \in \mathrm{DCyl}_X(m)} |\Delta_U| \\ &\geq \exp(c p_{\mathbf{x}}(2m - 7)) \qquad \text{(by (4.1))} \\ &\geq \exp(c p_{\mathbf{x}}(2c' r^{1/2} - 7)) \end{aligned}$$

for $c = \log(60)/9$. □

Remark 4.8. We can improve the constant c by modifying the construction so as to take $\Delta_U \cong \mathrm{Alt}(2d + 1)$ for large d, instead of $\mathrm{Alt}(5)$. To do this we would need to take $\mathrm{DCyl}_X(m)$ to be a family of m-cylinders U such that all sets $\sigma^i(U)$ are pairwise disjoint for $-d \leq i \leq d$, so that our construction in Lemma 4.2 would lead to a bound $|\mathrm{DCyl}_X(m)| \geq p_X(2m - C)/(2d - 1)$. We could nevertheless take $c = \log(d!/2)/(2d - 1)$, which grows in d.

Remark 4.9. The same argument also gives a lower bound on the LEF action growth of $[\![\sigma]\!]'$, which is a little stronger than that obtained by applying Remark 2.3 to the conclusion of Theorem 1.2. Suppose $\pi : B_S(r) \to \mathrm{Sym}(d)$ is a local embedding. Then by Propositions 2.6 and 4.7, $\mathrm{im}(\pi)$ contains a subgroup isomorphic to the direct product of $P = p_{\mathbf{x}}(2cr^{1/2} - 7)$ copies of $\mathrm{Alt}(5)$, which in turn contains the direct product of P copies of C_5. By [**9**, Theorem 2], the minimal degree of a faithful permutation representation of the latter is $5P$. Hence $\mathcal{LA}_{[\![\sigma]\!]'}(n) \succeq p_{\mathbf{x}}(n^{1/2})$.

5. *Systems of intermediate growth*

In this section we prove Theorem 1.5, and deduce Theorem 1.6. The following example is modelled on the construction in [10, §3]. In this section, 'large' means larger than a certain absolute constant (which we do not compute), so as to make true some needed inequalities. We start with a general observation, the utility of which is that it allows us to obtain an infinite word whose asymptotic features (such as the behaviour of the complexity or recurrence functions) can be controlled in terms of the sets of factors of a sequence of finite words $u^{(n)}$.

LEMMA 5.1. *Let (L_j) be an increasing sequence of positive integers; let $x^{(j)} \in A^{L_j}$. For each $j \in \mathbb{N}$, let $K_j \in \mathbb{N}$ with $2 \le K_j \le L_{j+1} - L_j$ and suppose that $x^{(j)}$ is the K_jth L_j-factor of $x^{(j+1)}$. Choose $M_0 \in \mathbb{N}$ with $1 \le M_0 \le L_0$, and define (M_j) recursively via $M_{j+1} = M_j + K_j - 1$. Then there is a unique point $\mathbf{x} \in A^{\mathbb{Z}}$ lying in the intersection of the cylinder sets $\ll x^{(j)} \gg_{M_j}$. Moreover, for all n, $F_n(\mathbf{x}) = \bigcup_j F_n(x^{(j)})$.*

Proof. By construction the $\ll x^{(j)} \gg_{M_n}$ form a nested descending sequence of non-empty closed sets in the compact space $A^{\mathbb{Z}}$, hence their intersection is non-empty. Let $\mathbf{x}$ lie in this intersection.

In $x^{(j)}$, there are $M_j - 1 = M_{j-1} + K_{j-1} - 2$ letters strictly to the left of the M_jth letter, and $L_j - M_j \ge (L_{j-1} - M_{j-1}) + 1$ strictly to the right. Since both these quantities tend to ∞, for every $i \in \mathbb{Z}$, the ith letter of $\mathbf{x}$ is uniquely determined by $\mathbf{x} \in \ll x^{(j)} \gg_{M_j}$, provided j is sufficiently large. Similarly, for any i and n, the ith n-factor of $\mathbf{x}$ lies in $x^{(j)}$ for j sufficiently large, so $F_n(\mathbf{x}) \subseteq \bigcup_j F_n(x^{(j)})$. Conversely, $\mathbf{x} \in \ll x^{(j)} \gg_{M_j}$ implies $x^{(j)}$ (and hence every n-factor of $x^{(j)}$) is a factor of $\mathbf{x}$. □

Choose a real number $r \ge 2$. First, fix a large x divisible by 3. We work over the alphabet $A = \{a, b\}$. Fix two words $w^{(0)}, w'^{(0)} \in A^*$ of length $x/3$. Among all words $w^{(0)} v w'^{(0)}$ for $|v| = x/3$, take a subset C_0 with $|C_0| = x$ and such that no factor of any element of C_0 is equal to $w^{(0)}, w'^{(0)}$ except for the prefixes and suffixes themselves. This is easily achieved: for instance, taking $w^{(0)} = a^{x/3}$, $w'^{(0)} = b^{x/3}$, we can form C_0 by choosing v from among those words starting in b and ending in a (of which there are $2^{(x/3)-2} \ge x$ for x large).

Assuming that we have already defined C_j, we will define C_{j+1}; for all i, we set $N_i = |C_i|$ and l_i the length of any element of C_i, so that $N_0 = l_0 = x$. We prove the following statements for all $j \ge 0$.

(i) N_j, l_j are large and increasing in j; $3|N_j, 3|l_j$; all words in C_j have the same prefix of length $l_j/3$, and the same suffix of length $l_j/3$.

(ii) $N_{j+2} \le \exp(2^r (\log N_j)^r)$, $N_{j+1} \ge N_j^2$, and, for j even, $N_{j+1} \ge \exp(\frac{1}{2}(\log N_j)^r)$.

(iii) $N_j < l_{j+1} \le N_j^2$.

We prove (i) and (iii) by induction with base case $j = 0$, for which all claims are true (for l_1 see below), and (ii) directly.

Fix an ordering of C_j, arbitrarily: the elements of C_j are words $u_i^{(j)}$, where the index i follows the ordering. Define the word

$$u^{(j+1)} = u_1^{(j)} u_2^{(j)} u_3^{(j)} \cdots u_{N_j-1}^{(j)} u_{N_j}^{(j)} = w^{(j+1)} u_{1/3N_j+1}^{(j)} u_{1/3N_j+2}^{(j)} \cdots u_{2/3N_j}^{(j)} w'^{(j+1)},$$

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where $w^{(j+1)}$ and $w'^{(j+1)}$ collect the first and last third of the $u_i^{(j)}$, respectively: this is possible since $3|N_j$ by induction, and it implies $3|l_{j+1}$ where $l_{j+1} = |u^{(j+1)}| = l_j N_j$ is large. Take a collection $P_j \subset \mathrm{Sym}(N_j/3)$: if j is even, choose $|P_j| = 3\lfloor \exp((\log N_j)^r)/3 \rfloor$, otherwise choose $|P_j| = N_j^2$ (note that this is possible for x large). Then define C_{j+1} to be the set of all words $\{w^{(j+1)} v_\pi^{(j+1)} w'^{(j+1)} : \pi \in P_j\}$ where, for $\pi \in P_j$, $v_\pi^{(j+1)}$ is obtained by permuting the factors $u_i^{(j)}$ of $u^{(j+1)}$ with $i \in (N_j/3, 2N_j/3]$ according to π; that is,

$$v_\pi^{(j+1)} = u_{1/3N_j+\pi(1)}^{(j)} u_{1/3N_j+\pi(2)}^{(j)} \cdots u_{1/3N_j+\pi(1/3N_j)}^{(j)}.$$

By definition we have $3|N_{j+1}$ and N_{j+1} large; $l_{j+1} > l_j$, $N_{j+1} > N_j$, and for all $j \geq 0$,

$$N_{j+2} \leq \begin{cases} \exp(2(\log N_j)^r) & (2 \mid j) \\ \exp((\log N_j^2)^r) & (2 \nmid j) \end{cases} \leq \exp(2^r (\log N_j)^r);$$

$$N_{j+1} \geq \min\{N_j^2, \exp((\log N_j)^r) - 3\} = N_j^2;$$

$$2|j \Rightarrow N_{j+1} \geq \exp((\log N_j)^r) - 3 \geq \exp((\log N_j)^r/2);$$

$$l_1 = x^2 = N_0^2 > N_0;$$

$$j \geq 1 \Rightarrow N_j < l_{j+1} = l_j N_j \leq N_{j-1}^2 N_j \leq N_j^2,$$

so (i)–(iii) do indeed hold. Two key features of this construction are that, for all j:

(a) all words in C_j have the same prefix $w^{(j)}$ and suffix $w'^{(j)}$ of length $\frac{1}{3} l_j$;

(b) every word in C_{j+1} is the product of all the words from C_j (in some order).

We now construct a uniformly recurrent non-periodic word $\mathbf{x} = \mathbf{x}(r)$, such that the subshift $X_r = \overline{O(\mathbf{x})}$ satisfies the conditions of Theorem 1.5. Consider the words $x^{(j)} = u_{N_j/3}^{(j)}$. Then $x^{(j)}$ is the $K_j = (l_j(N_j/3 - 1) + 1)$th l_j-factor of $x^{(j+1)}$, for all j, and $2 \leq K_j \leq l_{j+1} - l_j$. For any $1 \leq M_0 \leq l_0$, apply Lemma 5.1 to the sequence $x^{(j)}$ (with $L_j = l_j$) to obtain $\mathbf{x}$.

PROPOSITION 5.2. *The complexity function $p_{\mathbf{x}}$ of $\mathbf{x}$ satisfies $p_{\mathbf{x}}(l_j) \geq \exp((\log l_j)^r/2^{r+1})$, for all $j \in \mathbb{N}$ odd.*

Proof. Since $u_{N_j/3}^{(j+1)} \in C_{j+1}$ is a factor of $\mathbf{x}$, we have by (b) above that all elements of C_j are distinct factors of $\mathbf{x}$ of length l_j, so $p_{\mathbf{x}}(l_j) \geq N_j$ for all $j \in \mathbb{N}$. For j odd, it follows that

$$p_{\mathbf{x}}(l_j) \geq N_j \geq \exp((\log N_{j-1})^r/2) \geq \exp((\log \sqrt{l_j})^r/2) = \exp((\log l_j)^r/2^{r+1})$$

by (ii) and (iii) above. □

PROPOSITION 5.3. *The recurrence function $R_{\mathbf{x}}$ of $\mathbf{x}$ satisfies*

$$R_{\mathbf{x}}(n) \leq C \exp(4^{r+1} (\log n)^r)$$

for some $C > 0$ and all $n \in \mathbb{N}$.

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Proof. Fix any factor w of $\mathbf{x}$ of length n, and suppose that j is such that $n < 2l_j/3$. w is a factor of some $x^{(k)} = u^{(k)}_{N_k/3}$, for $k > j$, and applying (b) above repeatedly, $x^{(k)}$ is expressible as a product of words $u^{(j)}_i$. Therefore, there are indices i_1 and i_2 such that w is a factor of some $u^{(j)}_{i_1}u^{(j)}_{i_2}$; moreover, it intersects the middle third of at most one of $u^{(j)}_{i_1}$ or $u^{(j)}_{i_2}$, so by (a) above, it sits entirely inside either $u^{(j)}_{i_1}w^{(j)}$ or $w'^{(j)}u^{(j)}_{i_2}$.

Now let v be any factor $\mathbf{x}$ of length $3l_{j+1}$. As before, v is a factor of some $x^{(k')} = u^{(k')}_{N_{k'}/3}$, for $k' > j+1$, and applying (b) above repeatedly, $x^{(k')}$ is expressible as a product of words $u^{(j+1)}_i$. Therefore, there are indices i'_1 and i'_2 such that $u^{(j+1)}_{i'_1}u^{(j+1)}_{i'_2}$ is a factor of v. By (b) above, and the fact that every $u^{(j)}_i$ has $w^{(j)}$ as a prefix and $w'^{(j)}$ as a suffix, it follows that $u^{(j)}_{i_1}w^{(j)}$ and $w'^{(j)}u^{(j)}_{i_2}$ are factors of $u^{(j+1)}_{i'_1}u^{(j+1)}_{i'_2}$, hence w is a factor of v. That is, $R_{\mathbf{x}}(n) \le 3l_{j+1}$.

Partitioning the integers, we have $R_{\mathbf{x}}(n) \le 3l_{j+1}$ for any $n \in [2l_{j-1}/3, 2l_j/3)$. For $j \ge 2$, we have

$$\begin{aligned}\log R_{\mathbf{x}}(n) - \log 3 &\le \log l_{j+1} \le 2\log N_j \le 2^{r+1}(\log N_{j-2})^r \\ &< 2^{r+1}(\log l_{j-1})^r \le 2^{r+1}(\log(3n/2))^r \le 4^{r+1}(\log n)^r\end{aligned}$$

by (ii) and (iii), and our bounds on n, as required. □

Proof of Theorem 1.5. Let $\mathbf{x} = \mathbf{x}(r)$ be as above. Set $X_r = \overline{O(\mathbf{x})}$. By Remark 2.23, items (i) and (ii) follow from Propositions 5.3 and 5.2, respectively, with $n^{(r)}_i = l_{2i+1}$. □

Proof of Theorem 1.6. Recall that $N! \le \exp(N\log N)$ for all N. Applying Theorem 1.1 to $\Gamma^{(r)} = [\![\sigma]\!]'$, where (X_r, σ) is as in Theorem 1.5, we have

$$\mathcal{L}^S_{\Gamma^{(r)}}(n) \le \exp(2R_X(Cn)\log(2R_X(Cn))), \tag{5.1}$$

for all n and for some $C > 0$, where $X = X_r$, so that $R_X(n) \le \exp(C_r(\log n)^r)$. Hence, for $n \ge 2$,

$$\log(2R_X(Cn)) \le C_r(\log n + \log C)^r + \log 2 \le C'_r(\log n)^r$$

for a possibly larger constant C'_r, so that

$$\begin{aligned}\log(2R_X(Cn)) + \log\log(2R_X(Cn)) &\le C'_r(\log n)^r + r\log\log n + \log C'_r \\ &\le C''_r(\log n)^r\end{aligned}$$

again, for C''_r a possibly larger constant. Thus, by (5.1), $\mathcal{L}^S_{\Gamma^{(r)}}(n) \le \exp(\exp(C''_r(\log n)^r))$, and we have (i).

For (ii), suppose for a contradiction that $2 \le r' < r$ and that $C, C' > 0$ are such that, for all n sufficiently large,

$$\mathcal{L}^S_{\Gamma^{(r)}}(n) \le \exp(\exp(C(\log C'n)^{r'})). \tag{5.2}$$

By Theorems 1.2 and 1.5(ii), we have

$$\mathcal{L}^S_{\Gamma^{(r)}}(\lceil n^{(r)}_i/c\rceil^2) \ge \exp(cp_X(n^{(r)}_i)) \ge \exp(c\exp(c_r(\log n^{(r)}_i)^r))$$

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for some $c > 0$ and all $i \in \mathbb{N}$. Hence by (5.2),

$$C(2\log n_i^{(r)} + \log C' - 2\log c + 1)^{r'} \geq c_r(\log n_i^{(r)})^r + \log c$$

for all i sufficiently large, a contradiction.

For the final statement, let $\mathcal{F} = \{\Gamma^{(r)} : r \geq 2\}$. Let $2 \leq r' < r$, and suppose for a contradiction that $\mathcal{L}_{\Gamma^{(r)}} \preceq \mathcal{L}_{\Gamma^{(r')}}$. By (i),

$$\mathcal{L}_{\Gamma^{(r)}}(n) \preceq \exp(\exp(C_{r'}(\log n)^{r'})),$$

contradicting (ii). □

Acknowledgements. Both authors were supported by ERC grant no. 648329 'GRANT', during their permanence at Georg-August-Universität Göttingen. The second author was supported by the Emily Erskine Endowment Fund and Israel Science Foundation Grant No. 686/17 of Aner Shalev, during his permanence at the Hebrew University of Jerusalem.

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