Math. Proc. Camb. Phil. Soc. (2022), **172**, 685–703
doi:10.1017/S0305004121000372
First published online 17 May 2021

Multiplicatively badly approximable matrices up to logarithmic factors

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(*Received* 5 *April* 2020; *revised* 7 *April* 2021; *accepted* 22 *March* 2021)

Abstract

Let $\|x\|$ denote the distance from $x \in \mathbb{R}$ to the nearest integer. In this paper, we prove a new existence and density result for matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ satisfying the inequality

$$\liminf_{|\boldsymbol{q}|_\infty \to +\infty} \prod_{j=1}^{n} \max\{1, |q_j|\} \log\left(\prod_{j=1}^{n} \max\{1, |q_j|\}\right)^{m+n-1} \prod_{i=1}^{m} \|A_i \boldsymbol{q}\| > 0,$$

where $\boldsymbol{q}$ ranges in $\mathbb{Z}^n$ and A_i denote the rows of the matrix $\boldsymbol{A}$. This result extends previous work of Moshchevitin both to arbitrary dimension and to the inhomogeneous setting. The estimates needed to apply Moshchevitin's method to the case $m > 2$ are not currently available. We therefore develop a substantially different method, based on Cantor-like set constructions of Badziahin and Velani. Matrices with the above property also appear to have very small sums of reciprocals of fractional parts. This fact helps us to shed light on a question raised by Lê and Vaaler on such sums, thereby proving some new estimates in higher dimension.

2020 Mathematics Subject Classification: 11J13, 11J20, 11J83, 11H46 (Primary); 11H16 (Secondary)

1. *Introduction*

1·1. *Notation*

For $x \in \mathbb{R}$ we denote by $\|x\|$ the distance from x to the nearest integer. For a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ we denote by $A_i \in \mathbb{R}^n$ ($i = 1, \ldots, m$) the rows of $\boldsymbol{A}$, and by $A_i\boldsymbol{x}$ the sum $\sum_{j=1}^{n} A_{ij}x_j$, where $\boldsymbol{x} \in \mathbb{R}^n$. Given a set X and a pair of functions $f, g : X \to \mathbb{R}$, we write $f \ll g$ (or $f \gg g$) when there exists a constant $c > 0$ such that $f(x) \leq cg(x)$ (or $f(x) \geq cg(x)$) for all $x \in X$. If the constant c depends on some parameters, we write them under the symbol $\ll$ (or $\gg$). We denote by $|\cdot|_2$ the Euclidean norm and by $|\cdot|_\infty$ the supremum norm on $\mathbb{R}^n$. We denote by dist_2 and dist_∞ the Euclidean and supremum distances respectively. For a set $X \subset \mathbb{R}^n$ we denote by $\mathrm{diam}(X)$ its diameter and by $\mathrm{Vol}(X)$ its n-dimensional Hausdorff measure. If X is a (hyper)cube, we denote by $\mathrm{edge}(X)$ the length of its edges, i.e., its 1-dimensional faces. If $f : \mathbb{Z}^n \to [0, +\infty)$ is a function, we denote by $\liminf_{|\boldsymbol{q}|_\infty \to +\infty} f(\boldsymbol{q})$ the number $\liminf_{q \to +\infty} \min\{f(\boldsymbol{q}) : |\boldsymbol{q}|_\infty = q\}$. Finally, by any product $\prod_a^b$ where $b < a$, we signify the constant 1.

1·2. *Background*

It is well known that the set of real numbers $\alpha \in \mathbb{R}$ such that

$$\liminf_{q \to \infty} q \|q\alpha\| > 0 \tag{1·1}$$

is non empty and has full Hausdorff dimension. Such numbers are known as badly approximable and play a key role in the theory of Diophantine approximation. The notion of bad approximablity can be extended to a higher-dimensional setting, where it becomes significantly more varied and multi-faceted. In the standard setting, a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ (where m, n are positive integers) is said to be badly approximable if, in analogy to (1·1),

$$\liminf_{|\boldsymbol{q}|_\infty \to +\infty} |\boldsymbol{q}|_\infty^n \max_{i=1}^{m} \{\|A_i \boldsymbol{q}\|\}^m > 0.$$

Schmidt [16] showed that the set of such matrices has full Hausdorff dimension in $\mathbb{R}^{m \times n}$. In the multiplicative setting [5], a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is said to be badly approximable if

$$\liminf_{|\boldsymbol{q}|_\infty \to +\infty} \prod_{j=1}^{n} \max\{1, |q_j|\} \prod_{i=1}^{m} \|A_i \boldsymbol{q}\| > 0. \tag{1·2}$$

To simplify the notation, throughout this paper we shall write

$$\prod(\boldsymbol{q}) := \prod_{j=1}^{n} \max\{1, |q_j|\}$$

for all $\boldsymbol{q} \in \mathbb{Z}^n$.

The famous Littlewood conjecture states that for any pair of real numbers $\alpha, \beta \in \mathbb{R}$ it holds

$$\liminf_{q \to \infty} q \|q\alpha\| \|q\beta\| = 0, \tag{1·3}$$

or that, in other words, there exist no 2×1 multiplicatively badly approximable matrices. However, if the Littlewood conjecture were true, there would exist no $m \times n$ multiplicatively badly approximable matrices for any value of m and n (except when $n = m = 1$). This follows easily from the fact that every submatrix of a multiplicatively badly approximable matrix is itself multiplicatively badly approximable and from a well-known transference principle (see [12, theorem 2·2]). Proving or disproving the Littlewood conjecture (formulated around 1920) has eluded the efforts of the mathematical community to date, and constitutes a major challenge for the future. Nonetheless, partially due to the results achieved by Einsiedler, Katok and Lindenstrauss [7], who proved that the set of counterexamples to this conjecture has zero Hausdorff dimension, it is widely believed that the Littlewood conjecture is true.

Since the set of multiplicatively badly approximable matrices with the current definition is potentially empty, some authors (Badziahin, Velani, etc.) have suggested to introduce a different definition of multiplicative bad approximability, by weakening the Diophantine condition in (1·2). The most obvious way to do this is to increase the exponent of the factor $\prod(\boldsymbol{q})$ in (1·2). This modification, however, introduces "too many" new matrices, in consequence of the following two $0-1$ results.

THEOREM 1·1 (Gallagher). *Let m be a positive integer and let $\psi:\mathbb{N}\to(0,1]$ be a non-increasing function. Let also*

$$W^{\times}(m,1,\psi):=\left\{\boldsymbol{A}\in[0,1]^{m\times 1}:\prod_{i=1}^{m}\|A_i q\|<\psi\left(|q|\right)\ \textit{for infinitely many } q\in\mathbb{Z}\right\}.$$

Then, we have that

$$\mathscr{L}(W^{\times}(m,1,\psi))=\begin{cases}0 & \textit{if } \sum_{q=1}^{+\infty}\psi\left(q\right)\log\left(\psi(q)^{-1}\right)^{m-1}<+\infty\\ 1 & \textit{if } \sum_{q=1}^{+\infty}\psi\left(q\right)\log\left(\psi(q)^{-1}\right)^{m-1}=+\infty\end{cases},$$

where $\mathscr{L}$ stands for the m-dimensional Lebesgue measure.

THEOREM 1·2 (Sprindžuk). *Let m,n be positive integers and let $\psi:\mathbb{N}\to(0,1]$ be any function. Let also*

$$W^{\times}(m,n,\psi):=\left\{\boldsymbol{A}\in[0,1]^{m\times n}:\prod_{i=1}^{m}\|A_i\boldsymbol{q}\|<\psi\left(\prod(\boldsymbol{q})\right)\ \textit{for infinitely many } \boldsymbol{q}\in\mathbb{Z}^n\right\}.$$

Then, we have that

$$\mathscr{L}(W^{\times}(m,n,\psi))=\begin{cases}0 & \textit{if } \sum_{\boldsymbol{q}\in\mathbb{Z}^n}\psi\left(\prod(\boldsymbol{q})\right)\log\left(\psi\left(\prod(\boldsymbol{q})\right)^{-1}\right)^{m-1}<+\infty\\ 1 & \textit{if } \sum_{\boldsymbol{q}\in S}\psi\left(\prod(\boldsymbol{q})\right)\log\left(\psi\left(\prod(\boldsymbol{q})\right)^{-1}\right)^{m-1}=+\infty\end{cases},$$

where $\mathscr{L}$ stands for the mn-dimensional Lebesgue measure, and S is any infinite set of pairwise linearly independent vectors in $\mathbb{Z}^n$.

The reader may refer to [**10**, theorem 1] and to [**17**, chapter 1, theorem 13] for a proof of these two theorems. Note that there is a discrepancy between the cases $n=1$ and $n>1$. In particular, Theorem 1·2 does not imply Theorem 1·1, since for $n=1$ there are no infinite subsets of pairwise linearly independent vectors in $\mathbb{Z}$.

Gallagher and Sprindžuk's Theorems both imply that the set of matrices $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ such that

$$\liminf_{|\boldsymbol{q}|_\infty\to+\infty}\prod(\boldsymbol{q})^{1+\varepsilon}\prod_{i=1}^{m}\|A_i\boldsymbol{q}\|>0$$

has full Lebesgue measure in $\mathbb{R}^{m\times n}$ for all $\varepsilon>0$. Therefore, a finer alteration is required. A natural solution to this problem is to allow for logarithmic factors, i.e., to consider the set

$$\mathrm{Mad}^{\lambda}(m,n):=\left\{\boldsymbol{A}\in\mathbb{R}^{m\times n}:\liminf_{|\boldsymbol{q}|_\infty\to+\infty}\prod(\boldsymbol{q})\log\left(\prod(\boldsymbol{q})\right)^{\lambda}\prod_{i=1}^{m}\|A_i\boldsymbol{q}\|>0\right\}. \qquad (1\cdot4)$$

It follows from Theorems 1·1 and 1·2 that $\mathrm{Mad}^{\lambda}(m,n)$ has full Lebesgue measure for $\lambda>m+n-1$ and zero Lebesgue measure for $\lambda\leq m+n-1$. However, it could happen,

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for example, that the set $\mathrm{Mad}^{\lambda}(m,n)$ is empty for $\lambda \le m+n-1$. This is precisely the case that we treat in this paper.

To have a better understanding of the case $\lambda \le m+n-1$, we consider the analogue of the set $\mathrm{Mad}^{\lambda}(m,n)$ in the standard setting, i.e., the set

$$\mathrm{Bad}^{\lambda}(m,n) := \left\{ \boldsymbol{A} \in \mathbb{R}^{m\times n} : \liminf_{|\boldsymbol{q}|_\infty \to +\infty} |\boldsymbol{q}|_{\infty d}^{n} \log(|\boldsymbol{q}|_\infty)^{\lambda} \max\{\|A_1\boldsymbol{q}\|, \dots, \|A_m\boldsymbol{q}\|\}^m > 0 \right\}. \tag{1·5}$$

In this setting, Theorems 1·1 and 1·2 are replaced by the Khintchine–Groshev Theorem (see [4] and references therein), which we report here for the convenience of the reader.

THEOREM 1·3 (Khintchine–Groshev). *Let m, n be positive integers and let $\psi : \mathbb{N} \to (0, 1]$ be a non-increasing function. Let also*

$$W^{+}(m,n,\psi) := \left\{ \boldsymbol{A} \in [0,1]^{m\times n} : \max_{i=1}^{m}\{\|A_i\boldsymbol{q}\|\}^m < \psi(|\boldsymbol{q}|_\infty) \textit{ for infinitely many } \boldsymbol{q} \in \mathbb{Z}^n \right\}.$$

Then, we have that

$$\mathscr{L}(W^{+}(m,n,\psi)) = \begin{cases} 0 & \textit{if } \sum_{q=1}^{+\infty} \psi(q)q^{n-1} < +\infty \\ 1 & \textit{if } \sum_{q=1}^{+\infty} \psi(q)q^{n-1} = +\infty \end{cases},$$

where $\mathscr{L}$ stands for the mn-dimensional Lebesgue measure.

This theorem, in combination with Schmidt's dimensional result for badly approximable matrices [16] and Dirichlet's Theorem, implies that

$$\mathrm{Bad}^{\lambda}(m,n) = \begin{cases} \emptyset & \text{if } \lambda < 0 \\ \text{full Hausdorff dimension set} & \text{if } 0 \le \lambda \le 1 \\ \text{full Lebesgue measure set} & \text{if } \lambda > 1. \end{cases} \tag{1·6}$$

In particular, we observe that, in the standard setting, "shaving off" a logarithmic factor from the Lebesgue $0-1$ "switch over" ($\lambda = 1$) leads precisely to the set of badly approximable matrices (defined in (1·2)).

Let us move back to the multiplicative setting and draw a comparison. To keep things simple we set $m = 2$, $n = 1$. We note that Theorem 1·3 for $m = 2$, $n = 1$ and Theorem 1·1 for $m = 2$ differ only by the presence of a logarithmic factor in the sum. In particular, Gallagher's Theorem implies that

$$\mathscr{L}\left(\mathrm{Mad}^{\lambda}(2,1)\right) = \begin{cases} 0 & \text{if } \lambda \le 2 \\ +\infty & \text{if } \lambda > 2. \end{cases}$$

Drawing inspiration from (1·6) and from the "shaving off" phenomenon, Badziahin and Velani [2, statements L1-L3] made the following conjecture.

CONJECTURE 1·4 (Badziahin–Velani).

$$\mathrm{Mad}^{\lambda}(2,1) = \begin{cases} \emptyset & \textit{if } \lambda < 1 \\ \textit{full Hausdorff dimension set} & \textit{if } 1 \le \lambda \le 2 \\ \textit{full Lebesgue measure set} & \textit{if } \lambda > 2. \end{cases}$$

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This conjecture is also supported by heuristic volume arguments of Peck [14], and Pollington and Velani [15] (see references in [2]). If it were true, the set $\mathrm{Mad}^1(2, 1)$ would be rightfully regarded as the multiplicative analogue of the set $\mathrm{Bad}^0(2, 1)$.

Multiple authors have contributed towards a partial solution of Conjecture 1·4. Moshchevitin [13] was the first to show that the set $\mathrm{Mad}^2(2, 1)$ is non-empty, by using the so-called Peres–Schlag method. Subsequently, Bugeaud and Moschevitin [6] proved that $\dim \mathrm{Mad}^2(2, 1) = 2$, where dim denotes the Hausdorff dimension. Finally, Badziahin [1] made a significant breakthrough, showing that $\dim \mathrm{Mad}^\lambda(2, 1) = 2$ for all $\lambda > 1$. After Badziahin's result in 2013, no further progress has been made on this conjecture and, to date, the case $\lambda = 1$ remains unsolved.

1·3. *Main result*

Conjecture 1·4 has a natural extension to higher dimension.

CONJECTURE 1·5 (Generalised Mad Conjecture). *For all values of $m, n \in \mathbb{N}$ we have that*

$$\mathrm{Mad}^\lambda(m, n) = \begin{cases} \emptyset & \text{if } \lambda < m+n-1 \\ \text{full Hausdorff dimension set} & \text{if } m+n-2 \leq \lambda \leq m+n-1 \\ \text{full Lebesgue measure set} & \text{if } \lambda > m+n-1. \end{cases}$$

Note that the full Lebesgue measure part of Conjecture 1·5 follows directly from Theorems 1·1 and 1·2.

In analogy with Moshchevitin's result [13], we show in this paper that the set $\mathrm{Mad}^{m+n-1}(m, n)$ is dense and uncountable in $\mathbb{R}^{m\times n}$, and thus non-empty for all values of $m, n \in \mathbb{N}$. We furthermore generalise this result to the inhomogeneous setting.

Let $C \subset \mathbb{R}^{m\times n}$ be a cube of edge ℓ (a ball with respect to the supremum norm in $\mathbb{R}^{m\times n}$). For $f : [0, +\infty) \to [1, +\infty)$ non-decreasing, $\boldsymbol{\gamma} \in \mathbb{R}^m$, and $c > 0$ we consider the set

$$\mathrm{Mad}_{m,n}(C, \boldsymbol{\gamma}, f, c) := \left\{ \boldsymbol{A} \in C : \prod(\boldsymbol{q}) \|A_1\boldsymbol{q} + \gamma_1\| \cdots \|A_m\boldsymbol{q} + \gamma_m\| > \frac{c}{f(\prod(\boldsymbol{q}))} \right.$$
$$\left. \text{for all } \boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \right\}.$$

For $x \in [0, +\infty)$ we set $\log^*(x) := \log(\max\{e, x\})$, where $e = 2.71828\ldots$ is the base of the natural logarithm. With this notation, the following result holds.

PROPOSITION 1·6. *Let $m, n \in \mathbb{N}$, with $m + n \geq 3$, let C be a cube in $\mathbb{R}^{m\times n}$, and let $\boldsymbol{\gamma} \in \mathbb{R}^m$. Then, there exists a constant $c = c(m, n, \ell) > 0$, only depending on the integers m and n, and the length ℓ of the edge of the cube C, such that for any countable (possibly finite) family of hyperplanes $\mathscr{H}$ lying in $\mathbb{R}^{m\times n}$ we have that*

$$\mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H\in\mathscr{H}} H \neq \emptyset.$$

Proposition 1·6 immediately implies the following corollary.

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COROLLARY 1·7. *Let $m+n\geq 3$. Then, for all $\boldsymbol{\gamma}\in\mathbb{R}^m$ the set*

$$\mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma}) := \left\{A\in\mathbb{R}^{m\times n} : \liminf_{|\boldsymbol{q}|_\infty\to+\infty} \prod(\boldsymbol{q})\log\left(\prod(\boldsymbol{q})\right)^{m+n-1}\prod_{i=1}^m \|A_i\boldsymbol{q}+\gamma_i\|>0\right\}$$

is everywhere dense in $\mathbb{R}^{m\times n}$ and does not lie on a countable union of hyperplanes.

Note that, for certain choices of the vector $\boldsymbol{\gamma}\in\mathbb{R}^m$, the non-emptiness (uncountability) of the set $\mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma})$ becomes trivial (e.g., for $n=1$, $\gamma_1,\ldots,\gamma_{m-1}\notin\mathbb{Q}$, $\gamma_m=0$, $A_1,\ldots,A_{m-1}\in\mathbb{Z}$, and A_m badly approximable). In particular, the case $\lambda<m+n-1$ of Conjecture 1·5 in the inhomogeneous setting is clearly false. However, the fact that the set $\mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma})$ does not lie on a countable union of hyperplanes implies that there exist matrices $\boldsymbol{A}$ lying in $\mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma})$ whose entries, along with 1 and the entries of the vector $\boldsymbol{\gamma}$, are linearly independent over $\mathbb{Q}$. This additional linear independence condition excludes most of the trivial counterexamples (such as the one given above) and could potentially lead to a more meaningful generalisation of Conjecture 1·5 to the inhomogeneous setting.

It is worth observing that to prove Proposition 1·6 we do not follow the Peres–Schlag method, i.e., the method used by Moshchevitin to show that $\mathrm{Mad}^2(2,1)\neq\emptyset$ (see [13]). Moshchevitin's proof relies both on the one dimensional case ($m=n=1$), and on estimates for the sum

$$\sum_{q=1}^{Q}\frac{1}{q\|q\alpha\|}.$$

This sum is known to grow like $O(\log(Q)^2)$ for almost all $\alpha\in\mathbb{R}$ [11, theorem 6(b)]. However, to apply inductively Moshchevitin's argument in dimension, e.g., $m\times 1$, one would require an estimate of the form

$$\sum_{q=1}^{Q}\frac{1}{q\|q\alpha_1\|\ldots\|q\alpha_m\|}\ll_m(\log Q)^{m+1}$$

for at least some vectors $(\alpha_1,\ldots,\alpha_m)$. At present, such estimate is only known to hold for multiplicatively badly approximable vectors[1], which, according to the Littlewood Conjecture, do not exist. Hence, a different method is required.

To prove Proposition 1·6, we work directly in a higher-dimensional setting, without relying on induction. We generalise a construction introduced by Badziahin and Velani in [2], in order to produce a multi-dimensional Cantor-like set contained in $\mathrm{Mad}_{m,n}(C,\boldsymbol{\gamma},f,c)$. Such a construction requires to count lattice points contained in sets with "hyperbolic spikes", which arise naturally in the multiplicative setting. We do this through an elementary geometric argument that is the key to the whole proof. The core of this argument can be found in Lemma 4·1. We remark that Badziahin's proof [1] of the fact that $\dim\mathrm{Mad}^\lambda(2,1)=1$ for $\lambda>1$ also relies on an inductive argument, unlike ours.

We conclude by remarking that it would be equally desirable to prove a dimensional result for the set $\mathrm{Mad}^{m+n-1}(m,n)$. Unfortunately, the methods used in this paper do not seem powerful enough to obtain such a result, as the (suitably generalised) hypothesis

[1]To see this, it suffices to apply Abel's summation formula and [12, theorem 2·1].

in Badziahin and Velani's [2, theorem 4] does not hold for our construction. An adaptation of [2, theorem 4] to our setting appears equally challenging, due to an obstruction in [2, lemma 2].

1·4. *Applications*

Let $m, n \in \mathbb{N}$, let $\boldsymbol{Q} \in (0, +\infty)^n$, and let $X := \prod_{j=1}^{n}[-Q_j, Q_j]$. Let also $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ be a matrix whose entries $L_{i1}, \ldots, L_{in} \in \mathbb{R}$ together with 1 are linearly independent over $\mathbb{Z}$ for $i = 1, \ldots, m$. Consider the sum

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) := \sum_{\boldsymbol{q}\in X\cap\mathbb{Z}^n\setminus\{\boldsymbol{0}\}} \prod_{i=1}^{m} \|L_i\boldsymbol{q}\|^{-1}.$$

Sums of this shape, also known as sums of reciprocals of fractional parts, play a key role in Diophantine approximation as well as in the theory of uniform distibution (see, e.g., [3] or [9] for a deeper insight). In applications, it is often crucial to find upper or lower bounds for these sums as a function of the variable $\boldsymbol{Q}$, once the matrix $\boldsymbol{L}$ is fixed. Lê and Vaaler [12] proved very general lower bounds for the function $S_{\boldsymbol{L}}(\boldsymbol{Q})$ through Fourier analysis. More specifically, they showed [12, corollary 1·2] that for $Q := (Q_1 \cdots Q_n)^{1/n} \geq 1$ it holds

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \gg_{m,n} Q^n(\log Q)^m$$

independently of the choice of the matrix $\boldsymbol{L}$. They also asked whether this estimate is sharp, i.e., whether there exist matrices $\boldsymbol{L}$ such that

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{m,n} Q^n(\log Q)^m. \tag{1·7}$$

In [12, theorem 2], they proved that (1·7) holds true for multiplicatively badly approximable matrices. However, since these matrices are not known to exist, the question remains open. Proposition 1·6, in combination with previous estimates of the author [9], allows us to find matrices with "relatively small" (even though not optimal) upper bounds.

Let $\phi : [1, +\infty) \to (0, 1]$ be a non-increasing function. In [9, corollary 1·8] the author proved that if a matrix $\boldsymbol{L}$ is ϕ-semimultiplicatively badly approximable, i.e., if

$$|\boldsymbol{q}|_\infty^n \prod_{i=1}^{m} \|L_i\boldsymbol{q}\| \geq \phi(|\boldsymbol{q}|_\infty)$$

for all $\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$, then the following upper bound holds for $Q \geq 2$:

$$\sum_{\substack{\boldsymbol{q}\in[-Q,Q]^n\\ \cap\,\mathbb{Z}^n\setminus\{\boldsymbol{0}\}}} \prod_{i=1}^{m} \|L_i\boldsymbol{q}\|^{-1} \ll_{m,n} Q^n \log\left(\frac{Q^n}{\phi(Q)}\right)^m + \frac{Q^n}{\phi(Q)} \log\left(\frac{Q^n}{\phi(Q)}\right)^{m-1}. \tag{1·8}$$

Since $\prod_{j=1}^{n} \max\{1, |q_j|\} \leq |\boldsymbol{q}|_\infty^n$ for all $\boldsymbol{q} \in \mathbb{Z}^n$, from Proposition 1·6 we easily deduce the following.

COROLLARY 1·8. *Let $m, n \in \mathbb{N}$. Then, there exist uncountably many matrices $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ such that*

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{m,n} Q^n(\log Q)^{2m+n-2} \tag{1·9}$$

for all $\boldsymbol{Q} = (Q, \ldots, Q)$ with $Q \geq 2$.

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Note that the linear independence of the row entries of the matrix $\boldsymbol{L}$ together with 1 over $\mathbb{Z}$ follows directly from the definition of the set $\mathrm{Mad}^{m+n-1}(m, n)$.

This result is not best possible. In particular, for $m=2,\ n=1$ inequality (1·9) is not sharp. Indeed, by (1·8), we have that for all $\varepsilon>0$ and all $\boldsymbol{L}\in\mathrm{Mad}^{1+\varepsilon}(2, 1)$ it holds

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{\varepsilon} Q(\log Q)^{1+\varepsilon}.$$

Such matrices $\boldsymbol{L}$ exist thanks to the main result in [1], which implies that $\dim \mathrm{Mad}^{\lambda}(2, 1)=2$ for all $\lambda>1$. It is also well-known (see [8]) that set of $1\times n$ matrices $\boldsymbol{L}$ such that

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{n} Q^{n}\log Q$$

has full Hausdorff dimension in $\mathbb{R}^{1\times n}$. Thus, (1·9) is again not sharp for $m=1$. However, to the best of our knowledge, for $m\geq 3$ or $m=2,\ n\geq 2$ the existence of matrices satisfying (1·9) was not previously known.

2. *Generalised Cantor sets in higher dimension*

In this section we introduce a simple generalisation of a one-dimensional construction used by Badziahin and Velani in [2][2]. This generalisation will be useful in the proof of Proposition 1·6. Henceforth, the word cube will stand for ball in the supremum norm.

Let $l\in\mathbb{N}$ and let C be a closed cube in $\mathbb{R}^{l}$. For $k\geq 0$ let $\boldsymbol{R}:=(R_k)$ be a sequence of natural numbers, and let $\boldsymbol{r}:=(r_k)$ and $\boldsymbol{h}:=(h_k)$ be sequences of non-negative integers with $0\leq h_k\leq k$. Our goal is to construct a Cantor-like set contained in C depending on the sequences $\boldsymbol{R}$, $\boldsymbol{r}$, and $\boldsymbol{h}$. We denote such set by $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$. To this end, we introduce two sequences $\mathcal{I}_k$ and $\mathcal{J}_k$ of cube collections such that each cube lies in C $(k\geq 0)$. We set $\mathcal{I}_0=\mathcal{J}_0:=\{C\}$ and we define $\mathcal{I}_k$ and $\mathcal{J}_k$ by recursion on k. We do this in two steps. Suppose that we have constructed the sets $\mathcal{I}_h$ and $\mathcal{J}_h$ for $h=0, \ldots, k$. Then,

Step 1: we split each cube $J\in\mathcal{J}_k$ into R_k^l cubes of equal volume. We denote by $\mathcal{I}_{k+1}$ the family of all the cubes obtained via this splitting procedure for J ranging in $\mathcal{J}_k$; note that for $I\in\mathcal{I}_{k+1}$ we have that

$$\mathrm{edge}(I)=R_k^{-1}\mathrm{edge}(J) \quad \text{and} \quad \#\mathcal{I}_{k+1}=R_k^{l}\#\mathcal{J}_k;$$

Step 2: for each $J\in\mathcal{J}_{h_k}$ we remove from $\mathcal{I}_{k+1}$ at most r_k cubes $I\in\mathcal{I}_{k+1}$ such that $I\subset J$. We denote by $\mathcal{J}_{k+1}$ the family given by the remaining cubes in $\mathcal{I}_{k+1}$.

Finally, we set

$$\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r}) := \bigcap_{k=1}^{\infty}\bigcup_{J\in\mathcal{J}_k} J.$$

Note that the sequences $\boldsymbol{R}$, $\boldsymbol{r}$, and $\boldsymbol{h}$ do not determine a unique set, but a number of different sets obtained via the procedure described above. Indeed, we did not specify which cubes we remove in the second step (we only gave a bound on their number). We call every set

[2] To be precise, our construction is simplified compared to that of Badziahin and Velani. They use a double-indexed sequence $\boldsymbol{r}=(r_{h,k})$, whereas we use two separate single-indexed sequences $\boldsymbol{r}=(r_k)$ and $\boldsymbol{h}=(h_k)$.

constructed by using the sequences $\boldsymbol{R}$, $\boldsymbol{r}$, and $\boldsymbol{h}$, in the cube C, a $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set. We also observe that, by construction,

$$\#\mathcal{J}_{k+1} \geq R_k^l \#\mathcal{J}_k - r_k \#\mathcal{J}_{h_k} \tag{2·1}$$

for all $k \geq 0$.

Now, the following proposition extends [2, theorem 3].

PROPOSITION 2·1 (multidimensional Baziahin–Velani). *Let* $\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ *be a* $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ *-Cantor set, where* $C \subset \mathbb{R}^l$ *is a cube, and let*

$$t_k := R_k^l - \frac{r_k}{\prod_{i=h_k}^{k-1} t_i}$$

for $k \geq 1$. *If* $t_k > 0$ *for all* $k \geq 0$, *then we have that* $\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r}) \neq \emptyset$.

The proof is almost straightforward and we give it directly in this section.

Proof. We shall prove by induction on k that for $k \geq 1$

$$\#\mathcal{J}_k \geq t_{k-1} \#\mathcal{J}_{k-1}. \tag{2·2}$$

The fact that $t_k > 0$ for all k, along with (2·2), implies that

$$\#\mathcal{J}_k \geq \left(\prod_{h=0}^{k-1} t_h\right) \#\mathcal{J}_0 > 0.$$

Hence, every $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set is the intersection of a family of nested compact non-empty sets, and therefore non-empty.

We are left to prove that $\#\mathcal{J}_k \geq t_{k-1}\#\mathcal{J}_{k-1}$ for all $k \geq 1$. We do this by recursion on k. From (2·1), we deduce that $\#\mathcal{J}_1 \geq R_0^l \#\mathcal{J}_0 - r_0\#\mathcal{J}_0 = t_0\#\mathcal{J}_0$, and this proves the case $k = 1$. Now, let us assume that for all $1 \leq h \leq k$ it holds $\#\mathcal{J}_h \geq t_{h-1}\#\mathcal{J}_{h-1}$. Then, we have that

$$\#\mathcal{J}_k \geq \left(\prod_{i=h_k}^{k-1} t_i\right) \#\mathcal{J}_{h_k}.$$

This, combined with (2·1), gives

$$\#\mathcal{J}_{k+1} \geq R_k^l \#\mathcal{J}_k - r_k \#\mathcal{J}_{h_k} \geq \left(R_k^l - \frac{r_k}{\prod_{i=h_k}^{k-1} t_i}\right) \#\mathcal{J}_k = t_k \#\mathcal{J}_k,$$

whence the claim.

3. *Proof of Proposition 1·6*

The strategy is simple enough: by picking suitable parameters, we construct a non-empty $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set $\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ lying in $\mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$. To do so, we fix a non-decreasing sequence of integers $\boldsymbol{R} = (R_k)$ with $R_k \geq 1$, a sequence of non-negative integers $\boldsymbol{h}$, with $0 \leq h_k \leq k$, and a strictly increasing unbounded function F : $\{0\} \cup \mathbb{N} \to [1, +\infty)$. In the following technical lemma we specify the values of a sequence $\boldsymbol{r}$ (in terms of c, ℓ, F, $\boldsymbol{R}$, and $\boldsymbol{h}$) for which there exists a (possibly empty) $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set contained in $\mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$.

LEMMA 3·1. *Assume that*:

(i) $2^m c < e^{-1}$;
(ii) $F(0) = 1$ *and* $F(k+1)/F(k) \geq e$ *for all* $k \geq 0$;
(iii) $F(k+1)^2 \log^*(F(k+1))^{m+n-1} \leq c\ell^{-1} \prod_{h=0}^{k} R_h$ *for all* $k \geq 0$.

Then, there is a $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$*-Cantor set contained in* $\mathrm{Mad}_{m+n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$ *with* $\boldsymbol{r}$ *given by*

$$r_k := \mathrm{const}(m, n) \left[\mathfrak{f}(c, \ell, \boldsymbol{R}, \boldsymbol{h}, k) \prod_{h=h_k}^{k} R_h^{mn} + \prod_{h=h_k}^{k} R_h^{mn-1} \right], \tag{3·1}$$

where the factor $\mathfrak{f}(c, \ell, \boldsymbol{R}, \boldsymbol{h}, k)$ *has the form*

$$\mathfrak{f}(c, \ell, \boldsymbol{R}, \boldsymbol{h}, k) := c \log\left(\frac{1}{2^m c}\right)^{m-1} \frac{1}{\log^*(F(k))} \log\left(\frac{F(k+1)}{F(k)}\right)^{n-1} \tag{3·2}$$
$$\times \left(\log\left(\frac{F(k+1)}{F(k)}\right) + \ell^{-m} \left(2F(k)^{-m/n} - F(k+1)^{-m/n}\right) \prod_{h=0}^{h_k - 1} R_h^m \right),$$

and $\mathrm{const}(m, n) > 0$ *is a constant only depending on* m *and* n.

Lemma 3·1 is a key result in our method. Its proof, although quite technical, is essentially based on elementary geometric considerations. We prove Lemma 3·1 in Section 4.

Now, we need to show that the $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set constructed in Lemma 3·1 is non-empty. To do so, we use a non-emptiness condition involving the values of the sequence $\boldsymbol{r}$.

LEMMA 3·2. *Let* $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ *be a* $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$*-Cantor set. If for all* $k \geq 0$ *we have that*

$$r_k \leq \frac{g_k}{\max\{2, k\}} \prod_{h=h_k}^{k} R_h^{mn}, \tag{3·3}$$

where $g_k := \max\{2, h_k\}/(8 \max\{2, k-1\})$*, then the set* $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ *is non-empty.*

We prove this lemma in Section 5.

To conclude the proof of Proposition 1·6, it is enough to show that both the hypotheses of Lemma 3·1 and Lemma 3·2 simultaneously hold for an appropriate choice of the parameters c, F, $\boldsymbol{R}$, and $\boldsymbol{h}$. With this in mind, we fix a constant $R > 0$, and we set $R_k := R$, $F(k) := R^{k/3}$, and $h_k := \lfloor k/(3n) \rfloor$ for all $k \geq 0$. Then, we prove that, provided R is large enough, the constant c has enough room to satisfy both the hypotheses of Lemma 3·1 and Lemma 3·2.

With our choice of $\boldsymbol{R}$, F, and $\boldsymbol{h}$, condition $ii)$ in Lemma 3·1 becomes $R \geq e^3$, whereas condition (iii) becomes

$$R^{\frac{2(k+1)}{3}} \log^*\left(R^{\frac{k+1}{3}}\right)^{m+n-1} \leq c\ell^{-1} R^{k+1},$$

whence

$$\ell R^{-\frac{k+1}{3}} \log^*\left(R^{\frac{k+1}{3}}\right)^{m+n-1} \leq c. \tag{3·4}$$

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On the other hand, by substituting (3·1) into (3·3), we obtain

$$\mathrm{const}(m,n)\left[\mathfrak{f}(c,\ell,\boldsymbol{R},\boldsymbol{h},k)\prod_{h=h_k}^{k}R_h^{mn}+\prod_{h=h_k}^{k}R_h^{mn-1}\right]\leq\frac{g_k}{\max\{2,k\}}\prod_{h=h_k}^{k}R_h^{mn},$$

which, with our choice of $\boldsymbol{R}$, F, and $\boldsymbol{h}$, is equivalent to

$$\mathfrak{f}(c,\ell,R,k)+\frac{1}{R^{\left(k-\lfloor\frac{k}{3n}\rfloor+1\right)}}\leq\frac{g_k\mathrm{const}(m,n)^{-1}}{\max\{2,k\}}. \tag{3·5}$$

Since the sequence g_k is bounded away from 0 for all values of k, by choosing R suitably large in terms of m and n, we can ignore the second term at the left-hand side of (3·5). Hence, we are just left to prove that

$$\mathfrak{f}(c,\ell,R,k)\leq\frac{\mathrm{const}'(m,n)}{\max\{2,k\}},$$

where $\mathrm{const}'(m,n)$ is a constant only depending on m and n. By using (3·2), this can be written as

$$c\log^*\left(\frac{1}{2^mc}\right)^{m-1}\frac{1}{\max\{1,k\}}\log^*\left(R^{1/3}\right)^{n-1}\times\left(\log^*\left(R^{1/3}\right)+\ell^{-m}R^{\frac{-mk}{3n}}\left(2-R^{-\frac{m}{3n}}\right)R^{\lfloor\frac{k}{3n}\rfloor m}\right)\leq\frac{\mathrm{const}'(m,n)}{\max\{2,k\}}, \tag{3·6}$$

where we ignored a factor of $\log\left(R^{1/3}\right)$ at the denominator, coming from $\log^*(F(k))$ for $k\geq 1$. Assuming that $\ell<1$, condition (3·6) holds if we have that

$$c\log^*\left(\frac{1}{2^mc}\right)^{m-1}\leq\mathrm{const}''(m,n)\ell^m\log^*\left(R^{1/3}\right)^{-n}, \tag{3·7}$$

where $\mathrm{const}''(m,n)$ is some other positive constant only depending on m and n.

To conclude the proof, we fix a small real number $\varepsilon>0$ and we write $\log^*(1/2^mc)^{m-1}\ll_{m,\varepsilon}c^{-\varepsilon}$. Then, condition (3·7) is implied by

$$c\leq\mathrm{const}'''(m,n,\varepsilon)\ell^{\frac{m}{1-\varepsilon}}\log^*\left(R^{1/3}\right)^{-\frac{n}{1-\varepsilon}}, \tag{3·8}$$

where $\mathrm{const}'''(m,n,\varepsilon)$ is a suitably chosen positive constant only depending on m, n, and ε. The claim is then proved on noting that (3·4) and (3·8) can simultaneously hold for a sufficiently large value of R.

4. *Proof of Lemma* 3·1

4·1. *Construction of the Cantor-like set*

For each $P:=(\boldsymbol{p},\boldsymbol{q})\in\mathbb{Z}^m\times(\mathbb{Z}^n\setminus\{\boldsymbol{0}\})$, we introduce the following "bad" set:

$$\Delta(P):=\left\{\boldsymbol{X}\in\mathbb{R}^{m\times n}:\prod_{i=1}^{m}|X_i\boldsymbol{q}+\gamma_i+p_i|\leq\frac{c}{\prod(\boldsymbol{q})\log^*\left(\prod(\boldsymbol{q})\right)^{m+n-1}},\right.$$
$$\left.|X_i\boldsymbol{q}+\gamma_i+p_i|\leq\frac{1}{2}\ i=1,\dots,m\right\}, \tag{4·1}$$

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where we ignore the dependence on $\boldsymbol{\gamma}$ and c for simplicity. Further, we enumerate the hyperplanes in $\mathscr{H}$, indexing them for $k \in \{0\} \cup \mathbb{N}$. Then, we define the families $\mathcal{J}_k$ of a Cantor-like set so that the intersection of the cubes in $\mathcal{J}_k$ avoids all the "bad" sets $\Delta(P)$ and the hyperplanes H_k for $k \in \mathbb{N}$. More precisely, for each $J \in \mathcal{J}_k$ we require that $J \cap (\Delta(P) \cup H_h) = \emptyset$ for all the points P with $\prod(\boldsymbol{q}) < F(k)$ and all the indices h with $h \leq k$ (where we assume $H_0 = \emptyset$). If this condition is satisfied, we have that

$$\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r}) \subset \bigcap_{k=0}^{+\infty} \bigcap_{\prod(\boldsymbol{q})<F(k)} \bigcap_{h \leq k} C \setminus (\Delta(P) \cup H_h)$$

$$= C \Bigg\backslash \left(\bigcup_P \Delta(P) \cup \bigcup_{H \in \mathscr{H}} H \right) = \mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \Bigg\backslash \bigcup_{H \in \mathscr{H}} H, \quad (4\cdot 2)$$

whence the claim. Note that (4·2) holds true because the function $F(k)$ is unbounded.

We construct the families $\mathcal{J}_k$ by recursion on $k \geq 0$. For each k we need to ensure that

$$J \in \mathcal{J}_k \Rightarrow J \cap \left(\bigcup_{\prod(\boldsymbol{q})<F(k)} \Delta(P) \cup \bigcup_{h \leq k} H_h \right) = \emptyset. \quad (4\cdot 3)$$

If $k = 0$, we have that $\mathcal{J}_0 = \{C\}$ and $F(0) = 1$. Therefore, by definition,

$$\bigcup_{\prod(\boldsymbol{q})<1} \Delta(P) \cup \bigcup_{h \leq 0} H_h = \emptyset.$$

This shows that the family $\mathcal{J}_0$ satisfies (4·3). For $k \geq 1$ we subdivide the points $P \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})$ into "workable" families. Namely, we define

$$C(k) := \left\{ P \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\mathbf{0}\}) : F(k-1) \leq \prod(\boldsymbol{q}) < F(k) \right\}.$$

Suppose that we have constructed the family $\mathcal{J}_k$ in such a way that (4·3) holds for all $J \in \mathcal{J}_k$ (note that $\mathcal{J}_k$ can be empty). If $\mathcal{J}_k = \emptyset$, we set $\mathcal{J}_{k+1} := \emptyset$. If $\mathcal{J}_k \neq \emptyset$, we proceed as follows. Since any cube in $\mathcal{I}_{k+1}$ lies within some cube in $\mathcal{J}_k$, it is enough to construct the family $\mathcal{J}_{k+1}$ in such a way that if $J \in \mathcal{J}_{k+1}$ then $J \cap (\Delta(P) \cup H_{k+1}) = \emptyset$ for all $P \in C(k+1)$. To define $\mathcal{J}_{k+1}$, we therefore remove from $\mathcal{I}_{k+1}$ all the cubes I such that $I \cap (\Delta(P) \cup H_{k+1}) \neq \emptyset$ for some $P \in C(k+1)$. This procedure yields a possibly empty Cantor-like set $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$, contained in the set $\mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$.

To conclude the proof, we need to estimate the number of "small" cubes $I \in \mathcal{I}_{k+1}$ that need to be removed from each "big" cube $J \in \mathcal{J}_{h_k}$ to avoid the sets $\Delta(P)$ for $P \in C(k+1)$ and the hyperplane H_{k+1}, and show that such number is smaller than the constant r_k defined in (3·1). We start by counting the cubes intersecting the sets $\Delta(P)$ for $P \in C(k+1)$. For a fixed $J \in \mathcal{J}_{h_k}$ it is enough to estimate the quantity

$$\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\}.$$

If $\mathcal{I}_{k+1} = \emptyset$, there is nothing to prove. Otherwise, we write

$$\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\}$$

$$= \bigcup_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \bigcup_{\substack{P \in C(k+1) \\ \boldsymbol{q}(P) = \boldsymbol{q}}} \{I \in \mathcal{I}_{k+1} : I \cap J \cap \Delta(P) \neq \emptyset\}.$$

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From this, we deduce that

$$\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\} \leq \sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} A(\boldsymbol{q}) B(\boldsymbol{q}), \tag{4·4}$$

where

$$A(\boldsymbol{q}) := \max_{\substack{P \in C(k+1) \\ \boldsymbol{q}(P) = \boldsymbol{q}}} \#\{I \in \mathcal{I}_{k+1} : I \cap J \cap \Delta(P) \neq \emptyset\} \tag{4·5}$$

and

$$B(\boldsymbol{q}) := \#\left\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\right\}. \tag{4·6}$$

We estimate the factors $A(\boldsymbol{q})$ and $B(\boldsymbol{q})$ separately in the next two subsections.

4·2. *Estimate of* $A(\boldsymbol{q})$

To estimate $A(\boldsymbol{q})$, we rely on the following counting result.

LEMMA 4·1. *Let* $\boldsymbol{\gamma}' \in \mathbb{R}^m$, $\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$, *and* $\varepsilon, T \in (0, +\infty)$, *with* $\varepsilon/T^m < e^{-1}$, *where* $e = 2.71828\ldots$ *is the base of the natural logarithm. Let*

$$\mathscr{C} := \left\{ \boldsymbol{X} \in \mathbb{R}^{m\times n} : \prod_{i=1}^{m} |X_i \boldsymbol{q} + \gamma_i'| \leq \varepsilon,\ |X_i \boldsymbol{q} + \gamma_i'| \leq T,\ i = 1, \ldots, m \right\}$$

and let $\mathscr{D} \subset \mathbb{R}^{m\times n}$ *be a cube such that* $\mathscr{D} \cap \mathscr{C} \neq \emptyset$. *Finally, let* $\delta > 0$, $\boldsymbol{V} \in \mathbb{R}^{m\times n}$, *and* Λ *be the grid* $\delta\mathbb{Z}^{m\times n} + \boldsymbol{V}$. *Then, we have*

$$\delta^{mn}\#\{\text{tiles } \tau \text{ of the grid } \Lambda : \tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset\} \leq 2^{2m-1} \frac{\varepsilon + (T + n|\boldsymbol{q}|_\infty \delta)^m - T^m}{|\boldsymbol{q}|_\infty^m}$$
$$\times \log^*\left(\frac{(T + n|\boldsymbol{q}|_\infty\delta)^m}{\varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m}\right)^{m-1} (\mathrm{edge}(\mathscr{D}) + 2\delta)^{m(n-1)}, \tag{4·7}$$

where a tile is any set of the form $\{\boldsymbol{X} \in \mathbb{R}^{m\times n} : \delta S_{ij} + V_{ij} \leq X_{ij} \leq \delta(S_{ij}+1) + V_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$ *for some* $\boldsymbol{S} \in \mathbb{Z}^{m\times n}$.

We prove Lemma 4·1 in Section 6.

We note that if:

(a) $\varepsilon \gg_{m,n} (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m$;
(b) $T \gg_{m,n} |\boldsymbol{q}|_\infty \delta$;
(c) $\mathrm{edge}(\mathscr{D}) \gg_{m,n} \delta$;

then, Equation (4·7) implies

$$\#\{\text{tiles } \tau \text{ of the grid } \Lambda : \tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset\} \ll_{m,n} \frac{\varepsilon}{\delta^{mn}|\boldsymbol{q}|_\infty^m} \log^*\left(\frac{T^m}{\varepsilon}\right)^{m-1} \mathrm{edge}(\mathscr{D})^{m(n-1)}. \tag{4·8}$$

This is precisely the assertion that we need to prove the claim. We fix a point $P \in C(k+1)$ and a cube $J \in \mathcal{J}_{h_k}$, and we apply (4·8) to $\mathscr{C} = \Delta(P)$, $\mathscr{D} = J$, and to the grid Λ formed by the cubes $I \in \mathcal{I}_{k+1}$. We have $\varepsilon = c(\prod(\boldsymbol{q}) \log^*(\prod(\boldsymbol{q}))^{m+n-1})^{-1}$, $T = 1/2$, and

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$\delta = \ell \prod_{h=0}^{k} R_h^{-1}$ (note that by the hypothesis $\varepsilon/T^m < e^{-1}$). First, we need to show that conditions (a), (b), and (c) hold in this specific case. We start by noting that if condition (b) is satisfied, then to prove condition (a) it is enough to show that $\varepsilon \gg_{m,n} T^{m-1}|\boldsymbol{q}|_\infty \delta$. By the definition of $C(k+1)$ and part (iii) in the hypotheses of Lemma 3·1, we have that

$$\frac{\varepsilon}{|\boldsymbol{q}|_\infty} = \frac{c}{|\boldsymbol{q}|_\infty \prod(\boldsymbol{q}) \log^*\left(\prod(\boldsymbol{q})\right)^{m+n-1}} \geq \frac{c}{F(k+1)^2 \log^*(F(k+1))^{m+n-1}} \geq \ell \prod_{h=0}^{k} R_h^{-1} = \delta. \tag{4·9}$$

Hence, $\varepsilon \gg_{m,n} T^{m-1}|\boldsymbol{q}|_\infty \delta$, and we have (a). Condition (b) is equivalent to $1/|\boldsymbol{q}|_\infty \gg_{m,n} \delta$, which is again implied by (4·9). Finally, condition (c) is clearly satisfied since $\text{edge}(J) \geq \text{edge}(I)$ for any $I \in \mathcal{I}_{k+1}$. Thus, we can apply (4·8) to obtain

$$\begin{aligned} A(\boldsymbol{q}) = \max_{\substack{P \in C(k+1) \\ \boldsymbol{q}(P) = \boldsymbol{q}}} \#\{I \in \mathcal{I}_{k+1} : I \cap J \cap \Delta(P) \neq \emptyset\} &\ll_{m,n} \frac{c}{|\boldsymbol{q}|_\infty^m \prod(\boldsymbol{q}) \log^*\left(\prod(\boldsymbol{q})\right)^{m+n-1}} \\ &\times \log^*\left(\frac{\prod(\boldsymbol{q}) \log^*\left(\prod(\boldsymbol{q})\right)^{m+n-1}}{2^m c}\right)^{m-1} \ell^{-m} \prod_{h=0}^{h_k - 1} R_h^{-m(n-1)} \prod_{h=0}^{k} R_h^{mn} \\ &\ll_{m,n} \frac{c \log^*(1/(2^m c))^{m-1}}{|\boldsymbol{q}|_\infty^m \prod(\boldsymbol{q}) \log^*\left(\prod(\boldsymbol{q})\right)^n} \ell^{-m} \prod_{h=0}^{h_k - 1} R_h^{-m(n-1)} \prod_{h=0}^{k} R_h^{mn}. \end{aligned} \tag{4·10}$$

4·3. *Estimate of* $B(\boldsymbol{q})$

We are now left to estimate $\#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\}$ for each given $\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ such that $F(k) \leq \prod(\boldsymbol{q}) < F(k+1)$. For $P \in C(k+1)$ we consider the hyperspace

$$\pi(P) := \{\boldsymbol{X} \in \mathbb{R}^{m \times n} : X_i \boldsymbol{q} + \gamma_i + p_i = 0,\ i = 1, \ldots, m\},$$

i.e., the "core" of the set $\Delta(P)$. We show that to count the number of points $P \in C(k+1)$ such that $\Delta(P)$ intersects J, it is enough to count the number of points $P \in C(k+1)$ such that the thinner set $\pi(P)$ intersects an "inflation" of J. In particular, we claim that

$$\begin{aligned} &\#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\} \\ &\qquad \leq \#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset\}, \end{aligned} \tag{4·11}$$

where $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ is the "inflation" of the cube J by the quantity $\sqrt{m}/|\boldsymbol{q}|_\infty$, i.e., the set $\{\boldsymbol{X} \in \mathbb{R}^{m \times n} : \text{dist}_\infty(\boldsymbol{X}, J) \leq \sqrt{m}/|\boldsymbol{q}|_\infty\}$. To prove (4·11) we shall show that for any fixed point P in the left-hand side of (4·11) it holds

$$J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset.$$

We start by observing that, for any $\boldsymbol{Y} \in \Delta(P)$ we have that

$$|Y_i \boldsymbol{q} + \gamma_i + p_i| \leq 1/2 \text{ for } i = 1, \ldots, m.$$

Moreover, for $i = 1, \ldots, m$ the Euclidean distance in $\mathbb{R}^m$ between the vector Y_i and the hyperplane $\{X_i \boldsymbol{q} + \gamma_i + p_i = 0\}$ is given by $|Y_i \boldsymbol{q} + \gamma_i + p_i|/|\boldsymbol{q}|_2$. Hence, the Euclidean

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distance in $\mathbb{R}^{m\times n}$ between the vector $\boldsymbol{Y}$ and the hyperspace $\pi(P)$ is at most $\sqrt{m}/(2|\boldsymbol{q}|_2)$. This shows that for any point $\boldsymbol{Y} \in \Delta(P)$, we have that

$$\mathrm{dist}_2(\boldsymbol{Y}, \pi(P)) \leq \frac{\sqrt{m}}{2|\boldsymbol{q}|_2}.$$

Since $J \cap \Delta(P) \neq \emptyset$, we further deduce that

$$\mathrm{dist}_\infty(J, \pi(P)) \leq \mathrm{dist}_2(J, \pi(P)) \leq \mathrm{dist}_2(J \cap \Delta(P), \pi(P)) \leq \frac{\sqrt{m}}{2|\boldsymbol{q}|_2} \leq \frac{\sqrt{m}}{2|\boldsymbol{q}|_\infty}.$$

Hence, by the definition of distance, $J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset$, whence the claim.

We are now left to bound the right-hand side in (4·11). To do so, we notice that the distance between two hyperspaces $\pi(P)$ and $\pi(P')$ defined by the same vector $\boldsymbol{q}$ is at least $1/(n|\boldsymbol{q}|_\infty)$. Indeed, assume that two vectors $X_i, X_i' \in \mathbb{R}^n$ satisfy $X_i\boldsymbol{q} + \gamma_i + p_i = 0$ and $X_i'\boldsymbol{q} + \gamma_i + p_i' = 0$, with $p_i \neq p_i'$. Then, by the Cauchy-Schwartz inequality, we have that

$$\mathrm{dist}_\infty(X_i, X_i') \geq \frac{\mathrm{dist}_2(X_i, X_i')}{\sqrt{n}} \geq \frac{|(X_i - X_i')\boldsymbol{q}|}{\sqrt{n}|\boldsymbol{q}|_2} \geq \frac{|p_i - p_i'|}{\sqrt{n}|\boldsymbol{q}|_2} \geq \frac{1}{n|\boldsymbol{q}|_\infty}.$$

This shows that for all $\boldsymbol{X} \in \pi(P)$ and $\boldsymbol{X}' \in \pi(P')$ it holds

$$\mathrm{dist}_\infty(\boldsymbol{X}, \boldsymbol{X}') \geq \frac{1}{n|\boldsymbol{q}|_\infty}. \tag{4·12}$$

Now, if $J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset$ for some P, by an elementary dimensional argument[3], the hyperspace $\pi(P)$ must intersect at least one m-dimensional face of the cube $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$. For each point P such that $J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset$ we select a second point $Q(P)$ on an m-dimensional face of $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ lying in $\pi(P)$. We know, by (4·12), that all such points are at least at a distance $1/(n|\boldsymbol{q}|_\infty)$ away from each other in the supremum norm. To evaluate their number, we fix any m-dimensional face E of $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ and we enlarge it by $1/(2n|\boldsymbol{q}|_\infty)$ in all directions, i.e., we consider the set $E_{1/(2n|\boldsymbol{q}|_\infty)}$. Then, for each intersection point $Q(P) \in \pi(P) \cap J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ we take an mn-dimensional cube of edge $1/(n|\boldsymbol{q}|_\infty)$ centred at $Q(P)$. All these cubes are contained in the "inflated" face $E_{1/(2n|\boldsymbol{q}|_\infty)}$ and they all have disjoint interiors. Comparing the volume of these cubes and the volume of the inflated face, we find that

$$\begin{aligned}
&\#\{P \in C(k+1) : E \cap \pi(P) \neq \emptyset\} \left(\frac{1}{n|\boldsymbol{q}|_\infty}\right)^{mn} \\
&\leq \mathrm{Vol}\left(E_{1/(2n|\boldsymbol{q}|_\infty)}\right) = \left(\mathrm{edge}\left(J_{\sqrt{m}/|\boldsymbol{q}|_\infty}\right) + \frac{1}{n|\boldsymbol{q}|_\infty}\right)^m \left(\frac{1}{n|\boldsymbol{q}|_\infty}\right)^{m(n-1)} \\
&= \left(\mathrm{edge}\,(J) + \frac{1 + 2n\sqrt{m}}{n|\boldsymbol{q}|_\infty}\right)^m \left(\frac{1}{n|\boldsymbol{q}|_\infty}\right)^{m(n-1)}.
\end{aligned} \tag{4·13}$$

[3] We observe that the hyperspace $\pi(P)$ must intersect also the boundary of the cube J, i.e., some $(mn-1)$-dimensional face F of J. The intersection of $\pi(P)$ with the hyperspace $\mathrm{Span}(F)$, generated by F, has dimension at least $\dim(F) + \dim(\pi(P)) - \dim(F + \pi(P)) \geq \dim(\pi(P)) - 1$. Hence, we have that the hyperspace $\pi(P) \cap \mathrm{Span}(F)$ of dimension $\pi(P) - 1$ intersects a cube F of dimension $mn - 1$. The argument can be run recursively $\dim(\pi(P)) = mn - m$ times to show the claim.

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Now, since the number of m-dimensional faces of a cube only depends on m, from (4·11) and (4·13) we finally deduce that

$$B(\boldsymbol{q}) = \#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\} \leq \#\{P \in C(k+1) :$$
$$\boldsymbol{q}(P) = \boldsymbol{q},\ J_{\sqrt{m}/(2|\boldsymbol{q}|_\infty)} \cap \pi(P) \neq \emptyset\} \ll_{m,n} (|\boldsymbol{q}|_\infty \mathrm{edge}\,(J) + 1)^m \ll_{m,n} |\boldsymbol{q}|_\infty^m \ell^m \prod_{h=1}^{h_k-1} R_h^{-m} + 1. \tag{4·14}$$

4·4. *Conclusion*

To conclude the proof of Lemma 3·1, we combine (4·4), (4·10) and (4·14) to obtain

$$\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\}$$
$$\ll_{m,n} \sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \left(\frac{c \log^*(1/(2^m c))^{m-1}}{|\boldsymbol{q}|_\infty^m \prod(\boldsymbol{q}) \log^*\left(\prod(\boldsymbol{q})\right)^n} \ell^{-m} \prod_{h=0}^{h_k-1} R_h^{-m(n-1)} \prod_{h=0}^{k} R_h^{mn} \right)$$
$$\times \left(|\boldsymbol{q}|_\infty^m \ell^m \prod_{h=1}^{h_k-1} R_h^{-m} + 1 \right).$$

Hence, by using the fact that $|\boldsymbol{q}|_\infty^m \geq \prod(\boldsymbol{q})^{m/n}$, we find that

$$\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\} \ll_{m,n} c \log^*\left(\frac{1}{2^m c}\right)^{m-1} \prod_{h=h_k}^{k} R_h^{mn}$$
$$\times \frac{1}{\log^*\left(F(k)\right)^n} \sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \frac{1}{\prod(\boldsymbol{q})} \left(1 + \frac{\ell^{-m}}{\prod(\boldsymbol{q})^{m/n}} \prod_{h=0}^{h_k-1} R_h^m \right). \tag{4·15}$$

Now, a simple integration shows that

$$\sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \prod(\boldsymbol{q})^{-1} \ll_n \log^*(F(k+1))^{n-1} \log^*\left(\frac{F(k+1)}{F(k)}\right)$$

and

$$\sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \prod(\boldsymbol{q})^{-1-m/n} \ll_n \log^*(F(k+1))^{n-1} \left(2F(k)^{-m/n} - F(k+1)^{-m/n}\right).$$

Therefore, from (4·15) and from the trivial inequality $\log^*(F(k+1))^{n-1} \leq \log^*(F(k+1)/F(k))^{n-1} \log^*(F(k))^{n-1}$, we deduce that

$$\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\}$$
$$\ll_{m,n} c \log^*\left(\frac{1}{2^m c}\right)^{m-1} \frac{1}{\log^*\left(F(k)\right)} \log^*\left(\frac{F(k+1)}{F(k)}\right)^{n-1}$$
$$\times \left(\log^*\left(\frac{F(k+1)}{F(k)}\right) + \ell^{-m} \left(2F(k)^{-m/n} - F(k+1)^{-m/n}\right) \prod_{h=0}^{h_k-1} R_h^m \right) \prod_{h=h_k}^{k} R_h^{mn}. \tag{4·16}$$

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We are now left to count all the cubes in $\mathcal{I}_{k+1}$ lying in J and intersecting the hyperplane H_{k+1}. From the set of cubes $I \in \mathcal{I}_{k+1}$ such that $I \cap J \cap H_{k+1} \neq \emptyset$, we select a maximal subset S of pairwise disjoint cubes (with disjoint boundary). For each of these cubes I we pick a point lying in $I \cap H_{k+1}$. The points that we selected are, by construction, at least $\text{edge}(I)$ away from each other in the supremum norm. We then take $(mn-1)$-dimensional cubes in H_{k+1} of edge $\text{edge}(I)$ around each such point. By construction, these cubes are disjoint. Comparing the volume of the union of the cubes with the volume of the set $J \cap H_{k+1}$ inflated in the Euclidean distance by the quantity $\text{diam}(I)$ in H_{k+1}, i.e., the set $\{\boldsymbol{X} \in H_{k+1} : \text{dist}_2(\boldsymbol{X}, J \cap H_{k+1}) \leq \text{diam}(I)\}$, we find that

$$\#S \cdot \text{edge}(I)^{mn-1} \ll_{m,n} (\text{diam}(J \cap H_{k+1}) + \text{diam}(I))^{mn-1} \ll_{m,n} \text{edge}(J)^{mn-1},$$

whence

$$\#\{I \in \mathcal{I}_{k+1} : I \cap J \cap H_{k+1} \neq \emptyset\} \ll_{m,n} \#S \ll_{m,n} \left(\frac{\text{edge}(J)}{\text{edge}(I)}\right)^{mn-1} = \prod_{h=h_k}^{k} R_h^{mn-1}. \tag{4·17}$$

Combining (4·16) and (4·17), the proof of Lemma 3·1 is concluded.

5. *Proof of Lemma* 3·2

We show by induction on k that

$$t_k \geq R_k^{mn}\left(1 - \frac{1}{\max\{2, k\}}\right) > 0 \tag{5·1}$$

for all $k \geq 0$. By Proposition 2·1, this is enough to prove the claim. If $k = 0$ we have that

$$t_0 = R_0^{mn} - r_0 \geq R_0^{mn} - \frac{g_0}{2} R_0^{mn}.$$

Thus, the base case is proved, given that $g_0 = 1/8$. Now, let us assume that (5·1) holds for all $0 \leq h \leq k$. Then, we have that

$$\begin{aligned} t_{k+1} &= R_{k+1}^{mn} - \frac{r_{k+1}}{\prod_{i=h_{k+1}}^{k} t_i} \\ &\geq R_{k+1}^{mn} - \frac{r_{k+1}}{\prod_{i=h_{k+1}}^{k} R_h^{mn}\left(1 - \max\{2, i\}^{-1}\right)}. \end{aligned} \tag{5·2}$$

Moreover, for $k \geq 0$ we have that

$$\prod_{i=h_{k+1}}^{k} \left(1 - \max\{2, i\}^{-1}\right) \geq \frac{\max\{2, h_{k+1}\} - 1}{4 \max\{2, k\}} \geq \frac{\max\{2, h_{k+1}\}}{8 \max\{2, k\}} = g_{k+1}.$$

Hence, by (5·2) and by the hypothesis, we deduce that

$$t_{k+1} \geq R_{k+1}^{mn} - \frac{g_{k+1}^{-1} r_{k+1}}{\prod_{h=h_{k+1}}^{k} R_h^{mn}} \geq R_{k+1}^{mn}\left(1 - \frac{1}{\max\{2, k+1\}}\right),$$

concluding the proof.

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6. *Proof of Lemma* 4·1

For a set $\mathscr{A} \subset \mathbb{R}^{m\times n}$ we denote by $\mathscr{A}_\delta$ the "inflation" of $\mathscr{A}$ by the quantity δ, i.e., the set $\{\boldsymbol{X} \in \mathbb{R}^{m\times n} : \mathrm{dist}_\infty(\boldsymbol{X}, \mathscr{A}) \leq \delta\}$. We start by showing that

$$\delta^{mn}\#\{\text{tiles } \tau \text{ of the grid } \Lambda : \tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset\} \leq \mathrm{Vol}(\mathscr{D}_\delta \cap \mathscr{C}_\delta). \tag{6·1}$$

This follows from the fact that for any tile τ of the grid Λ we have that

$$\tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset \longrightarrow \tau \subset \mathscr{D}_\delta \cap \mathscr{C}_\delta. \tag{6·2}$$

To see why (6·2) holds, it is enough to observe that for all points $P \in \tau \cap \mathscr{C}$ and all points $Q \in \tau$ we have $\mathrm{dist}_\infty(P, Q) \leq \delta$. Hence, $\tau \subset \mathscr{C}_\delta$. The same is true for $\mathscr{D}$, whence (6·2).

To conclude the proof, we need to estimate the volume of the set $\mathscr{D}_\delta \cap \mathscr{C}_\delta$. By definition, if a point $\boldsymbol{X}$ lies in the set $\mathscr{C}_\delta$, there exists some point $\boldsymbol{X}' \in \mathscr{C}$ such that $\mathrm{dist}_\infty(\boldsymbol{X}, \boldsymbol{X}') \leq \delta$. Hence, we have that

$$\prod_{i=1}^m |X_i\boldsymbol{q} + \gamma_i'| \leq \prod_{i=1}^m \left(|X_i'\boldsymbol{q} + \gamma_i'| + n|\boldsymbol{q}|_\infty\delta\right)$$
$$\leq \prod_{i=1}^m |X_i'\boldsymbol{q} + \gamma_i'| + \sum_{I \subsetneq \{1,\ldots,m\}} \left(\prod_{i\in I} |X_i'\boldsymbol{q} + \gamma_i'|\right)(n|\boldsymbol{q}|_\infty\delta)^{m-|I|} \leq \varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m. \tag{6·3}$$

Now, let $\boldsymbol{\mu}$ be the centre of the cube $\mathscr{D}$. From (6·3), we deduce that

$$\mathscr{D}_\delta \cap \mathscr{C}_\delta \subset$$
$$\left\{\boldsymbol{X} \in \mathbb{R}^{m\times n} : \begin{cases} \prod_{i=1}^m |X_i\boldsymbol{q} + \gamma_i'| \leq \varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m & \\ |X_i\boldsymbol{q} + \gamma_i'| \leq T + n|\boldsymbol{q}|_\infty\delta & i = 1, \ldots, m \\ |X_{ij} - \mu_{ij}| \leq \mathrm{edge}(\mathscr{D})/2 + \delta & i = 1, \ldots, m, \ \ j = 1, \ldots, n \end{cases}\right\}. \tag{6·4}$$

Without loss of generality, we can assume that $|\boldsymbol{q}|_\infty = |q_n|$. We consider the linear transformation $\xi : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ defined by

$$\xi(\boldsymbol{X})_{ij} = \begin{cases} X_{ij} & \text{if } j \neq n \\ X_i\boldsymbol{q} & \text{if } j = n \end{cases}.$$

Under the action of ξ, the right-hand side of (6·4) is sent into a subset of the set

$$\left\{\boldsymbol{X} \in \mathbb{R}^{m\times n} : \begin{cases} \prod_{i=1}^m |X_{in} + \gamma_i'| \leq \varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m & \\ |X_{in} + \gamma_i'| \leq T + n|\boldsymbol{q}|_\infty\delta & i = 1, \ldots, m \\ |X_{ij} - \mu_{ij}| \leq \mathrm{edge}(\mathscr{D})/2 + \delta & i = 1, \ldots, m \ j = 1, \ldots, n-1 \end{cases}\right\}. \tag{6·5}$$

Since the determinant of ξ is $|q_n|^m = |\boldsymbol{q}|_\infty^m \neq 0$, to obtain an estimate of the quantity $\mathrm{Vol}(\mathscr{D}_\delta \cap \mathscr{C}_\delta)$, it is enough to estimate the volume of the set in (6·5) and to divide it by

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the absolute value of the determinant of ξ. A simple integration shows that the volume of this set is bounded above by

$$2^{2m-1}(\varepsilon+(T+n|\boldsymbol{q}|_\infty\delta)^m-T^m)\log^*\left(\frac{(T+n|\boldsymbol{q}|_\infty\delta)^m}{\varepsilon+(T+n|\boldsymbol{q}|_\infty\delta)^m-T^m}\right)^{m-1}(\mathrm{edge}(\mathscr{D})+2\delta)^{m(n-1)}.$$

Hence, $\mathrm{Vol}(\mathscr{D}_\delta\cap\mathscr{C}_\delta)$ is bounded above by this quantity divided by the absolute value of the determinant of ξ, that is $|q_n|^m=|\boldsymbol{q}|_\infty^m$.

Acknowledgements. I am most grateful to Nikolay Moshchevitin for the useful discussions that we had via email and at the Baikal Number Theory Conference (2019). I am thankful to my supervisor, Martin Widmer, for his invaluable advice and constant encouragement. I would also like to thank Royal Holloway, University of London, for funding my position here and the anonymous referee for their careful checking.

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