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Saturated Graphs of Prescribed Minimum Degree

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A graph G is H-saturated if it contains no copy of H as a subgraph but the addition of any new edge to G creates a copy of H. In this paper we are interested in the function $\mathrm{sat}_t(n,p)$, defined to be the minimum number of edges that a K_p-saturated graph on n vertices can have if it has minimum degree at least t. We prove that $\mathrm{sat}_t(n,p) = tn - O(1)$, where the limit is taken as n tends to infinity. This confirms a conjecture of Bollobás when $p = 3$. We also present constructions for graphs that give new upper bounds for $\mathrm{sat}_t(n,p)$.

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1. Introduction

We say a graph G is H-saturated if it contains no copy of H as a subgraph but the addition of any new edge to G creates a copy of H. In this paper we are interested in the case where H is the complete graph on p vertices, denoted K_p. For further results on saturated graphs see surveys by either Faudree, Faudree and Schmitt [6] or Pikhurko [12]. Erdős, Hajnal and Moon [4] showed that if G is a K_p-saturated graph on n vertices then $e(G) \geqslant n(p-2) - \binom{p-1}{2}$, and that the unique graph achieving equality is formed by taking a clique on $p-2$ vertices and fully connecting it to an independent set of size $n-(p-2)$. This extremal graph has minimum degree $p-2$, and no K_p-saturated graph on at least p vertices can have smaller minimum degree. Thus it is natural to ask: How few edges can a K_p-saturated graph have if it has minimum degree at least t for $t \geqslant p-2$?

Observe that any K_3-saturated graph on n vertices must be connected and so cannot have fewer than $n-1$ edges. The graph comprising a single vertex connected to all other vertices is K_3-saturated, has minimum degree 1 and has this minimum number of edges. Duffus and Hanson [3] showed that any K_3-saturated graph on n vertices with minimum degree 2 has at least $2n-5$ edges. Moreover, they showed that the unique graphs achieving this are obtained by taking a 5-cycle and repeatedly duplicating vertices of degree 2, that is, picking a vertex of degree 2 and adding a new vertex to the graph with the same neighbourhood as the chosen vertex. They also showed that any K_3-saturated graph on $n \geqslant 10$ vertices with minimum degree 3 has at least $3n-15$ edges and that any graph achieving this contains the Petersen graph as a subgraph.

In this paper we consider the function

$$\mathrm{sat}_t(n,p) = \min\{e(G) : |V(G)| = n, G \text{ is } K_p\text{-saturated}, \delta(G) \geqslant t\},$$

where $\delta(G)$ is the minimum degree of G. We also define the set $\mathrm{Sat}_t(n,p)$ to be

$$\{G : |V(G)| = n, G \text{ is } K_p\text{-saturated}, \delta(G) \geqslant t, e(G) = \mathrm{sat}_t(n,p)\}.$$

The complete bipartite graph $K_{t,n-t}$ shows that for $n \geqslant 2t$ we have $\mathrm{sat}_t(n,3) \leqslant tn - t^2$. This upper bound and Duffus and Hanson's results led Bollobás [8] to conjecture that for fixed t we have $\mathrm{sat}_t(n,3) = tn - O(1)$.

For more general values of p, Duffus and Hanson [3] showed that $\mathrm{sat}_t(n,p) \geqslant n(t+p-2)/2 - O(1)$. Writing $\alpha(G)$ for the size of the largest independent set in G, Alon, Erdős, Holzman and Krivelevich [1] showed that any K_p-saturated graph on n vertices with at most $O(n)$ edges has $\alpha(G) \geqslant n - O(n/\log\log n)$. This shows that $\mathrm{sat}_t(n,p) \geqslant tn - O(n/\log\log n)$ as $e(G) \geqslant \alpha(G)\delta(G)$. Pikhurko [12] improved this result to show that $\mathrm{sat}_t(n,p) \geqslant tn - O(n \log\log n/\log n)$.

Our main result in this paper improves these results by confirming and generalizing Bollobás's conjecture.

Theorem 1.1. *Let $t \in \mathbb{N}$. There exists a constant $c = c(t)$ such that, for all $3 \leqslant p \in \mathbb{N}$ and all $n \in \mathbb{N}$, if G is a K_p-saturated graph of order n and minimum degree at least t then $e(G) \geqslant tn - c$.*

The proof of Theorem 1.1 is presented in Section 2. To see that this result is best possible (up to the value of the constant), consider the graph obtained from fully connecting a clique of size $p-3$ to the complete bipartite graph $K_{t-(p-3),n-t}$. This graph is K_p-saturated and has minimum degree t, showing that

$$\mathrm{sat}_t(n,p) \leqslant tn - t^2 + t(p-3) - \binom{p-2}{2} \tag{1.1}$$

for $n \geqslant 2t - (p-3)$ and $t \geqslant p-2$.

We remark that although it may seem surprising that the constant $c(t)$ in the statement of Theorem 1.1 does not depend on p, it is a consequence of the fact that any K_p-saturated graph (on at least $p-1$ vertices) has minimum degree at least $p-2$. As a result, Theorem 1.1 is trivially true whenever $p \geqslant 2t+2$, and so, for fixed t, there are only a finite number of values of p we need to consider. The independence of $c(t)$ from p is also reflected in our proof of Theorem 1.1, which only makes use of the fact that our graph is K_p-saturated for some $3 \leqslant p \in \mathbb{N}$ and does not make use of p's value in any way.

On the other hand, Theorem 1.1 can be used to show the following: for all $t, p \in \mathbb{N}$ with $t \geqslant p-2 \geqslant 1$, there exists a constant $c(t,p)$ such that, for all sufficiently large $n \in \mathbb{N}$, we have $\mathrm{sat}_t(n,p) = tn - c(t,p)$. Indeed, Theorem 1.1 together with (1.1) shows that, for n sufficiently large, all $G \in \mathrm{Sat}_t(n,p)$ have $\delta(G) = t$. Duplicating a vertex of degree t in such a graph G gives a K_p-saturated graph on $n+1$ vertices with minimum degree t and $\mathrm{sat}_t(n,p) + t$ edges. Thus, as n increases, the integer sequence $tn - \mathrm{sat}_t(n,p)$ becomes non-decreasing but bounded above by $c(t)$, and so is eventually constant.

The proof of Theorem 1.1 can be used to show that $c(t,p) \leqslant t^{(t^{(2t^2)})}$. In Section 3 we discuss constructing K_p-saturated graphs and prove a lower bound for $c(t,p)$.

Theorem 1.2. *Let $3 \leqslant p \in \mathbb{N}$. There exists a constant $C = C(p) > 0$ such that, for all sufficiently large $t \in \mathbb{N}$, we have $c(t,p) \geqslant C2^t t^{3/2}$.*

The large distance between these upper and lower bounds for $c(t,p)$ naturally leads to the problem of improving these bounds, or perhaps even determining $c(t,p)$ for all t and p. Our proof of Theorem 1.1 seems to be inefficient for the purposes of bounding $c(t,p)$, and so we believe $c(t,p)$ is likely to be closer to the lower bound we give in Theorem 1.2 than the upper bound obtained from Theorem 1.1.

We remark that one may also ask how few edges a K_p-saturated graph can have if restrictions are placed on its maximum degree rather than its minimum degree. Results on this problem for $p = 3$ can be found in the paper of Füredi and Seress [7] and also in the paper of Erdős and Holzman [5]. Results for the case $p = 4$ can be found in the paper of Alon, Erdős, Holzman and Krivelevich [1]. There are currently no known results for $p \geqslant 5$.

2. Proof of Theorem 1.1

For a graph G and a vertex $v \in V(G)$, let $N(v)$ be the set of vertices in G that are adjacent to v. For $X \subseteq V(G)$ let $N_X(v) = N(v) \cap X$, let $d_X(v) = |N_X(v)|$ and let $e(X)$ be the number of edges in the graph $G[X]$. For another set $Y \subseteq V(G)$ that is disjoint from X, let $N_Y(X)$ be the set of vertices in Y adjacent to X and let $e(X,Y)$ be the number of edges between X and Y.

Proof of Theorem 1.1. Let G be a K_p-saturated graph on vertex set V with $|V| = n$ and $\delta(G) \geqslant t$. Given a set $R \subseteq V$, let $\overline{R}$ be the closure of R under t-neighbour bootstrap percolation on G. That is, let $\overline{R} = \bigcup_{i \geqslant 0} R_i$ where $R_0 = R$ and

$$R_i = R_{i-1} \cup \{v \in V : d_{R_{i-1}}(v) \geqslant t\}$$

for $i \geqslant 1$. Any vertex $x \in R_i \setminus R_{i-1}$ sends at least t edges to R_{i-1} and so $e(\overline{R}) \geqslant t(|\overline{R}| - |R|)$. Let $Y(R) = V \setminus \overline{R}$ and for a vertex $v \in V$ let

$$w_R(v) = d_{\overline{R}}(v) + \frac{1}{2} d_{Y(R)}(v).$$

We call $w_R(v)$ the weight of v (with respect to R). Within $Y(R)$, we define $B(R)$ to be the set $\{v \in Y(R) : w_R(v) < t\}$, which we call the set of bad vertices. Our aim will be to prove the following claim.

Claim 2.1. *There exists a constant $c_1 = c_1(t)$ and a set $R \subseteq V$ with $|R| \leqslant c_1(t)$ such that $B(R) = \emptyset$.*

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If we can prove Claim 2.1 then we have proved the theorem as

$$\begin{aligned} e(G) &= e(\overline{R}) + e(\overline{R}, Y(R)) + e(Y(R)) \\ &\geqslant t(|\overline{R}| - |R|) + \sum_{y \in Y(R)} w_R(y) \\ &\geqslant t(|\overline{R}| - c_1) + t|Y(R)| \\ &= t(n - c_1), \end{aligned}$$

as required. To prove Claim 2.1, we would like to show that if a set $R \subseteq V$ does lead to $B(R)$ being non-empty, then we can move a small number of vertices into R so that the remaining vertices in $B(R)$ have strictly larger weight. If so, we can start with some initial small set of vertices R and keep moving small numbers of vertices into R until $B(R)$ is empty. This idea of moving vertices into R fits naturally with our set-up so far. Indeed, suppose that S is a set of vertices with $R \subseteq S$. We have that $\overline{R} \subseteq \overline{S}$ and $Y(R) \supseteq Y(S)$ and so $w_R(v) \leqslant w_S(v)$ for all $v \in V$. Thus, we have that $B(R) \supseteq B(S)$.

It turns out that dealing with $w_R(v)$ directly is difficult and so we introduce a control function $l_R(v) = \sum_{x \in N(v)} f_R(x)$ defined for all $v \in V$, where for all $x \in V$

$$f_R(x) = \begin{cases} 1 & \text{if } x \in R, \\ \dfrac{1}{2} & \text{if } x \in \overline{R} \setminus R, \\ \dfrac{1}{2t} d_R(x) & \text{if } x \in Y(R). \end{cases}$$

Observe that $l_R(v) \leqslant w_R(v)$ for every $v \in V$, since $d_R(X) \leqslant t-1$ for every $x \in Y(R)$. Similarly, we have $f_R(v) \leqslant f_S(v)$ for every $R \subseteq S$ and every $v \in V$, since $Y(S) \subseteq Y(R)$.

We use our control function $l_R(v)$ to make the following claim.

Claim 2.2. *For every set $R \subseteq V$, there exists a set $S \subseteq V$ such that $R \subseteq S$, $|S| \leqslant |R| + t2^{|R|}$ and $l_S(v) \geqslant l_R(v) + 1/2t$ for all $v \in B(S)$.*

We note that Claim 2.2 is enough to prove Claim 2.1 and hence our theorem. Indeed, begin by taking $R = \{v\}$ for any $v \in V$ and repeatedly replace R with S. After at most $2t^2$ such replacements, we will have that $B(R)$ is empty: any bad vertex $v \in B(R)$ would have $w_R(v) \geqslant l_R(v) \geqslant t$, which is not possible by the definition of $B(R)$. Moreover, each time we replace R with S we have $|S| \leqslant |R| + t2^{|R|}$, and so our final set will have size bounded above by some function $c_1(t)$, as required.

We now describe how to find a suitable set S given some set R. Suppose that $B(R)$ is non-empty. Let $\mathcal{C}$ be the set

$$\{C \subseteq R : C = N_R(y) \text{ for some } y \in B(R)\}$$

and label its elements $\mathcal{C} = \{C_1, \dots, C_k\}$. The set $\mathcal{C}$ is a collection of subsets of R and so $k \leqslant 2^{|R|}$. For each $C_i \in \mathcal{C}$ pick a representative $y_i \in B(R)$ such that $C_i = N_R(y_i)$. As $y_i \in Y(R)$, we have that $d_{\overline{R}}(y_i) < t$ and so, as $d(y_i) \geqslant t$, we can pick some $x_i \in Y(R)$ such that y_i and x_i are adjacent. Let

$X = \{x_1, \ldots, x_k\}$ and let

$$S = R \cup X \cup N_{\overline{R}}(X).$$

Clearly $R \subseteq S$. Noting that $d_{\overline{R}}(x) \leqslant t-1$ for each $x \in X$, which holds as $X \subseteq Y(R)$, it follows that $|S| \leqslant |R| + tk \leqslant |R| + t2^{|R|}$. It remains to check that $l_S(y) \geqslant l_R(y) + 1/2t$ for all $y \in B(S)$. Recall that for each $v \in V$ we have $f_S(v) \geqslant f_R(v)$. Thus, to show that $l_S(y) \geqslant l_R(y) + 1/2t$ for $y \in B(S)$ it is sufficient to find $v \in N(y)$ with $f_S(v) \geqslant f_R(v) + 1/2t$.

Given $y \in B(S)$ let $C_i \in \mathcal{C}$ be such that $N_R(y) = C_i$. We have two cases to deal with depending on whether or not y is adjacent to x_i. If y is not adjacent to x_i then there are a few further subcases to deal with.

Case 1: $x_i \in N(y)$. If y is adjacent to x_i then, as $x_i \in Y(R) \cap S$, we have $f_R(x_i) < 1/2$ while $f_S(x_i) = 1$, and so we are done.

Case 2: $x_i \notin N(y)$. If y is not adjacent to x_i then there exists some clique $Z \subseteq V$ of order $p-2$ such that adding an edge between y and x_i turns $Z \cup \{x_i, y\}$ into a copy of K_p. Recalling that $N_R(y) = N_R(y_i)$, we note that $Z \nsubseteq R$, as otherwise $Z \cup \{x_i, y_i\}$ would be an example of a copy of K_p in G. Thus there exists some $z \in Z \setminus R$ such that z is adjacent to x_i and y. We conclude the proof by showing that $f_S(z) \geqslant f_R(z) + 1/2t$.

Case 2a: $z \in \overline{R} \setminus R$. If $z \in \overline{R} \setminus R$ then $z \in S$ (as it is adjacent to x_i) and so $f_S(z) = 1$, while $f_R(z) = 1/2$.

Case 2b: $z \in Y(R) \cap \overline{S}$. If $z \in Y(R) \cap \overline{S}$ then $f_S(z) \geqslant 1/2$, while $f_R(z) \leqslant (t-1)/2t$, as $d_R(z) \leqslant t-1$.

Case 2c: $z \in Y(R) \cap Y(S)$. If $z \in Y(R) \cap Y(S)$ then $f_R(z) = d_R(z)/2t$ and $f_S(z) = d_S(z)/2t$. As $x_i \in Y(R) \cap S$ and $R \subseteq S$, we have that $d_S(z) \geqslant d_R(z) + 1$, and so $f_S(z) \geqslant f_R(z) + 1/2t$.

In all cases, we have shown that there is some $v \in N(y)$ with $f_S(v) \geqslant f_R(v) + 1/2t$. As a result, we have that $l_S(y) \geqslant l_R(y) + 1/2t$ for all $y \in B(S)$. This completes the proof of Claim 2.2, which in turn proves Claim 2.1 and hence our theorem. □

As proved in the Introduction, Theorem 1.1 can be used to show that there exists a constant $c(t,p)$ such that, for n sufficiently large, we have $\mathrm{sat}_t(n,p) = tn - c(t,p)$. From a quantitative perspective, Theorem 1.1 gives an upper bound for $c(t,p)$ that is larger than a tower of exponentials of height $2t^2$. This upper bound can be greatly improved by, in the proof of Theorem 1.1, replacing $\mathcal{C}$ with its set of maximal elements (with respect to set inclusion). Under this change, $\mathcal{C}$ becomes an antichain (meaning that if $A, B \in \mathcal{C}$ then $A \nsubseteq B$) whose elements have size at most $t-1$. From this, the LYMB-inequality, due to Lubell [10], Yamamoto [13], Meshalkin [11] and Bollobás [2], shows us that $|\mathcal{C}| \leqslant \binom{|R|}{t-1}$. As a result, it is possible to prove

$$c(t,p) \leqslant t^{\left(t^{(2t^2)}\right)}. \tag{2.1}$$

The nature of the proof of Theorem 1.1 leads us to believe that (2.1) is not a good upper bound for $c(t,p)$. For example, the proof only used that G is K_p-saturated for some $3 \leqslant p \in \mathbb{N}$ and did not make any use of p's actual value. Moreover, in the proof of Claim 2.2 we only used the K_p-saturated condition on missing edges in $Y(R)$ rather than on all missing edges in G. In Section 3

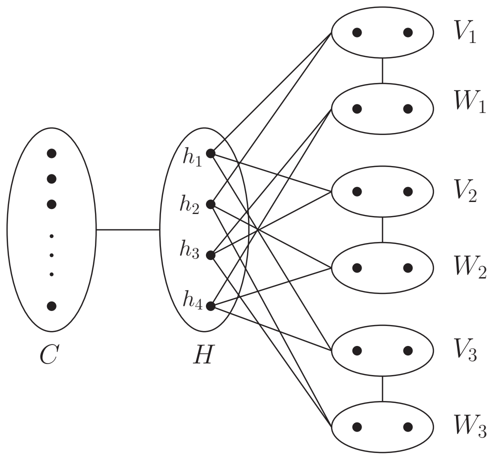

Figure 1. $G(n,4)$ where $X_1 = \{1,2\}, X_2 = \{1,3\}, X_3 = \{1,4\}$. An edge between two sets (or between a vertex and a set) represents that the two sets (or vertex and set) are fully connected.

we construct graphs that give a lower bound for $c(t,p)$. We believe this lower bound to be closer to the behaviour of $c(t,p)$ than the upper bound (2.1) obtained from Theorem 1.1.

3. Constructing K_p-saturated graphs

Proof of Theorem 1.2. Let $n,t \in \mathbb{N}$ with $t \geqslant 4$ and $n \geqslant t\left(1+\binom{t-1}{\lfloor t/2\rfloor -1}\right)$. We begin by constructing a graph $G(n,t)$ on n vertices that is K_3-saturated and has minimum degree t. Let $\mathcal{X} = \{X \subseteq [t] : 1 \in X, |X| = \lfloor t/2 \rfloor\}$ and label its elements $\mathcal{X} = \{X_1,\dots,X_r\}$. The vertices of $G(n,t)$ are split into vertex classes $C,H,V_1,\dots,V_r,W_1,\dots,W_r$, where

- $H = \{h_1,\dots,h_t\}$,
- each V_i has $\lfloor t/2 \rfloor$ vertices,
- each W_i has $\lceil t/2 \rceil$ vertices,
- C has the remaining $n - t\left(1+\binom{t-1}{\lfloor t/2\rfloor -1}\right)$ vertices.

The edges of $G(n,t)$ are as follows:

- C is fully connected to H,
- each V_i is fully connected to the set $\{h_k : k \in X_i\}$,
- each W_i is fully connected to the set $\{h_k : k \notin X_i\}$,
- each V_i is fully connected to W_i.

See Figure 1 for an example of the construction when $t = 4$. It is easy to verify that $G(n,t)$ has minimum degree t, is K_3-saturated and has $tn - f(t)$ edges, for some function $f(t) = \Omega(2^t t^{3/2})$. We now use $G(n,t)$ to create K_p-saturated graphs for $p > 3$.

Given a graph G, let G^* be the graph obtained by adding a new vertex to G and fully connecting it to all other vertices. If G is a K_p-saturated graph with minimum degree at least t, then G^* is a K_{p+1}-saturated graph with minimum degree at least $t+1$. Applying this construction $p-3$ times to the graph $G(n-p+3, t-p+3)$ (where $t \geqslant p-2$ and n is sufficiently large) gives a K_p-saturated graph on n vertices with minimum degree t and fewer than $tn - f(t-(p-3))$ edges. Thus, for fixed p, we have $c(t,p) = \Omega(2^t t^{3/2})$. □

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The idea of forming a new graph G^* from G can also be considered in the other direction. We say a vertex in a graph is a *conical vertex* if it is connected to all other vertices. Suppose G is a K_p-saturated graph with minimum degree t. If G has a conical vertex, then removing this vertex leaves a K_{p-1}-saturated graph with minimum degree $t-1$. Hajnal [9] showed that if G is a K_p-saturated graph without a conical vertex then $\delta(G) \geqslant 2(p-2)$. Recall that a consequence of Theorem 1.1 is that, for n sufficiently large, if $G \in \mathrm{Sat}_t(n,p)$ then $\delta(G) = t$. Thus, if $t < 2(p-2)$, these graphs must have a conical vertex and so are of the form G^* for some $G \in \mathrm{Sat}_{t-1}(n-1,p-1)$. This leads us to the following question.

Question 3.1. For which $n,t,p \in \mathbb{N}$ are all graphs in $\mathrm{Sat}_t(n,p)$ of the form G^* for some $G \in \mathrm{Sat}_{(t-1)}(n-1,p-1)$?

We remark that there do exist values of n,t and p where $\mathrm{Sat}_t(n,p)$ contains graphs without a conical vertex. For example, $\mathrm{Sat}_4(6,4)$ consists of only the complete tripartite graph $K_{2,2,2}$. On the other hand Alon, Erdős, Holzman and Krivelevich [1] showed that $\mathrm{sat}_4(n,4) = 4n-19$ for $n \geqslant 11$, and that all graphs achieving equality have a conical vertex. Perhaps it is the case that, for all fixed t, all fixed $p \geqslant 4$ and all n sufficiently large, all graphs in $\mathrm{Sat}_t(n,p)$ have a conical vertex.

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