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A CONVENIENT NOTION OF COMPACT SET FOR GENERALIZED FUNCTIONS

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Abstract We introduce the notion of functionally compact sets into the theory of nonlinear generalized functions in the sense of Colombeau. The motivation behind our construction is to transfer, as far as possible, properties enjoyed by standard smooth functions on compact sets into the framework of generalized functions. Based on this concept, we introduce spaces of compactly supported generalized smooth functions that are close analogues to the test function spaces of distribution theory. We then develop the topological and functional–analytic foundations of these spaces.

Keywords: functionally compact sets; Colombeau generalized functions; generalized smooth functions; locally convex modules

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1. Introduction

A main advantage of nonlinear generalized functions in the sense of Colombeau as compared to Schwartz distributions is the fact that they can be viewed as set-theoretic maps on domains consisting of generalized points. This change of perspective allows for the development of several branches of the theory in close analogy to classical analysis, and thereby has become increasingly important in recent years (see, for example, [**1**, **2**, **4**, **5**, **11**, **12**, **16**, **21**, **26**, **27**]). In particular, appropriate topologies on spaces of nonlinear generalized functions, the so-called sharp topologies (see below for the definition), were introduced in [**28**, **29**] and have since been studied by many authors. Apart from their central position in the structure theory of Colombeau algebras, they also supply the foundation for applications in the theory of nonlinear partial differential equations (for example, for a suitable concept of well-posedness).

From the point of view of analysis, a key notion underlying many existence results is that of compactness. It turns out, however, that sharply compact subsets of generalized points display certain unwanted properties For example, no infinite subset of $\mathbb{R}^n$ is sharply

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compact since the trace of the sharp topology on subsets of $\mathbb{R}^n$ is discrete. In fact, this is a necessary consequence of the set of generalized numbers $\widetilde{\mathbb{R}}$ containing actual infinitesimals, and hence seems unavoidable also in alternative approaches (see [**17**, Proposition 2.1] and [**21**, Theorem 25].

The importance of a convenient notion of compactness for nonlinear generalized functions has been recognized by several authors, most recently in [**6**]. The approach we take in the present paper is to introduce an appropriate concept of compactly supported generalized function, and then to study spaces consisting of such functions, which, in analogy to the test function space $\mathcal{D}(\Omega)$ in distribution theory, we denote by $\mathcal{GD}(U)$. The domain U here is a set of generalized points. Based on Garetto's theory of locally convex $\widetilde{\mathbb{C}}$-modules [**11**, **12**, **14**] we then develop the topological and functional analytic foundations of these spaces. We find that they are indeed close analogues of the classical spaces of test functions in that they are countable strict inductive limits of complete metric spaces $\mathcal{GD}_K(U)$ (analogues of $\mathcal{D}_K(\Omega)$ in distribution theory) satisfying properties paralleling those of the classical strict (LF)-spaces $\mathcal{D}(\Omega)$.

The plan of the paper is as follows. In the remainder of this introduction we fix some basic notions used throughout this work. Section 2 introduces what we call *functionally compact sets*, based on work by Oberguggenberger and Vernaeve in [**27**]. Building on this, in Section 3 we define compactly supported generalized smooth functions (GSFs), as well as the corresponding spaces $\mathcal{GD}(U)$ and $\mathcal{GD}_K(U)$. We also show that every Colombeau generalized function $f \in \mathcal{G}^s(\Omega)$ (in particular, every Schwartz distribution) defines a compactly supported GSF $\bar{f} : \widetilde{\mathbb{R}} \longrightarrow \widetilde{\mathbb{R}}$ that coincides with f on $\widetilde{\Omega}_c$. In order to obtain appropriate topologies on these spaces, we define so-called generalized norms in Section 4. These are maps that share the basic properties of classical norms, yet take values in $\widetilde{\mathbb{R}}$, thereby generalizing a standard alternative description of the sharp topology on generalized numbers (see [**1**, **17**]). In Sections 5 and 6 these generalized norms are employed to endow the spaces $\mathcal{GD}_K(U)$ with metric topologies. In particular, in Section 5.1 we study connections between non-Archimedean properties and Hausdorff topological vector spaces of generalized functions, proving an impossibility theorem: there does not exist a Hausdorff topological vector subspace of the Colombeau special algebra that contains the Dirac delta and even a single trace of an open set of the sharp topology. The completeness of the spaces $\mathcal{GD}_K(U)$ is established in Section 7. Finally, in Section 8 we derive the fundamental functional analytic properties of the space $\mathcal{GD}(U)$.

1.1. Basic notions

Our main references for Colombeau's theory are [**8**, **9**, **22**, **25**]. The special Colombeau algebra $\mathcal{G}^s(\Omega)$ over an open subset Ω of $\mathbb{R}^n$ is defined as the quotient $\mathcal{E}^s_M(\Omega)/\mathcal{N}^s(\Omega)$, where (setting $I := (0,1]$ and noting that in the naturals $\mathbb{N} = \{0,1,2,3\ldots\}$ we include zero)

$$\mathcal{E}^s_M(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\},$$

$$\mathcal{N}^s(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \forall m \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m})\}.$$

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Elements of $\mathcal{E}^s_M(\Omega)$ are called moderate, those of $\mathcal{N}^s(\Omega)$ are called negligible. Nets in $\mathcal{E}^s_M(\Omega)$ are written as (u_ε), and $u = [u_\varepsilon]$ denotes the corresponding equivalence class in $\mathcal{G}^s(\Omega)$. For $(u_\varepsilon) \in \mathcal{N}^s(\Omega)$ we also write $(u_\varepsilon) \sim 0$. We will abbreviate 'Colombeau generalized function' by CGF. $\mathcal{G}^s(\cdot)$ is a fine sheaf of differential algebras and there exist sheaf embeddings (based on smoothing via convolution) of the space of Schwartz distributions $\mathcal{D}'$ into $\mathcal{G}^s$ (see [**22**]).

Given that $\Omega \subseteq \mathbb{R}^n$ is open, the space of generalized points in Ω is $\widetilde{\Omega} = \Omega_M/\sim$, where $\Omega_M = \{(x_\varepsilon) \in \Omega^I \mid \exists N \in \mathbb{N} : |x_\varepsilon| = O(\varepsilon^{-N})\}$ is called the set of moderate nets and $(x_\varepsilon) \sim (y_\varepsilon)$ if $|x_\varepsilon - y_\varepsilon| = O(\varepsilon^m)$ for every $m \in \mathbb{N}$. In the particular case $\Omega = \mathbb{R}$ we obtain the ring of Colombeau generalized numbers (CGNs) $\widetilde{\mathbb{R}} = \mathbb{R}_M/\sim$ (and analogously for $\widetilde{\mathbb{C}}$), which can also be written as $\widetilde{\mathbb{R}} = \mathbb{R}_M/\mathcal{N}^s$, where $\mathcal{N}^s$ is the set of all negligible nets of real numbers $(x_\varepsilon) \in \mathbb{R}^I$, i.e. such that $(x_\varepsilon) \sim 0$. $\widetilde{\mathbb{R}}$ is an ordered ring with respect to its natural order relation: $x \leq y$ if and only if there are representatives (x_ε) and (y_ε) such that $x_\varepsilon \leq y_\varepsilon$ for ε sufficiently small. We point out that, in the present work, the notion $x > y$ does *not* mean $x \geq y$ and $x \neq y$. Rather, it is to be understood as $x - y \geq 0$ and $x - y$ invertible. By [**22**, Theorem 1.2.38] and [**24**, Propostion 3.2] we have the following lemma.

Lemma 1.1. *Let $x \in \widetilde{\mathbb{R}}$. Then the following are equivalent:*

(i) $x > 0$,

(ii) *for each representative (x_ε) of x there exists some ε_0 and some m such that $x_\varepsilon > \varepsilon^m$ for all $\varepsilon < \varepsilon_0$,*

(iii) *for each representative (x_ε) of x there exists some ε_0 such that $x_\varepsilon > 0$ for all $\varepsilon < \varepsilon_0$.*

We shall use the notation $\mathrm{d}\varepsilon^m := [\varepsilon^m] \in \widetilde{\mathbb{R}}$ for any $m \in \mathbb{R}$. Hence $x > 0$ is equivalent to $x \geq \mathrm{d}\varepsilon^m$ for some $m > 0$. If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we will also sometimes use the notation $\forall^0\varepsilon :\ \mathcal{P}(\varepsilon)$ to denote $\exists \varepsilon_0 \in I\, \forall \varepsilon \in (0, \varepsilon_0] :\ \mathcal{P}(\varepsilon)$.

The space of compactly supported generalized points $\widetilde{\Omega}_c$ is defined by $\Omega_c/\sim$, where $\Omega_c := \{(x_\varepsilon) \in \Omega^I \mid \exists K \Subset \Omega\, \forall^0\varepsilon :\ x_\varepsilon \in K\}$ and $\sim$ is the same equivalence relation as in the case of $\widetilde{\Omega}$.

Concerning intervals, we use the following notations: $[a, b] := \{x \in \widetilde{\mathbb{R}} \mid a \leq x \leq b\}$, $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$. Also, for $x, y \in \widetilde{\mathbb{R}}^n$ we write $x \approx y$ if $x - y$ is infinitesimal, i.e. if $|x - y| \leq r$ for all $r \in \mathbb{R}_{>0}$.

As already indicated above, the natural topology for Colombeau-type spaces is the so-called sharp topology (see [**1**, **3**, **7**, **16**, **23**, **28**, **29**]). This topology is generated by balls $B_\rho(x) = \{y \in \widetilde{\mathbb{R}}^n \mid |y - x| < \rho\}$, where $|\cdot|$ is the natural extension of the Euclidean norm to $\widetilde{\mathbb{R}}^n$, $|[x_\varepsilon]| := [|x_\varepsilon|] \in \widetilde{\mathbb{R}}$, and $\rho \in \widetilde{\mathbb{R}}_{>0}$ is strictly positive (see [**2**, **4**, **17**]). For Euclidean balls, we will write

$$B^{\mathrm{E}}_\rho(x) = \{y \in \mathbb{R}^n \mid |y - x| < \rho\}.$$

On the other hand, the so-called Fermat-topology on $\widetilde{\mathbb{R}}^n$ (see [**17**, **21**]) is generated by the balls $B_r(x)$ for $x \in \widetilde{\mathbb{R}}^n$ and $r \in \mathbb{R}_{>0}$. Originally, the sharp topology was introduced

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using an ultrametric as follows. The map

$$v : \mathbb{R}_M \longrightarrow (-\infty, \infty],$$
$$v((u_\varepsilon)) := \sup\{b \in \mathbb{R} \mid |u_\varepsilon| = O(\varepsilon^b)\}$$

gives a pseudovaluation on $\widetilde{\mathbb{R}}$. Then setting$|\cdot|_e : \widetilde{\mathbb{R}} \to [0, \infty)$, $|u|_e := \exp(-v(u))$ provides a translation-invariant complete ultrametric

$$d_s : \widetilde{\mathbb{R}} \times \widetilde{\mathbb{R}} \longrightarrow \mathbb{R}_+,$$
$$d_s(u, v) := |u - v|_e$$

on $\widetilde{\mathbb{R}}$, which induces the sharp topology on $\widetilde{\mathbb{R}}$.

Garetto in [**11**, **12**] extended the above construction to arbitrary locally convex spaces by functorially assigning a space of CGFs $\mathcal{G}_E$ to any given locally convex space E. In this approach, the seminorms of E are used to define pseudovaluations that induce a generalized locally convex topology on the $\widetilde{\mathbb{C}}$-module $\mathcal{G}_E$, again called sharp topology. In the present paper, we will exclusively work with $\widetilde{\mathbb{R}}$-modules. We note, however, that all our constructions trivially carry over to the $\widetilde{\mathbb{C}}$-case.

For any $S \subseteq I$, e_S denotes the equivalence class in $\widetilde{\mathbb{R}}$ of the characteristic function of S (see [**1**, **30**]). Any e_S is an idempotent, and $e_S + e_{S^c} = 1$. Also, $e_S \neq 0$ if and only if $0 \in \overline{S}$. For any subset A of $\widetilde{\mathbb{R}}^n$, its interleaving (See [**27**]) is defined as

$$\operatorname{interl}(A) := \left\{ \sum_{j=1}^{m} e_{S_j} a_j \mid m \in \mathbb{N}, \{S_1, \ldots, S_m\} \text{ a partition of } I,\ a_j \in A \right\}.$$

If (A_ε) is a net of subsets of $\mathbb{R}^n$ then the internal set ([**27**, **31**]) generated by (A_ε) is

$$[A_\varepsilon] = \left\{ [x_\varepsilon] \in \widetilde{\mathbb{R}}^n \mid x_\varepsilon \in A_\varepsilon \text{ for } \varepsilon \text{ small} \right\},$$

and the strongly internal set (see [**21**]) generated by (A_ε) is

$$\langle A_\varepsilon \rangle := \left\{ [x_\varepsilon] \in \widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\varepsilon A_\varepsilon \right\}.$$

Here, $x_\varepsilon \in_\varepsilon A_\varepsilon$ means that $x_\varepsilon \in A_\varepsilon$ for ε small and that the same property holds for any representative of $[x_\varepsilon]$. The net (A_ε) is called sharply bounded if there exists some $N \in \mathbb{R}_{>0}$ such that for ε sufficiently small we have $\sup_{x \in A_\varepsilon} |x| \leq \varepsilon^{-N}$. Equivalently, we have that (A_ε) is sharply bounded if there exists $\rho \in \widetilde{\mathbb{R}}_{>0}$ such that $[A_\varepsilon] \subseteq B_\rho(0)$.

Finally, given $X \subseteq \widetilde{\mathbb{R}}^n$ and $Y \subseteq \widetilde{\mathbb{R}}^d$, then (see [**21**])

$$f : X \longrightarrow Y \text{ is a generalized smooth function (GSF)}$$

if there exists a net $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defining f in the sense that $X \subseteq \langle \Omega_\varepsilon \rangle$, $f([x_\varepsilon]) = [u_\varepsilon(x_\varepsilon)] \in Y$ and $(\partial^\alpha u_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_M$ for all $x = [x_\varepsilon] \in X$ and all $\alpha \in \mathbb{N}^n$. The space of GSFs from X to Y is denoted by $\mathcal{GC}^\infty(X, Y)$ (in contrast to [**21**], where the notation $\widetilde{\mathcal{G}}(X, Y)$ was used). GSFs are a natural generalization of CGFs to general domains. In particular, for any $\Omega \subseteq \mathbb{R}^n$ open, $\mathcal{GC}^\infty(\widetilde{\Omega}_c) \simeq \mathcal{G}^s(\Omega)$. GSFs on subsets of $\widetilde{\mathbb{R}}^n$, endowed with the sharp topology, form a subcategory of the category of topological spaces. In particular, they can be composed unrestrictedly.

2. A new notion of compact subset for nonlinear generalized functions

Even though the intervals $[a, b] \subseteq \widetilde{\mathbb{R}}$, a, $b \in \mathbb{R}$, are neither compact in the sharp nor in the Fermat topology (see [21, Theorem 25]), analogously to the case of smooth functions, a GSF satisfies an extreme-value theorem on such sets. In fact, the following proposition.

Proposition 2.1. *Let $f \in \mathcal{GC}^\infty(X, \widetilde{\mathbb{R}})$ be a generalized smooth function defined on the subset X of $\widetilde{\mathbb{R}}^n$. Let $\emptyset \neq K = [K_\varepsilon] \subseteq X$ be an internal set generated by a sharply bounded net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$, then*

$$\exists m, M \in K \, \forall x \in K : \; f(m) \leq f(x) \leq f(M). \tag{2.1}$$

Proof. By [21, Lemma 28], f can be represented by a net $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$. Since $K \neq \emptyset$, for ε sufficiently small, say for $\varepsilon \in (0, \varepsilon_0]$, K_ε is non-empty and, by assumption, it is also compact. For all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\exists m_\varepsilon, M_\varepsilon \in K_\varepsilon \, \forall x \in K_\varepsilon : \; u_\varepsilon(m_\varepsilon) \leq u_\varepsilon(x) \leq u_\varepsilon(M_\varepsilon).$$

Since the net (K_ε) is sharply bounded, both the nets (m_ε) and (M_ε) are moderate. Therefore $m = [m_\varepsilon]$, $M = [M_\varepsilon] \in K \subseteq X$. Take any $x \in [K_\varepsilon]$. Then there exists a representative (x_ε) such that $x_\varepsilon \in K_\varepsilon$ for ε small. Therefore $f(m) = [u_\varepsilon(m_\varepsilon)] \leq [u_\varepsilon(x_\varepsilon)] = f(x) \leq f(M)$. □

We shall use the assumptions on K and (K_ε) given in this theorem to introduce a new notion of a 'compact subset' that behaves better than the usual classical notion of compactness in the sharp topology.

Definition 2.2. A subset K of $\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

(i) $K = [K_\varepsilon] \subseteq \widetilde{\mathbb{R}}^n$,

(ii) (K_ε) is sharply bounded,

(iii) $\forall \varepsilon \in I : \; K_\varepsilon \Subset \mathbb{R}^n$.

If, in addition, $K \subseteq U \subseteq \widetilde{\mathbb{R}}^n$ then we write $K \Subset_{\mathrm{f}} U$. Finally, we write $[K_\varepsilon] \Subset_{\mathrm{f}} U$ if (ii), (iii) and $[K_\varepsilon] \subseteq U$ hold.

We note that in (iii) it suffices to ask that K_ε be closed since it is bounded by (ii), at least for ε small. In fact, we have the following lemma.

Lemma 2.3. *A subset K of $\widetilde{\mathbb{R}}^n$ is functionally compact if and only if it is internal and sharply bounded.*

Proof. By [27, Lemma 2.4 and Corollary 2.2], every sharply bounded internal set K has a sharply bounded representative (K_ε) consisting of closed (hence compact) subsets of $\mathbb{R}^n$. □

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We motivate the name *functionally compact subset* by anticipating that on this type of subsets, GSFs have properties very similar to those that ordinary smooth functions have on standard compact sets.

Remark 2.4.

(i) By [27, Proposition 2.3], any internal set $K = [K_\varepsilon]$ is closed in the sharp topology. In particular, the open interval $(0,1) \subseteq \widetilde{\mathbb{R}}$ is not functionally compact since it is not closed.

(ii) If $H \Subset \mathbb{R}^n$ is a non-empty ordinary compact set, then $\widetilde{H} = [H]$ is functionally compact. In particular, $[0,1] = \widetilde{[0,1]_{\mathbb{R}}} = [[0,1]_{\mathbb{R}}]$ is functionally compact.

(iii) The empty set $\emptyset = \widetilde{\emptyset} \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}$.

(iv) By Lemma 2.3, $\widetilde{\mathbb{R}}^n$ is not functionally compact since it is not sharply bounded.

(v) The set of compactly supported points $\widetilde{\mathbb{R}}_c$ is not functionally compact because the GSF $f(x) = x$ does not satisfy (2.1).

We start the study of functionally compact sets by proving suitable generalizations of theorems from classical analysis.

Theorem 2.5. *Let $K \subseteq X \subseteq \widetilde{\mathbb{R}}^n$, and $f \in \mathcal{GC}^\infty(X, \widetilde{\mathbb{R}}^d)$. Then $K \Subset_f \widetilde{\mathbb{R}}^n$ implies that $f(K) \Subset_f \widetilde{\mathbb{R}}^d$.*

Proof. Let (K_ε) be as in Definition 2.2 and let the GSF f be defined by the net $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$. Let us first prove that $f(K) = [u_\varepsilon(K_\varepsilon)]$. In fact, $y \in f(K) = f([K_\varepsilon])$ is equivalent to

$$\exists (x_\varepsilon) \in \mathbb{R}^n_M \, \forall^0 \varepsilon : \; x_\varepsilon \in K_\varepsilon \text{ and } y = [u_\varepsilon(x_\varepsilon)]. \tag{2.2}$$

This necessary entails $y \in [u_\varepsilon(K_\varepsilon)]$. Conversely, if $y \in [u_\varepsilon(K_\varepsilon)]$, then there exists $(y_\varepsilon) \in \mathbb{R}^d_M$ such that $y_\varepsilon \in u_\varepsilon(K_\varepsilon)$ for ε small. Hence, for each of these ε there also exists $x_\varepsilon \in K_\varepsilon$ such that $y_\varepsilon = u_\varepsilon(x_\varepsilon)$, which implies $y = [u_\varepsilon(x_\varepsilon)]$, i.e. (2.2) holds since (K_ε) is sharply bounded. Clearly, $u_\varepsilon(K_\varepsilon) \Subset \mathbb{R}^d$, so it remains to prove that the net $(u_\varepsilon(K_\varepsilon))$ is sharply bounded. If $\forall \varepsilon_0 \, \exists \varepsilon \le \varepsilon_0 : \; K_\varepsilon = \emptyset$, then $[K_\varepsilon] = K = \emptyset$, so $f(K) = \emptyset$ and the conclusion is trivial. Otherwise, assume that $K_\varepsilon \neq \emptyset$ for $\varepsilon \le \varepsilon_0$ and proceed by contradiction assuming that

$$\forall k \in \mathbb{N} \, \exists (\varepsilon_{kn})_n \downarrow 0 \, \forall n \, \exists y_{kn} \in u_{\varepsilon_{kn}}(K_{\varepsilon_{kn}}) : \; |y_{kn}| > \varepsilon_{kn}^{-k}. \tag{2.3}$$

We can write $y_{kn} = u_{\varepsilon_{kn}}(x_{kn})$ for some $x_{kn} \in K_{\varepsilon_{kn}}$. Next, set $\varepsilon_0 := \varepsilon_{00}$ and for $k > 0$ pick n_k such that $\varepsilon_{kn_k} < \min(1/k, \varepsilon_{k-1})$ and set $\varepsilon_k := \varepsilon_{kn_k}$. Take any $\bar{x}_\varepsilon \in K_\varepsilon$ for each $\varepsilon \le \varepsilon_0$, and set $x_\varepsilon := x_{kn_k}$ if $\varepsilon = \varepsilon_k$ and $x_\varepsilon := \bar{x}_\varepsilon$ otherwise. Then $x_\varepsilon \in K_\varepsilon$ for $\varepsilon \le \varepsilon_0$, so $x = [x_\varepsilon] \in K \subseteq X$ and $(u_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_M$ by the definition of GSF, which contradicts (2.3). □

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As a corollary of this theorem and Remark 2.4 (ii), we get the following.

Corollary 2.6. *If a, $b \in \widetilde{\mathbb{R}}$ and $a \leq b$, then $[a, b] \Subset_f \widetilde{\mathbb{R}}$.*

Let us note that a, $b \in \widetilde{\mathbb{R}}$ can also be infinite, for example, $a = [-\varepsilon^{-N}]$, $b = [\varepsilon^{-M}]$ or $a = [\varepsilon^{-N}]$, $b = [\varepsilon^{-M}]$ with $M > N$.

Lemma 2.7. *Let K, $H \Subset_f \widetilde{\mathbb{R}}^n$. Then we have that have:*

(i) $K \cup H \subseteq \operatorname{interl}(K \cup H) \Subset_f \widetilde{\mathbb{R}}^n$*;*

(ii) If $K \cup H$ is internal, then it is functionally compact;

(iii) If $K \cap H$ is internal, then it is functionally compact.

Proof. Property (i) follows from [**27**, Proposition 2.8], which implies $K \cup H \subseteq \operatorname{interl}(K \cup H) = [K_\varepsilon \cup H_\varepsilon]$, where the nets (K_ε) and (H_ε) satisfy Definition 2.2. Property (ii) follows from [**27**, Lemma 2.7] which implies that if the union of internal sets is internal, then it is equal to its interleaving. Property (iii) is a consequence of Lemma 2.3. □

If $H \subseteq K \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^n$, then also H is sharply bounded. So, another consequence of Lemma 2.3 is the following corollary.

Corollary 2.8. *Let $H \subseteq K \Subset_f \widetilde{\mathbb{R}}^n$, Then H internal implies that $H \Subset_f \widetilde{\mathbb{R}}^n$.*

Finally, in the following result we consider the product of functionally compact sets.

Proposition 2.9. *Let $K \Subset_f \widetilde{\mathbb{R}}^n$ and $H \Subset_f \widetilde{\mathbb{R}}^d$, Then $K \times H \Subset_f \widetilde{\mathbb{R}}^{n+d}$. In particular, if $a_i \leq b_i$ for $i = 1, \ldots, n$, then $\prod_{i=1}^n [a_i, b_i] \Subset_f \widetilde{\mathbb{R}}^n$.*

Proof. From [**27**, Proposition 2.13] if follows that $K \times H$ is internal, in fact for $K = [K_\varepsilon]$, $H = [H_\varepsilon]$, $K \times H = [K_\varepsilon \times H_\varepsilon]$. From this representation it immediately follows that $H \times K$ is sharply bounded as well, so Lemma 2.3 gives the claim. □

3. Compactly supported generalized smooth functions

Our main goal in this section is to define and study an analogue within the framework of GSFs of the space $\mathcal{D}_K(\Omega)$ of smooth functions supported in a fixed compact set $K \Subset \Omega$. In order to define this space, we first try to define the concept of support of a GSF. Clearly, if $\varphi \in \mathcal{D}_{[-a,a]_{\mathbb{R}}}(\mathbb{R})$, one would expect that φ should have compact support also if we think of φ as a GSF. In fact, $\operatorname{supp}(\varphi)$ should be contained in $[-a, a] \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}$. Already this basic requirement implies that if $f \in \mathcal{GC}^\infty(X, Y)$, the natural definition

$$S(f) := X \setminus \bigcup \Big\{ B_\rho(x) \cap X \mid x \in X,\ \rho \in \widetilde{\mathbb{R}}_{>0},\ f|_{B_\rho(x) \cap X} = 0 \Big\}$$

does not fit with our intuition. Indeed, if we take the aforementioned φ so that $\varphi(0) = 1$, and $S \subseteq (0, 1]$ such that $0 \in \bar{S}$ and $0 \in \overline{S^c}$, then $\varphi(e_S \, \mathrm{d}\varepsilon^{-1}) = e_{S^c} \neq 0$ and $e_S \, \mathrm{d}\varepsilon^{-1} \in S(\varphi) \setminus [-a, a]$. This motivates the following definition.

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Definition 3.1. Let $X \subseteq \widetilde{\mathbb{R}}^n$, $Y \subseteq \widetilde{\mathbb{R}}^d$ and $f \in \mathcal{GC}^\infty(X,Y)$, then

$$\operatorname{supp}(f) := \overline{\{x \in X \mid |f(x)| > 0\}},$$

where here $\overline{(\cdot)}$ denotes the relative closure in X with respect to the sharp topology.

Using this concept, we have $\operatorname{supp}(\varphi) \subseteq [-a,a]$. In fact, $|\varphi(x)| = [|\varphi(x_\varepsilon)|] > 0$ implies that $|\varphi(x_\varepsilon)| > \varepsilon^q$ for some $q \in \mathbb{R}_{>0}$ and for ε small, and hence $x_\varepsilon \in [-a,a]_{\mathbb{R}}$.

Remark 3.2.

(i) In the setting of Colombeau algebras, one usually defines the support of some $f \in \mathcal{G}^s(\Omega)$ as a subset of $\Omega \subseteq \mathbb{R}^n$, i.e., as a set of classical points, namely as $\operatorname{supp}_{\mathcal{G}^s}(f) := \Omega \setminus \bigcup \{B_r(x) \cap \Omega \mid x \in \Omega,\ r \in \mathbb{R}_{>0},\ f|_{B_r(x)\cap\Omega} = 0\}$, where the last equality is understood to hold in $\mathcal{G}^s(B_r(x) \cap \Omega)$. Using $X = \widetilde{\Omega}_c$ as the natural domain of any $f \in \mathcal{G}^s(\Omega)$ (see [21, Theorem 37]), it is then immediate that $\operatorname{supp}(f) \cap \Omega \subseteq \operatorname{supp}_{\mathcal{G}^s}(f)$.

(ii) Let $u \in \mathcal{D}'(\Omega)$ be a Schwartz distribution and denote by $\iota : \mathcal{D}'(\Omega) \to \mathcal{GC}^\infty(\widetilde{\Omega}_c, \mathbb{R})$ a standard embedding via convolution. Then $\operatorname{supp}(\iota(u)) \cap \Omega \subseteq \operatorname{supp}(u)$, as follows from (i) and the fact that ι is a sheaf-morphism.

(iii) Assume that the embedding $\iota : \mathcal{D}'(\Omega) \to \mathcal{GC}^\infty(\widetilde{\Omega}_c, \mathbb{R})$ has been defined by using a mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$ that is identically equal to 1 in the ball $B_p^{\mathrm{E}}(0)$, $p \in \mathbb{R}_{>0}$. Then $\delta(x) = \mathrm{d}\varepsilon^{-n}$ for each $x \in B_{p\cdot \mathrm{d}\varepsilon}(0)$ and hence $B_{p\cdot \mathrm{d}\varepsilon}(0) \subseteq \operatorname{supp}(\iota(\delta))$, whereas $\operatorname{supp}(\iota(\delta)) \cap \mathbb{R}^n = \{0\}$.

(iv) In general, $\operatorname{supp}(f)$ is not an internal set because it is not generally closed by finite interleaving (see [27, Lemma 2.7]). Consider, for example, X with only near standard points and $f \in \mathcal{GC}^\infty(X, \widetilde{\mathbb{R}})$ which is strictly positive on two disjoint intervals. However, if X itself is closed under finite interleaving then so is $\operatorname{supp}(f)$.

If (u_ε) defines $f \in \mathcal{GC}^\infty(X,Y)$, the internal set $[\operatorname{supp}(u_\varepsilon)]$ is not intrinsically defined since it depends on the defining net (u_ε). Consider, for example, $u_\varepsilon(x) := \varphi(x) + \varepsilon^{1/\varepsilon} > 0$ where $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$.

In our further analysis we will repeatedly make use of the following notion:

Definition 3.3. For $A \subseteq \widetilde{\mathbb{R}}^n$ we call the set

$$\operatorname{ext}(A) := \{x \in \widetilde{\mathbb{R}}^n \mid \forall a \in A :\ |x - a| > 0\}$$

the *strong exterior* of A.

This set can also be described in the following way:

Lemma 3.4. *If $A \subseteq \widetilde{\mathbb{R}}^n$, then* $\operatorname{ext}(A) = \{x \in \widetilde{\mathbb{R}}^n \mid \forall S \subseteq I :\ e_S \neq 0 \ \Rightarrow\ xe_S \notin Ae_S\}$.

Proof. $\subseteq$: Let $x = [x_\varepsilon] \in \operatorname{ext}(A)$ and suppose that there exists some $S \subseteq I$ with $e_S \neq 0$ and some $a = [a_\varepsilon] \in A$ such that $xe_S = ae_S$. Then there exists some $q > 0$ such that

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$|x_\varepsilon - a_\varepsilon| > \varepsilon^q$ for ε small. However, $xe_S = ae_S$ implies that $|x_\varepsilon - a_\varepsilon| = O(\varepsilon^{q+1})$ for $\varepsilon \to 0$, $\varepsilon \in S$, yielding a contradiction.

$\supseteq$ If there exists some $a = [a_\varepsilon] \in A$ such that $|x - a| \not> 0$ then there is a sequence $\varepsilon_k \downarrow 0$ with $|x_{\varepsilon_k} - a_{\varepsilon_k}| < \varepsilon_k^k$ for all k. Letting $S := \{\varepsilon_k \mid k \in \mathbb{N}\}$ implies $xe_S = ae_S$. $\square$

For non-trivial internal sets we have the following characterization of the strong exterior:

Lemma 3.5. *Let $\emptyset \neq [K_\varepsilon] = K \subseteq \widetilde{\mathbb{R}}^n$. Then*

$$\operatorname{ext}(K) = \langle K_\varepsilon^c \rangle.$$

Proof. Suppose first that $x \in \langle K_\varepsilon^c \rangle$ and let $S \subseteq I$ with $e_S \neq 0$. Suppose that there existed some $a \in K$ with $xe_S = ae_S$. Since $a \in K$ there exists a representative (a_ε) of a with $a_\varepsilon \in K_\varepsilon$ for all ε. Then $xe_S = ae_S$ implies that there exists a representative (x_ε) of x and a sequence $\varepsilon_k \downarrow 0$ in S with $x_{\varepsilon_k} = a_{\varepsilon_k} \in K_{\varepsilon_k}$ for all k. But this contradicts the fact that $x \in \langle K_\varepsilon^c \rangle$.

Conversely, if $x \notin \langle K_\varepsilon^c \rangle$ then there exists a representative (x_ε) of x and a sequence $\varepsilon_m \downarrow 0$ with $x_{\varepsilon_m} \in K_{\varepsilon_m}$ for all m. Since $K \neq \emptyset$, there exists some $w = [w_\varepsilon] \in K$ with $w_\varepsilon \in K_\varepsilon$ for all ε. Now let

$$a_\varepsilon := \begin{cases} x_{\varepsilon_m} & \text{if } \varepsilon = \varepsilon_m, \\ w_\varepsilon & \text{otherwise} \end{cases}$$

and set $S := \{\varepsilon_m \mid m \in \mathbb{N}\}$. Then $a = [a_\varepsilon] \in [K_\varepsilon]$ and $xe_S = ae_S$ by construction. Thus $xe_S \in Ke_S$, and so $x \notin \operatorname{ext}(K)$. $\square$

As an immediate conclusion we obtain the following corollary.

Corollary 3.6. *Let $K = [K_\varepsilon] = [L_\varepsilon] \neq \emptyset$. Then $\langle K_\varepsilon^c \rangle = \langle L_\varepsilon^c \rangle$.*

The next result relates the support of a GSF to the exterior of certain internal sets. To formulate it concisely, we introduce the following notations: Denote by $\mathcal{K}_f$ the set of all internal $\emptyset \neq K \subseteq \widetilde{\mathbb{R}}^n$ with $\operatorname{ext}(K) \neq \emptyset$ and such that there exists a net $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ that defines f and such that $[u_\varepsilon(x_\varepsilon)] = 0$ for all $[x_\varepsilon] \in \operatorname{ext}(\mathrm{K})$. Also, denote by $\mathcal{H}_f$ the set of all the internal sets of the form $K = [\operatorname{supp}(u_\varepsilon)] \subseteq \widetilde{\mathbb{R}}^n$ for some net $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ that defines f and such that both K and $\operatorname{ext}(K)$ are non-empty.
Then we have the following lemma.

Lemma 3.7. *Let $X \subseteq \widetilde{\mathbb{R}}^n$, $Y \subseteq \widetilde{\mathbb{R}}^d$ and $f \in \mathcal{GC}^\infty(X, Y)$. Then*

$$\operatorname{supp}(f) \subseteq X \cap \bigcap_{K \in \mathcal{K}_f} K \subseteq X \cap \bigcap_{K \in \mathcal{H}_f} K. \tag{3.1}$$

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Proof. Since $X \cap \bigcap_{K\in\mathcal{K}_f} K$ is a sharply closed subset of X, in order to show the first inclusion in (3.1), it suffices to prove that

$$\{x \in X \mid |f(x)| > 0\} \subseteq X \cap \bigcap_{K\in\mathcal{K}_f} K.$$

Let $x \in X$ be such that $|f(x)| > 0$, so that

$$\exists r \in \mathbb{R}_{>0} : \ |f(x)| > \mathrm{d}\varepsilon^r. \tag{3.2}$$

Let $K = [K_\varepsilon] \in \mathcal{K}_f$, and assume, by contradiction, that $x = [x_\varepsilon] \notin K$. We first prove that

$$\exists q \in \mathbb{N}_{>r}\, \exists (\varepsilon_k)_{k\in\mathbb{N}} \downarrow 0\, \forall k \in \mathbb{N} : \ B^{\mathrm{E}}_{\varepsilon_k^q}(x_{\varepsilon_k}) \subseteq K^c_{\varepsilon_k}, \tag{3.3}$$

where r comes from (3.2). In fact, suppose to the contrary that

$$\forall q \in \mathbb{N}_{>r}\, \exists \varepsilon_q\, \forall \varepsilon \le \varepsilon_q\, \exists y_\varepsilon^{(q)} \in B^{\mathrm{E}}_{\varepsilon^q}(x_\varepsilon) : \ y_\varepsilon^{(q)} \in K_\varepsilon.$$

We may assume that $(\varepsilon_q)_{q\in\mathbb{N}} \downarrow 0$. Setting $\tilde{y}_\varepsilon := y_\varepsilon^{(q)}$ for $\varepsilon \in (\varepsilon_{q+1}, \varepsilon_q]$, we have $x = [\tilde{y}_\varepsilon] \in K$, which contradicts $x \notin K$.

By assumption $\exists z = [z_\varepsilon] \in \mathrm{ext}(K)$, and hence

$$\exists s \in \mathbb{N}_{>q}\, \forall^0 \varepsilon : \ d(z_\varepsilon, K_\varepsilon) > \varepsilon^s,$$

where q comes from (3.3). Using $(\varepsilon_k)_{k\in\mathbb{N}}$ from (3.3), we set $\tilde{x}_\varepsilon := x_{\varepsilon_k}$ if $\varepsilon = \varepsilon_k$ and $\tilde{x}_\varepsilon := z_\varepsilon$ otherwise. Then $\tilde{x} := [\tilde{x}_\varepsilon] \in \mathrm{ext}(K)$. But $K \in \mathcal{K}_f$, so there exists a net $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ that defines f and such that $[u_\varepsilon(\tilde{x}_\varepsilon)] = 0$. In particular, $|u_{\varepsilon_k}(\tilde{x}_{\varepsilon_k})| = |u_{\varepsilon_k}(x_{\varepsilon_k})| = O(\varepsilon_k^{2s})$ as $k \to +\infty$, which contradicts $|f(x)| = [|u_\varepsilon(x_\varepsilon)|] > \mathrm{d}\varepsilon^r > \mathrm{d}\varepsilon^s$.

Turning now to the second inclusion, assume that $x \in X \setminus \bigcap_{K\in\mathcal{H}_f} K$, so that there exists a net (u_ε) that defines f, with $\emptyset \neq K := [\mathrm{supp}(u_\varepsilon)] \subseteq \widetilde{\mathbb{R}}^n$ and $\mathrm{ext}(K) \neq \emptyset$, but such that $x \notin K$. We want to show that $x \notin \bigcap_{K'\in\mathcal{K}_f} K'$. But for all $[y_\varepsilon] \in \mathrm{ext}(K) = \langle \mathrm{supp}(u_\varepsilon)^c \rangle$, we have $y_\varepsilon \notin \mathrm{supp}(u_\varepsilon)$ for ε small. Hence, $u_\varepsilon(y_\varepsilon) = 0$ for these ε, and this yields $[u_\varepsilon(y_\varepsilon)] = 0$. This proves that $K \in \mathcal{K}_f$, but $x \notin K$. □

Remark 3.8.

(i) Using methods from non-standard analysis, one can prove the converse of the first inclusion in (3.1) in the following case: If $X = \widetilde{\mathbb{R}}^n$ and the sharp interior of $\left\{x \in \widetilde{\mathbb{R}}^n \mid f(x) = 0\right\}$ is non-empty, then

$$\mathrm{supp}(f) = \bigcap_{K\in\mathcal{K}_f} K.$$

To see this, let $x \in \bigcap_{K\in\mathcal{K}_f} K$ and choose some z in the interior of $\left\{y \in \widetilde{\mathbb{R}}^n \mid f(y) = 0\right\}$ and some $q > 0$ such that $B_{d\varepsilon^q}(z) \subseteq \left\{y \in \widetilde{\mathbb{R}}^n \mid f(y) = 0\right\}$. Given a representative (v_ε) of f, let (χ_ε) be a moderate net of smooth functions such that χ_ε

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vanishes on $B^{\mathrm{E}}_{\varepsilon^{q+2}}(z_\varepsilon)$ and is identically equal to 1 on $\mathbb{R}^n \setminus B^{\mathrm{E}}_{\varepsilon^{q+1}}(z_\varepsilon)$. Then setting $u_\varepsilon := \chi_\varepsilon v_\varepsilon$ gives a new representative of f such that $[\{y \in \mathbb{R}^n \mid u_\varepsilon(y) = 0\}]$ has non-empty sharp interior. Let $m_\varepsilon \in \mathbb{N}$, $m_\varepsilon \to \infty$ as $\varepsilon \to 0$. Then set $K_\varepsilon := \{y \in \mathbb{R}^n \mid |u_\varepsilon(y)| \geq \varepsilon^{m_\varepsilon}\}$ and $K := [K_\varepsilon]$. It follows that $\emptyset \neq \mathrm{ext}(K) \subseteq \{y \in \mathbb{R}^n \mid f(y) = 0\}$, so $x \in K$. Fix any $k \in \mathbb{N}$. Now using non-standard notation, letting $\rho := [\varepsilon]$, $u := [u_\varepsilon]$, and fixing a representative of x, which we temporarily simply denote by x, the above in particular implies that

$$^*\mathbb{N}_\infty \subseteq \{m \in {}^*\mathbb{N} \mid \exists y_k \in {}^*\mathbb{R}^n : |y_k - x| \leq \rho^k \wedge |u(y_k)| \geq \rho^m\}.$$

By the underspill principle, therefore, there also exists some $m \in \mathbb{N}$ and some $y_k \in {}^*\mathbb{R}^n$ such that $|y_k - x| \leq \rho^k$ and $|u(y_k)| \geq \rho^m$. Taking equivalence classes of these nets, it follows that there exist $y_k \in \widetilde{\mathbb{R}}^n$ such that $|y_k - x| \leq \mathrm{d}\varepsilon^k$ and $|f(y_k)| > 0$. Since $y_k \to x$ in the sharp topology, this implies that $x \in \mathrm{supp}(f)$.

(ii) The assumption $\mathrm{ext}(K) \neq \emptyset$ in the definition of $\mathcal{K}_f$ is essential. To illustrate this, take φ as defined before Definition 3.1, pick any sequence $\varepsilon_k \downarrow 0$ and set $K_\varepsilon := \{0\}$ for $\varepsilon = \varepsilon_k$, and $K_\varepsilon = \mathbb{R}^n$ otherwise. Then $\mathrm{ext}(K) = \emptyset$, and obviously $\mathrm{supp}(\varphi) \not\subseteq K$.

(iii) The question of whether the reverse of the last inclusion in (3.1) also holds remains open.

In the following section we shall see that even though the notion of support introduced above may not be entirely satisfactory, there nevertheless is a very convenient notion of being compactly supported for GSFs.

3.1. The spaces $\mathcal{GD}_K(U, Y)$ and $\mathcal{GD}(U, Y)$

An idea frequently used idea to solve problems like the previous ones comes from considering a family of GSFs having 'good representatives', i.e. possessing a defining net (u_ε) that conforms to our intuition and includes the examples we have in mind. In the following, we denote by $(u^1, \ldots, u^d)$ the components of a function u that takes values, for example, $\mathbb{R}^d$.

Definition 3.9. Let $\emptyset \neq K \Subset_{\mathrm{f}} U \subseteq \widetilde{\mathbb{R}}^n$ and $Y \subseteq \widetilde{\mathbb{R}}^d$, then we say that f is a *GSF compactly supported in* K, and we write

$$f \in \mathcal{GD}_K(U, Y)$$

if $f \in \mathcal{GC}^\infty(U, Y)$ and there exists a net (u_ε) such that

(i) (u_ε) defines f, where $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ for all ε;

(ii) $\forall \alpha \in \mathbb{N}^n \, \forall [x_\varepsilon] \in \mathrm{ext}(K) : \; [\partial^\alpha u_\varepsilon(x_\varepsilon)] = 0$.

Moreover, we set

$$\mathcal{GD}(U, Y) := \bigcup_{\emptyset \neq K \Subset_{\mathrm{f}} U} \mathcal{GD}_K(U, Y).$$

We will simply use the symbols $\mathcal{GD}_K(U)$ and $\mathcal{GD}(U)$ if $Y = \widetilde{\mathbb{R}}$.

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Remark 3.10.

(i) Lemma 3.7 implies that if $f \in \mathcal{GD}_K(U,Y)$, then $\mathrm{supp}(f) \subseteq K$ because $K \in \mathcal{K}_f$. The converse implication for an arbitrary subset U remains an open problem. For the case $U = \widetilde{\mathbb{R}}^n$, and if $\mathrm{supp}(f)$ is not empty, see Theorem 3.17.

(ii) It is clear that, in general, another net defining $f : U \longrightarrow \widetilde{\mathbb{R}}^d$ will not necessarily satisfy Definition 3.9 (ii) because such a net is not bound to have any particular behavior outside of U.

(iii) Set $K_\varepsilon := \overline{B_1^{\mathrm{E}}(0)}$ and $L_\varepsilon := K_\varepsilon \setminus B^{\mathrm{E}}_{e^{-1/\varepsilon}}(0)$, then for the Hausdorff distance of K_ε and L_ε we obtain $d_{\mathcal{H}}(K_\varepsilon, L_\varepsilon) = e^{-1/\varepsilon}$. By [**27**, Corollary 2.10], it follows that $[K_\varepsilon] = [L_\varepsilon]$. If we consider a net of smooth functions such that $u_\varepsilon|_{B^{\mathrm{E}}_{1/2}(0)} = 1$, $u_\varepsilon|_{\mathbb{R}^2 \setminus K_\varepsilon} = 0$, $u_\varepsilon \geq 0$, then $\sup_{y \in \mathbb{R}^2 \setminus K_\varepsilon} |\partial^\alpha u_\varepsilon(y)| = 0$ but $\sup_{y \in \mathbb{R}^2 \setminus L_\varepsilon} |\partial^\alpha u_\varepsilon(y)| = 1$. This motivates the use of the strongly internal set $\langle K_\varepsilon^c \rangle$ in (ii) instead of the simpler $[K_\varepsilon^c]$. Analogously, we can consider as L_ε an $e^{-1/\varepsilon}$-mesh of points for $K_\varepsilon = \overline{B_1^{\mathrm{E}}(0)}$. L_ε may also contain points in K_ε^c, but so that $d(x, K_\varepsilon) = e^{-1/\varepsilon}$. This example shows clearly that we need to be sufficiently 'far' from ∂K_ε to be sure that $[\partial^\alpha u_\varepsilon(x_\varepsilon)] = 0$, i.e. at points $x = [x_\varepsilon]$ such that $[d(x_\varepsilon, K_\varepsilon)] > 0$, as stated in (ii). Since $\mathrm{ext}(K)$ is independent of the choice of representative of K by Corollary 3.6, so is Definition 3.9. This is essential to prove the completeness of the spaces $\mathcal{GD}_K(U)$ and $\mathcal{GD}(U)$.

The following result will turn out to be useful when proving results by contradiction in several instances below. It permits to restrict the analysis to only two cases: points in K or in $\mathrm{ext}(K)$. To state it more clearly, we say that a generalized point $[y_\varepsilon]$ *joins points of the sequence* $(y_k)_{k\in\mathbb{N}}$ *at* $(\varepsilon_k)_{k\in\mathbb{N}}$ if $\forall N \in \mathbb{N}\, \exists k \geq N :\ y_{\varepsilon_k} = y_k$.

Lemma 3.11. *Let $K = [K_\varepsilon] \Subset_f \widetilde{\mathbb{R}}^n$, let $(\varepsilon_k)_{k\in\mathbb{N}}$ a sequence in $(0,1]$ that strictly decreases to 0, and let $(y_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}^n$. For each $k \in \mathbb{N}$, let $x_k \in K_{\varepsilon_k}$ be such that $d(y_k, x_k) = d(y_k, K_{\varepsilon_k})$. Then either*

(i) $\exists [y_\varepsilon] \in \mathrm{ext}(K) :\ [y_\varepsilon]$ *joins points of the sequence* $(y_k)_{k\in\mathbb{N}}$ *at* $(\varepsilon_k)_{k\in\mathbb{N}}$

or

(ii) $\exists [\bar{y}_\varepsilon], [\bar{x}_\varepsilon]$ *respectively joining points of the sequences* $(y_k)_{k\in\mathbb{N}}$ *and* $(x_k)_{k\in\mathbb{N}}$ *at* $(\varepsilon_k)_{k\in\mathbb{N}}$ *such that* $[\bar{y}_\varepsilon] = [\bar{x}_\varepsilon]$ *and* $\bar{x}_\varepsilon \in K_\varepsilon\ \forall \varepsilon$.

Proof. We can always pick a point $e_\varepsilon \in K_\varepsilon^c$ so that $e := [e_\varepsilon] \in \mathrm{ext}(K)$; in fact, since (K_ε) is sharply bounded, we can find $e_\varepsilon \in \mathbb{R}^n \setminus K_\varepsilon$ so that $d(e_\varepsilon, K_\varepsilon) > 1$ and (e_ε) is moderate. We can also take a point $i_\varepsilon \in K_\varepsilon$ for each ε because, without loss of generality, we can assume that $K_\varepsilon \neq \emptyset$ for all ε.

The first alternative in the statement is realized if

$$\exists b \in \mathbb{R}_{>0}\, \exists N \in \mathbb{N}\, \forall k \geq N :\ |y_k - x_k| > \varepsilon_k^b. \tag{3.4}$$

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Set $y_\varepsilon := y_k$ if $\varepsilon = \varepsilon_k$ and $y_\varepsilon := e_\varepsilon$ otherwise. Then, if $\varepsilon = \varepsilon_k$, by (3.4) we have $d(y_\varepsilon, K_\varepsilon) = |y_k - x_k| > \varepsilon^b$; otherwise $d(y_\varepsilon, K_\varepsilon) = d(e_\varepsilon, K_\varepsilon) > 1 \geq \varepsilon^b$. Therefore, $[y_\varepsilon] \in \langle K_\varepsilon^c \rangle = \mathrm{ext}(K)$ and, of course, $[y_\varepsilon]$ joins points of the sequence $(y_k)_{k\in\mathbb{N}}$ at $(\varepsilon_k)_{k\in\mathbb{N}}$.

Conversely, if

$$\forall h \in \mathbb{N}\, \exists k_h > h : \ |y_{k_h} - x_{k_h}| \leq \varepsilon_{k_h}^h, \tag{3.5}$$

then we can set $\bar{y}_\varepsilon := y_{k_h}$, $\bar{x}_\varepsilon := x_{k_h}$ if $\varepsilon = \varepsilon_{k_h}$ and $\bar{y}_\varepsilon := \bar{x}_\varepsilon := i_\varepsilon$ otherwise. Then, if $\varepsilon = \varepsilon_{k_h}$, by (3.5) we have $|\bar{y}_\varepsilon - \bar{x}_\varepsilon| = |y_{k_h} - x_{k_h}| \leq \varepsilon_{k_h}^h = \varepsilon^h$; otherwise $|\bar{y}_\varepsilon - \bar{x}_\varepsilon| = |i_\varepsilon - i_\varepsilon| = 0$, and (ii) follows. □

We use the above result to guarantee that the maximum values of any partial derivative $\partial^\alpha f$ are attained on K and not outside, as precisely stated in the following lemma

Lemma 3.12. *Let (u_ε) and $K = [K_\varepsilon]$ satisfy Definition 3.9. Then*

$$\forall \alpha \in \mathbb{N}^n\, \forall i = 0, \ldots, n : \ \left[\sup_{y\in\mathbb{R}^n} |\partial^\alpha u_\varepsilon^i(y)|\right] = \left[\sup_{x\in K_\varepsilon} |\partial^\alpha u_\varepsilon^i(x)|\right]. \tag{3.6}$$

Proof. By contradiction, assume that

$$\left[\sup_{y\in\mathbb{R}^n} |\partial^\alpha u_\varepsilon^i(y)|\right] \neq \left[\sup_{x\in K_\varepsilon} |\partial^\alpha u_\varepsilon^i(x)|\right]. \tag{3.7}$$

For simplicity of notation, set $v_\varepsilon := \partial^\alpha u_\varepsilon^i$. Inequality (3.7) means that

$$\exists a \in \mathbb{R}_{>0}\, \exists \varepsilon_k \searrow 0\, \forall k \in \mathbb{N} : \ \left|\sup_{y\in\mathbb{R}^n} |v_{\varepsilon_k}(y)| - \sup_{x\in K_\varepsilon} |v_{\varepsilon_k}(x)|\right| > \varepsilon_k^a.$$

Thus

$$\forall k \in \mathbb{N}\, \exists y_k \in \mathbb{R}^n : \ \varepsilon_k^a + \sup_{x\in K_{\varepsilon_k}} |v_{\varepsilon_k}(x)| < |v_{\varepsilon_k}(y_k)|. \tag{3.8}$$

We can hence apply Lemma 3.11. In case (i) we have $y := [y_\varepsilon] \in \mathrm{ext}(K)$ so that $[v_\varepsilon(y_\varepsilon)] = 0$ by Definition 3.9 (ii). Since $[y_\varepsilon]$ joins points of $(y_k)_{k\in\mathbb{N}}$ at $(\varepsilon_k)_{k\in\mathbb{N}}$, from (3.8) and $[v_\varepsilon(y_\varepsilon)] = 0$ we get

$$\varepsilon_k^a \leq \varepsilon_k^a + \sup_{x\in K_{\varepsilon_k}} |v_{\varepsilon_k}(x)| < |v_{\varepsilon_k}(y_k)| = |v_{\varepsilon_k}(y_{\varepsilon_k})| < \varepsilon_k^{a+1},$$

for k sufficiently big, which gives a contradiction in the first case. In Lemma 3.11 (ii) we have $\bar{y} := [\bar{y}_\varepsilon] = \bar{x} := [\bar{x}_\varepsilon] \in K \subseteq U$. Therefore $[v_\varepsilon(\bar{y}_\varepsilon)] = [v_\varepsilon(\bar{x}_\varepsilon)]$ by the definition of GSF. Since $[\bar{y}_\varepsilon]$ and $[\bar{x}_\varepsilon]$ join points of $(y_k)_{k\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$, respectively, at $(\varepsilon_k)_{k\in\mathbb{N}}$, for k sufficiently big we have

$$\begin{aligned}\varepsilon_k^a + \sup_{x\in K_{\varepsilon_k}} |v_{\varepsilon_k}(x)| < |v_{\varepsilon_k}(y_k)| &\leq |v_{\varepsilon_k}(y_k) - v_{\varepsilon_k}(x_k)| + |v_{\varepsilon_k}(x_k)| \\ &\leq \varepsilon_k^{a+1} + \sup_{x\in K_{\varepsilon_k}} |v_{\varepsilon_k}(x)|,\end{aligned}$$

again leading to a contradiction. □

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The previous result will be essential to prove that any compactly supported GSF can be extended to the whole of $\widetilde{\mathbb{R}}^n$, and to define an $\widetilde{\mathbb{R}}$-valued norm of f that does not depend on K.

Using Lemma 3.12, we can prove that any derivative of a compactly supported GSF is globally bounded in an appropriate sense:

Lemma 3.13. *Let the net (u_ε) satisfies Definition 3.9. Then*

$$\forall \alpha \in \mathbb{N}^n \, \exists C \in \widetilde{\mathbb{R}} \, \forall \beta \in \mathbb{N}^n : \ |\beta| \leq |\alpha| \Rightarrow \left[\sup_{y \in \mathbb{R}^n} |\partial^\beta u_\varepsilon(y)| \right] \leq C.$$

Proof. By the extreme-value property Proposition 2.1, we can set

$$C_{i\beta} := \left[\sup_{y \in K_\varepsilon} |\partial^\beta u^i_\varepsilon(y)| \right],$$

where $\beta \in \mathbb{N}^n$, $|\beta| \leq |\alpha|$, and $i = 1, \ldots, n$.

Set

$$C := 1 + \sqrt{n} \max_{\substack{|\beta| \leq |\alpha| \\ 1 \leq i \leq n}}, C_{i\beta}.$$

Then $C > C_{i\beta}$ and property (3.6) yield $\left[\sup_{y \in \mathbb{R}^n} |\partial^\beta u^i_\varepsilon(y)|\right] = \left[\sup_{x \in K_\varepsilon} |\partial^\beta u^i_\varepsilon(x)|\right] = C_{i\beta}$, and hence $\left[\sup_{y \in \mathbb{R}^n} |\partial^\beta u_\varepsilon(y)|\right] < C$. □

Moreover, compactly supported GSF can be extended in a unique way to the entire $\widetilde{\mathbb{R}}^n$:

Theorem 3.14. *Let $\emptyset \neq K \Subset_f U \subseteq \widetilde{\mathbb{R}}^n$ and $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$ be defined by (u_ε) which satisfies Definition 3.9. Then (u_ε) defines a GSF of the type $\widetilde{\mathbb{R}}^n \longrightarrow \widetilde{\mathbb{R}}^d$ and there exists one and only one $\bar{f} \in \mathcal{GD}_K(\widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^d)$ such that:*

(i) $\bar{f}|_K = f|_K$,

(ii) $\bar{f}|_{\mathrm{ext}(K)} = 0$.

Moreover, this $\bar{f}$ satisfies the following conditions:

(i) *If U is sharply open and $\alpha \in \mathbb{N}^n$, then $\partial^\alpha \bar{f}|_U = \partial^\alpha f$;*

(ii) *$\bar{f}|_{\widetilde{\mathbb{R}}^n_c}$ can be identified with a Colombeau generalized function.*

Proof. The existence part follows by showing that for all $\alpha \in \mathbb{N}^n$ and all $[x_\varepsilon] \in \widetilde{\mathbb{R}}^n$ we have $(\partial^\alpha u_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_M$. This follows since Lemma 3.13 yields that for any $\alpha \in \mathbb{N}^n$ there exists some $(C_{\alpha\varepsilon}) \in \mathbb{R}^n_M$ such that $|\sup_{x \in \mathbb{R}^n} \partial^\alpha u_\varepsilon(x)| \leq C_{\alpha\varepsilon}$ for all ε small. To prove uniqueness, let $g \in \mathcal{GD}_K(\widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^d)$ be such that $g|_K = \bar{f}|_K = f$ and $g|_{\mathrm{ext}(K)} = \bar{f}|_{\mathrm{ext}(K)} = 0$. By contradiction, assume that $\bar{f}(y) = [u_\varepsilon(y_\varepsilon)] \neq g(y) = [v_\varepsilon(y_\varepsilon)]$, for some $y = [y_\varepsilon] \in \widetilde{\mathbb{R}}^n$,

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where (v_ε) defines g. Thus

$$\exists a \in \mathbb{R}_{>0}\, \exists \varepsilon_k \searrow 0\, \forall k \in \mathbb{N}: \ |u_{\varepsilon_k}(y_{\varepsilon_k}) - v_{\varepsilon_k}(y_{\varepsilon_k})| > \varepsilon_k^a. \tag{3.9}$$

By Lemma 3.11, this leaves two possibilities. In the first one, there exists a point $z = [z_\varepsilon] \in \operatorname{ext}(K)$ that joins points of the sequence $(y_{\varepsilon_k})_{k\in\mathbb{N}}$ at $(\varepsilon_k)_{k\in\mathbb{N}}$. Therefore $g(z) = [v_\varepsilon(z_\varepsilon)] = \bar{f}(z) = [u_\varepsilon(z_\varepsilon)] = 0$, which gives a contradiction at $\varepsilon = \varepsilon_k$ when compared with (3.9). In the second one, there exists a point $\bar{z} = [\bar{z}_\varepsilon] \in K$ joining points of the sequence $(y_{\varepsilon_k})_{k\in\mathbb{N}}$ at $(\varepsilon_k)_{k\in\mathbb{N}}$. Once again, we have $g(\bar{z}) = [v_\varepsilon(\bar{z}_\varepsilon)] = \bar{f}(\bar{z}) = [u_\varepsilon(\bar{z}_\varepsilon)] = f(z)$, in contradiction to (3.9).

Furthermore, [**21**, Theorem 31] implies claim (iii). Finally, (iv) follows from the isomorphism $\mathcal{GC}^\infty(\widetilde{\mathbb{R}}_c, \widetilde{\mathbb{R}}^d) \simeq \mathcal{G}^s(\mathbb{R})^d$. □

We also have this simple but useful result:

Lemma 3.15. *Let $\emptyset \neq K \Subset_f U \subseteq \widetilde{\mathbb{R}}^n$, U be a sharply open set, $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$ and $\alpha \in \mathbb{N}^n$. Then $\partial^\alpha f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$.*

Theorem 3.14 provides the possibility to restrict our attention to compactly supported GSF whose domain is the entire $\widetilde{\mathbb{R}}^n$, as stated in the following theorem

Theorem 3.16. *For $\emptyset \neq K \Subset_f \widetilde{\mathbb{R}}^n$ and $Y \subseteq \widetilde{\mathbb{R}}^d$ set*

$$\mathcal{GD}^{\mathrm{g}}(K, Y) := \left\{ f \in \mathcal{GC}^\infty(\widetilde{\mathbb{R}}^n, Y) \mid f|_{\operatorname{ext}(K)} = 0 \right\}, \tag{3.10}$$

where the g *superscript means* globally defined. *Let $K \subseteq U \subseteq \widetilde{\mathbb{R}}^n$. Then*

(i) *If $f \in \mathcal{GD}^{\mathrm{g}}(K, Y)$, then $f|_U \in \mathcal{GD}_K(U, Y)$;*

(ii) *If $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$, then $\exists! \bar{f} \in \mathcal{GD}^{\mathrm{g}}(K, \widetilde{\mathbb{R}}^d): \ \bar{f}|_K = f|_K$;*

(iii) *$\mathcal{GD}^{\mathrm{g}}(K, Y) = \mathcal{GD}_K(\widetilde{\mathbb{R}}^n, Y)$.*

Proof.

(i) Let (u_ε) be any net that defines f, so that $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$. Clearly $f|_U \in \mathcal{GC}^\infty(U, Y)$, so for all $\alpha \in \mathbb{N}^n$ and $x = [x_\varepsilon] \in \operatorname{ext}(K)$ it remains to show that $[\partial^\alpha u_\varepsilon(x_\varepsilon)] = 0$. By Lemma 3.5, we get $\operatorname{ext}(K) = \langle K_\varepsilon^c \rangle$, which is a sharply open set. So $x \in \operatorname{ext}(K)$ yields $B_r(x) \subseteq \operatorname{ext}(K)$ for some $r \in \widetilde{\mathbb{R}}_{>0}$, and hence $f|_{B_r(x)} = 0$. Thereby $\partial^\alpha f(x) = [\partial^\alpha u_\varepsilon(x_\varepsilon)] = 0$.

(ii) This is exactly Theorem 3.14.

(iii) This follows directly from (i) by setting $U = \widetilde{\mathbb{R}}^n$, and by Definition 3.9 which yields $f|_{\operatorname{ext}(K)} = 0$.

□

For the extension of property (ii) to arbitrary codomains $Y \subseteq \widetilde{\mathbb{R}}^d$ (provided that U is strongly internal) see Theorem 8.14.

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Note explicitly that in (3.10), instead of the more technical properties of Definition 3.9, we have a concise and simpler pointwise condition. Notwithstanding this and several other positive aspects of (3.10) (see the second part of Lemma 3.7 and Theorem 3.17), in the present work, we prefer not to change Definition 3.9 in favor of (3.10): On the one hand, Definition 3.9 is nearer to the classical definition of $\mathcal{D}_K(\Omega)$, where the domain is $\Omega \subseteq \mathbb{R}^n$; on the other hand, for applications of these notions to geometry, locally defined functions are a more natural setting. We can also summarize these results by saying that compactly supported GSFs give rise to a good notion of being compactly supported, and globally defined compactly supported GSFs are in addition well behaved with respect to the concept of support introduced above. This is also clearly confirmed by the following theorem.

Theorem 3.17. *Let $Y \subseteq \widetilde{\mathbb{R}}^d$, and let $f \in \mathcal{GC}^\infty(\widetilde{\mathbb{R}}^n, Y)$ such that $\operatorname{supp}(f) \neq \emptyset$. Then $f \in \mathcal{GD}_K(\widetilde{\mathbb{R}}^n, Y)$ if and only if $\operatorname{supp}(f) \subseteq K$.*

Proof. One implication is immediate from Remark 3.10 (i). Conversely, if $\operatorname{supp}(f) \subseteq K$, by Theorem 3.16 (iii) we have to show that $f|_{\operatorname{ext}(K)} = 0$. Let $x \in \operatorname{ext}(K)$ but assume that $f(x) \neq 0$. Pick any $y \in \left\{y' \in \widetilde{\mathbb{R}}^n \mid |f(y')| > 0\right\}$, which is non-empty because $\operatorname{supp}(f) \neq \emptyset$. Let (u_ε) be a net that defines f and let $y = [y_\varepsilon]$. Since $f(x) \neq 0$, there exists $S \subseteq I$ such that $e_S \neq 0$ and $|f(x)e_S| > 0$. Then setting $z := xe_S + ye_{S^c}$, we have $|f(z)| > 0$ and hence $z \in \operatorname{supp}(f) \subseteq K$. But then $xe_S = ze_S \in Ke_S$ which, by Lemma 3.4, yields $x \notin \operatorname{ext}(K)$, a contradiction. □

3.2. Examples of compactly supported GSFs

A first class of examples comes by considering each $\varphi \in \mathcal{D}_K(\Omega)$, $K \Subset \Omega \subseteq \mathbb{R}^n$. Indeed, it suffices to set $K_\varepsilon := K$ and $u_\varepsilon(x) := \varphi(x)$ if $x \in \Omega$ and $u_\varepsilon(x) := 0$ otherwise to have that $\varphi \in \mathcal{GD}_{\widetilde{K}}(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$, where we recall that $\widetilde{K} = [K]$. Therefore $\mathcal{D}_K(\Omega) \subseteq \mathcal{GD}_{\widetilde{K}}(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$.

Moreover, since any given CGF can be defined by a net (u_ε) of maps with sharply bounded compact supports, we have the following result.

Theorem 3.18. *Let Ω be an open subset of $\mathbb{R}^n$ and let $J = [J_\varepsilon] \in \widetilde{\mathbb{R}}$ be a CGN such that $\lim_{\varepsilon \to 0^+} J_\varepsilon = +\infty$. Set $K_\varepsilon := \{x \in \Omega \mid |x| \leq J_\varepsilon\}$ and $K := [K_\varepsilon]$. Then for all $f \in \mathcal{GC}^\infty(\widetilde{\Omega}_c, \widetilde{\mathbb{R}}^d)$ there exists $\bar{f} \in \mathcal{GD}_K(\widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^d)$ such that $\bar{f}|_{\widetilde{\Omega}_c} = f$.*

Proof. Set $U_\varepsilon := \{x \in \Omega \mid |x| < \frac{1}{2}J_\varepsilon\}$ so that $U_\varepsilon \subseteq K_\varepsilon$ for ε small. Let $\chi_\varepsilon \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ be such that $\chi|_{U_\varepsilon} = 1$ and $\operatorname{supp}(\chi_\varepsilon) \subseteq K_\varepsilon$. Let $f \in \mathcal{GC}^\infty(\widetilde{\Omega}_c, \widetilde{\mathbb{R}}^d)$ be represented by (v_ε), with $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$, and set $u_\varepsilon := \chi_\varepsilon \cdot v_\varepsilon$. Then each u_ε is compactly supported and any $x = [x_\varepsilon] \in \widetilde{\Omega}_c$ satisfies $x_\varepsilon \in U_\varepsilon$ for ε small because $\lim_{\varepsilon \to 0^+} J_\varepsilon = +\infty$. Therefore $\bar{f} := [u_\varepsilon(\cdot)] \in \mathcal{GD}_K(\widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^d)$, and if $x_\varepsilon \in U_\varepsilon$ then $u_\varepsilon(x_\varepsilon) = v_\varepsilon(x_\varepsilon)$, so $\bar{f}|_{\widetilde{\Omega}_c} = f$. □

This theorem gives an infinity of non-trivial examples of compactly supported GSFs. Moreover, even though $\bar{f}$ depends on the fixed infinite number $J \in \widetilde{\mathbb{R}}$, every such $\bar{f}$ contains all the information of the original CGF f because $\bar{f}|_{\widetilde{\Omega}_c} = f$.

Finally, the constant function $f(x) = 1$ for all $x \in \widetilde{\mathbb{R}}$ is not compactly supported. In fact, by contradiction, assume that f admits (u_ε) and (K_ε) such that Definition 3.9 holds. Then choosing r large enough that $\mathrm{d}\varepsilon^{-r} \in \widetilde{\mathbb{R}} \setminus K$ we arrive at $f(\mathrm{d}\varepsilon^{-r}) = [u_\varepsilon(\varepsilon^{-r})] = 0$.

4. Generalized norms on $\mathcal{GD}_K$ and $\mathcal{GD}$

As a first step to topologizing the spaces $\mathcal{GD}_K$ and $\mathcal{GD}$ we prove the following theorem.

Theorem 4.1. *Let $\emptyset \neq K \Subset_f U \subseteq \widetilde{\mathbb{R}}^n$, Then*

(i) *$\mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$ is an $\widetilde{\mathbb{R}}$-module*

(ii) *for all non-empty $H \Subset_f U$, the inclusion $K \subseteq H$ implies $\mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d) \subseteq \mathcal{GD}_H(U, \widetilde{\mathbb{R}}^d)$.*

Proof.

(i) is immediate from Definition 3.9.

(ii) Take $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$. Since $K \subseteq H$, by [**27**, Proposition 2.8] for each representative (K_ε) of K we get the existence of a representative (H_ε) of H such that $K_\varepsilon \subseteq H_\varepsilon$ for all ε. Therefore, $\mathrm{ext}(K) = \langle K_\varepsilon^c \rangle \supseteq \langle H_\varepsilon^c \rangle = \mathrm{ext}(H)$ and hence the conclusion follows.

□

From the extreme-value property, Proposition 2.1, it is natural to expect that the following CGNs could serve as generalized $\widetilde{\mathbb{R}}$-valued norms.

Definition 4.2. Let $\emptyset \neq K \Subset_{\mathrm{f}} U \subseteq \widetilde{\mathbb{R}}^n$, where U is a sharply open set. Let $m \in \mathbb{N}$ and $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$. Then

$$\|f\|_{m,K} := \max_{\substack{|\alpha| \leq m, \\ 1 \leq i \leq d}} \max\left(\left|\partial^\alpha f^i(M_{\alpha i})\right|, \left|\partial^\alpha f^i(m_{\alpha i})\right|\right) \in \widetilde{\mathbb{R}},$$

where $m_{\alpha i}$, $M_{\alpha i} \in K$ satisfy

$$\forall x \in K: \ \partial^\alpha f^i(m_{\alpha i}) \leq \partial^\alpha f^i(x) \leq \partial^\alpha f^i(M_{\alpha i}).$$

The following result permits to calculate the (generalized) norm $\|f\|_{m,K}$ using any net (v_ε) that defines f. In case the net (v_ε) satisfies Definition 3.9, it also permits to prove that this norm does not depend on K, as is the case for any ordinary compactly supported smooth function.

Even though $\|f\|_{m,K} \in \widetilde{\mathbb{R}}$, using an innocuous abuse of language, in the following we will simply call $\|f\|_{m,K}$ a norm.

Proposition 4.3. *Under the assumptions of Definition 4.2, let the set $K = [K_\varepsilon] \Subset_f \widetilde{\mathbb{R}}^n$. Then we have that*

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(i) *if the net* (v_ε) *defines* f, *then*

$$\|f\|_{m,K} = \left[\max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \sup_{x\in K_\varepsilon} \left|\partial^\alpha v^i_\varepsilon(x)\right|\right];$$

(ii) *if* (u_ε) *defines* f *and* (u_ε) *satisfies Definition 3.9, then*

$$\|f\|_{m,K} = \left[\max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \sup_{x\in \mathbb{R}^n} \left|\partial^\alpha u^i_\varepsilon(x)\right|\right]. \tag{4.1}$$

Proof. In proving (i) we will also prove that the norm $\|f\|_{m,K}$ is well defined, i.e. it does not depend on the particular choice of points $m_{\alpha i}$, $M_{\alpha i}$ as in Definition 4.2. As in the proof of Proposition 2.1, we get the existence of $\bar{m}_{\alpha i\varepsilon}$, $\bar{M}_{\alpha i\varepsilon} \in K_\varepsilon$ such that

$$\forall x \in K_\varepsilon:\ \partial^\alpha v^i_\varepsilon(\bar{m}_{\alpha i\varepsilon}) \le \partial^\alpha v^i_\varepsilon(x) \le \partial^\alpha v^i_\varepsilon(\bar{M}_{\alpha i\varepsilon}).$$

Hence $\left|\partial^\alpha v^i_\varepsilon(x)\right| \le \max\left(\left|\partial^\alpha v^i_\varepsilon(\bar{m}_{\alpha i\varepsilon})\right|, \left|\partial^\alpha v^i_\varepsilon(\bar{M}_{\alpha i\varepsilon})\right|\right)$. Thus

$$\max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \sup_{x\in K_\varepsilon} \left|\partial^\alpha v^i_\varepsilon(x)\right| \le \max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \max\left(\left|\partial^\alpha v^i_\varepsilon(\bar{m}_{\alpha i\varepsilon})\right|, \left|\partial^\alpha v^i_\varepsilon(\bar{M}_{\alpha i\varepsilon})\right|\right).$$

But $\bar{m}_{\alpha i\varepsilon}$, $\bar{M}_{\alpha i\varepsilon} \in K_\varepsilon$, so

$$\begin{aligned}\left[\max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \sup_{x\in K_\varepsilon} \left|\partial^\alpha v^i_\varepsilon(x)\right|\right] &= \left[\max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \max\left(\left|\partial^\alpha v^i_\varepsilon(\bar{m}_{\alpha i\varepsilon})\right|, \left|\partial^\alpha v^i_\varepsilon(\bar{M}_{\alpha i\varepsilon})\right|\right)\right]\\ &= \max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \max\left(\left|\partial^\alpha f^i(\bar{M}_{\alpha i})\right|, \left|\partial^\alpha f^i(\bar{m}_{\alpha i})\right|\right).\end{aligned}$$

From this, both the fact that the norm $\|f\|_{m,K}$ is well defined and claim (i) follow.

(ii) By Lemma 3.12, we have that

$$\begin{aligned}\left[\max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \sup_{x\in K_\varepsilon} \left|\partial^\alpha u^i_\varepsilon(x)\right|\right] &= \max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \left[\sup_{x\in K_\varepsilon} \left|\partial^\alpha u^i_\varepsilon(x)\right|\right]\\ &= \max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \left[\sup_{x\in \mathbb{R}^n} \left|\partial^\alpha u^i_\varepsilon(x)\right|\right]\\ &= \left[\max_{\substack{|\alpha|\le m,\\ 1\le i\le d}} \sup_{x\in \mathbb{R}^n} \left|\partial^\alpha u^i_\varepsilon(x)\right|\right].\end{aligned}$$

□

Corollary 4.4. *Let* $\emptyset \ne K \Subset_f U \subseteq \widetilde{\mathbb{R}}^n$, *where* U *is a sharply open set. Let* $\emptyset \ne H \Subset_f U$ *and* $m \in \mathbb{N}$. *If* $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d) \cap \mathcal{GD}_H(U, \widetilde{\mathbb{R}}^d)$, *then* $\|f\|_{m,K} = \|f\|_{m,H} =: \|f\|_m$.

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Proof. The right hand side of (4.1) does not depend on K. □

Another consequence of Proposition 4.3 is the following corollary.

Corollary 4.5. *Let $U \subseteq \widetilde{\mathbb{R}}^n$, $f \in \mathcal{GD}(U, \widetilde{\mathbb{R}}^d)$ and $\bar{f} \in \mathcal{GD}(\widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^d)$ be the extension of f defined in Theorem 3.14. Then for all $m \in \mathbb{N}$, $\|f\|_m = \|\bar{f}\|_m$.*

Our use of the term 'norm' is justified by the following proposition.

Proposition 4.6. *Let $\emptyset \neq K \Subset_f U \subseteq \widetilde{\mathbb{R}}^n$, where U is a sharply open set. Let f, $g \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$ and $m \in \mathbb{N}$. Then*

(i) $\|f\|_m \geq 0$,

(ii) $\|f\|_m = 0$ *if and only if* $f = 0$,

(iii) $\forall c \in \widetilde{\mathbb{R}} : \ \|cf\|_m = |c|\|f\|_m$,

(iv) $\|f + g\|_m \leq \|f\|_m + \|g\|_m$,

(v) $\|fg\|_m \leq c_m \|f\|_m \|g\|_m$ *for some* $c_m \in \mathbb{R}_{>0}$.

Proof. Properties (i), (iii) and (iv) follow directly from Proposition 4.3, as does (v), using the Leibniz rule. The 'if' part of property (ii) follows from (4.1). □

We now prove that also the space $\mathcal{GD}(U, \widetilde{\mathbb{R}}^d)$ is an $\widetilde{\mathbb{R}}$-module, at least for certain U.

Proposition 4.7. *Let $U \subseteq \widetilde{\mathbb{R}}^n$ be a non-empty sharply open set. Assume that*

$$\forall K, H \Subset_f U : \ \mathrm{interl}(H \cup K) \subseteq U. \tag{4.2}$$

Then $\mathcal{GD}(U, \widetilde{\mathbb{R}}^d)$ is an $\widetilde{\mathbb{R}}$-module.

Proof. Since in Theorem 4.1 we already proved that $\mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$ is closed with respect to products by scalars, we only need to prove that $\mathcal{GD}(U, \widetilde{\mathbb{R}}^d)$ is closed with respect to sum. Let $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$, $g \in \mathcal{GD}_H(U, \widetilde{\mathbb{R}}^d)$ and let (u_ε), (v_ε) satisfy Definition 3.9 for f and g, respectively. Lemma 2.7 and [**27**, Propostion 2.8] imply that $\mathrm{interl}(H \cup K) = [H_\varepsilon \cup K_\varepsilon] \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^n$. But $\mathrm{ext}(H \cup K) = \langle H_\varepsilon^c \cap K_\varepsilon^c \rangle = \langle H_\varepsilon^c \rangle \cap \langle K_\varepsilon^c \rangle = \mathrm{ext}(H) \cap \mathrm{ext}(K)$. Therefore, $\partial^\alpha(u_\varepsilon + v_\varepsilon)$ is zero on $\mathrm{ext}(H \cup K)$. By our assumption (4.2) $\emptyset \neq \mathrm{interl}(H \cup K) \Subset_{\mathrm{f}} U$, so that $f + g \in \mathcal{GD}(U, \widetilde{\mathbb{R}}^d)$. □

In the following result, we give two general sufficient conditions for (4.2) to hold.

Proposition 4.8. *Let $U \subseteq \widetilde{\mathbb{R}}^n$ be a non-empty sharply open set. If U is $\widetilde{\mathbb{R}}$-convex or U is a strongly internal set, then (4.2) holds.*

Proof. Assume that U is $\widetilde{\mathbb{R}}$-convex, i.e. $xh + (1-x)k \in U$ for all h, $k \in U$ and all $x \in [0, 1]$. Then for all H, $K \subseteq U$ (even if we do not assume them to be functionally

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compact), and all $y \in \mathrm{interl}(H \cup K)$, we can write $y = e_S \cdot h + e_{S^c} \cdot k$ for some $S \subseteq I$ and $h \in H$, $k \in K$. Thus $y = e_S \cdot h + (1 - e_S) \cdot k \in U$ since $e_S \in [0,1]$ and h, $k \in U$.

Now, assume that U is a strongly internal set, i.e. for some net (U_ε) of subsets of $\mathbb{R}^n$, we have $U = \langle U_\varepsilon \rangle$. We continue to use the notations for y as above. Since h, $k \in U$, [**21**, Theorem 8] entails that $d(h_\varepsilon, U_\varepsilon^c)$, $d(k_\varepsilon, U_\varepsilon^c) > \varepsilon^q$ for some $q \in \mathbb{R}_{>0}$ and ε small, where $h = [h_\varepsilon]$, $k = [k_\varepsilon]$. But $y = e_S \cdot h + e_{S^c} \cdot k$, so for all ε small, if $\varepsilon \in S$ then $y_\varepsilon = h_\varepsilon$ and if $\varepsilon \notin S$ then $y_\varepsilon = k_\varepsilon$. In any case, $d(y_\varepsilon, U_\varepsilon^c) > \varepsilon^q$, hence $y \in \langle U_\varepsilon \rangle = U$. □

Example 4.9. If $U = (-1,1) \cup (2,4) \subseteq \widetilde{\mathbb{R}}$, then U is a sharply open set, but it does not satisfy condition (4.2) of Proposition 4.7. Let $H := \left[\left[-\frac{1}{2}, \frac{1}{2}\right]_{\mathbb{R}}\right]$, $K := \left[\left[\frac{5}{2}, \frac{7}{2}\right]_{\mathbb{R}}\right]$ and $x_\varepsilon := 0$ if $\varepsilon \in [1/n, 1/(n+1))$ and $n \in \mathbb{N}_{>0}$ is even, and $x_\varepsilon := 3$ otherwise. Then $x := [x_\varepsilon] \in \mathrm{interl}(H \cup K)$ but $x \notin U$. Moreover, let $\varphi \in \mathcal{D}_{[-1/2,1/2]_{\mathbb{R}}}(\mathbb{R}) \subseteq \mathcal{GD}_H(\widetilde{\mathbb{R}}_c)$, $\psi \in \mathcal{D}_{[5/2,7/2]_{\mathbb{R}}}(\mathbb{R}) \subseteq \mathcal{GD}_K(\widetilde{\mathbb{R}}_c)$ be positive non-trivial functions. Then, as we showed in the proof of Proposition 4.7, the GSF $\varphi + \psi \in \mathcal{GC}^\infty(U, \widetilde{\mathbb{R}})$ is compactly supported in

$$\left[\left[-\frac{1}{2}, \frac{1}{2}\right]_{\mathbb{R}} \cup \left[\frac{5}{2}, \frac{7}{2}\right]_{\mathbb{R}}\right] = \mathrm{interl}(H \cup K) \not\subseteq U.$$

Finally, let $f := \varphi|_U$ and $g := \psi|_U$, so that $f \in \mathcal{GD}_H(U, \widetilde{\mathbb{R}})$ and $g \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}})$. Remark 3.10.(i) yields that $f + g \notin \mathcal{GD}_J(U, \widetilde{\mathbb{R}})$ for all $J \Subset_{\mathrm{f}} U$. Otherwise, taking suitable sub-intervals L of $\left(-\frac{1}{2}, \frac{1}{2}\right)_{\mathbb{R}}$ and M of $\left(\frac{5}{2}, \frac{7}{2}\right)_{\mathbb{R}}$ where f and g, respectively, do not vanish, we would have $[L \cup M] \subseteq \mathrm{supp}(f+g) \subseteq J \subseteq U$ (here $\mathrm{supp}(f+g)$ is the support as in Definition 3.1). But the inclusion $[L \cup M] \subseteq U$ is impossible — a counterexample can be constructed similar to the above x.

This example shows that an assumption like (4.2) is necessary to have the closure of the space $\mathcal{GD}(U, \widetilde{\mathbb{R}})$ with respect to sum.

5. Topological structure on $\mathcal{GD}_K$

Using our $\widetilde{\mathbb{R}}$-valued norms, it is now natural to give the following definition.

Definition 5.1. Let $\emptyset \neq K \Subset_{\mathrm{f}} U \subseteq \widetilde{\mathbb{R}}^n$, where U is a sharply open set. Let $f \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$, $m \in \mathbb{N}$, $\rho \in \widetilde{\mathbb{R}}_{>0}$. Then the following hold.

(i) We set $B_\rho^m(f) := \left\{g \in \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d) \mid \|f - g\|_m < \rho\right\}$. In any case where there is potential for confusion, we use the more precise symbol $B_\rho^m(f, K) := B_\rho^m(f)$.

(ii) If $V \subseteq \mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$, then we say that V is a *sharply open set* if

$$\forall v \in V\, \exists m \in \mathbb{N}\, \exists \rho \in \widetilde{\mathbb{R}}_{>0} :\ B_\rho^m(v) \subseteq V.$$

Moreover, we say that V is *Fermat open* if

$$\forall v \in V\, \exists m \in \mathbb{N}\, \exists r \in \mathbb{R}_{>0} :\ B_r^m(v) \subseteq V.$$

As in [**21**, Theorem 2] it follows that sharply open sets as well as Fermat open sets form topologies on $\mathcal{GD}_K(U, \widetilde{\mathbb{R}}^d)$.

On the other hand, it is also natural to view the space $\mathcal{GD}_K(U,\widetilde{\mathbb{R}}^d)$ inside Garetto's theory [**11**, **12**] of $\widetilde{\mathbb{R}}$-locally convex algebras. In this section, we will realize this comparison, proving that the space $\mathcal{GD}_K(U,\widetilde{\mathbb{R}})$ is a Fréchet $\widetilde{\mathbb{R}}$-module and a topological algebra. For this purpose, we will only consider the sharp topology. Indeed, as we will see below, the Fermat topology is less interesting in this context since it doesn't permit to prove the continuity of the product by scalars $(r,f)\in\widetilde{\mathbb{R}}\times\mathcal{GD}_K(U,\widetilde{\mathbb{R}}^d)\mapsto r\cdot f\in\mathcal{GD}_K(U,\widetilde{\mathbb{R}}^d)$.

In the following, we will always assume that $\emptyset\neq K\Subset_{\mathrm{f}} U\subseteq\widetilde{\mathbb{R}}^n$, where U is a non-empty sharply open set. The main problem in performing this comparison, which does not permit to view our space $\mathcal{GD}_K(U,\widetilde{\mathbb{R}})$ as a particular case of the theory developed in [**11**, **12**], is that the domain U contains generalized points.

Using the valuation v on $\widetilde{\mathbb{R}}$, it is natural to introduce the following notions.

Definition 5.2. Let $m\in\mathbb{N}$ and $f\in\mathcal{GD}(U,\widetilde{\mathbb{R}})$. Then

(i) $v_m(f):=v(\|f\|_m)\in\mathbb{R}$,

(ii) $\mathcal{P}_m(f):=e^{-v_m(f)}$.

From the properties of the valuation v and of the e-norm $|\cdot|_e=e^{-v(\cdot)}$ on $\widetilde{\mathbb{R}}$ (see [**1**]), the following result directly follows.

Proposition 5.3. *For each $m\in\mathbb{N}$, we have the following.*

(i) $v_m:\mathcal{GD}(U)\longrightarrow\mathbb{R}\cup\{+\infty\}$ *is a valuation, i.e. for all f, $g\in\mathcal{GD}(U)$:*
- $v_m(0)=+\infty$,
- $v_m(\lambda\cdot f)\geq v(\lambda)+v_m(f)\quad\forall\lambda\in\widetilde{\mathbb{R}}$,
- $v_m(\,\mathrm{d}\varepsilon^a\cdot f)=v(\,\mathrm{d}\varepsilon^a)+v_m(f)=a+v_m(f)\quad\forall a\in\mathbb{R}$
- $v_m(f+g)\geq\min\,[v_m(f),v_m(g)]$.

(ii) $\mathcal{P}_m:\mathcal{GD}(U)\longrightarrow\mathbb{R}$ *is an ultra-pseudo-norm, i.e. for all f, $g\in\mathcal{GD}(U)$:*
- $\mathcal{P}_m(f)=0$ *if and only if* $f=0$,
- $\mathcal{P}_m(\lambda\cdot f)\leq|\lambda|_e\cdot\mathcal{P}_m(f)\quad\forall\lambda\in\widetilde{\mathbb{R}}$,
- $\mathcal{P}_m(\,\mathrm{d}\varepsilon^a\cdot f)=|\,\mathrm{d}\varepsilon^a|_e\cdot\mathcal{P}_m(f)=e^{-a}\cdot\mathcal{P}_m(f)\quad\forall a\in\mathbb{R}$,
- $\mathcal{P}_m(f+g)\leq\max\,[\mathcal{P}_m(f),\mathcal{P}_m(g)]$.

The following result states that to define the sharp topology, instead of the above ultra-pseudo-norms we can equivalently use the countable family of generalized norms $(\|f\|_m)_{m\in\mathbb{N}}$.

Theorem 5.4.

(i) *Sum and product in $\mathcal{GD}_K(U)$ are continuous in the sharp topology. Therefore, $\mathcal{GD}_K(U)$ is a topological $\widetilde{\mathbb{R}}$-algebra.*

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(ii) *The product in $\mathcal{GD}_K(U)$ is continuous in the Fermat topology only on the subspace* $\{(f,g) \in \mathcal{GD}_K(U) \times \mathcal{GD}_K(U) \mid \forall\, m \in \mathbb{N}:\ \|f\|_m,\ \|g\|_m < \infty\}$.

(iii) *For $f \in \mathcal{GD}_K(U)$, set*

$$C_r^m(f) := \{g \in \mathcal{GD}_K(U) \mid \mathcal{P}_m(f-g) < r\} \quad (r \in \mathbb{R}_{>0},\ m \in \mathbb{N}).$$

Then for each q, $s \in \mathbb{R}_{>0}$ we have:

(i) *If $q \leq -\log r$, then $C_r^m(f) \subseteq B^m_{\mathrm{d}\varepsilon^q}(f)$;*

(ii) *If $q \geq -\log s$ and $s < r$, then $B^m_{\mathrm{d}\varepsilon^q}(f) \subseteq C_r^m(f)$.*

(i) *The sharp topology on $\mathcal{GD}_K(U)$ is the coarsest topology such that each $\mathcal{P}_m$ is continuous.*

(ii) *$\mathcal{GD}_K(U)$ is a separated locally convex topological $\widetilde{\mathbb{R}}$-module.*

Proof. (i), (ii) Continuity of the sum in the sharp (and in the Fermat) topology follows directly from the triangle inequality Proposition 4.6 (iv). The continuity of the product at (f_0, g_0) and property (ii) follow from Proposition 4.6 (v) via

$$\|f \cdot g - f_0 \cdot g_0\|_m \leq c_m(\|f - f_0\|_m \cdot \|g - g_0\|_m + \|f - f_0\|_m \cdot \|g_0\|_m + \\ + \|f_0\|_m \cdot \|g - g_0\|_m).$$

(iii) Let us first assume that $q \leq -\log r$ and $g \in C_r^m(f)$, so that $\mathcal{P}_m(f-g) < r$ and $v_m(f-g) > -\log r$. This implies that

$$\exists b > -\log r: \max_{\substack{|\alpha| \leq m \\ i \leq d}} \sup_{x \in K_\varepsilon} \left|\partial^\alpha u_\varepsilon^i(x) - \partial^\alpha v_\varepsilon^i(x)\right| = O(\varepsilon^b), \tag{5.1}$$

where (u_ε) and (v_ε) define f and g, respectively. Property (5.1) yields the existence of some $M > 0$ such that for ε sufficiently small we obtain

$$\max_{\substack{|\alpha| \leq m \\ i \leq d}} \sup_{x \in K_\varepsilon} \left|\partial^\alpha u_\varepsilon^i(x) - \partial^\alpha v_\varepsilon^i(x)\right| + \varepsilon^b \leq (M+1) \cdot \varepsilon^b < \varepsilon^{-\log r} \leq \varepsilon^q.$$

Therefore, $\|f-g\|_m + \mathrm{d}\varepsilon^b \leq \mathrm{d}\varepsilon^q$, so $\|f-g\|_m < \mathrm{d}\varepsilon^q$.

Now, let us assume $q \geq -\log s$, $s < r$, and $g \in B^m_{\mathrm{d}\varepsilon^q}(f)$, so that $\|f-g\|_m < \mathrm{d}\varepsilon^q$. Therefore,

$$\forall^0 \varepsilon: \max_{\substack{|\alpha| \leq m \\ i \leq d}} \sup_{x \in K_\varepsilon} \left|\partial^\alpha u_\varepsilon^i(x) - \partial^\alpha v_\varepsilon^i(x)\right| < \varepsilon^q \leq \varepsilon^{-\log s}.$$

Taking the $|\cdot|_e$-norm we get

$$\left| \max_{\substack{|\alpha| \leq m \\ i \leq d}} \sup_{x \in K_\varepsilon} \left|\partial^\alpha u_\varepsilon^i(x) - \partial^\alpha v_\varepsilon^i(x)\right| \right|_e \leq e^{\log s} = s < r,$$

that is $\mathcal{P}_m(f-g) < r$ as claimed.

(iv) follows directly from (iii) and Proposition 5.3 (ii).

(v) follows from Proposition 4.6 (ii)and [**11**, Proposition 1.11]. □

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5.1. Generalized functions and non-Archimedean properties

In this section, we want to clarify some relationships between the classical notion of convexity, the notion of $\widetilde{\mathbb{R}}$-convexity of [**11**] and the use of $\widetilde{\mathbb{R}}$-valued norms.

We have seen that balls $B^m_\rho(0)$, $\rho \in \widetilde{\mathbb{R}}_{>0}$, define a neighborhood system of 0 for $\mathcal{GD}_K(U)$; they are convex in the usual sense, i.e. if f, $g \in B^m_\rho(0)$, $t \in [0,1]$ (in particular if $t \in [0,1]_{\mathbb{R}}$), then

$$\|tf + (1-t)g\|_m \leq t\|f\|_m + (1-t)\|g\|_m < t\rho + (1-t)\rho = \rho.$$

Moreover, each ball $B^m_\rho(0)$ is also balanced: if $t \in \widetilde{\mathbb{R}}$, $|t| \leq 1$, then $t \cdot B^m_\rho(0) \subseteq B^m_\rho(0)$. However, this space is not a classical locally convex topological vector space over the field $\mathbb{R}$ because of two reasons: (i) the product by scalars is not continuous with respect to the Euclidean topology on $\mathbb{R}$, (ii) Lemma 3.13 implies that the property

$$\forall f \in \mathcal{GD}_K(U)\, \exists t \in \widetilde{\mathbb{R}} :\ f \in t \cdot B^m_\rho(0)$$

holds for $t \in \widetilde{\mathbb{R}}$ but it cannot be extended to $t \in \mathbb{R}$. As we have seen in the proof of Theorem 5.4 (ii), this is a necessary consequence of the existence of generalized functions with infinite $\widetilde{\mathbb{R}}$-valued $\|\cdot\|_m$-norm.

On the other hand, even though the sets $C^m_r(0)$ are defined using $\mathbb{R}$ only, i.e. without mentioning any non-Archimedean property, they satisfy

$$\forall f \in C^m_r(0)\, \forall \lambda \in \mathbb{R} :\ \lambda \cdot f \in C^m_r(0), \tag{5.2}$$

and this is possible only because they are infinitesimal sets. In fact, we have seen in Theorem 5.4 that $C^m_r(0) \subseteq B^m_{\mathrm{d}\varepsilon^q}(0)$ for $q \leq -\log r$.

More generally, a set $A \subseteq \mathcal{GD}_K(U)$ can be $\widetilde{\mathbb{R}}$-balanced (see [**11**]), i.e.

$$\lambda A \subseteq A \quad \forall \lambda \in \widetilde{\mathbb{R}} :\ |\lambda|_e \leq 1,$$

and at the same time can be thought of as 'small' only in the case which A consists infinitesimal points. For example, the ball $B^m_{\mathrm{d}\varepsilon^b}(0)$ is $\widetilde{\mathbb{R}}$-balanced, but $B^m_1(0)$ is not. In fact, we have the following lemma.

Lemma 5.5. *Suppose that $A \subseteq \mathcal{GD}_K(U)$ and $m \in \mathbb{N}$ are such that*

$$A + A \subseteq A \tag{5.3}$$

and

$$\exists r \in \mathbb{R}_{>0} :\ A \subseteq B^m_r(0).$$

Then every element $u \in A$ has infinitesimal norm: $\|u\|_m \approx 0$.

Proof. In fact, (5.3) implies $n \cdot u \in A \subseteq B^m_r(0)$ for all $n \in \mathbb{N}_{\neq 0}$. Therefore, $\|u\|_m < r/n$ for all $n \in \mathbb{N}_{\neq 0}$, which proves our claim. □

Let us note that condition (5.3) holds both for A which is $\widetilde{\mathbb{R}}$-balanced or $\widetilde{\mathbb{R}}$-convex.

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These remarks permit to show that in dealing with generalized functions, we are naturally induced to consider a topology on $\widetilde{\mathbb{R}}$ which contains infinitesimal neighborhoods (hence inducing the discrete topology on $\mathbb{R}$, see [**17**]). This is due to the coexistence of a continuous product by scalars and of an infinite element in $\mathcal{GD}_K(U)$, as stated in the following general result. In a possible interpretation of its statement, we can think of R as $\mathbb{R}$ with a topology τ, $\widetilde{R}$ as $\widetilde{\mathbb{R}}$ with the sharp topology $\tilde{\tau}$, and $<_{\tilde{R}}$ as the strict order relation $<$ of Lemma 1.1.

Theorem 5.6. *Let $(R, +_R, \cdot_R, <_R, \tau)$ and $(\widetilde{R}, +_{\tilde{R}}, \cdot_{\tilde{R}}, <_{\tilde{R}}, \tilde{\tau})$ be Hausdorff topological ordered rings such that $(\mathbb{R}, +, \cdot, <)$ is a substructure of $(R, +_R, \cdot_R, <_R)$, which in turn is a substructure of $(\widetilde{R}, +_{\tilde{R}}, \cdot_{\tilde{R}}, <_{\tilde{R}})$ and such that*

$$\forall r \in R\ \forall s \in \mathbb{R}:\ r <_{\tilde{R}} s \quad \Longrightarrow \quad r <_R s. \tag{5.4}$$

Let $(G, +_G, \cdot_G, \sigma)$ be a Hausdorff topological R-module, and $|\cdot|_G : G \to \widetilde{R}$, $|\cdot|_R : R \to R$ be maps such that $|r \cdot_G g|_G = |r|_R \cdot_{\tilde{R}} |g|_G$ for all $r \in R$ and all $g \in G$. Assume that any τ-neighborhood of $0 \in R$ contains a ball $B^R_\eta(0) := \{s \in R \mid |s|_R <_R \eta\}$ for some $\eta \in R$, $\eta >_R 0$, and that there exists some $\rho \in \widetilde{R}$ with $\rho >_{\tilde{R}} 0$ such that the ball $B^G_\rho(0) := \{g \in G \mid |g|_G <_{\tilde{R}} \rho\}$ is σ-open. Finally, assume that

$$\exists g \in G:\ |g|_G \text{ is invertible},\ \forall M \in \mathbb{R}_{>0}:\ |g|_G >_{\tilde{R}} \rho \cdot_{\tilde{R}} M. \tag{5.5}$$

Then the induced topology $\tau \cap \mathbb{R}$ is discrete.

Proof. Since G is a topological R-module, the product by scalars is $\tau \times \sigma$-continuous, and

$$\lim_{\substack{r \to 0 \\ r \in R}} r \cdot_G g = 0, \tag{5.6}$$

where $g \in G$ comes from assumption (5.5). By hypothesis, the ball $B^G_\rho(0) \in \sigma$ and every τ-neighborhood of $r = 0 \in R$ contains some ball $B^R_\eta(0)$. Therefore (5.6) entails that there exists some $\eta >_R 0$ such that

$$\forall r \in R:\ |r|_R <_R \eta \quad \Longrightarrow \quad \rho >_{\tilde{R}} |r \cdot_G g|_G = |r|_R \cdot_{\tilde{R}} |g|_G. \tag{5.7}$$

For each $s \in \mathbb{R}_{>0}$ take $M \in \mathbb{R}_{>0}$ such that $\frac{1}{M} < s$, so that $|g|_G >_{\tilde{R}} \rho \cdot_{\tilde{R}} M >_{\tilde{R}} 0$ from (5.5). For all $r \in R$ such that $|r|_R <_R \eta$, we have

$$|r|_R <_{\tilde{R}} \frac{\rho}{|g|_G} <_{\tilde{R}} \frac{1}{M} < s$$

because $|g|_G$ is invertible in $\widetilde{R}$. Therefore $|r|_R <_{\tilde{R}} s$ and hence $|r|_R <_R s$ by (5.4). This means that r is infinitesimal in the ring R, i.e. the ball $B^R_\rho(0)$ is contained in the monad of 0 (see e.g. [**17**] for the notion of monad) and so also every ball $B^R_\eta(\bar{r})$ is contained in the monad of $\bar{r} \in R$. Therefore, [**17**, Prop. 2.1] implies the conclusion. □

We can therefore say that if we want to find a space G of generalized functions which is an ordinary Hausdorff topological vector space on $\mathbb{R}$, then we cannot define the topologies

τ and σ using seminorms valued in a non-Archimedean (see (5.5)) extension of $\mathbb{R}$. This results confirms [**10**, Remark 43].

As a consequence, we have the following impossibility result.

Corollary 5.7. *There does not exist any real Hausdorff topological vector space* $(G, +_G, \cdot_G, \sigma)$ *such that*

(i) $(G, +_G, \cdot_G)$ *is a linear subspace of* $\mathcal{GD}_K(U)$ *for some* $\emptyset \neq K \Subset_f U \subseteq \widetilde{\mathbb{R}}^n$

(ii) G *contains the Dirac delta* $\delta \in G$,

(iii) $\exists m \in \mathbb{N}\, \exists \rho \in \widetilde{\mathbb{R}}_{>0} : \rho < 1,\ B^m_\rho(0) \cap G \in \sigma$,

In particular, the Colombeau algebra $\mathcal{G}^s(\Omega)$ *does not contain any real Hausdorff topological vector subspace* G *such that some* $B^m_\rho(0) \cap G$ *is open and* $\delta \in G$.

Proof. By contradiction, in Thm. 5.6, set $R := \mathbb{R}$ with the usual Euclidean topology τ, $\widetilde{R} := \widetilde{\mathbb{R}}$ with the sharp topology; set $|g|_G := \|g\|_m$, where $m \in \mathbb{N}$ comes from (iii) and we used the inclusion (i); set $|r|_R := |r|$ the usual absolute value in $\mathbb{R}$. Note also that $B^m_\rho(0) \cap G = \{g \in G \mid \|g\|_m < \rho\} = \{g \in G \mid |g|_G <_{\tilde{R}} \rho\}$. If $\delta \in G$, then $|\delta|_G = \|\delta\|_m$ is infinite and invertible in $\widetilde{\mathbb{R}}$, so that Theorem 5.6 implies that the Euclidean topology would be discrete. The second part of the claim follows from Theorem 3.18. □

We can summarize Corollary 5.7 by saying that a real Hausdorff topological vector space structure G for a space of generalized functions cannot contain even a single trace $B^m_\rho(0) \cap G$, $\rho < 1$, of a sharply open ball. This result does not contradict [**6**, Propostion 4], where it is stated that the sharp topology induces on *bounded* sets of the real locally convex space $\mathcal{G}_a(\Omega) \subseteq \mathcal{G}^s(\Omega)$ a topology which is *finer* than the topology σ_a on $\mathcal{G}_a(\Omega)$. On the other hand, Corollary 5.7 implies that $B^m_\rho(0) \cap \mathcal{G}_a(\Omega) \notin \sigma_a$ for all $\rho \in \widetilde{\mathbb{R}}_{>0}$, $\rho < 1$.

6. Metric structure on $\mathcal{GD}_K$

In this section, we want to use [**11**, Theorem 1.14] to prove metrizability of $\mathcal{GD}_K(U)$. However, we will apply this result using an explicit and simple countable base of neighborhoods of the origin which consists of $\widetilde{\mathbb{R}}$-absorbent and absolutely $\widetilde{\mathbb{R}}$-convex sets. In this way, we will arrive at a simpler metric.

The idea is to consider only points of balls $f \in B^m_\rho(0)$ whose norm $\|f\|_m$ are infinitely smaller than ρ. To formally express this idea, we introduce the following definition

Definition 6.1. Let $\rho \in \widetilde{\mathbb{R}}_{>0}$, $m \in \mathbb{N}$ and $g \in \mathcal{GD}_K(U)$. Then

$$U^m_\rho(g) := \left\{ f \in \mathcal{GD}_K(U) \mid \frac{1}{\rho} \cdot \|f - g\|_m \approx 0 \right\}.$$

In any case confusion might arise, we will use the more precise symbol $U^m_\rho(g, K) := U^m_\rho(g)$.

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Proposition 6.2.

(i) *$U^m_\rho(0)$ is $\widetilde{\mathbb{R}}$-absorbent and absolutely $\widetilde{\mathbb{R}}$-convex.*

(ii) *Both the system*

$$\left\{U^m_\rho(v) \mid v \in \mathcal{GD}_K(U),\ m \in \mathbb{N},\ \rho \in \widetilde{\mathbb{R}}_{>0},\ \rho \approx 0\right\}$$

and the system

$$\{U^n_{\mathrm{d}\varepsilon^n}(v) \mid v \in \mathcal{GD}_K(U),\ n \in \mathbb{N}_{>0}\}$$

generate the sharp topology on $\mathcal{GD}_K(U)$.

Proof. To see that $U^m_\rho(0)$ is $\widetilde{\mathbb{R}}$-absorbent, by [11, Definition 1.1] we have to show that

$$\forall u \in \mathcal{GD}_K(U)\, \exists a \in \mathbb{R}\, \forall b \in \mathbb{R}_{\leq a} :\ u \in \mathrm{d}\varepsilon^b \cdot U^m_\rho(0), \tag{6.1}$$

i.e. $\frac{u}{\mathrm{d}\varepsilon^b} \in U^m_\rho(0)$, which is equivalent to

$$\left\|\frac{u}{\mathrm{d}\varepsilon^b}\right\|_m \cdot \frac{1}{\rho} = \frac{\|u\|_m}{\mathrm{d}\varepsilon^b \cdot \rho} \approx 0.$$

But ρ is strictly positive, so $\rho \geq \mathrm{d}\varepsilon^p$ for some $p \in \mathbb{R}$. Moreover, $\|u\|_m \in \widetilde{\mathbb{R}}$, so $\|u\|_m \leq \mathrm{d}\varepsilon^q$ for some $q \in \mathbb{R}$. Therefore,

$$\frac{\|u\|_m}{\mathrm{d}\varepsilon^b \cdot \rho} \leq \mathrm{d}\varepsilon^{q-b-p},$$

and we have $\mathrm{d}\varepsilon^{q-b-p} \approx 0$ if and only if $b < q - p$. This proves (6.1).

To prove that $U^m_\rho(0)$ is balanced, assume $\lambda \in \widetilde{\mathbb{R}}$ with $|\lambda|_e \leq 1$. Then $v(\lambda) \geq 0$, so $|\lambda| \leq c$ for some $c \in \mathbb{R}_{>0}$. Therefore, if $u \in U^m_\rho(0)$ then

$$\frac{\|\lambda u\|_m}{\rho} = |\lambda| \frac{\|u\|_m}{\rho} \approx 0,$$

so $\lambda u \in U^m_\rho(0)$.

Finally, we show $\widetilde{\mathbb{R}}$-convexity: for all a, $b \in \mathbb{R}_{\geq 0}$ and all u, $v \in U^m_\rho(0)$, we have

$$\frac{1}{\rho} \cdot \|\,\mathrm{d}\varepsilon^a \cdot u + \mathrm{d}\varepsilon^b \cdot v\|_m \leq \mathrm{d}\varepsilon^a \cdot \frac{\|u\|_m}{\rho} + \mathrm{d}\varepsilon^b \cdot \frac{\|v\|_m}{\rho} \approx 0.$$

In order to prove (ii), we note that $U^m_\rho(v) \subseteq B^m_\rho(v)$ because $\|u - v\|_m/\rho \approx 0$ implies $\|u - v\|_m/\rho < 1$. Also, if $\rho \approx 0$, then $B^m_\rho(v) \subseteq U^m_{\sqrt{\rho}}(v)$ because $\|u - v\|_m < \rho$ implies that $\cdot\frac{1}{\sqrt{\rho}}\|u - v\|_m \leq \sqrt{\rho} \approx 0$. Finally, if $\rho \approx 0$, every $U^m_\rho(v)$ is sharply open: if $u \in U^m_\rho(v)$ and $w \in B^m_{\rho^2}(u)$, then

$$\frac{1}{\rho} \cdot \|w - v\|_m \leq \frac{1}{\rho} \cdot \|w - u\|_m + \frac{1}{\rho} \cdot \|u - v\|_m \approx 0.$$

The proof for the second system in (ii) follows by observing that given $\rho > 0$, $\rho \approx 0$, there exists $q \in \mathbb{N}$ such that $\rho \geq \mathrm{d}\varepsilon^q$, and setting $n := \max(m, q)$ we have $U^n_{\mathrm{d}\varepsilon^n}(v) \subseteq U^m_\rho(v)$. $\square$

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From [11, Theorem 1.14], we have that $\mathcal{GD}_K(U)$ is metrizable with metric

$$d_2(u,v) = \sum_{n=1}^{+\infty} 2^{-n} \cdot \min\left\{\mathcal{P}_{U^n_{\mathrm{d}\varepsilon^n}(0)}(u-v), 1\right\}. \tag{6.2}$$

Concerning (6.2) we recall (see [11]) that if $A \subseteq \mathcal{GD}_K(U)$ is $\widetilde{\mathbb{R}}$-absorbent, then, for all $u \in \mathcal{GD}_K(U)$, we define

$$V_A(u) := \sup\left\{b \in \mathbb{R} \mid u \in \mathrm{d}\varepsilon^b \cdot A\right\},$$

$$\mathcal{P}_A(u) := e^{-V_A(u)}. \tag{6.3}$$

The following result gives a metric which is equivalent to (6.2) but is defined by a simpler formula.

Proposition 6.3. *Set $A_n := U^n_{\mathrm{d}\varepsilon^n}(0)$ for $n \in \mathbb{N}_{>0}$, and let $u \in \mathcal{GD}_K(U)$. Then*

(i) *$V_{A_n}(u) = v_n(u) - n$ and*

(ii) *the map*

$$d_e(u,v) = \sum_{n=1}^{+\infty} e^{\min[n - v_n(u-v), 0] - n}.$$

is a metric on $\mathcal{GD}_K(U)$ that is equivalent to d_2.

Proof. Concerning (i), we note that

$$u \in \mathrm{d}\varepsilon^b \cdot A_n \Rightarrow \frac{u}{\mathrm{d}\varepsilon^b} \in U^n_{\mathrm{d}\varepsilon^n}(0) \Rightarrow \frac{\|u\|_n}{\mathrm{d}\varepsilon^{b+n}} \approx 0 \Rightarrow v\left(\mathrm{d}\varepsilon^{-b-n} \cdot \|u\|_n\right) \geq 0$$
$$\Rightarrow b \leq v_n(u) - n,$$

so $V_{A_n}(u) \leq v_n(u) - n$. Conversely, if we had $V_{A_n}(u) < v_n(u) - n$, we could pick $V_{A_n}(u) < b < v_n(u) - n$. Then as above it would follow that $\|u\|_n/\mathrm{d}\varepsilon^{b+n} \approx 0$, contradicting the definition of $V_{A_n}(u)$. This proves (i).

In order to prove (ii), we use (i) in (6.2): $\mathcal{P}_{A_n}(u-v) = e^{-v_n(u-v)+n}$ and

$$\begin{aligned} d_2(u,v) &= \sum_{n=1}^{+\infty} 2^{-n} \cdot \min\left\{e^{-v_n(u-v)+n}, 1\right\} \\ &= \sum_{n=1}^{+\infty} 2^{-n} \cdot e^{\min[n - v_n(u-v), 0]} \\ &\geq \sum_{n=1}^{+\infty} e^{\min[n - v_n(u-v), 0] - n} = d_e(u,v). \end{aligned}$$

But we also have $e^{-n} \geq 2^{-n-1}$ for all n, so that $d_e(u,v) \geq \frac{1}{2} d_2(u,v)$. Using Proposition 5.3, it is easily checked that d_e is a metric, which we have just proved to be equivalent to d_2. □

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7. Completeness of $\mathcal{GD}_K$

In order to prove the completeness of $\mathcal{GD}_K(U)$, we generalize the proof of [**11**, Proposition 3.4] (based in turn on [**28**]) to the present context.

Theorem 7.1. *The space $\mathcal{GD}_K(U)$ with the sharp topology is complete.*

Proof. By Proposition 6.3 $\mathcal{GD}_K(U)$ with the sharp topology is metrizable. Hence, it suffices to consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in this topology, i.e.,

$$\forall q \in \mathbb{R}_{>0}\, \forall i \in \mathbb{N}\, \exists N \in \mathbb{N}\, \forall m, n \geq N : \ \|u_n - u_m\|_i < \mathrm{d}\varepsilon^q.$$

Setting $i = q = k \in \mathbb{N}_{>0}$, this implies the existence of a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\|u_{n_{k+1}} - u_{n_k}\|_k < \mathrm{d}\varepsilon^k$. Hence picking any representatives $(u_{n\varepsilon})$ of u_n as in Definition 3.9 we have

$$\left[\max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} \left|\partial^\alpha u_{n_{k+1},\varepsilon}(x) - \partial^\alpha u_{n_k,\varepsilon}(x)\right|\right] < \left[\varepsilon^k\right] \quad \forall k \in \mathbb{N}_{>0}.$$

By Lemma 1.1 this yields that for each $k \in \mathbb{N}_{>0}$ there exists an ε_k such that $\varepsilon_k \searrow 0$ and

$$\forall \varepsilon \in (0, \varepsilon_k) : \ \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} \left|\partial^\alpha u_{n_{k+1},\varepsilon}(x) - \partial^\alpha u_{n_k,\varepsilon}(x)\right| < \varepsilon^k. \tag{7.1}$$

Now set

$$h_{k\varepsilon} := \begin{cases} u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \text{if } \varepsilon \in (0, \varepsilon_k) \\ 0 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \text{if } \varepsilon \in [\varepsilon_k, 1) \end{cases} \tag{7.2}$$

$$u_\varepsilon := u_{n_0\varepsilon} + \sum_{k=0}^{\infty} h_{k\varepsilon} \quad \forall \varepsilon \in I.$$

Since $\varepsilon_k \searrow 0$, for all $\varepsilon \in I$ we have $\varepsilon \notin (0, \varepsilon_k)$ for all $k \geq \bar{k}$, with $\bar{k}$ sufficiently big. Therefore, $u_\varepsilon = u_{n_{\bar{k}+1},\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. In order to prove that (u_ε) defines a GSF of the type $U \to \widetilde{\mathbb{R}}$, take $[x_\varepsilon] \in U$ and $\alpha \in \mathbb{N}$. We claim that $(\partial^\alpha u_\varepsilon(x_\varepsilon)) \in \mathbb{R}_M$. Now if $p \in \mathbb{N}$ satisfies $|\alpha| \leq p$, then for any $x \in \mathbb{R}^n$ we have

$$|\partial^\alpha u_\varepsilon(x)| \leq \left|\partial^\alpha u_{n_{p+1},\varepsilon}(x)\right| + \sum_{k=p+1}^{\infty} |\partial^\alpha h_{k\varepsilon}(x)|.$$

From (7.1) and (7.2) we get that $|\partial^\alpha h_{k\varepsilon}(x)| \leq \varepsilon^k$ for all $k \geq p+1$, $x \in \mathbb{R}^n$ and all $\varepsilon \in (0, 1]$. Hence for $\varepsilon \in (0, 1]$, $|\alpha| \leq p$ and all $x \in \mathbb{R}^n$ we obtain

$$|\partial^\alpha u_\varepsilon(x)| \leq \left|\partial^\alpha u_{n_{p+1},\varepsilon}(x)\right| + \frac{\varepsilon^{p+1}}{1-\varepsilon}. \tag{7.3}$$

Inserting $x = x_\varepsilon$ and noting that $(\partial^\alpha u_{n_{p+1},\varepsilon}(x_\varepsilon)) \in \mathbb{R}_M$ proves our claim. Moreover, since all $(u_{n\varepsilon})$ satisfy Definition 3.9, we also conclude from (7.3) that for any α and any $[x_\varepsilon] \in \mathrm{ext}(K)$ we have $[\partial^\alpha u_\varepsilon(x_\varepsilon)] = 0$, and hence the GSF $[u_\varepsilon(-)]\,|_U \in \mathcal{GD}_K(U)$.

Finally, $\|u - u_{n_p}\|_i < \mathrm{d}\varepsilon^{p-1}$ for all $p \in \mathbb{N}_{>1}$ and all $i \leq p$. This yields that $(u_{n_k})_k$ tends to u in the sharp topology, and hence so does (u_n). □

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8. The space $\mathcal{GD}$ as inductive limit of $\mathcal{GD}_K$

In this section, we always assume that $U \subseteq \widetilde{\mathbb{R}}^n$ is a non-empty strongly internal set. By Proposition 4.7 and Proposition 4.8, this entails that $\mathcal{GD}(U)$ is an $\widetilde{\mathbb{R}}$-module.

In order to define a natural topology on $\mathcal{GD}(U)$ we will employ [**11**, Theorem 1.18], which we restate here for the reader's convenience.

Theorem 8.1. *Let $\mathcal{G}$ be an $\widetilde{\mathbb{R}}$-module, $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ be a family of locally convex topological $\widetilde{\mathbb{R}}$-modules and, for each $\gamma \in \Gamma$, let $i_\gamma : \mathcal{G}_\gamma \longrightarrow \mathcal{G}$ be an $\widetilde{\mathbb{R}}$-linear map. Assume that*

$$\mathcal{G} = \operatorname{span}\left(\bigcup_{\gamma\in\Gamma} i_\gamma\left(\mathcal{G}_\gamma\right)\right)$$

*and let $V \in \mathcal{V}$ if and only if $V \subseteq \mathcal{G}$, V is absolutely $\widetilde{\mathbb{R}}$-convex and $i_\gamma^{-1}(V)$ is a neighborhood of 0 in $\mathcal{G}_\gamma$ for all $\gamma \in \Gamma$. Then each $V \in \mathcal{V}$ is $\widetilde{\mathbb{R}}$-absorbent and the topology τ induced by the gauges $\{\mathcal{P}_V\}_{V\in\mathcal{V}}$ (see [**11**] and (6.3)) is the finest $\widetilde{\mathbb{R}}$-locally convex topology on $\mathcal{G}$ such that i_γ is continuous for all $\gamma \in \Gamma$. Endowed with this topology, $\mathcal{G}$ is called the inductive limit (colimit) of the spaces $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ and we write $\mathcal{G} = \varinjlim \mathcal{G}_\gamma$.*

Since $\mathcal{GD}(U) = \bigcup_{\emptyset\neq K\Subset_f U} \mathcal{GD}_K(U)$, we may therefore equip it with the inductive limit topology with respect to the inclusions $\iota_K : \mathcal{GD}_K(U) \hookrightarrow \mathcal{GD}(U)$. We call the resulting $\widetilde{\mathbb{R}}$-locally convex topology the *sharp topology* on $\mathcal{GD}(U)$. Hence,

$$\mathcal{GD}(U) = \varinjlim \mathcal{GD}_K(U) \qquad (\emptyset \neq K \Subset_{\mathrm{f}} U).$$

Henceforth we will denote the sharp topology on $\mathcal{GD}_K(U)$ by $\sigma_K(U)$ (or, for short, by σ_K). Also, the inductive limit topology on $\mathcal{GD}(U)$ will be denoted by $\sigma(U)$ (or by σ). Setting

$$U^n_{\mathrm{d}\varepsilon^n}(0) := \left\{ f \in \mathcal{GD}(U) \mid \frac{\|f\|_n}{\mathrm{d}\varepsilon^n} \approx 0 \right\},$$

where $n \in \mathbb{N}_{>0}$, we obtain an $\widetilde{\mathbb{R}}$-absorbent and absolutely $\widetilde{\mathbb{R}}$-convex subset of $\mathcal{GD}(U)$ such that

$$i_K^{-1}\left(U^n_{\mathrm{d}\varepsilon^n}(0)\right) = U^n_{\mathrm{d}\varepsilon^n}(0) \cap \mathcal{GD}_K(U) = U^n_{\mathrm{d}\varepsilon^n}(0, K) \in \sigma_K.$$

Therefore, these sets generate a coarser topology than the sharp topology σ. The proof is identical to that of Proposition 6.2 (i).

From the (co)universal property of inductive limits ([**11**, Proposition 1.19]) we immediately conclude that the following proposition holds.

Proposition 8.2. *Let $\mathcal{H}$ be a locally convex topological $\widetilde{\mathbb{R}}$-module. For each non-empty $K \Subset_f U$, let $T_K : \mathcal{GD}_K(U) \longrightarrow \mathcal{H}$ be an $\widetilde{\mathbb{R}}$-linear and continuous map. Assume that $T_K(f) = T_H(f)$ if $f \in \mathcal{GD}_K(U) \cap \mathcal{GD}_H(U)$. Then there exists one and only one map $T : \mathcal{GD}(U) \longrightarrow \mathcal{H}$ which is $\widetilde{\mathbb{R}}$-linear and continuous and such that $T \circ \iota_K = T_K$ for all non-empty $K \Subset_f U$.*

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The work [**11**] includes a detailed analysis of countable inductive limits $\mathcal{G} = \varinjlim \mathcal{G}_n$ ($n \in \mathbb{N}$), where $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is increasing, $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$, and where the topology on $\mathcal{G}_n$ is that induced by $\mathcal{G}_{n+1}$. Such inductive limits are called *strict*. As in the case of classical function spaces like $\mathcal{D}(\Omega)$ for Ω open in $\mathbb{R}^n$, the importance of strict inductive limits in the theory of $\widetilde{\mathbb{R}}$-locally convex models ultimately stems from the possibility of covering every open set $\Omega \subseteq \mathbb{R}^n$ by a countable increasing family of compact sets. The key point in the structure theory of strict inductive limits as above is that a countable family $(\mathcal{G}_n)_{n\in\mathbb{N}}$ permits us to define recursively a family of neighborhoods of 0. Using the latter, one can prove that the topology on $\mathcal{G}$ induces on each $\mathcal{G}_n$ its given topology. We will show below that similar properties hold for $\mathcal{GD}(U)$. We shall see that the assumption of U being strongly internal and sharply open are essential for this task. To begin with, we prove a strengthening of Theorem 4.1.

Proposition 8.3. *Let H, $K \Subset_f U$ be non-empty sets, with $H \subseteq K$. Then $\mathcal{GD}_H(U)$ is a topological subspace of $\mathcal{GD}_K(U)$, i.e.,*

$$\sigma_K(U)|_{\mathcal{GD}_H(U)} = \sigma_H(U).$$

Proof. In this proof we will use the more precise notation $B^m_\rho(u, K)$ for balls (see Definition 5.1).

Let $V \in \sigma_K$. We claim that $V \cap \mathcal{GD}_H(U) \in \sigma_H$. For each $u \in V \cap \mathcal{GD}_H(U)$ there exist $m \in \mathbb{N}$ and $\rho \in \widetilde{\mathbb{R}}_{>0}$ such that $B^m_\rho(u, K) \subseteq V$. But $B^m_\rho(u, H) \subseteq B^m_\rho(u, K)$ because $\mathcal{GD}_H(U) \subseteq \mathcal{GD}_K(U)$ and because the norm $\|\cdot\|_m$ does not depend on H, K. Therefore, $B^m_\rho(u, H) \subseteq V \cap \mathcal{GD}_H(U)$, which proves our claim.

Conversely, if $W \in \sigma_H$, then we set

$$V := \bigcup \left\{ B^m_\rho(u, K) \mid u \in W \ , \ m \in \mathbb{N} \ , \ \rho \in \widetilde{\mathbb{R}}_{>0} \ , \ B^m_\rho(u, H) \subseteq W \right\} \in \sigma_K, \tag{8.1}$$

and we claim that $W = V \cap \mathcal{GD}_H(U)$. In fact, since $W \in \sigma_H$, for all $u \in W$ we have $B^m_\rho(u, H) \subseteq W$ for some ρ and m. By (8.1) $B^m_\rho(u, K) \subseteq V$, and so $u \in V \cap \mathcal{GD}_H(U)$. Conversely, if $u \in V \cap \mathcal{GD}_H(U)$, then $u \in B^m_\rho(v, K)$ for some $v \in W$, m, ρ, such that $B^m_\rho(v, H) \subseteq W$. So $\|u - v\|_m < \rho$ and hence $u \in B^m_\rho(v, H) \subseteq W$. □

We now show that the space $\mathcal{GD}(U)$ can be seen as a strict inductive limit of a countable increasing family of subspaces $\mathcal{GD}_K(U)$. Indeed, since U is strongly internal, we can write $U = \langle U_\varepsilon \rangle$ for some net (U_ε) of subsets of $\mathbb{R}^n$. Since U is non-empty, by [**21**, Theorem 8], fixing any $x = [x_\varepsilon] \in U$ we obtain:

$$\exists N \in \mathbb{N} \forall^0 \varepsilon : \ d(x_\varepsilon, U^c_\varepsilon) > \varepsilon^N \ , \ |x_\varepsilon| \le \varepsilon^{-N}. \tag{8.2}$$

With N as in (8.2), we define

$$K_{j\varepsilon} := \left\{ x \in \mathbb{R}^n \mid d(x, U^c_\varepsilon) \ge \varepsilon^j \ , \ |x| \le \varepsilon^{-j} \right\}$$
$$K_j := [K_{j\varepsilon}] \quad \forall j \in \mathbb{N}_{\ge N}.$$

Using this notation, the following theorem.

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Theorem 8.4. *If $N \in \mathbb{N}$ satisfies (8.2) for some $x \in U$, then*

(i) $\emptyset \neq K_j \Subset_f U$ *for all* $j \in \mathbb{N}_{\geq N}$,

(ii) $K_j \subseteq K_{j+1}$ *for all* $j \in \mathbb{N}_{\geq N}$,

(iii) $U = \bigcup_{j \geq N} K_j$,

(iv) *for every* $\emptyset \neq K \Subset_f U = \langle U_\varepsilon \rangle$ *there exists some* $j \geq N$ *such that* $K \subseteq K_j$,

(v) $\mathcal{GD}(U)$ *is the strict inductive limit of the family* $\mathcal{GD}_{K_j}(U)$, $j \geq N$:

$$\mathcal{GD}(U) = \varinjlim \mathcal{GD}_{K_j}(U) \quad (j \geq N).$$

Proof. (i), (ii): It follows immediately from the definition that each $(K_{j\varepsilon})$ is compact and that $(K_{j\varepsilon})_\varepsilon$ is sharply bounded, so $K_j \Subset_f U$. Moreover, $K_N \neq \emptyset$ by (8.2), hence $K_j \neq \emptyset$ follows by (ii),which again is immediate from the definition.

(iii) If $x = [x_\varepsilon] \in U = \langle U_\varepsilon \rangle$, then $d(x_\varepsilon, U_\varepsilon^c) > \varepsilon^{j_1}$ for ε small and for some $j_1 \in \mathbb{N}_{>0}$. Since (x_ε) is moderate, $|x_\varepsilon| \leq \varepsilon^{-j_2}$ for ε small and some $j_2 \in \mathbb{N}$. Setting $j := \max(j_1, j_2, N)$ we hence have that $x \in K_j$.

(iv) In order to prove (iv), we need the following strengthening of [**21**, Theorem 11].

Lemma 8.5. *Let* $H = [H_\varepsilon] \Subset_f V = \langle V_\varepsilon \rangle$. *Then*

$$\exists j \in \mathbb{N}\, \forall [x_\varepsilon] \in [H_\varepsilon]\, \forall^0 \varepsilon : \ d(x_\varepsilon, V_\varepsilon^c) \geq \varepsilon^j. \tag{8.3}$$

Proof. Equation (8.3) expresses that for all representatives $(x_\varepsilon) \in \mathbb{R}^n_M$, if $x_\varepsilon \in H_\varepsilon$ for ε small, then $\forall^0 \varepsilon : \ d(x_\varepsilon, V_\varepsilon^c) \geq \varepsilon^j$.

By contradiction, assume that

$$\forall j \in \mathbb{N}\, \exists (x_{j\varepsilon}) \in \mathbb{R}^n_M : \ \left(\forall^0 \varepsilon : \ x_{j\varepsilon} \in H_\varepsilon\right) \ , \ \exists (\varepsilon_{jk})_k \downarrow 0\, \forall k : \ d(x_{j\varepsilon_{jk}}, V^c_{\varepsilon_{jk}}) < \varepsilon^j_{jk}.$$

By recursively applying this condition, we get that for all $j \in \mathbb{N}$ there exists a moderate $(x_{j\varepsilon})$ and some $\varepsilon_j \in (0, 1]$ such that $x_{j\varepsilon} \in H_\varepsilon$ for $\varepsilon \leq \varepsilon_j$ and

$$d(x_{j\varepsilon_{jk}}, V^c_{\varepsilon_{jk}}) < \varepsilon^j_{jk}, \tag{8.4}$$

where $(\varepsilon_{jk})_k \downarrow 0$. Since $(\varepsilon_{jk})_k \downarrow 0$, without loss of generality we can assume to have defined recursively $(\varepsilon_j)_j$ so that $(\varepsilon_j)_j \downarrow 0$ and $\varepsilon_j > \varepsilon_{jk_j} > \varepsilon_{j+1}$ for some subsequence $(k_j)_j \uparrow +\infty$. Set $x_\varepsilon := x_{j\varepsilon} \in H_\varepsilon$ for $\varepsilon \in (\varepsilon_{j+1}, \varepsilon_j]$, so that $x_{\varepsilon_{jk_j}} = x_{j\varepsilon_{jk_j}}$ for all j. Then $(x_\varepsilon) \in \mathbb{R}^n_M$ since H is sharply bounded and $x := [x_\varepsilon] \in H \subseteq \langle V_\varepsilon \rangle$, which entails

$$\exists q \in \mathbb{R}_{>0}\, \forall^0 \varepsilon : \ d(x_\varepsilon, V_\varepsilon^c) > \varepsilon^q. \tag{8.5}$$

Therefore, for $j \in \mathbb{N}$ sufficiently big, (8.5) holds at $\varepsilon = \varepsilon_{jk_j} < 1$ and $j > q$. Thus $d(x_{j\varepsilon_{jk_j}}, V^c_{\varepsilon_{jk_j}}) = d(x_{\varepsilon_{jk_j}}, V^c_{\varepsilon_{jk_j}}) > \varepsilon^q_{jk_j} > \varepsilon^j_{jk_j}$, which contradicts (8.4). □

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Continuing the proof of (iv), if $K \Subset_{\mathrm{f}} U$ is non-empty, by applying Lemma 8.5 we obtain

$$\exists j_1 \in \mathbb{N}\, \forall [x_\varepsilon] \in K\, \forall^0 \varepsilon : \; d(x_\varepsilon, U_\varepsilon^c) \geq \varepsilon^{j_1}.$$

On the other hand, sharp boundedness of (K_ε), where $K = [K_\varepsilon]$, implies that

$$\exists j_2 \in \mathbb{N}\, \forall [x_\varepsilon] \in K\, \forall^0 \varepsilon : \; |x_\varepsilon| \leq \varepsilon^{-j_2}.$$

Therefore, for $j := \max(j_1, j_2, N)$, we get $K \subseteq K_j$, hence (iv), and thereby also $\mathcal{GD}_K(U) \subseteq \mathcal{GD}_{K_j}(U)$ (using Theorem 4.1). It follows that $\mathcal{GD}(U) \subseteq \bigcup_{j \geq N} \mathcal{GD}_{K_j}(U)$. The converse inclusion follows directly from (i).

It remains to prove that the topology σ on $\mathcal{GD}(U)$ coincides with the inductive $\widetilde{\mathbb{R}}$-locally convex topology generated by $\big(\mathcal{GD}_{K_j}(U)\big)_{j \geq N}$. Let us denote the latter topology by σ'. We have $\sigma \subseteq \sigma'$ by definition of the inductive topology and by Thm. 8.4 (i). To see that, conversely, $\sigma' \subseteq \sigma$ we show that for every $K \Subset_{\mathrm{f}} U$ the inclusion $(\mathcal{GD}_K(U), \sigma_K) \hookrightarrow (\mathcal{GD}(U), \sigma')$ is continuous (see Theorem 8.1). Now given any $K \Subset_{\mathrm{f}} U$, by what we have proved above there exists some $j \geq N$ with $K \subseteq K_j$. But then Proposition 8.3 implies the continuity of

$$(\mathcal{GD}_K(U), \sigma_K) \hookrightarrow (\mathcal{GD}_{K_j}(U), \sigma_{K_j}) \hookrightarrow (\mathcal{GD}(U), \sigma')$$

and thereby our claim. □

As a consequence of this result, we can now prove a series of corollaries by applying the general theorems of [**11**] concerning countable strict inductive limits.

Corollary 8.6. *If $\emptyset \neq K \Subset_{\mathrm{f}} U = \langle U_\varepsilon \rangle$, then $\mathcal{GD}_K(U)$ is a topological subspace of $\mathcal{GD}(U)$, i.e., $\sigma(U)|_{\mathcal{GD}_K(U)} = \sigma_K(U)$.*

Proof. According to Theorem 8.4 (iv) we may pick $j \geq N$ such that $K \subseteq K_j$. By [**11**, Proposition 1.21], $\mathcal{GD}_{K_j}(U)$ carries the trace topology of $\mathcal{GD}(U)$. Since, in turn, $\mathcal{GD}_K(U)$ is a topological subspace of $\mathcal{GD}_{K_j}(U)$ by Proposition 8.3, the claim follows. □

Corollary 8.7. *$\mathcal{GD}(U)$ is separated.*

Proof. This follows from Theorem 5.4 (v) and [**11**, Corollary 1.24]. □

Lemma 8.8. *If $\emptyset \neq H$, $K \Subset_{\mathrm{f}} U$ and $H \subseteq K$, then $\mathcal{GD}_H(U)$ is closed in $\mathcal{GD}_K(U)$.*

Proof. This is immediate from Proposition 8.3 and Theorem 7.1. □

Corollary 8.9. *If $\emptyset \neq K \Subset_{\mathrm{f}} U$, then $\mathcal{GD}_K(U)$ is closed in $\mathcal{GD}(U)$.*

Proof. This follows from Theorem 7.1, Corollary 8.7, and Corollary 8.6. □

Corollary 8.10. *Let $B \subseteq \mathcal{GD}(U)$, then B is bounded in $\mathcal{GD}(U)$ if and only if there exists a non-empty $K \Subset_{\mathrm{f}} U$ such that B is bounded in $\mathcal{GD}_K(U)$.*

Proof. $\Rightarrow$: This follows from [**11**, Theorem 1.26] and Lemma 8.8.

$\Leftarrow$: If B is bounded in $\mathcal{GD}_K(U)$, then [**11**, Lemma 1.27] yields

$$\forall (u_n)_n \in B^{\mathbb{N}}\, \forall (\lambda_n)_n \in \widetilde{\mathbb{R}}^{\mathbb{N}} :\ \lambda_n \to 0 \text{ in } \widetilde{\mathbb{R}} \quad \Longrightarrow \quad \lambda_n u_n \to 0 \text{ in } \mathcal{GD}_K(U). \tag{8.6}$$

Pick $j \geq N$ such that $K \subseteq K_j$. Since generalized norms $\|\cdot\|_m$ do not depend on K, K_j, condition (8.6) holds also in $\mathcal{GD}_{K_j}(U) \supseteq \mathcal{GD}_K(U)$. Therefore, from [**11**, Lemma 1.27] we get that B is bounded in $\mathcal{GD}_{K_j}(U)$ and [**11**, Theorem 1.26] yields that B is bounded in $\mathcal{GD}(U)$. $\square$

A similar proof applies to this corollary, which is a consequence of [**11**, Corollary 1.29]:

Corollary 8.11. *Let $(u_n)_n \in \mathcal{GD}(U)^{\mathbb{N}}$, then $u_n \to 0$ in $\mathcal{GD}(U)$ if and only if there exists a non-empty $K \Subset_f U$ such that $u_n \in \mathcal{GD}_K(U)$ and $u_n \to 0$ in $\mathcal{GD}_K(U)$.*

Finally, from [**11**, Theorem 1.32], Lemma 8.8 and Theorem 7.1 we obtain the following corollary.

Corollary 8.12. *$\mathcal{GD}(U)$ with the sharp topology is complete.*

Using Lemma 8.5, we can also show that any compactly supported generalized smooth function $f \in \mathcal{GD}_K(U,Y)$ on a sharply open set $U \subseteq \widetilde{\mathbb{R}}^n$ is defined by a net (u_ε) of smooth functions that are compactly supported in an arbitrarily small extension $\left[K_\varepsilon + \overline{B^{\mathrm{E}}_{\varepsilon^a}(0)}\right]$ of $K = [K_\varepsilon]$. We recall that in this section we are assuming that U is a strongly internal set.

Theorem 8.13. *Let $\emptyset \neq K \Subset_f U$. Let $Y \subseteq \widetilde{\mathbb{R}}^d$, $f \in \mathcal{GD}_K(U,Y)$ and $K = [K_\varepsilon] \Subset_f \widetilde{\mathbb{R}}^n$. Let $j \in \mathbb{N}$ be as in (8.3) and let $a \in \mathbb{R}$ such that $a \geq j$. Then there exist nets (u_ε), (H_ε) such that:*

(i) *$[H_\varepsilon] \Subset_f U$,*

(ii) *(u_ε) defines f and $u_\varepsilon \in \mathcal{D}_{H_\varepsilon}(\mathbb{R}^n, \mathbb{R}^d)$ for all ε,*

(iii) *$H_\varepsilon \subseteq K_\varepsilon + \overline{B^{\mathrm{E}}_{\varepsilon^a}(0)}$ for all ε.*

Proof. Let $U = \langle U_\varepsilon \rangle$, where each $U_\varepsilon \subseteq \mathbb{R}^n$ is an open set (see [**18**, Corollary 9]), and let (v_ε) satisfy Definition 3.9 for f and $K = [K_\varepsilon]$. By [**18**, Theorem 11], we can assume that $K_\varepsilon \subseteq U_\varepsilon$ for all ε. Let $L_\varepsilon := K_\varepsilon + \overline{B^{\mathrm{E}}_{\varepsilon^a/2}(0)}$, and denote by χ_{L_ε} the characteristic function of L_ε. Let $\psi \in \mathcal{D}(B^{\mathrm{E}}_1(0))$ have unit integral and set $\psi_\varepsilon := (\varepsilon^a/2)^{-n}\psi(2x/\varepsilon^a)$. Then $(\psi_\varepsilon) \in \mathcal{E}^s_M(\mathbb{R}^n)$, and $(\chi_{L_\varepsilon} * \psi_{\varepsilon^a})|_{K_\varepsilon} = 1$. Set $u_\varepsilon := (\chi_{L_\varepsilon} * \psi_\varepsilon) \cdot v_\varepsilon$. Then $(\chi_{L_\varepsilon} * \psi_\varepsilon)$ defines a GSF of the type $\widetilde{\mathbb{R}}^n \longrightarrow \widetilde{\mathbb{R}}$ and hence (u_ε) defines a GSF of the type $U \longrightarrow \widetilde{\mathbb{R}}^d$. Moreover,

$$H_\varepsilon := \operatorname{supp}(u_\varepsilon) \subseteq \operatorname{supp}(\chi_{L_\varepsilon} * \psi_\varepsilon) \subseteq K_\varepsilon + \overline{B^{\mathrm{E}}_{\varepsilon^a}(0)}.$$

Since $a \geq j$, we have $[H_\varepsilon] \subseteq \langle U_\varepsilon \rangle = U$.

It remains to prove that $f(x) = [u_\varepsilon(x_\varepsilon)]$ for all $x = [x_\varepsilon] \in U$. Suppose this was not the case. Then there would exist some $y = [y_\varepsilon] \in U$, some $b > 0$ and a sequence $\varepsilon_k \searrow 0$ such

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that

$$|u_{\varepsilon_k}(y_{\varepsilon_k}) - v_{\varepsilon_k}(y_{\varepsilon_k})| \geq \varepsilon_k^b \tag{8.7}$$

for all $k \in \mathbb{N}$. By Lemma 3.11, we may without loss of generality assume that either $y_\varepsilon \in K_\varepsilon$ for all ε or that $y \in \mathrm{ext}(K)$. In the first case, $v_\varepsilon(y_\varepsilon) = u_\varepsilon(y_\varepsilon)$ for all ε, contradicting (8.7). In the second case,

$$|v_\varepsilon(y_\varepsilon) - u_\varepsilon(y_\varepsilon)| \leq 2|v_\varepsilon(y_\varepsilon)|.$$

Since $(v_\varepsilon(y_\varepsilon)))$ is negligible, we again arrive at a contradiction to (8.7). □

Assume that we have an operator $I : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ with the property that if (u_ε) and (v_ε) define $f \in \mathcal{GD}(U)$, where u_ε, $v_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$, then $[I(u_\varepsilon)] = [I(v_\varepsilon)] \in \widetilde{\mathbb{R}}$. Then Theorem 8.13 permits to extend I to the whole of $\mathcal{GD}(U)$.

Using this result, we can now prove the extension of Theorem 3.16 (ii) to arbitrary codomains $Y \subseteq \widetilde{\mathbb{R}}^d$.

Theorem 8.14. *Let $\emptyset \neq K \Subset_f U$, $Y \subseteq \widetilde{\mathbb{R}}^d$ and $f \in \mathcal{GD}_K(U,Y)$. Then $\exists! \bar{f} \in \mathcal{GD}^{\mathrm{g}}(K,Y) : \bar{f}|_K = f|_K$.*

Proof. We only have to prove that $\bar{f}(x) \in Y$ for all $x \in \widetilde{\mathbb{R}}^n$. Let (u_ε) and (H_ε) as in Theorem 8.13. By Theorem 3.14, we have that (u_ε) also defines $\bar{f}$. For each ε pick any point $h_\varepsilon \in \partial H_\varepsilon$ and set $\bar{x}_\varepsilon := x_\varepsilon$ if $x_\varepsilon \in H_\varepsilon$, and $\bar{x}_\varepsilon := h_\varepsilon$ otherwise. Therefore $\bar{x} := [\bar{x}_\varepsilon] \in [H_\varepsilon] \subseteq U$. Then, if $x_\varepsilon \notin H_\varepsilon$, $|u_\varepsilon(x_\varepsilon) - u_\varepsilon(\bar{x}_\varepsilon)| = |u_\varepsilon(x_\varepsilon) - u_\varepsilon(h_\varepsilon)| = 0$ because $u_\varepsilon \in \mathcal{D}_{H_\varepsilon}(\mathbb{R}^n, \mathbb{R}^d)$. Thus $\bar{f}(x) = f(\bar{x}) \in Y$. □

9. Conclusions and further developments

The notion of functionally compact set we introduced in the present work permits to show that compactly supported GSFs are close analogues of classical compactly supported smooth functions. In particular, their functional analytic properties parallel those of the test function space of distribution theory. At the same time, for suitable K, the space $\mathcal{GD}_K(\widetilde{\mathbb{R}}^n)$ contains extensions to all CGF $\mathcal{G}^s(\Omega)$ and hence also all Schwartz distributions.

The theory developed here opens the door to addressing several central topics in the theory of nonlinear generalized functions from a new angle. As indicated after Theorem 8.13, a direct generalization of the integral of compactly supported functions to compactly supported GSFs is feasible. An immediate application of this lies in a theory of integration for GSFs that we hope will allow to harmonize the Schwartz view of generalized functions as functionals with that prevalent in Colombeau's theory of considering generalized functions as pointwise maps. Our approach will take inspiration from Garetto's very fruitful duality theory of locally convex $\widetilde{\mathbb{C}}$-modules [**12**, **13**, **15**].

A further natural development of the present article goes in the direction of a generalization to suitable types of asymptotic gauges (see [**18**, **19**]) and hence to the full and the diffeomorphism invariant Colombeau algebras.

Moreover, one can ask whether $\widetilde{\mathbb{R}}$-valued generalized norms in $\mathcal{GD}_K(U)$ permit us to generalize results from classical analysis, like a Picard-Lindelöf theorem for ordinary differential equations with a GSF right-hand side, or a Hahn-Banach theorem for functionals $I : \mathcal{GD}_K(U) \longrightarrow \widetilde{\mathbb{R}}$ defined by diffeologically smooth functionals (see [**20**]) of the

type $I_\varepsilon : \mathcal{D}_{K_\varepsilon}(U_\varepsilon) \longrightarrow \mathbb{R}$, analogously to the way a GSF is defined by a net of smooth functions.

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