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Trudinger's inequalities for Riesz potentials in Morrey spaces of double phase functionals on half spaces

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Abstract. Our aim in this paper is to establish Trudinger's exponential integrability for Riesz potentials in weighted Morrey spaces on the half space. As an application, we obtain Trudinger's inequality for Riesz potentials in the framework of double phase functionals.

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and $B(x,r)$ denote the open ball in $\mathbb{R}^n$ centered at x of radius $r > 0$. We consider the Riesz potential of order α on the half space $\mathbb{H} = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 : x_n > 0\}$ defined by

$$I_{\mathbb{H},\alpha} f(x) = \int_{B(x,x_n)} |x-y|^{\alpha-n} f(y)\,dy$$

for $0 < \alpha < n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{H})$. For $f \in L^p_{\mathrm{loc}}(\mathbb{H})$ with $1 < p < \infty$, Trudinger type inequalities for Riesz potentials of order α have been studied in the limiting case $\alpha p = n$ (see e.g., [8–11, 17, 18, 28]).

Our first aim in this paper is to establish Trudinger's exponential integrability for $I_{\mathbb{H},\alpha} f$ of functions f satisfying the weighted L^p condition

$$\int_{\mathbb{H}} |f(y) y_n^{\beta}|^p\,dy \le 1, \tag{1.1}$$

when $\alpha p = n$ and $\beta < (n+1)/(2p')$, where $1/p + 1/p' = 1$ (see Theorem 3.1). Note that $\omega(y) = |y_n|^{\beta p}$ is not always Muckenhoupt A_p weight; more precisely, ω is not Muckenhoupt A_p weight when $\beta \notin (-1/p, 1/p')$ (see Remarks 2.2 and 3.3). For this purpose, we apply the technique by Hedberg in [1] using the central Hardy–Littlewood maximal function $M_{\mathbb{H}} f$ defined by

$$M_{\mathbb{H}} f(x) = \sup_{\{r>0: B(x,r)\subset\mathbb{H}\}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy,$$

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where $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$. We show the boundedness of the maximal operator $M_{\mathbb{H}}$ (Theorem 2.5), as an improvement of [23, Theorem 2.1]. We also give a Sobolev type inequality for $I_{\mathbb{H},\alpha}f$ of functions f satisfying (1.1) when $\alpha p < n$ and $\beta < (n+1)/(2p')$ (Theorem 3.2). Compare Theorem 3.2 with [23, Theorem 2.2] which is a Sobolev type inequality for the fractional maximal function.

In the previous paper [24, Theorem 3.4], we proved a Sobolev type inequality for $I_{\mathbb{H},,\alpha}f$ of functions f satisfying the weighted Morrey condition

$$\sup_{r>0,x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} \left(|f(y)|y_n^\beta\right)^p dy \le 1, \tag{1.2}$$

when $\alpha p < \sigma < (n+1)/2$ and $\beta < (n+1)/(2p')$. We refer to [25] and [26] for Morrey spaces, which were introduced to estimate solutions of partial differential equations. See also [5, 12]. Applying our discussions in Theorem 3.1, we study Trudinger's exponential integrability for $I_{\mathbb{H},\alpha}f$ of functions f satisfying (1.2) when $\alpha p = \sigma \le n$ and $\beta < (n+1)/(2p')$ (see Theorem 4.1), as an improvement of [24, Theorem 3.4].

Further, as an application, we establish Trudinger's inequality for $I_{\mathbb{H},\alpha}f$ in the framework of double phase functionals

$$\Phi(x,t) = t^p + (b(x)t)^q, \tag{1.3}$$

where $1 < p < q$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ (see Theorems 5.1 and 5.2). Double phase functionals are studied by Baroni, Colombo, and Mingione [2, 3, 6, 7] regarding the regularity theory of differential equations. See [24, Theorem 4.1] for Sobolev's inequality of $I_{\mathbb{H},\alpha}f$ in the framework of (1.3). We refer to [16, 20, 21] for related results. Other double phase problems were studied e.g., in [4, 13–15, 19, 22, 27].

Throughout this paper, let C denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

2 Boundedness of the maximal operator in the half space

For later use, it is convenient to see the following result.

Lemma 2.1 [23, Lemma 2.3] *For $\varepsilon > (n-1)/2$ and $x \in \mathbb{H}$, set*

$$I(x) = \int_{B(x,x_n)} y_n^{\varepsilon-n}\,dy.$$

Then there exists a constant $C > 0$ such that

$$I(x) \le C x_n^\varepsilon.$$

Remark 2.2 Let $\beta > (n+1)/(2p')$. If $f(y) = |y_n|^{-a}$, then:

(1) $\int_{B(x,x_n)} |f(y)y_n^\beta|^p dy < \infty$ for $x \in \mathbb{H}$ when $(\beta - a)p + n > (n-1)/2$ and

(2) $\int_{B(x,x_n)} f(y)dy = \infty$ for $x \in \mathbb{H}$ when $-a + n \le (n-1)/2$.

If $(n+1)/2 \le a < \beta + (n+1)/(2p)$, then both (1) and (2) hold.

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For $f \in L^1_{\rm loc}(\mathbb{H})$, the central Hardy–Littlewood maximal function $M_{\mathbb{H}} f$ is defined by

$$M_{\mathbb{H}} f(x) = \sup_{\{r>0: B(x,r)\subset \mathbb{H}\}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

The mapping $f \mapsto M_{\mathbb{H}} f$ is called the fractional central maximal operator.

The usual fractional maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} |f(y)|\, dy.$$

It is known that the maximal operator $f \to Mf$ is bounded in Morrey spaces as follows:

Lemma 2.3 [12, Lemma 4] *Let $0 < \sigma \le n$ and $p > 1$. Then there exists a constant $C > 0$ such that*

$$\sup_{z\in\mathbb{R}^n, r>0} \frac{r^\sigma}{|B(z,r)|} \int_{B(z,r)} \{Mf(x)\}^p dx \le C \sup_{z\in\mathbb{R}^n, r>0} \frac{r^\sigma}{|B(z,r)|} \int_{B(z,r)} |f(y)|^p dy$$

for all measurable functions f on $\mathbb{R}^n$.

Throughout this paper, let $1 < p < \infty$ and $1/p + 1/p' = 1$. We extend Lemma 2.3 to $M_{\mathbb{H}}$. For this purpose, we prepare the following result.

Lemma 2.4 *Let $\beta < (n+1)/(2p')$. Then there exists a constant $C > 0$ such that*

$$M_{\mathbb{H}} f(x) \le C x_n^{-\beta} (Mg(x))^{1/p}$$

for all $x \in \mathbb{H}$ and measurable functions f on $\mathbb{H}$, where $g(y) = (|f(y)||y_n|^\beta)^p \chi_{\mathbb{H}}(y)$.

Proof Let f be a non-negative measurable function on $\mathbb{H}$. For $0 < r < x_n/2$,

$$\int_{B(x,r)} y_n^{-\beta p'} dy \le C x_n^{-\beta p'} r^n$$

and for $x_n/2 < r < x_n$ and $-\beta p' + n > (n-1)/2$

$$\int_{B(x,r)} y_n^{-\beta p'} dy \le C x_n^{-\beta p' + n} \le C x_n^{-\beta p'} r^n$$

by Lemma 2.1. Hence, we have by Hölder's inequality

$$\begin{aligned}\int_{B(x,r)} f(y) dy &\le \left(\int_{B(x,r)} y_n^{-\beta p'} dy\right)^{1/p'} \left(\int_{B(x,r)} (f(y) y_n^\beta)^p dy\right)^{1/p} \\ &\le C x_n^{-\beta} r^{n/p'} \left(\int_{B(x,r)} (f(y) y_n^\beta)^p dy\right)^{1/p},\end{aligned}$$

so that

$$M_{\mathbb{H}} f(x) \le C x_n^{-\beta} \sup_{0<r<x_n} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} (f(y) y_n^\beta)^p dy\right)^{1/p},$$

as required. ■

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By Lemmas 2.3 and 2.4, we obtain the following result, which is an improvement of [23, Theorem 2.1].

Theorem 2.5 *Let $\beta < (n+1)/(2p')$ and $0 < \sigma \le n$. Then there exists a constant $C > 0$ such that*

$$\sup_{r>0, z\in\mathbb{H}} \frac{r^\sigma}{|B(z,r)|} \int_{\mathbb{H}\cap B(z,r)} |x_n^\beta M_{\mathbb{H}} f(x)|^p dx \le C \sup_{r>0, z\in\mathbb{H}} \frac{r^\sigma}{|B(z,r)|} \int_{\mathbb{H}\cap B(z,r)} |f(y) y_n^\beta|^p dy$$

for all measurable functions f on $\mathbb{H}$.

Proof Let f be a measurable function on $\mathbb{H}$, and take q such that $1 < q < p$ and $\beta < (n+1)/(2q')$. Lemma 2.4 with p replaced by q and Lemma 2.3 give

$$\begin{aligned} &\sup_{r>0, z\in\mathbb{H}} \frac{r^\sigma}{|B(z,r)|} \int_{\mathbb{H}\cap B(z,r)} |x_n^\beta M_{\mathbb{H}} f(x)|^p dx \\ &\le C \sup_{r>0, z\in\mathbb{H}} \frac{r^\sigma}{|B(z,r)|} \int_{\mathbb{H}\cap B(z,r)} |Mg(x)|^{p/q} dx \\ &\le C \sup_{r>0, z\in\mathbb{H}} \frac{r^\sigma}{|B(z,r)|} \int_{\mathbb{H}\cap B(z,r)} |g(y)|^{p/q} dy \\ &= C \sup_{r>0, z\in\mathbb{H}} \frac{r^\sigma}{|B(z,r)|} \int_{\mathbb{H}\cap B(z,r)} |f(y) y_n^\beta|^p dy, \end{aligned}$$

where $g(y) = (|f(y)||y_n|^\beta)^q \chi_{\mathbb{H}}(y)$. ∎

3 Trudinger's inequality for Riesz potentials in L^p

For $0 < \alpha < n$ and $f \in L^1_{loc}(\mathbb{H})$, let us consider the Riesz potential of order α on $\mathbb{H}$ defined by

$$I_{\mathbb{H},\alpha} f(x) = \int_{B(x,x_n)} |x-y|^{\alpha-n} f(y) dy.$$

We are now ready to show Trudinger's exponential integrability for Riesz potentials on $\mathbb{H}$.

Theorem 3.1 *Let $\alpha p = n$ and $\beta < (n+1)/(2p')$. Then there exist constants $c_1 > 0$, $c_2 > 0$ such that*

$$\frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(\left| x_n^\beta I_{\mathbb{H},\alpha} f(x)/c_1 \right|^{p'} \right) dx \le c_2$$

for all $R > 0$ and measurable functions f satisfying (1.1).

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Proof Let $\alpha p = n$ and f be a non-negative measurable function on $\mathbb{H}$ satisfying (1.1). Write

$$I_{\mathbb{H},\alpha} f(x) = \int_{B(x,r)} |x-y|^{\alpha-n} f(y)dy + \int_{B(x,x_n)\setminus B(x,r)} |x-y|^{\alpha-n} f(y)dy$$
$$= T_1(x) + T_2(x).$$

First note that

$$T_1(x) \le Cr^{\alpha} M_{\mathbb{H}} f(x).$$

Next, we have by Hölder's inequality for $0 < r < x_n/2$

$$\begin{aligned} T_{21}(x) &= \int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{\alpha-n} f(y)dy \\ &\le Cx_n^{-\beta} \int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{\alpha-n} f(y) y_n^{\beta} dy \\ &\le Cx_n^{-\beta} \left(\int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{(\alpha-n)p'} dy \right)^{1/p'} \\ &\quad \times \left(\int_{B(x,x_n/2)\setminus B(x,r)} \{f(y)y_n^{\beta}\}^p dy \right)^{1/p} \\ &\le Cx_n^{-\beta} \left(\log(x_n/r)\right)^{1/p'}, \end{aligned}$$

since $\alpha p = n$. Moreover, we have by Hölder's inequality and Lemma 2.1 for $x_n/2 \le r < x_n$

$$\begin{aligned} T_{22}(x) &= \int_{B(x,x_n)\setminus B(x,r)} |x-y|^{\alpha-n} f(y)dy \\ &\le Cx_n^{\alpha-n} \int_{B(x,x_n)\setminus B(x,r)} f(y)dy \\ &\le Cx_n^{\alpha-n} \left(\int_{B(x,x_n)\setminus B(x,r)} y_n^{-\beta p'} dy \right)^{1/p'} \left(\int_{B(x,x_n)\setminus B(x,r)} \{f(y)y_n^{\beta}\}^p dy \right)^{1/p} \\ &\le Cx_n^{\alpha-n} \left(x_n^{-\beta p'+n} \right)^{1/p'} \left(\int_{B(x,x_n)\setminus B(x,r)} \{f(y)y_n^{\beta}\}^p dy \right)^{1/p} \\ &\le Cx_n^{-\beta}, \end{aligned}$$

since $-\beta p' + n > (n-1)/2$ and $\alpha p = n$. Therefore,

$$T_2(x) \le Cx_n^{-\beta} \{\log(e + (x_n/r))\}^{1/p'},$$

so that

$$I_{\mathbb{H},\alpha} f(x) \le Cr^{\alpha} M_{\mathbb{H}} f(x) + Cx_n^{-\beta} \{\log(e + (x_n/r))\}^{1/p'}$$

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for every $r > 0$. Letting $r = \{x_n^\beta M_{\mathbb{H}} f(x)\}^{-1/\alpha}$, we obtain

$$x_n^\beta I_{\mathbb{H},\alpha} f(x) \le C + C\left(\log(e + x_n\{x_n^\beta M_{\mathbb{H}} f(x)\}^{1/\alpha})\right)^{1/p'}.$$

Hence, there exists a constant $c_1 > 0$ such that

$$x_n^\beta I_{\mathbb{H},\alpha} f(x) \le c_1\left(\log(e + x_n^{\alpha p}\{x_n^\beta M_{\mathbb{H}} f(x)\}^{p})\right)^{1/p'},$$

so that

$$\exp\left[\left\{x_n^\beta I_{\mathbb{H},\alpha} f(x)/c_1\right\}^{p'}\right] \le e + x_n^n\{x_n^\beta M_{\mathbb{H}} f(x)\}^p,$$

since $\alpha p = n$. Now it follows from Theorem 2.5 that

$$\begin{aligned}
&\frac{1}{|B(0,R)|}\int_{\mathbb{H}\cap B(0,R)} \exp\left[\left\{x_n^\beta I_{\mathbb{H},\alpha} f(x)/c_1\right\}^{p'}\right] dx \\
&\le e + \frac{1}{|B(0,R)|}\int_{\mathbb{H}\cap B(0,R)} x_n^n\{x_n^\beta M_{\mathbb{H}} f(x)\}^p dx \\
&\le e + C\int_{\mathbb{H}\cap B(0,R)} \{x_n^\beta M_{\mathbb{H}} f(x)\}^p dx \\
&\le c_2
\end{aligned}$$

for $R > 0$. ∎

In the same manner as the previous proof, we obtain Sobolev's inequality in weighted L^p spaces.

Theorem 3.2 (cf. [23, Theorem 2.2]) *Let $1/p^* = 1/p - \alpha/n > 0$ and $\beta < (n+1)/(2p')$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{H}} \left|x_n^\beta I_{\mathbb{H},\alpha} f(x)\right|^{p^*} dx \le C$$

for all measurable functions f satisfying (1.1).

In fact, as in the proof of Theorem 3.1, we have by Hölder's inequality

$$T_{21}(x) \le C x_n^{-\beta} r^{\alpha - n/p}$$

for $0 < r < x_n/2$, and

$$T_{22}(x) \le C x_n^{-\beta} x_n^{\alpha - n/p}$$

for $x_n/2 \le r < x_n$. Hence,

$$I_{\mathbb{H},\alpha} f(x) \le C r^\alpha M_{\mathbb{H}} f(x) + C x_n^{-\beta} r^{\alpha - n/p}$$

for every $r > 0$. Letting $r = \{x_n^\beta M_{\mathbb{H}} f(x)\}^{-p/n}$, we obtain

$$\begin{aligned}
x_n^\beta I_{\mathbb{H},\alpha} f(x) &\le C\{x_n^\beta M_{\mathbb{H}} f(x)\}^{1-\alpha p/n} \\
&= C\{x_n^\beta M_{\mathbb{H}} f(x)\}^{p/p^*}.
\end{aligned}$$

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Now it follows from Theorem 2.5 that

$$\int_{\mathbb{H}} \left\{x_n^{\beta} I_{\mathbb{H},\alpha} f(x)\right\}^{p^*} dx \le C \int_{\mathbb{H}} \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^p dx \le C \int_{\mathbb{H}} |y_n^{\beta} f(y)|^p dy.$$

Remark 3.3 Let $\beta + \alpha - b + n/q \le a < \beta - b + n/p$ and $(n-1)/q < (n-1)/p < b$. If $f(y) = |y_n|^{-a}|y|^{-b}\chi_{B(0,1)}(y)$, where χ_E denotes the characteristic function of E, then:

(1) $\int_{\mathbb{H}\cap B(0,1)} |f(y) y_n^{\beta}|^p dy < \infty$ when $-bp + (n-1) < 0$ and $(-a+\beta)p + (-bp + n - 1) + 1 > 0$;

(2) $I_\alpha f(x) = \int_{\mathbb{H}} |x-y|^{\alpha-n} f(y) dy = \infty$ for all $x \in \mathbb{H}$ when $a \ge 1$;

(3) $I_{\mathbb{H},\alpha} f(x) \ge C x_n^{\alpha-a}|x|^{-b}$ for all $x \in \mathbb{H}\cap B(0,1)$;

(4) $\int_{\mathbb{H}\cap B(0,1)} \{x_n^{\beta} I_{\mathbb{H},\alpha} f(x)\}^q dx = \infty$ when $-bq + (n-1) < 0$ and $(\beta - a + \alpha)q + (-bq + n - 1) + 1 \le 0$.
In particular, it happens that

$$\int_{\mathbb{H}\cap B(0,1)} \{x_n^{\beta} I_{\mathbb{H},\alpha} f(x)\}^q dx = \infty,$$

when $q > p^*$.

For (3), it suffices to see that

$$\begin{aligned} I_{\mathbb{H},\alpha} f(x) &\ge \int_{B(x,x_n/2)} |x-y|^{\alpha-n} f(y) dy \\ &\ge C x_n^{-a}|x|^{-b} \int_{B(x,x_n/2)\cap B(0,1)} |x-y|^{\alpha-n} dy \\ &\ge C x_n^{-a+\alpha}|x|^{-b}. \end{aligned}$$

4 Trudinger's inequality for Riesz potentials in Morrey spaces

In this section, we are concerned with Trudinger's exponential integrability in weighted Morrey spaces.

Theorem 4.1 *Let $\alpha p = \sigma \le n$ and $\beta < (n+1)/(2p')$. Then there exist constants $c_1 > 0, c_2 > 0$ such that*

$$\frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(|x_n^{\beta} I_{\mathbb{H},\alpha} f(x)/c_1|\right) dx \le c_2$$

for all $R > 0$ and measurable functions f on $\mathbb{H}$ satisfying (1.2).

Proof Let f be a non-negative measurable function on $\mathbb{H}$ satisfying (1.2). Write

$$\begin{aligned} I_{\mathbb{H},\alpha} f(x) &= \int_{B(x,r)} |x-y|^{\alpha-n} f(y) dy + \int_{B(x,x_n)\setminus B(x,r)} |x-y|^{\alpha-n} f(y) dy \\ &= T_1(x) + T_2(x). \end{aligned}$$

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By (1.2), we have for $0 < r < x_n/2$

$$\begin{aligned} T_{21}(x) &= \int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy \\ &\le C x_n^{-\beta} \int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{\alpha-n} f(y) y_n^{\beta}\,dy \\ &\le C x_n^{-\beta} \int_r^{x_n/2} t^{\alpha-1} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} f(y) y_n^{\beta}\,dy \right) dt \\ &\le C x_n^{-\beta} \int_r^{x_n/2} t^{\alpha-1} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \{f(y) y_n^{\beta}\}^p\,dy \right)^{1/p} dt \\ &\le C x_n^{-\beta} \int_r^{x_n/2} t^{-1}\,dt \\ &\le C x_n^{-\beta} \log(x_n/r), \end{aligned}$$

since $\alpha p = \sigma$. Moreover, as in the proof of Theorem 3.1, by Hölder's inequality and Lemma 2.1, we have for $x_n/2 \le r < x_n$

$$\begin{aligned} T_{22}(x) &= \int_{B(x,x_n)\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy \\ &\le C x_n^{\alpha-n} \int_{B(x,x_n)\setminus B(x,r)} f(y)\,dy \\ &\le C x_n^{\alpha-n} \left(x_n^{-\beta p'+n} \right)^{1/p'} \left(\int_{B(x,x_n)\setminus B(x,r)} \{f(y) y_n^{\beta}\}^p\,dy \right)^{1/p} \\ &\le C x_n^{\alpha-n} \left(x_n^{-\beta p'+n} \right)^{1/p'} (x_n^{n-\sigma})^{1/p} \\ &= C x_n^{-\beta}, \end{aligned}$$

since $-\beta p' + n > (n-1)/2$ and $\alpha p = \sigma$. Therefore,

$$T_2(x) \le C x_n^{-\beta} \log(x_n/r),$$

so that

$$I_{\mathbb{H},\alpha} f(x) \le C r^{\alpha} M_{\mathbb{H}} f(x) + C x_n^{-\beta} \log(e + (x_n/r))$$

for every $r > 0$. Letting $r = \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^{-1/\alpha}$, we obtain

$$x_n^{\beta} I_{\mathbb{H},\alpha} f(x) \le C + C \log(e + x_n \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^{1/\alpha}).$$

Hence, there exists a constant $c_1 > 0$ such that

$$x_n^{\beta} I_{\mathbb{H},\alpha} f(x) \le c_1 \log(e + x_n^{\alpha p} \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^p),$$

so that

$$\exp\left(x_n^{\beta} I_{\mathbb{H},\alpha} f(x)/c_1 \right) \le e + x_n^{\alpha p} \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^p.$$

Hence, in view of Theorem 2.5, we obtain

$$\begin{aligned}
&\frac{1}{|B(0,R)|}\int_{\mathbb{H}\cap B(0,R)} \exp\left(x_n^{\beta} I_{\mathbb{H},\alpha} f(x)/c_1\right) dx \\
&\le e + C\frac{1}{|B(0,R)|}\int_{\mathbb{H}\cap B(0,R)} x_n^{\sigma}\{x_n^{\beta} M_{\mathbb{H}} f(x)\}^p dx \\
&\le e + C\frac{R^{\sigma}}{|B(0,R)|}\int_{\mathbb{H}\cap B(0,R)} \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^p dx \\
&\le c_2
\end{aligned}$$

for $R > 0$. ■

In the same manner as the previous proof, we obtain Sobolev's inequality in weighted Morrey spaces, which is an improvement of [24, Theorem 3.4].

Theorem 4.2 *Let $1/p_\sigma = 1/p - \alpha/\sigma > 0$, $0 < \sigma \le n$ and $\beta < (n+1)/(2p')$. Then there exists a constant $C > 0$ such that*

$$\frac{r^{\sigma}}{|B(z,r)|}\int_{\mathbb{H}\cap B(z,r)} \left|x_n^{\beta} I_{\mathbb{H},\alpha} f(x)\right|^{p_\sigma} dx \le C$$

for all $z \in \mathbb{H}$, $r > 0$ and measurable functions f on $\mathbb{H}$ satisfying (1.2).

In fact, as in the proof of Theorem 4.1, we have by Hölder's inequality

$$T_{21}(x) \le C x_n^{-\beta} r^{\alpha-\sigma/p}$$

for $0 < r < x_n/2$, and

$$T_{22}(x) \le C x_n^{-\beta} x_n^{\alpha-\sigma/p}$$

for $x_n/2 \le r < x_n$. Hence,

$$I_{\mathbb{H},\alpha} f(x) \le C r^{\alpha} M_{\mathbb{H}} f(x) + C x_n^{-\beta} r^{\alpha-\sigma/p}$$

for every $r > 0$. Letting $r = \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^{-p/\sigma}$, we obtain

$$\begin{aligned}
x_n^{\beta} I_{\mathbb{H},\alpha} f(x) &\le C\{x_n^{\beta} M_{\mathbb{H}} f(x)\}^{1-\alpha p/\sigma} \\
&= C\{x_n^{\beta} M_{\mathbb{H}} f(x)\}^{p/p_\sigma}.
\end{aligned}$$

Now it follows from Theorem 2.5 that

$$\begin{aligned}
\frac{r^{\sigma}}{|B(z,r)|}\int_{\mathbb{H}\cap B(z,r)} \left\{x_n^{\beta} I_{\mathbb{H},\alpha} f(x)\right\}^{p_\sigma} dx &\le C\frac{r^{\sigma}}{|B(z,r)|}\int_{\mathbb{H}\cap B(z,r)} \{x_n^{\beta} M_{\mathbb{H}} f(x)\}^p dx \\
&\le C\frac{r^{\sigma}}{|B(z,r)|}\int_{\mathbb{H}\cap B(z,r)} |y_n^{\beta} f(y)|^p dy
\end{aligned}$$

for all $z \in \mathbb{H}$ and $r > 0$.

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5 Double phase functionals

In this section, we consider the double phase functional

$$\Phi(x,t) = t^p + (b(x)t)^q,$$

where $1 < p < q$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ [16].

We obtain Trudinger's inequality for $I_{\mathbb{H},\alpha} f$ in weighted Morrey spaces of the double phase functional $\Phi(x,t)$ using Theorem 4.1.

Theorem 5.1 *Let* $0 < \sigma \le n$, $1/q = 1/p - \theta/\sigma$, $1/p_\sigma = 1/p - \alpha/\sigma > 0$ *and* $1/q_\sigma = 1/q - \alpha/\sigma = 0$. *Suppose* $\beta < (n+1)/(2p')$. *Then there exist constants* $c_1 > 0, c_2 > 0$ *such that*

$$\frac{R^\sigma}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \left| x_n^\beta I_{\mathbb{H},\alpha} f(x) \right|^{p_\sigma} dx + \frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(|x_n^\beta b(x) I_{\mathbb{H},\alpha} f(x)/c_1| \right) dx \le c_2$$

for all $R > 0$ *and measurable functions* f *on* $\mathbb{H}$ *satisfying*

$$\sup_{x\in\mathbb{H}, r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} \Phi\left(y, |f(y)| y_n^\beta\right) dy \le 1. \tag{5.1}$$

Proof Let f be a non-negative measurable function on $\mathbb{H}$ satisfying (5.1). First, we see from Theorem 4.2 that

$$\sup_{r>0: x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (z_n^\beta I_{\mathbb{H},\alpha} f(z))^{p_\sigma} dz \le C,$$

since $\alpha p < \sigma$.

Note that

$$\begin{aligned} b(x) I_{\mathbb{H},\alpha} f(x) &= \int_{B(x,x_n)} \{b(x) - b(y)\} |x-y|^{\alpha-n} f(y) dy + \int_{B(x,x_n)} b(y) |x-y|^{\alpha-n} f(y) dy \\ &\le C \int_{B(x,x_n)} |x-y|^{\alpha+\theta-n} f(y) dy + \int_{B(x,x_n)} |x-y|^{\alpha-n} b(y) f(y) dy \\ &= C I_{\mathbb{H},\alpha+\theta} f(x) + I_{\mathbb{H},\alpha}[bf](x), \end{aligned}$$

when $x \in \mathbb{H}$. We find by Theorem 4.1

$$\frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(|x_n^\beta I_{\mathbb{H},\alpha+\theta} f(x)/c_1| \right) dx \le c_2$$

and

$$\frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(|x_n^\beta I_{\mathbb{H},\alpha}[bf](x)/c_1| \right) dx \le c_2$$

for all $R > 0$ since $1/q_\sigma = 1/q - \alpha/\sigma = 1/p - (\alpha+\theta)/\sigma = 0$. Thus, we complete the proof. ■

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We obtain the following theorem using Theorem 3.1.

Theorem 5.2 *Let* $1/q = 1/p - \theta/n$, $\alpha p < n = \alpha q$ *and* $\beta < (n+1)/(2p')$. *Then there exist constants* $c_1 > 0, c_2 > 0$ *such that*

$$\int_{\mathbb{H}} \left| x_n^{\beta} I_{\mathbb{H},\alpha} f(x) \right|^{p^*} dx + \frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(\left| x_n^{\beta} b(x) I_{\mathbb{H},\alpha} f(x)/c_1 \right|^{q'} \right) dx \le c_2$$

for all $R > 0$ *and measurable functions* f *satisfying*

$$\int_{\mathbb{H}} \Phi\left(y, |f(y)| y_n^{\beta}\right) dy \le 1. \tag{5.2}$$

Proof Let f be a non-negative measurable function on $\mathbb{H}$ satisfying (5.2). Then Theorem 3.2 gives

$$\int_{\mathbb{H}} \left| x_n^{\beta} I_{\mathbb{H},\alpha} f(x) \right|^{p^*} dx \le C,$$

since $\alpha p < n$. Further, Theorem 3.1 gives

$$\frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(\left| x_n^{\beta} I_{\mathbb{H},\alpha+\theta} f(x)/c_1 \right|^{p'} \right) dx \le c_2$$

and

$$\frac{1}{|B(0,R)|} \int_{\mathbb{H}\cap B(0,R)} \exp\left(\left| x_n^{\beta} I_{\mathbb{H},\alpha} [bf](x)/c_1 \right|^{q'} \right) dx \le c_2$$

for all $R > 0$ since $(\alpha + \theta)p = n = \alpha q$ and $1/p' < 1/q'$. Thus, we obtain the result. ∎

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