



# Torsion in the space of commuting elements in a Lie group

Daisuke Kishimoto  and Masahiro Takeda

*Abstract.* Let  $G$  be a compact connected Lie group, and let  $\text{Hom}(\mathbb{Z}^m, G)$  be the space of pairwise commuting  $m$ -tuples in  $G$ . We study the problem of which primes  $p$   $\text{Hom}(\mathbb{Z}^m, G)_1$ , the connected component of  $\text{Hom}(\mathbb{Z}^m, G)$  containing the element  $(1, \dots, 1)$ , has  $p$ -torsion in homology. We will prove that  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$  has  $p$ -torsion in homology if and only if  $p$  divides the order of the Weyl group of  $G$  for  $G = SU(n)$  and some exceptional groups. We will also compute the top homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  and show that  $\text{Hom}(\mathbb{Z}^m, G)_1$  always has 2-torsion in homology whenever  $G$  is simply-connected and simple. Our computation is based on a new homotopy decomposition of  $\text{Hom}(\mathbb{Z}^m, G)_1$ , which is of independent interest and enables us to connect torsion in homology to the combinatorics of the Weyl group.

## 1 Introduction

Let  $\pi$  be a discrete group, and let  $G$  be a compact connected Lie group. Let  $\text{Hom}(\pi, G)$  denote the space of homomorphisms from  $\pi$  to  $G$ , having the induced topology of the space of continuous maps from  $\pi$  to  $G$ . In this paper, we study torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$ , the connected component of  $\text{Hom}(\mathbb{Z}^m, G)$  containing the trivial homomorphism.

The space  $\text{Hom}(\pi, G)$  has connections to diverse contexts of mathematics and physics [10, 15, 20, 27–29], and the topology of  $\text{Hom}(\pi, G)$  has been intensely studied in recent years, especially when  $\pi$  is a free abelian group. The space  $\text{Hom}(\mathbb{Z}^m, G)$  is identified with the space of commuting  $m$ -tuples in  $G$ , so that it is often called the *space of commuting elements* (see [1–3, 5, 6, 8, 13, 16, 21, 24–26] and the references therein for the topology of  $\text{Hom}(\mathbb{Z}^m, G)$ ). In particular, Baird [6] described the cohomology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  over a field of characteristic not dividing the order of the Weyl group of  $G$  or zero as a certain ring of invariants of the Weyl group. Based on this result, Ramras and Stafa [24] gave a formula for the Poincaré series of  $\text{Hom}(\mathbb{Z}^m, G)_1$ . We start with recalling this formula. Let  $W$  denote the Weyl group of  $G$ , and let  $\mathbb{F}$  be a field of characteristic not dividing the order of  $W$  or zero. Then Ramras and Stafa [24]

---

Received by the editors August 7, 2022; accepted May 8, 2023.

Published online on Cambridge Core May 22, 2023.

The first author was supported by JSPS KAKENHI (Grant Nos. JP17K05248 and JP19K03473), and the second author was supported by JSPS KAKENHI (Grant No. JP21J10117).

AMS subject classification: 57S05, 55P65.

Keywords: Space of commuting elements, Lie group, Weyl group, homotopy colimit, Bousfield–Kan spectral sequence, extended Dynkin diagram.



proved that the Poincaré series of the cohomology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  over  $\mathbb{F}$  is given by

$$\frac{\prod_{i=1}^r (1 - t^{2d_i})}{|W|} \sum_{w \in W} \frac{\det(1 + tw)^m}{\det(1 - t^2w)},$$

where  $d_1, \dots, d_r$  are the characteristic degrees of  $W$ , that is, the rational cohomology of  $G$  is an exterior algebra generated by elements of degrees  $2d_1 - 1, \dots, 2d_r - 1$ . A more explicit formula for the Poincaré series was obtained by the authors [21], and a minimal generating set of the cohomology over  $\mathbb{F}$  was also obtained there. An explicit description of the cohomology of  $\text{Hom}(\mathbb{Z}^2, G)_1$  over  $\mathbb{F}$  for  $G$  of rank two was obtained by the second author [26]. Notice that the Poincaré series is independent of the ground field  $\mathbb{F}$  as long as its characteristic does not divide the order of  $W$  or is zero. Then we immediately get the nonexistence of torsion in homology.

**Lemma 1.1** *The homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  has  $p$ -torsion in homology only when  $p$  divides the order of  $W$ .*

On the other hand, as for the existence of torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$ , there are only a few results, the proofs of which do not extend to more general cases. Adem and Cohen [1] proved a stable splitting of  $\text{Hom}(\mathbb{Z}^m, G)$ , and Baird, Jeffrey, and Selick [5] and Crabb [13] described the splitting summands for  $G = SU(2)$  explicitly. As a result, we can conclude that  $\text{Hom}(\mathbb{Z}^m, SU(2))_1$  has 2-torsion in homology for  $m \geq 2$ . Recently, Adem, Gómez, and Gritschacher [3] computed the second homology group of  $\text{Hom}(\mathbb{Z}^m, G)_1$ , and so by combining with the result on the fundamental group by Gómez, Pettet, and Souto [16],  $\text{Hom}(\mathbb{Z}^m, Sp(n))_1$  has 2-torsion in homology for  $m \geq 3$ . These are all known torsion in homology so far.

## 1.1 Results

By Lemma 1.1, we must know the order of the Weyl group of a Lie group. Then we give a table of the order of the Weyl groups of compact simply-connected simple Lie groups.

Now, we state our results.

**Theorem 1.2** *The homology of  $\text{Hom}(\mathbb{Z}^m, SU(n))_1$  for  $m \geq 2$  has  $p$ -torsion if and only if  $p \leq n$ .*

Since the Weyl group of  $SU(n)$  is of order  $n!$ , it follows from Lemma 1.1 that  $\text{Hom}(\mathbb{Z}^m, SU(n))_1$  has  $p$ -torsion in homology for all possible primes  $p$ . We will also prove a similar result for some exceptional groups.

**Theorem 1.3** *Let  $G = G_2, F_4, E_6$ . Then  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$  has  $p$ -torsion in homology if and only if  $p$  divides the order of the Weyl group of  $G$ .*

Then for  $G = G_2, F_4, E_6$ ,  $\text{Hom}(\mathbb{Z}^m, G)_1$  with  $m \geq 2$  has all possible torsion in homology. So the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  with  $m \geq 2$  for  $G = SU(n), G_2, F_4, E_6$  are quite complicated. We will also show the existence of some torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  for other Lie groups  $G$ , though incomplete (see Section 7 and Corollary 1.5 for details). Our next result is on the top homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  (see [21] for the top rational homology).

**Theorem 1.4** Let  $G$  be a compact simply-connected simple Lie group of rank  $n$ , and let

$$t = \begin{cases} \dim G + n(m - 1) - 1, & m \text{ is even,} \\ \dim G + n(m - 1), & m \text{ is odd.} \end{cases}$$

Then the top homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  is given by

$$H_t(\text{Hom}(\mathbb{Z}^m, G)_1) \cong \begin{cases} \mathbb{Z}/2, & m \text{ is even,} \\ \mathbb{Z}, & m \text{ is odd.} \end{cases}$$

Since  $\text{Hom}(\mathbb{Z}^2, G)_1$  is a retract of  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$ , we immediately obtain the following corollary.

**Corollary 1.5** Let  $G$  be a compact simply-connected simple Lie group. Then  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$  has 2-torsion in homology.

Let  $\pi$  be a finitely generated nilpotent group whose abelianization is of rank  $m$ . We can extend our results to  $\text{Hom}(\pi, G)_1$  as follows. Let  $G(\mathbb{C})$  be a complexification of  $G$ . Then Bergeron [7] proved that  $\text{Hom}(\pi, G)$  is a deformation retract of  $\text{Hom}(\pi, G(\mathbb{C}))$ . Moreover, Bergeron and Silberman [8] proved that there is a homotopy equivalence  $\text{Hom}(\pi, G(\mathbb{C}))_1 \simeq \text{Hom}(\mathbb{Z}^m, G(\mathbb{C}))_1$ . Then we get a homotopy equivalence

$$\text{Hom}(\pi, G)_1 \simeq \text{Hom}(\mathbb{Z}^m, G)_1,$$

and so all the results above also hold for  $\text{Hom}(\pi, G)_1$ .

### 1.2 Summary of computation

We compute the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  in three steps: the first step is to give a new homotopy decomposition of  $\text{Hom}(\mathbb{Z}^m, G)_1$ , namely, we will describe  $\text{Hom}(\mathbb{Z}^m, G)_1$  as a homotopy colimit, the second step is to extract the top line of (a variant of) the Bousfield–Kan spectral sequence for the homotopy colimit in the first step, and the third step is to encode the information of the top line extracted in the second step into the combinatorial data of the extended Dynkin diagram of  $G$ .

Let  $G$  act on  $\text{Hom}(\pi, G)$  by conjugation. Then the quotient space, denoted by  $\text{Rep}(\pi, G)$ , is called the *representation space* or the *character variety*, which has been studied in a variety of contexts [4, 14, 18]. We will show that if  $G$  is simply-connected and simple, then  $\text{Rep}(\mathbb{Z}, G)$ , the quotient of  $\text{Hom}(\mathbb{Z}, G)_1 = G$ , is naturally identified with the closure of a Weyl alcove which is an  $n$ -simplex whose facets are defined by simple roots and the highest root, where  $G$  is of rank  $n$ . We consider the composite

$$\pi: \text{Hom}(\mathbb{Z}^m, G)_1 \rightarrow \text{Hom}(\mathbb{Z}, G) \rightarrow \text{Rep}(\mathbb{Z}, G) = \Delta^n,$$

where the first map is the  $m$ th projection and the second map is the quotient map. We will see that the fiber of  $\pi$  is constant as long as the point belongs to the interior of some face of  $\Delta^n$ . Let  $\sigma_0$  denote the barycenter of a face  $\sigma$  of  $\Delta^n$ , and let  $P(\Delta^n)$  denote the face poset of  $\Delta^n$ . Then we get a functor

$$F_m: P(\Delta^n) \rightarrow \mathbf{Top}, \quad \sigma \mapsto \pi^{-1}(\sigma_0),$$

and a new homotopy decomposition in the first step is the following.

**Theorem 1.6** (Theorem 3.2) *Let  $G$  be a compact simply-connected simple Lie group. Then there is a homeomorphism*

$$\text{Hom}(\mathbb{Z}^m, G)_1 \cong \text{hocolim } F_m.$$

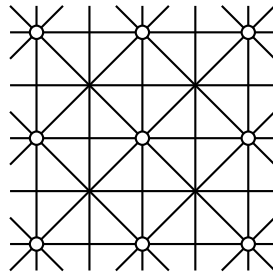
This homotopy decomposition seems to be of independent interest. We will see that if  $m$  even, then for each  $\sigma \in P(\Delta^n)$ ,  $F_m(\sigma)$  is of dimension  $\dim G + n(m - 2)$  and its top homology is isomorphic with  $\mathbb{Z}$ . Thus, we can consider the pinch map onto the top cell of  $F_m(\sigma)$ . This enables us to extract (a variant of) the Bousfield–Kan spectral sequence for  $\text{hocolim } F_m$ , which is the second step. The pinch map onto the top cell can be explicitly described in terms faces of  $\Delta^n$ . Then, since faces of  $\Delta^n$  are defined by simple roots and the highest weight, the computation of the extracted top line can be connected to the extended Dynkin diagram, which is the third step.

## 2 Triangulation of a maximal torus

Hereafter, let  $G$  denote a compact simply-connected simple Lie group such that  $\text{rank } G = n$  and  $\dim G = d$ . Let  $T$  and  $W$  denote a maximal torus and the Weyl group of  $G$ , respectively. This section constructs a  $W$ -equivariant triangulation of a maximal torus  $T$ , which will play the fundamental role in our study. Let  $\mathfrak{t}$  be the Lie algebra of  $T$ , and let  $\Phi$  be the set of roots of  $G$ . Recall that the Stiefel diagram is defined by

$$\bigcup_{\substack{\alpha \in \Phi \\ i \in \mathbb{Z}}} \alpha^{-1}(i),$$

which is a union of hyperplanes in  $\mathfrak{t}$ , where each  $\alpha^{-1}(i)$  is called a wall in the Stiefel diagram. For example, the Stiefel diagram of  $Sp(2)$  is given as follows, where integer points are indicated by white points.



Since  $G$  is simple, its Stiefel diagram is a simplicial complex such that every  $k$ -face is included in an intersection of exactly  $n - k$  walls.

**Lemma 2.1** *If two vertices  $v$  and  $v + w$  of the Stiefel diagram of  $G$  are joined by an edge, then  $w$  is a vertex of the Stiefel diagram which is joined with the vertex  $0$  by an edge.*

**Proof** Since  $v$  and  $v + w$  are joined by an edge,

$$\{v\} = \theta_1^{-1}(k_1) \cap \cdots \cap \theta_n^{-1}(k_n) \quad \text{and} \quad \{v + w\} = \theta_1^{-1}(k_1) \cap \cdots \cap \theta_{n-1}^{-1}(k_{n-1}) \cap \theta_n^{-1}(k_n + \varepsilon)$$

for some roots  $\theta_1, \dots, \theta_n$  and integers  $k_1, \dots, k_n$ , where  $\varepsilon = \pm 1$ . Then

$$\{w\} = \theta_1^{-1}(0) \cap \dots \cap \theta_{n-1}^{-1}(0) \cap \theta_n^{-1}(\varepsilon),$$

which is a vertex of the Stiefel diagram. Moreover, since  $\varepsilon = \pm 1, 0$  and  $w$  are joined by an edge on the line  $\theta_1^{-1}(0) \cap \dots \cap \theta_{n-1}^{-1}(0)$ , completing the proof. ■

Each connected component of the complement of the Stiefel diagram is called a *Weyl alcove*. Since  $G$  is simple and of rank  $n$ , the closure of each Weyl alcove is an  $n$ -simplex. Let  $\alpha_1, \dots, \alpha_n$  be simple roots, and let  $\tilde{\alpha}$  be the highest root of  $G$ . We shall consider the following closure of a Weyl alcove:

$$\Delta = \{x \in \mathfrak{t} \mid \alpha_1(x) \geq 0, \dots, \alpha_n(x) \geq 0, \tilde{\alpha}(x) \leq 1\}.$$

Let  $L$  denote the group generated by coroot shifts. Since  $G$  is simply-connected,  $L$  is identified with the integer lattice of  $\mathfrak{t}$ . The affine Weyl group of  $G$  is defined by

$$W_{\text{aff}} = W \rtimes L.$$

Then  $W_{\text{aff}}$  acts on  $\mathfrak{t}$ . Since this action fixes the Stiefel diagram,  $W_{\text{aff}}$  permutes Weyl alcoves. By [19, Theorem 4.5, Part I], we have the following.

**Lemma 2.2** *The affine Weyl group  $W_{\text{aff}}$  permutes Weyl alcoves of  $G$  simply transitively.*

Let  $\mathcal{P}$  be the union of all closures of Weyl alcoves around the origin. Then  $\mathcal{P}$  is a simplicial convex  $n$ -polytope.

**Lemma 2.3** *If  $\sigma$  is a face of  $\mathcal{P}$  such that  $\sigma + a$  is also a face of  $\mathcal{P}$  for some  $0 \neq a \in L$ , then both  $\sigma$  and  $\sigma + a$  must be faces of the boundary of  $\mathcal{P}$ .*

**Proof** Since  $W$  permutes Weyl chambers simply transitively, it follows from Lemma 2.2 that  $\sigma$  and  $\sigma + a$  must not include the vertex  $0 \in \mathcal{P}$ , completing the proof. ■

By Lemma 2.3, we can define

$$\Omega = \mathcal{P} / \sim,$$

where  $\sigma \sim \sigma + a$  if  $\sigma$  and  $\sigma + a$  are faces of the boundary of  $\mathcal{P}$  for some  $a \in L$ . Clearly, the inclusion  $\mathcal{P} \rightarrow \mathfrak{t}$  induces a homeomorphism

$$(2.1) \quad \Omega \xrightarrow{\cong} \mathfrak{t}/L.$$

Since  $G$  is simply-connected,  $L$  is the integer lattice of  $\mathfrak{t}$ , so a torus  $\mathfrak{t}/L$  coincides with a maximal torus  $T$ . Then we get the following.

**Proposition 2.4** *The homeomorphism (2.1) is a  $W$ -equivariant triangulation of  $T$ .*

**Proof** The homeomorphism (2.1) is obviously  $W$ -equivariant, so it remains to prove  $\Omega$  is a simplicial complex. It suffices to show that there is no vertex  $v$  of the boundary of  $\mathcal{P}$  such that  $v$  and  $v + a$  are joined by an edge of  $\mathcal{P}$  for  $0 \neq a \in L$ . This has been already proved in Lemma 2.1. ■

**Proposition 2.5** *The quotient space  $T/W$  is naturally identified with  $\Delta$ .*

**Proof** This follows from [19, Theorem 4.8, Part I]. ■

A maximal torus  $T$  will always be equipped with a  $W$ -equivariant triangulation (2.1). We consider objects related to the triangulation of  $T$ . Let  $\sigma$  be a face of  $T$ . Then there is a face  $\tilde{\sigma}$  of  $\mathcal{Q}$  which is mapped onto  $\sigma$ . We can associate with  $\tilde{\sigma}$  roots corresponding to walls including  $\tilde{\sigma}$ . Lifts of  $\sigma$  are related by translations in  $L$ , so that the associated roots are equal. Then we can associate roots to  $\sigma$ . Let  $\Phi(\sigma)$  denote the set of roots associated with  $\sigma$ . Clearly, we have the following.

**Lemma 2.6** *If faces  $\sigma, \tau$  of  $T$  satisfy  $\sigma < \tau$ , then*

$$\Phi(\sigma) \supset \Phi(\tau).$$

Let  $\sigma$  be a face of  $T$ . We define two groups associated with  $\sigma$ . Let  $W(\sigma)$  be a subgroup of  $W$  generated by reflections corresponding to roots in  $\Phi(\sigma)$ , and let

$$Z(\sigma) = \{x \in G \mid xy = yx \text{ for each } y \in \sigma\}.$$

We have  $W(\sigma) = 1$  and  $Z(\sigma) = T$  for  $\dim \sigma = n$ . Notice that since we may assume  $\Delta$  is a face of  $T$ , we can consider the groups  $W(\sigma)$  and  $Z(\sigma)$  for a face  $\sigma$  of  $\Delta$ .

**Lemma 2.7** *If faces  $\sigma, \tau$  of  $T$  satisfy  $\sigma < \tau$ , then*

$$W(\sigma) > W(\tau) \quad \text{and} \quad Z(\sigma) > Z(\tau).$$

**Proof** The first statement follows from Lemma 2.6. Since  $Z(\sigma)$  is the union of all maximal tori including  $\sigma$ , the second statement is true. ■

### 3 Homotopy decomposition

This section proves a new homotopy decomposition of  $\text{Hom}(\mathbb{Z}^m, G)_1$  (Theorem 1.6). Let  $\pi$  be a discrete group, and let  $\text{Rep}(\pi, G)$  be the quotient of the conjugation action of  $G$  on  $\text{Hom}(\pi, G)$  as in Section 1. Then we have

$$\text{Rep}(\mathbb{Z}, G) = \text{Hom}(\mathbb{Z}, G)/G = T/W,$$

which is identified with  $\Delta$  by Proposition 2.5. We will consider the composite

$$\pi: \text{Hom}(\mathbb{Z}^m, G)_1 \rightarrow \text{Hom}(\mathbb{Z}, G)_1 = \text{Hom}(\mathbb{Z}, G) \rightarrow \text{Rep}(\mathbb{Z}, G) = \Delta,$$

where the first map is the  $m$ th projection and the second map is the quotient map. We aim to identify the fibers of the map  $\pi$ . We consider a map

$$\phi: G/T \times T^m \rightarrow \text{Hom}(\mathbb{Z}^m, G)_1, \quad (gT, t_1, \dots, t_m) \mapsto (gt_1g^{-1}, \dots, gt_mg^{-1}).$$

It is proved in [9] that the map  $\phi$  is surjective. Let the Weyl group  $W$  act on  $G/T \times T^m$  by

$$(gT, t_1, \dots, t_m) \cdot w = (gwT, w^{-1}t_1w, \dots, w^{-1}t_mw)$$

for  $(gT, t_1, \dots, t_m) \in G/T \times T^m$  and  $w \in W$ . Then the map  $\phi$  is invariant under the action of  $W$ , and so it induces a surjective map

$$(3.1) \quad G/T \times_w T^m \rightarrow \text{Hom}(\mathbb{Z}^m, G)_1.$$

**Lemma 3.1** *If  $x, y \in \Delta$  belong to the interior of a common face, then*

$$\pi^{-1}(x) \cong \pi^{-1}(y).$$

**Proof** Suppose that  $x, y \in \Delta$  belong to the interior of a common face  $\sigma$ , and consider the adjoint action of  $G$  on  $G^{m-1} \times \mathfrak{t}$ . Then for each  $w \in W$  and  $(t_1, \dots, t_{m-1}) \in T^{m-1}$ , the isotropy subgroups of  $(t_1, \dots, t_{m-1}, wx)$  and  $(t_1, \dots, t_{m-1}, wy)$  are equal. Thus, since

$$(\phi \circ \pi)^{-1}(x) = G/T \times T^{m-1} \times W \cdot x \quad \text{and} \quad (\phi \circ \pi)^{-1}(y) = G/T \times T^{m-1} \times W \cdot y,$$

where  $W \cdot x \cong W \cdot y \cong W/W(\sigma)$ , we obtain  $\pi^{-1}(x) \cong \pi^{-1}(y)$ , as stated. ■

Let  $P(\Delta)$  denote the face poset of  $\Delta$ , and let  $\sigma_0$  denote the barycenter of a face  $\sigma \in P(\Delta)$ . For  $\sigma \in P(\Delta)$ , let  $\phi_\sigma: G/T \times T^{m-1} \rightarrow \pi^{-1}(\sigma_0)$  denote the restriction of the quotient map (3.1). Observe that for  $\sigma < \tau \in P(\Delta)$ , there is a natural map  $q_{\sigma,\tau}: \pi^{-1}(\tau_0) \rightarrow \pi^{-1}(\sigma_0)$  satisfying a commutative diagram

$$\begin{array}{ccc} G/T \times T^{m-1} & \xrightarrow{\phi_\tau} & \pi^{-1}(\tau_0) \\ \phi_\sigma \downarrow & & \downarrow q_{\sigma,\tau} \\ \pi^{-1}(\sigma_0) & \xlongequal{\quad} & \pi^{-1}(\sigma_0). \end{array}$$

Clearly, we have

$$q_{\sigma,\tau} \circ q_{\tau,\nu} = q_{\sigma,\nu}$$

for  $\sigma < \tau < \nu \in P(\Delta)$ . Let  $\iota_{\tau,\sigma}: \sigma \rightarrow \tau$  denote the inclusion for  $\sigma < \tau \in P(\Delta)$ . Then the above observation implies that

$$(3.2) \quad \text{Hom}(\mathbb{Z}^m, G)_1 = \left( \coprod_{\sigma \in P(\Delta)} \pi^{-1}(\sigma_0) \times \sigma \right) / \sim,$$

where  $(x, \iota_{\tau,\sigma}(y)) \sim (q_{\sigma,\tau}(x), y)$  for  $(x, y) \in \pi^{-1}(\tau_0) \times \sigma \subset \pi^{-1}(\tau_0) \times \tau$ . Define a functor

$$F_m: P(\Delta) \rightarrow \mathbf{Top}, \quad \sigma \mapsto \pi^{-1}(\sigma_0)$$

such that  $F_m(\tau > \sigma) = q_{\sigma,\tau}$ , where we understand a poset  $P$  as a category by assuming an inequality  $x > y \in P$  as a unique morphism  $x \rightarrow y$ . Thus, by (3.2), we obtain the following.

**Theorem 3.2** *There is a homeomorphism*

$$\text{Hom}(\mathbb{Z}^m, G)_1 \cong \text{hocolim } F_m.$$

We further look into  $F_m(\sigma)$  for  $\sigma \in P(\Delta)$ . Let  $P(X)$  denote the face poset of a regular CW complex  $X$ , and let  $\sigma$  be a face of  $\Delta$ . For  $\tau = \tau_1 \times \dots \times \tau_{m-1} \in P(T^{m-1})$ , let  $Z(\tau) = Z(\tau_1) \cap \dots \cap Z(\tau_{m-1})$ . For  $\tau < \mu \in P(T^{m-1})$ , let  $\iota_{\mu,\tau}: \tau \rightarrow \mu$  denote the inclusion, and let

$$q_{\tau,\mu}^\sigma: G/Z(\mu) \cap Z(\sigma) \rightarrow G/Z(\tau) \cap Z(\sigma)$$

be the natural projection. Then we have

$$\pi^{-1}(\sigma_0) = \left( \coprod_{\tau \in P(T^{m-1})} G/Z(\tau) \cap Z(\sigma) \times \tau \right) / \sim / W(\sigma),$$

where  $(x, \iota_{\mu, \tau}(y)) \sim (q_{\tau, \mu}^\sigma(x), y)$  for  $(x, y) \in G/Z(\mu) \cap Z(\sigma) \times \tau \subset G/Z(\mu) \cap Z(\sigma) \times \mu$  and the quotient by  $W(\sigma)$  is taken by the action of  $W$  on  $G/T \times T^{m-1}$ . Now, we define a functor

$$F_m^\sigma: P(T^{m-1}) \rightarrow \mathbf{Top}, \quad \tau \mapsto G/Z(\tau) \cap Z(\sigma),$$

where  $F_m^\sigma(\mu > \tau)$  is the projection  $q_{\tau, \mu}^\sigma$ . Then we obtain the following.

**Proposition 3.3** For  $\sigma \in P(\Delta)$ , there is a natural homeomorphism

$$F_m(\sigma) \cong (\text{hocolim } F_m^\sigma) / W(\sigma).$$

Hereafter, we let

$$q_m = d + n(m - 2).$$

Since the maximal dimension of  $F_m^\sigma(\tau) \times \tau$  for  $\tau \in P(T^{m-1})$  is  $q_m$ , we get the following.

**Corollary 3.4** For each  $\sigma \in P(\Delta)$ ,  $\dim F_m(\sigma) = q_m$ .

**Example 3.5** We examine Theorem 3.2 in the  $G = SU(2)$  case. Since  $\text{rank } SU(2) = 1$ ,  $\Delta$  is a 1-simplex. Let  $v_0, v_1$  be vertices of  $\Delta$ , and let  $e$  be an edge of  $\Delta$ . Since  $G = SU(2)$ ,  $\{v_0, v_1\}$  corresponds to the center, so we have

$$F_m(v_i) = \text{Hom}(\mathbb{Z}^{m-1}, SU(2))_1$$

for  $i = 0, 1$ . By Proposition 3.3, we also have

$$F_m(e) = SU(2)/T \times T^{m-1} = S^2 \times (S^1)^{m-1}.$$

Then Theorem 3.2 for  $G = SU(2)$  is equivalent to that there is a homotopy pushout

$$\begin{array}{ccc} S^2 \times (S^1)^{m-1} & \xrightarrow{g_m} & \text{Hom}(\mathbb{Z}^{m-1}, SU(2))_1 \\ g_m \downarrow & & \downarrow \\ \text{Hom}(\mathbb{Z}^{m-1}, SU(2))_1 & \longrightarrow & \text{Hom}(\mathbb{Z}^m, SU(2))_1. \end{array}$$

For  $m = 2$ , the map  $g_2: S^2 \times S^1 \rightarrow S^3$  is of degree 2. On the other hand, the map  $\text{Hom}(\mathbb{Z}^{m-1}, SU(2))_1 \rightarrow \text{Hom}(\mathbb{Z}^m, SU(2))_1$  has a retraction. Then the homotopy pushout above for  $m = 2$  splits after a suspension, and so we get a stable homotopy equivalence

$$(3.3) \quad \text{Hom}(\mathbb{Z}^2, SU(2))_1 \underset{S}{\cong} S^2 \vee S^3 \vee (S^3 \cup_2 e^4),$$

which was previously proved by Baird, Jeffrey, and Selick [5] and Crabb [13] in different ways.



### 4 The functor $\widehat{F}_m$

This section defines a functor  $\widehat{F}_m: P(\Delta) \rightarrow \mathbf{Top}$  which extracts the top line of (a variant of) the Bousfield–Kan spectral sequence for  $\text{hocolim } F_m$ . First, we compute some homology of  $F_m(\sigma)$ . To this end, we recall the work of Baird [6] on the cohomology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  over a field whose characteristic does not divide the order of  $W$ . Let  $K$  be a topological group acting on a space  $X$ , and let  $f: X \rightarrow Y$  be a  $K$ -equivariant map, where  $K$  acts trivially on  $Y$ . Let  $\bar{f}: X/K \rightarrow Y$  be the induced map from  $f$ . Baird [6] defined that a map  $f: X \rightarrow Y$  is an  $\mathbb{F}$ -cohomological principal  $K$ -bundle if  $f$  is a closed surjection and

$$\widetilde{H}^*(\bar{f}^{-1}(y); \mathbb{F}) = 0$$

for each  $y \in Y$ , where  $\mathbb{F}$  is a field. The main result of Baird’s work [6] is the following.

**Theorem 4.1** *Let  $K$  be a finite group, and let  $\mathbb{F}$  be a field of characteristic prime to  $|K|$ . If a map  $f: X \rightarrow Y$  is an  $\mathbb{F}$ -cohomological principal  $K$ -bundle, where  $X$  is paracompact and Hausdorff, then there is an isomorphism*

$$H^*(X; \mathbb{F})^K \cong H^*(Y; \mathbb{F}).$$

This theorem is applicable to  $\text{Hom}(\mathbb{Z}^m, G)_1$  as in [6].

**Theorem 4.2** *Let  $\mathbb{F}$  be a field of characteristic prime to  $|W|$ . Then the map (3.1) is an  $\mathbb{F}$ -cohomological principal  $W$ -bundle, so that there is an isomorphism*

$$H^*(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F}) \cong H^*(G/T \times T^m; \mathbb{F})^W.$$

We also apply Theorem 4.1 to  $F_m(\sigma)$ . The following lemma is immediate from the definition of a cohomological principal bundle.

**Lemma 4.3** *If  $f: X \rightarrow Y$  is an  $\mathbb{F}$ -cohomological principal  $K$ -bundle, then for any closed subset  $Z \subset Y$ , the natural map*

$$f^{-1}(Z) \rightarrow Z$$

*is an  $\mathbb{F}$ -cohomological principal  $K$ -bundle.*

We consider special representations of  $W$ .

- Lemma 4.4** (1) *The  $W$ -representation  $H^n(T; \mathbb{Q})$  is the sign representation.*  
 (2) *For  $n \geq 2$ , the  $W$ -representation  $H^{n-1}(T; \mathbb{Q})$  does not include the trivial representation.*  
 (3) *The  $W$ -representation  $H^{\dim G-n}(G/T; \mathbb{Q})$  is the sign representation.*

**Proof** (1) Since each reflection of  $W$  changes the orientation of  $\mathfrak{t}$  and  $H^1(T; \mathbb{R}) \cong \mathfrak{t}$  as a  $W$ -module,  $H^n(T; \mathbb{Q}) \cong \Lambda^n H^1(T; \mathbb{Q})$  is the sign representation of  $W$ .

(2) By [12, Theorem III.2.4], there is an isomorphism

$$H^*(T/W; \mathbb{Q}) \cong H^*(T; \mathbb{Q})^W.$$

Then, by Proposition 2.5,  $H^{n-1}(T; \mathbb{Q})^W = 0$  for  $n \geq 2$ , completing the proof.

(3) By Theorem 4.1, there is an isomorphism

$$H^*(G/T \times T; \mathbb{Q})^W \cong H^*(G; \mathbb{Q}),$$

because  $\text{Hom}(\mathbb{Z}, G)_1 = G$ . Then we get

$$(H^{d-n}(G/T; \mathbb{Q}) \otimes H^n(T; \mathbb{Q}))^W \cong H^d(G/T \times T; \mathbb{Q})^W \cong H^d(G; \mathbb{Q}) \cong \mathbb{Q}.$$

So since  $H^{d-n}(G/T; \mathbb{Q}) \cong H^n(T; \mathbb{Q}) \cong \mathbb{Q}$ ,  $H^{d-n}(G/T; \mathbb{Q}) \otimes H^n(T; \mathbb{Q})$  is the trivial  $W$ -representation. Thus, the statement follows from (1). ■

Now, we compute the homology of  $F_m(\sigma)$ .

**Lemma 4.5** For  $\sigma \in P(\Delta)$ , we have

$$H_{q_m}(F_m(\sigma)) \cong \begin{cases} \mathbb{Z}, & m \text{ is even or } \dim \sigma = n, \\ 0, & m \text{ is odd and } \dim \sigma < n. \end{cases}$$

**Proof** By Corollary 3.4,  $H_{q_m}(F_m(\sigma))$  is a free abelian group. Then we compute  $\dim H^{q_m}(F_m(\sigma); \mathbb{Q})$  because  $\text{rank} H_{q_m}(F_m(\sigma)) = \dim H^{q_m}(F_m(\sigma); \mathbb{Q})$ . By Theorem 4.2 and Lemma 4.3, the map

$$\phi^{-1}(\pi^{-1}(\sigma_0)) \rightarrow \pi^{-1}(\sigma_0)$$

is a  $\mathbb{Q}$ -cohomological principal  $W$ -bundle. The space  $\phi^{-1}(\pi^{-1}(\sigma_0)) = G/T \times T^{m-1} \times W/W(\sigma)$  has  $|W|/|W(\sigma)|$  connected components and permutes these components transitively such that each component is fixed by the action of  $W(\sigma)$ . Then the map

$$G/T \times T^{m-1} \rightarrow \pi^{-1}(\sigma_0) = F_m(\sigma)$$

is a  $\mathbb{Q}$ -cohomological principal  $W(\sigma)$ -bundle. Thus, by Theorem 4.1, we obtain an isomorphism

$$H^*(F_m(\sigma); \mathbb{Q}) \cong H^*(G/T \times T^{m-1}; \mathbb{Q})^{W(\sigma)}.$$

If  $\dim \sigma = n$ , then  $W(\sigma) = 1$ , implying  $\dim H^{q_m}(F_m(\sigma); \mathbb{Q}) = 1$ . Now, we assume  $\dim \sigma < n$ , or equivalently,  $W(\sigma) \neq 1$ . By Lemma 4.4,  $H^{d-n}(G/T; \mathbb{Q})$  and  $H^n(T; \mathbb{Q})$  are the sign representation of  $W$ . Then it follows from the Künneth theorem that  $H^{q_m}(G/T \times T^{m-1}; \mathbb{Q})$  is the tensor product of  $m$  copies of the sign representation of  $W$ . Thus, since  $W(\sigma) \neq 1$ , we obtain

$$H^{q_m}(G/T \times T^{m-1}; \mathbb{Q})^{W(\sigma)} \cong \begin{cases} \mathbb{Q}, & m \text{ is even,} \\ 0, & m \text{ is odd.} \end{cases}$$

Therefore, the proof is complete. ■

Now, we define a functor  $\widehat{F}_m: P(\Delta) \rightarrow \mathbf{Top}$  by

$$\widehat{F}_m(\sigma) = \begin{cases} S^{q_m}, & m \text{ is even or } \dim \sigma = n, \\ *, & m \text{ is odd and } \dim \sigma < n, \end{cases}$$

such that the map  $F_m(\sigma > \tau)$  is the constant map for  $m$  odd and a map of degree  $|W(\tau)|/|W(\sigma)|$  for  $m$  even. Since

$$(|W(\mu)|/|W(\tau)|) \cdot (|W(\tau)|/|W(\sigma)|) = |W(\mu)|/|W(\sigma)|$$

for  $\sigma > \tau > \mu \in P(\Delta)$ ,  $\widehat{F}_m$  is well defined.

Next, we define a natural transformation  $\rho: F_m \rightarrow \widehat{F}_m$ . For  $m$  odd,  $\rho$  is defined by the pinch map onto the top cell  $G/T \times T^{m-1} \rightarrow S^{q_m}$  and the constant map. Suppose  $m$  is even. For  $\sigma \in P(\Delta)$ , let  $\mathcal{Q}(\sigma)$  be the union of the boundary of  $\mathcal{P}$  and the image of all walls of the Stiefel diagram including  $\sigma$  under the projection  $\mathfrak{t} \rightarrow \mathcal{Q}$ , where  $\mathcal{Q}$  is the triangulation of  $T$  in Section 2 and  $\mathfrak{t}$  is the Lie algebra of  $T$ . Then, by (2.1),

$$\mathcal{Q}/\mathcal{Q}(\sigma) = \bigvee_{|W(\sigma)|} S^n.$$

Moreover, for  $\sigma > \tau \in P(\Delta)$ , we have  $\mathcal{Q}(\sigma) \subset \mathcal{Q}(\tau)$ , implying there is a commutative diagram

$$(4.1) \quad \begin{array}{ccc} \mathcal{Q}/\mathcal{Q}(\sigma) & \xlongequal{\quad} & \bigvee_{|W(\sigma)|} S^n \\ \downarrow & & \downarrow \bigvee_{|W(\sigma)|} \nabla \\ \mathcal{Q}/\mathcal{Q}(\tau) & \xlongequal{\quad} & \bigvee_{|W(\sigma)|} \bigvee_{|W(\tau)|/|W(\sigma)|} S^n \xlongequal{\quad} \bigvee_{|W(\tau)|} S^n, \end{array}$$

where  $\nabla: S^n \rightarrow \bigvee_{|W(\tau)|/|W(\sigma)|} S^n$  is the pinch map. On the other hand, a face  $\tau$  of  $\mathcal{Q}$  satisfies

$$Z(\tau) \cap Z(\sigma) = T$$

whenever  $\text{Int}(\tau)$  is in  $\mathcal{Q} - \mathcal{Q}(\sigma)$ , where  $Z(\tau) \cap Z(\sigma)$  always includes  $T$ . Then, by Proposition 3.3, there is a projection

$$(4.2) \quad \begin{aligned} F_m(\sigma) &\rightarrow ((G/T \times T^{m-2}) \wedge (\mathcal{Q}/\mathcal{Q}(\sigma)))/W(\sigma) \\ &= \left( (G/T \times T^{m-2}) \wedge \bigvee_{|W(\sigma)|} S^n \right) / W(\sigma). \end{aligned}$$

Since  $W(\sigma)$  permutes spheres in  $\bigvee_{|W(\sigma)|} S^n$ , we get

$$\left( (G/T \times T^{m-2}) \wedge \bigvee_{|W(\sigma)|} S^n \right) / W(\sigma) = (G/T \times T^{m-2}) \wedge S^n.$$

Then, by (4.1), the map (4.2) satisfies the commutative diagram

$$\begin{array}{ccc} F_m(\sigma) & \longrightarrow & (G/T \times T^{m-2}) \wedge S^n \\ F_m(\sigma > \tau) \downarrow & & \downarrow |W(\tau)|/|W(\sigma)| \\ F_m(\tau) & \longrightarrow & (G/T \times T^{m-2}) \wedge S^n. \end{array}$$

Thus, composing with the pinch map onto the top cell  $(G/T \times T^{m-2}) \wedge S^n \rightarrow S^{q_m}$ , we obtain a natural transformation  $\rho: F_m \rightarrow \widehat{F}_m$ .

We show properties of the natural transformation  $\rho: F_m \rightarrow \widehat{F}_m$  in homology. By the construction and Lemma 4.5, we have the following.

**Proposition 4.6** Let  $\sigma \in P(\Delta)$ . If  $m$  is even or  $\dim \sigma = n$ , then the map  $\rho_\sigma: F_m(\sigma) \rightarrow \widehat{F}_m(\sigma)$  is an isomorphism in  $H_{q_m}$ .

The following variant of the Bousfield–Kan spectral sequence for a homotopy colimit is constructed in [17] (see [11, XII 4.5] for the original Bousfield–Kan spectral sequence).

**Proposition 4.7** Let  $F: P(K) \rightarrow \mathbf{Top}$  be a functor for a simplicial complex  $K$ . Then there is a spectral sequence

$$E_{p,q}^1 = \bigoplus_{\sigma \in P_p(K)} H_q(F(\sigma)) \implies H_{p+q}(\text{hocolim } F),$$

where  $P_p(K)$  denotes the set of  $p$ -simplices of  $K$ .

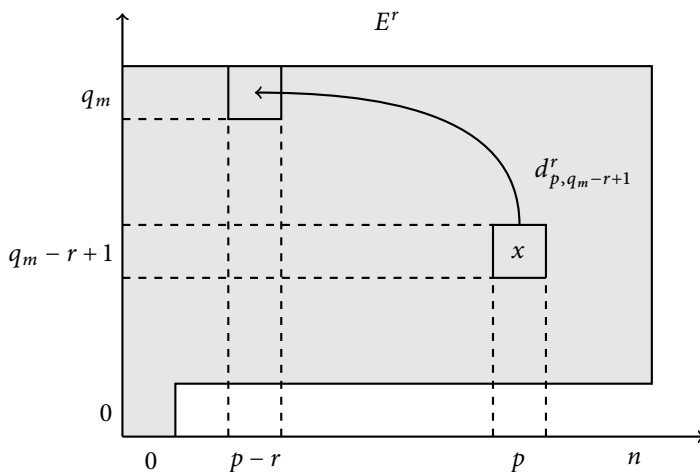
By Proposition 4.6, we get the following.

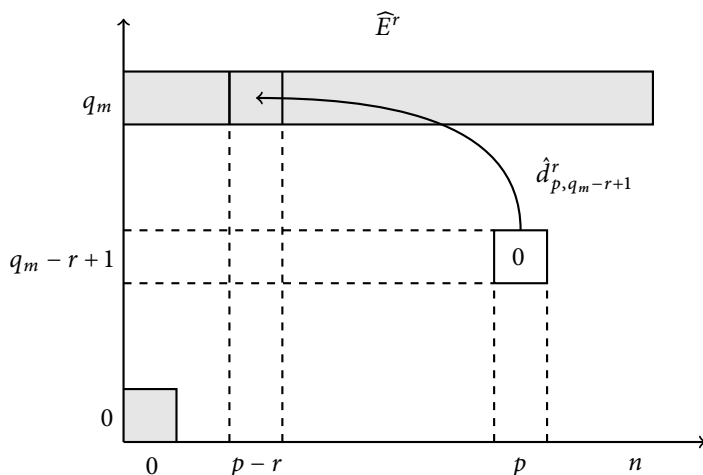
**Lemma 4.8** Let  $E^r$  and  $\widehat{E}^r$  be the spectral sequences of Proposition 4.7 for  $\text{hocolim } F_m$  and  $\text{hocolim } \widehat{F}_m$ , respectively. Then the natural transformation  $\rho: F_m \rightarrow \widehat{F}_m$  induces an isomorphism of the top lines

$$\rho_*: E_{*,q_m}^1 \xrightarrow{\cong} \widehat{E}_{*,q_m}^1.$$

**Proposition 4.9**  $H_*(\text{hocolim } \widehat{F}_m)$  is a direct summand of  $H_*(\text{Hom}(\mathbb{Z}^m, G)_1)$  for  $* \geq q_m$ .

**Proof** Let  $(E^r, d^r)$  and  $(\widehat{E}^r, \widehat{d}^r)$  denote the spectral sequences of Proposition 4.7 for  $\text{hocolim } F_m$  and  $\text{hocolim } \widehat{F}_m$ , respectively. Let  $r$  be the smallest integer  $\geq 2$  such that there is a nontrivial differential  $d_{p,q_m-r+1}^r: E_{p,q_m-r+1}^r \rightarrow E_{p-r,q_m}^r$  for some  $p \geq 0$ . Suppose that  $d_{p,q_m-r+1}^r(x) \neq 0$  for  $x \in E_{p,q_m-r+1}^r$ . Then  $E^r$  and  $\widehat{E}^r$  are illustrated below, where possibly nontrivial parts are shaded.





By Lemma 4.8, the natural map  $\rho_*: E_{p,q_m}^r \rightarrow \widehat{E}_{p,q_m}^r$  is an isomorphism, implying

$$0 \neq \rho_*(d_{p,q_m-r+1}^r(x)) = \widehat{d}_{p,q_m-r+1}^r(\rho_*(x)) = 0.$$

This is a contradiction. Thus, we obtain  $E_{p,q_m}^2 \cong E_{p,q_m}^\infty$ . On the other hand, we have  $\widehat{E}_{p,q_m}^2 \cong \widehat{E}_{p,q_m}^\infty \cong H_{*+q_m}(\text{hocolim } \widehat{F}_m)$ . Then the composite

$$E_{p,q_m}^\infty \rightarrow H_{p+q_m}(\text{hocolim } F_m) \xrightarrow{\rho_*} H_{p+q_m}(\text{hocolim } \widehat{F}_m) \cong \widehat{E}_{p,q_m}^\infty$$

is identified with  $\rho_*: E_{p,q_m}^2 \rightarrow \widehat{E}_{p,q_m}^2$ , and so it is an isomorphism. Therefore, by Theorem 3.2, the proof is finished. ■

**Example 4.10** We examine  $\text{hocolim } \widehat{F}_m$  for  $G = SU(2)$ . In this case,  $\Delta$  is a 1-simplex, and so as in Example 3.5, there is a homotopy pushout involving  $\text{hocolim } \widehat{F}_m$  which yields a homotopy equivalence

$$\text{hocolim } \widehat{F}_m \simeq \begin{cases} S^{m+2}, & m \text{ is odd,} \\ S^{m+1} \vee (S^{m+1} \cup_2 e^{m+2}), & m \text{ is even.} \end{cases}$$

In particular, we can see from (3.3) that  $\text{hocolim } \widehat{F}_m$  computes the top homology of  $\text{Hom}(\mathbb{Z}^m, SU(2))_1$ . This will be generalized in the next section to an arbitrary compact simply-connected simple Lie group  $G$ .

### 5 Top homology

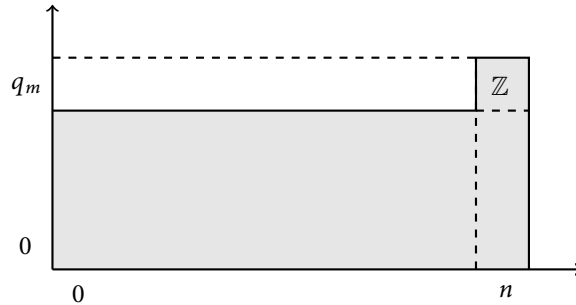
This section computes the top homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  and proves Theorem 1.4. The result depends on the parity of  $m$ . We start with the case  $m$  is odd.

**Theorem 5.1** *If  $m$  is odd, then the top homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  is*

$$H_{q_m+n}(\text{Hom}(\mathbb{Z}^m, G)_1) \cong \mathbb{Z}.$$

**Proof** We present two proofs.

**First proof.** By Corollary 3.4, the  $E^1$ -term of the spectral sequence of Proposition 4.7 for hocolim  $F_m$  is given below, where a possibly nontrivial part is shaded. Then, by degree reasons, the statement is proved.



**Second proof.**  $\text{Hom}(\mathbb{Z}^m, G)_1$  is of dimension  $q_m + n$  as mentioned above, implying  $H_{q_m+n}(\text{Hom}(\mathbb{Z}^m, G)_1)$  is a free abelian group. Then it suffices to compute the dimension of the rational cohomology  $H^{q_m+n}(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{Q})$ . By Theorem 4.2 and the Künneth theorem,

$$H^{q_m+n}(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{Q}) \cong (H^{\dim G-n}(G/T; \mathbb{Q}) \otimes \underbrace{H^n(T; \mathbb{Q}) \otimes \cdots \otimes H^n(T; \mathbb{Q})}_m)^W.$$

By Lemma 4.4,  $H^{\dim G-n}(G/T; \mathbb{Q})$  and  $H^n(T; \mathbb{Q})$  are the sign representation of  $W$ , and so we get  $\dim H^{q_m+n}(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{Q}) = 1$ , completing the proof. ■

Next, we consider the case  $m$  is even.

**Lemma 5.2** *If  $m$  is even and  $n \geq 2$ , then  $H_{q_m+n-1}(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{Q}) = 0$ .*

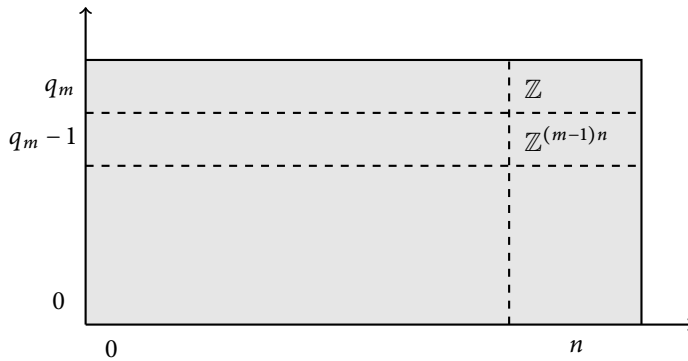
**Proof** By Lemma 4.4,  $H^{n-1}(T; \mathbb{Q})$  does not include the trivial representation of  $W$ . Then, by arguing as in the second proof of Theorem 5.1, the statement is proved. ■

**Remark 5.3** For  $n = 1$ ,  $H^{n-1}(T; \mathbb{Q})$  is the trivial representation of  $W$ , so that the proof of Lemma 5.2 does not work for  $n = 1$ . The top homology of  $\text{Hom}(\mathbb{Z}^m, \text{SU}(2))_1$  for  $m$  even can be deduced from the results of Baird, Jeffrey, and Selick [5] and Crabb [13]; it is given by  $H_{m+1}(\text{Hom}(\mathbb{Z}^m, G)_1) \cong \mathbb{Z}/2$ .

**Theorem 5.4** *If  $m$  is even and  $n \geq 2$ , then the top homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  is*

$$H_{q_m+n-1}(\text{Hom}(\mathbb{Z}^m, G)_1) \cong \mathbb{Z}/2.$$

**Proof** Let  $E^r$  and  $\widehat{E}^r$  denote the spectral sequences of Proposition 4.7 for hocolim  $F_m$  and hocolim  $\widehat{F}_m$ , respectively. By Proposition 3.3,  $E_{n,*}^1 \cong H_*(G/T \times T^{m-1})$ . Then, by Corollary 3.4,  $E^1$  is given below, where a possibly nontrivial part is shaded.



Thus,  $H_*(\text{Hom}(\mathbb{Z}^m, G)_1) = 0$  for  $*$   $>$   $q_m + n$ , and by Corollary 4.8 and Lemma 5.2,

$$H_{q_m+n-i}(\text{Hom}(\mathbb{Z}^m, G)_1) \cong E_{n-i, q_m}^2 \cong \widehat{E}_{n-i, q_m}^2$$

for  $i = 0, 1$ . Let  $\sigma$  be the only one  $n$ -face of  $\Delta$ , and let  $\tau_0, \dots, \tau_n$  be  $(n - 1)$ -faces of  $\Delta$ . Then

$$\widehat{E}_{n, q_m}^1 = \mathbb{Z}\langle u \otimes \sigma \rangle \quad \text{and} \quad \widehat{E}_{n-1, q_m}^1 = \mathbb{Z}\langle u \otimes \tau_i \mid i = 0, \dots, n \rangle,$$

where  $u$  is a generator of  $H_{q_m}(S^{q_m})$ . Since  $|W(\sigma)| = 1$  and  $|W(\tau_i)| = 2$  for each  $i$ , we have

$$d^1: \widehat{E}_{n, q_m}^1 \rightarrow \widehat{E}_{n-1, q_m}^1, \quad u \otimes \sigma \mapsto 2 \sum_{i=0}^n (-1)^i u \otimes \tau_i,$$

implying  $\widehat{E}_{n, q_m}^2 = 0$ , which is proved by the same way as the second proof of Theorem 5.1. Since  $\widehat{E}_{n-1, q_m}^1$  is a free abelian group,  $\sum_{i=0}^n (-1)^i u \otimes \tau_i \in \widehat{E}_{n-1, q_m}^1$  is a nontrivial cycle. Then, by Lemma 5.2,  $\widehat{E}_{n-1, q_m}^2 \cong \mathbb{Z}/2$ , completing the proof. ■

**Proof of Theorem 1.4** Combine Theorems 5.1 and 5.4 and Remark 5.3. ■

### 6 The complex $\Delta_p(k)$

This section provides a combinatorial way to detect torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$ . Define a subcomplex of  $\Delta$  by

$$\Delta_p(k) = \{ \sigma \in \Delta \mid p^{k+1} \text{ does not divide } |W|/|W(\sigma)| \}.$$

Then there is a sequence of subcomplexes

$$\Delta_p(0) \subset \Delta_p(1) \subset \dots \subset \Delta_p(r) = \Delta,$$

where  $r$  is given by  $|W| = p^r q$  with  $(p, q) = 1$ .

**Example 6.1** Let  $G = SU(3)$ , so that  $\Delta$  is a 2-simplex and  $|W| = 6$ . Then possibly nontrivial  $\Delta_p(k)$  are  $\Delta_2(0)$  and  $\Delta_3(0)$ . It is easy to see that  $\Delta_2(0)$  is the 1-skeleton of  $\Delta$  and  $\Delta_3(0)$  is the 0-skeleton of  $\Delta$ .

The following lemma will play a fundamental role in connecting the mod  $p$  homology of  $\Delta_p(k)$  to  $p$ -torsion in the homology of  $\text{Hom}(\mathbb{Z}^2, G)_1$ .

**Lemma 6.2** *The homology  $H_*(\text{hocolim } \widehat{F}_2)$  is a finite abelian group for each  $* > q_2$  and a finitely generated abelian group of rank 1 for  $* = q_2$ .*

**Proof** Since  $\text{hocolim } \widehat{F}$  is a CW complex of finite type, the statement is equivalent to that  $H_*(\text{hocolim } \widehat{F}_2; \mathbb{Q})$  is trivial for each  $* > q_2$  and isomorphic with  $\mathbb{Q}$  for  $* = q_2$ . Then we compute the rational homology of  $\text{hocolim } \widehat{F}_2$ . Let  $C_*(-; \mathbb{F})$  denote the cellular chain complex over a field  $\mathbb{F}$ . We assume that a sphere  $S^{q_2}$  is given a cell decomposition  $S^{q_2} = e^0 \cup e^{q_2}$ . Then for  $0 \leq * \leq n$ , we can define a map

$$\phi: C_*(\Delta; \mathbb{Q}) \rightarrow C_{*+q_2}(\text{hocolim } \widehat{F}_2; \mathbb{Q}), \quad \sigma \mapsto \frac{|W(\sigma)|}{|W|} u \times \sigma,$$

where  $\sigma$  is a face of  $\Delta$  and  $u$  is a generator of  $C_{q_2}(S^{q_2}; \mathbb{Q}) \cong \mathbb{Q}$ . Then

$$\phi(\partial\sigma) = \sum_{i=0}^{\dim \sigma} (-1)^i \frac{|W(\tau_i)|}{|W|} u \times \tau_i = \frac{|W(\sigma)|}{|W|} \sum_{i=0}^{\dim \sigma} (-1)^i \frac{|W(\tau_i)|}{|W(\sigma)|} u \times \tau_i = \partial\phi(\sigma),$$

where  $\partial\sigma = \sum_{i=0}^{\dim \sigma} (-1)^i \tau_i$ . Thus,  $\phi$  is a chain map. Clearly,  $\phi$  is bijective, and so  $H_*(\Delta; \mathbb{Q}) \cong H_{*+q_2}(\text{hocolim } \widehat{F}_2; \mathbb{Q})$ . Since  $\Delta$  is contractible, the proof is done. ■

Now, we prove the main theorem of this section.

**Theorem 6.3** *The mod  $p$  homology of  $\Delta_p(k)$  is nontrivial for some  $k$  if and only if  $\text{hocolim } \widehat{F}_2$  has  $p$ -torsion in homology.*

**Proof** We assume that a sphere  $S^{q_2}$  is given a cell decomposition  $S^{q_2} = e^0 \cup e^{q_2}$ . Consider a map

$$\phi_k: C_*(\Delta_p(k); \mathbb{Z}/p) \rightarrow C_{*+q_2}(\text{hocolim } \widehat{F}_2; \mathbb{Z}/p), \quad \sigma \mapsto \frac{|W(\sigma)|}{p^{r-k}} u \times \sigma,$$

where  $\sigma$  is a face of  $\Delta_p(k)$ ,  $u$  is a generator of  $C_{q_2}(S^{q_2}; \mathbb{Z}/p) \cong \mathbb{Z}/p$  and  $r$  is given by  $|W| = p^r q$  with  $(p, q) = 1$ . Then

$$\phi_k(\partial\sigma) = \sum_{i=0}^{\dim \sigma} (-1)^i \frac{|W(\tau_i)|}{p^{r-k}} u \times \tau_i = \frac{|W(\sigma)|}{p^{r-k}} \sum_{i=0}^{\dim \sigma} (-1)^i \frac{|W(\tau_i)|}{|W(\sigma)|} u \times \tau_i = \partial\phi_k(\sigma),$$

and so  $\phi_k$  is a chain map.

Let  $P(K)$  denote the face poset of a simplicial complex  $K$  as in Section 4. Suppose that  $\widetilde{H}_*(\Delta_p(k); \mathbb{Z}/p) \neq 0$ . Then there is a non-boundary cycle

$$\alpha = \sum_{\sigma \in P(\Delta_p(k))} a_\sigma \sigma$$



in the reduced cellular chain complex  $\widetilde{C}_*(\Delta_p(k); \mathbb{Z}/p)$ . We may assume that the homology class  $[\alpha]$  does not lie in the image of the natural map

$$H_*(\Delta_p(k-1); \mathbb{Z}/p) \rightarrow H_*(\Delta_p(k); \mathbb{Z}/p),$$

because we can replace  $\Delta_p(k)$  with  $\Delta_p(l)$  for some  $l < k$  otherwise. Since  $\phi_k$  annihilates  $C_*(\Delta_p(k-1); \mathbb{Z}/p)$ ,

$$\phi_k(\alpha) = \sum_{\sigma \in P(\Delta_p(k)) - P(\Delta_p(k-1))} \frac{|W(\sigma)|}{p^{r-k}} a_\sigma(u \times \sigma) \in C_{*+q_2}(\text{hocolim } \widehat{F}_2; \mathbb{Z}/p),$$

which is a cycle because  $\alpha$  is a cycle and  $\phi_k$  is a chain map. Suppose

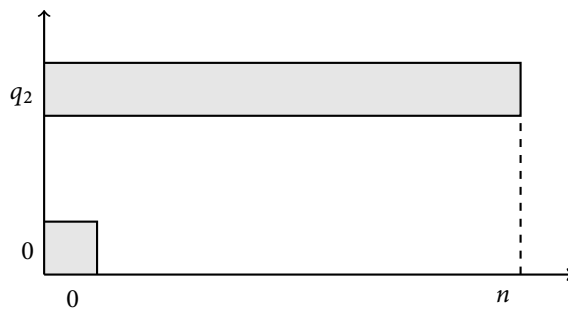
$$\phi_k(\alpha) = \partial \left( \sum_{\tau \in P(\Delta_p(k))} b_\tau(u \times \tau) \right).$$

Then we have

$$\begin{aligned} \phi_k(\alpha) &= \partial \left( \sum_{\tau \in P(\Delta_p(k)) - P(\Delta_p(k-1))} b_\tau(u \times \tau) \right) \\ &= \partial \phi_k \left( \sum_{\tau \in P(\Delta_p(k)) - P(\Delta_p(k-1))} \frac{p^{r-k}}{|W(\tau)|} b_\tau \tau \right). \end{aligned}$$

By definition,  $\phi_k$  is injective on the subgroup of  $C_*(\Delta_p(k); \mathbb{Z}/p)$  generated by faces in  $P(\Delta_p(k)) - P(\Delta_p(k-1))$ . So we obtain that  $\alpha$  is homologous to a cycle in  $\Delta_p(k-1)$ . Since  $\alpha$  is not homologous to a non-boundary cycle in  $\Delta_p(k-1)$  by assumption,  $\alpha$  is homologous to a boundary in  $\Delta_p(k)$ . Then  $\alpha$  itself is a boundary, which is a contradiction. Then  $\phi_k(\alpha)$  is a non-boundary cycle in  $C_{*+q_2}(\text{hocolim } \widehat{F}_2; \mathbb{Z}/p)$ . If  $|\alpha| = 0$ , then the proof of Lemma 6.2 implies that  $\phi_k(\alpha)$  is not the mod  $p$  reduction of a representative of an integral homology class of infinite order. Thus, by Lemma 6.2,  $H_*(\text{hocolim } \widehat{F}_2)$  has  $p$ -torsion in homology.

Assume  $\widetilde{H}_*(\Delta_p(k); \mathbb{Z}/p) = 0$  for each  $k$ . Then we suppose  $\text{hocolim } \widehat{F}_2$  has  $p$ -torsion in homology and derive a contradiction. The  $E^2$ -term of the spectral sequence of Proposition 4.7 for  $\text{hocolim } \widehat{F}_2$  is illustrated as follows, where possibly nontrivial parts are shaded.



Then  $H_*$  (hocolim  $\widehat{F}_2$ ) has  $p$ -torsion only for  $* \geq q_2$ . So there is a cycle representing a  $p$ -torsion element in the homology of hocolim  $\widehat{F}_2$ . We may assume that its mod  $p$  reduction

$$\widehat{\alpha} = \sum_{\sigma \in P(\Delta)} \bar{a}_\sigma(u \times \sigma)$$

represents a nontrivial mod  $p$  homology class of hocolim  $\widehat{F}_2$ . Let

$$\widehat{\alpha}_k = \sum_{\sigma \in P(\Delta_p(k)) - P(\Delta_p(k-1))} \bar{a}_\sigma(u \times \sigma).$$

By definition,  $\widehat{\alpha} = \widehat{\alpha}_0 + \dots + \widehat{\alpha}_r$ . Clearly,  $\partial \widehat{\alpha}_k$  is a linear combination of simplices in  $P(\Delta_p(k))$ . For  $\sigma \in P(\Delta_p(k)) - P(\Delta_p(k-1))$  and  $\tau \in P(\Delta_p(k-1))$ , if  $\sigma > \tau$ , then  $|W(\tau)|/|W(\sigma)| \equiv 0 \pmod p$ , implying that  $\partial \widehat{\alpha}_k$  is actually a linear combination of simplices in  $P(\Delta_p(k)) - P(\Delta_p(k-1))$ . Thus, since  $\partial \widehat{\alpha} = 0$ , we get

$$\partial \widehat{\alpha}_k = 0$$

for each  $k$ . Let

$$\alpha_k = \sum_{\sigma \in P(\Delta_p(k)) - P(\Delta_p(k-1))} \frac{p^{r-k}}{|W(\sigma)|} a_\sigma \sigma \in \widetilde{C}_*(\Delta_p(k); \mathbb{Z}/p).$$

Then  $\phi_k(\alpha_k) = \widehat{\alpha}_k$ , implying  $\partial \phi_k(\alpha_k) = 0$ . So we get  $\partial \alpha_k \in \widetilde{C}_*(\Delta_p(k-1); \mathbb{Z}/p)$ . Since  $\widetilde{H}_*(\Delta_p(k-1); \mathbb{Z}/p) = 0$ , there is  $\beta_k \in \widetilde{C}_*(\Delta_p(k-1); \mathbb{Z}/p)$  such that  $\partial \alpha_k = \partial \beta_k$ . Then, since the map  $\phi_k$  annihilates  $C_*(\Delta_p(k-1); \mathbb{Z}/p)$ , we get

$$\phi_k(\alpha_k - \beta_k) = \phi_k(\alpha_k) = \widehat{\alpha}_k.$$

So since  $\widetilde{H}_*(\Delta_p(k-1); \mathbb{Z}/p) = 0$ ,  $\alpha_k - \beta_k$  is a boundary, implying  $\widehat{\alpha}_k$  is a boundary. Thus, since  $k$  is arbitrary,  $\widehat{\alpha}$  is a boundary, which is a contradiction. Therefore, hocolim  $\widehat{F}_2$  does not have  $p$ -torsion in homology, completing the proof. ■

**Corollary 6.4** *If the mod  $p$  homology of  $\Delta_p(k)$  is nontrivial for some  $k$ , then  $\text{Hom}(\mathbb{Z}^m, G)_1$  has  $p$ -torsion in homology for each  $m \geq 2$ .*

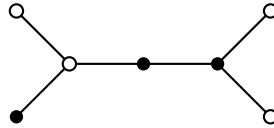
**Proof** By Theorem 6.3, if the mod  $p$  homology of  $\Delta_p(k)$  is nontrivial for some  $k$ , then hocolim  $\widehat{F}_2$  has  $p$ -torsion in homology. Thus, by Proposition 4.9,  $\text{Hom}(\mathbb{Z}^2, G)_1$  has  $p$ -torsion in homology too. Since  $\text{Hom}(\mathbb{Z}^2, G)_1$  is a retract of  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$ , the statement is proved. ■

## 7 Computation of torsion in homology

This section computes torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  when  $G$  is  $SU(n)$ ,  $Spin(2n)$  and exceptional groups by describing the complex  $\Delta_p(k)$  in terms of the extended Dynkin diagram of  $G$ .

Note that every facet of  $\Delta$  corresponds to a simple root or the highest root, and every  $i$ -face is an intersection of  $n - i$  facets, where a facet of  $\Delta$  means a face of codimension one. Then there is a one-to-one correspondence between  $i$ -faces of  $\Delta$  and choices of  $n - i$  roots from the simple roots and the highest root. Recall that the

extended Dynkin diagram of  $G$  is a graph whose vertices are simple roots and the highest root. We will mean by a colored extended Dynkin diagram of  $G$  an extended Dynkin diagram of  $G$  whose vertices are colored by black and white. Here is an example of a colored extended Dynkin diagram of  $Spin(12)$ .



Let  $\mathbb{D}_i$  be the set of all colored extended Dynkin diagram with  $i + 1$  white vertices and  $n - i$  black vertices, where the extended Dynkin diagram of  $G$  has  $n + 1$  vertices. Then, by the observation above, we get the following.

**Lemma 7.1** *There is a bijection*

$$\Psi_i: P_i(\Delta) \xrightarrow{\cong} \mathbb{D}_i,$$

which sends an  $i$ -face  $\sigma \in P_i(\Delta)$  to a colored extended Dynkin diagram such that  $n - i$  vertices corresponding to  $\sigma$  are black.

We will compute the mod  $p$  homology of  $\Delta_p(k)$  by specifying  $\Psi_i(P_i(\Delta_p(k)))$ . Let  $\Gamma$  be a colored extended Dynkin diagram of  $G$ . For an induced subgraph  $\Theta$  of  $\Gamma$ , let  $W_\Theta$  denote the subgroup of  $W$  generated by simple reflections corresponding to the vertices of  $\Theta$ , where we put  $W_\emptyset = 1$ . By definition, we have the following.

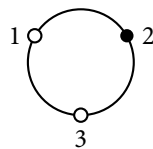
**Lemma 7.2** (1) *If a colored extended Dynkin diagram  $\Gamma$  is the disjoint union of induced subgraphs  $\Gamma_1, \dots, \Gamma_k$  after removing all white vertices, then*

$$W_\Gamma = W_{\Gamma_1} \times \dots \times W_{\Gamma_k}.$$

(2) *For an  $i$ -face  $\sigma$  of  $\Delta$ , there is an isomorphism*

$$W(\sigma) \cong W_{\Psi_i(\sigma)}.$$

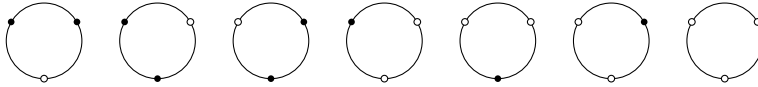
Let  $v_1, \dots, v_i$  be vertices of the extended Dynkin diagram. We denote by  $v_1 \dots v_i$  an  $(i - 1)$ -face  $\sigma$  such that white vertices of the extended Dynkin diagram  $\Psi_{i-1}(\sigma)$  are  $v_1, \dots, v_i$ . For example, as for  $G = SU(3)$ , 13 corresponds the following colored extended Dynkin diagram.



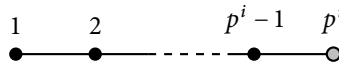
### 7.1 The $SU(n)$ case

Throughout this subsection, let  $G = SU(n + 1)$ . Recall that the extended Dynkin diagram of  $SU(n + 1)$  is the cycle graph with  $n + 1$  vertices, denoted by  $C_{n+1}$ .

**Example 7.3** The following figure shows all colored extended Dynkin diagrams of  $SU(3)$ , where  $\Delta$  is a 2-simplex. The left three graphs correspond to vertices, the middle three graphs correspond to edges, and the most right graph corresponds to the 2-simplex.



Now, we describe faces of  $\Delta_p(0)$ . Let  $C(i)$  be the following graph with  $p^i - 1$  black vertices and one gray vertex.



**Proposition 7.4** For  $n > 1$ , let  $n + 1 = \sum_{j=0}^l a_j p^j$  be the  $p$ -adic expansion, where  $0 \leq a_j < p$  for each  $j$ . A colored extended Dynkin diagram of  $SU(n + 1)$  is in  $\Psi_i(P_i(\Delta_p(0)))$  if and only if it is obtained by gluing  $a_j$  copies of  $C(j)$  for  $j = 0, \dots, l$  such that  $i + 1$  gray vertices are replaced by white vertices and the remaining gray vertices are replaced by black vertices.

**Proof** First, we prove the if part. Let  $\Gamma$  be the colored extended Dynkin diagram specified in the statement, and let  $\Gamma'$  be a colored extended Dynkin diagram which is constructed by recoloring the originally gray vertices of  $\Gamma$  by white. Then, by Lemma 7.2,

$$|W_{\Gamma'}| = \prod_{j=0}^l (p^j!)^{a_j},$$

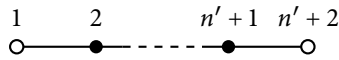
and so we get

$$\frac{|W|}{|W_{\Gamma'}|} = \prod_{j=1}^l \prod_{j'=1}^{a_j-1} \binom{n + 1 - j' p^j \sum_{k>j} a_k p^k}{p^j}.$$

By Lucas's theorem,  $\binom{n+1-j'p^j \sum_{k>j} a_k p^k}{p^j}$  is prime to  $p$  for each  $j$  and  $j'$ , implying that  $\frac{|W|}{|W_{\Gamma'}|}$  is prime to  $p$ . Then, since  $|W_{\Gamma'}|$  divides  $|W_{\Gamma}|$ ,  $\frac{|W|}{|W_{\Gamma}|}$  is prime to  $p$ , so that we obtain  $\Gamma \in \Psi_i(P_i(\Delta_p(0)))$ .

Next, we prove the only if part. The  $n = 2$  case can be easily deduced from Example 7.3. Suppose we have the one-to-one correspondence in the statement for  $SU(k + 1)$  with  $k \leq n - 1$ . Since  $W(v) = W$  for each vertex  $v$  of  $\Delta$ , vertices of  $\Delta$  are vertices of  $\Delta_p(0)$ . Then, by Lemma 7.1, there is a one-to-one correspondence between  $\Psi_0(P_0(\Delta_p(0)))$  and vertices of  $\Delta_p(0)$ .

Let  $\Gamma \in \Psi_i(P_i(\Delta_p(0)))$  for  $i > 0$ . Then  $\Gamma$  includes the following subgraph  $\Theta$  with  $n'$  black vertices and two white vertices for  $0 \leq n' \leq n - 2$ .



Let  $n' + 1 = \sum_{j=0}^l a'_j p^j$  be the  $p$ -adic expansion. By Lemma 7.2,

$$|W_\Gamma| = |W_\Theta| |W_{\Gamma-\Theta}|$$

such that  $|W_\Theta| = (n' + 1)!$  and  $|W_{\Gamma-\Theta}|$  divides  $(n - n')!$ . Then, since  $|W|/|W_\Gamma|$  is prime to  $p$ ,

$$\frac{|W|}{(n' + 1)!(n - n')!} = \frac{(n + 1)!}{(n' + 1)!(n - n')!} = \binom{n + 1}{n' + 1}$$

is also prime to  $p$ . Thus, by Lucas's theorem, we obtain

$$a'_j \leq a_j$$

for each  $j$ . Let  $\bar{\Theta}$  be a cycle graph with  $n - n' + 1$  vertices which is obtained from  $\Gamma$  by contracting  $\Theta$  to a single white vertex. Then, by the induction hypothesis,  $\bar{\Theta}$  belongs to  $\Psi_{i-1}(\Delta_p(0))$  for  $G = SU(n - n')$ , and so  $\bar{\Theta}$  is obtained by gluing  $C(j)$ . By definition, the graph  $\Theta$  is also obtained by gluing  $C(j)$ . Therefore,  $\Gamma$  itself is obtained by gluing  $C(j)$ , completing the proof. ■

Now, we compute torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, SU(n + 1))_1$ .

**Theorem 7.5** *The homology of  $\text{Hom}(\mathbb{Z}^m, SU(n + 1))_1$  for  $m \geq 2$  has  $p$ -torsion in homology if and only if  $p \leq n + 1$ .*

**Proof** By Lemma 1.1 and  $|W| = (n + 1)!$ ,  $\text{Hom}(\mathbb{Z}^m, SU(n + 1))_1$  has no  $p$ -torsion in homology for  $p > n + 1$ . So we assume  $p \leq n + 1$  and prove that  $\text{Hom}(\mathbb{Z}^m, SU(n + 1))_1$  has  $p$ -torsion in homology. By Corollary 6.4, it suffices to show that the mod  $p$  homology of  $\Delta_p(0)$  for  $SU(n + 1)$  is nontrivial. To this end, we aim to prove  $\chi(\Delta_p(0)) \neq 1$ , where  $\chi(K)$  denotes the Euler characteristic of a simplicial complex  $K$ . By Lemma 7.1 and Proposition 7.4, rotations of a cycle graph induce the action of a cyclic group  $\mathbb{Z}/(n + 1)$  on  $\Psi_i(P_i(\Delta_p(0)))$ . Let  $n + 1 = \sum_{j=0}^l a_j p^j$  be the  $p$ -adic expansion, where  $a_l \neq 0$ . Then every element of the stabilizer of a face  $\sigma \in P_i(\Delta_p(0))$  permutes  $C(l)$ -parts of  $\Psi_i(\sigma)$  because  $C(l)$  is not obtained by gluing  $a_i$  copies of  $C(i)$  for  $i < l$ . Thus, the order of the stabilizer of  $\Psi_i(\sigma)$  is at most  $(n + 1, a_l)$ , implying  $|P_i(\Delta_p(0))|$  is divisible by

$$1 < \frac{n + 1}{(n + 1, a_l)} < n + 1$$

for each  $i$ . Therefore, since  $a_l < n + 1$  and  $\chi(\Delta_p(0)) = \sum_{i \geq 0} (-1)^i |P_i(\Delta_p(0))|$ , we obtain that  $\chi(\Delta_p(0))$  is divisible by  $\frac{n+1}{(n+1, a_l)}$ , implying  $\chi(\Delta_p(0)) \neq 1$ . ■

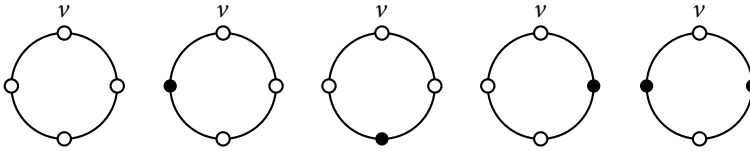
### 7.2 The Spin(2n) case

Throughout this subsection, let  $G = Spin(2n)$ , and let  $r$  be the integer such that  $|W| = p^r q$  for  $(p, q) = 1$ . We aim to prove the non-triviality of the homology of  $\Delta_p(r - 1)$ .

Fix a vertex  $v$  of the extended Dynkin diagram of  $SU(n + 1)$ . Let  $f_i(n, p)$  denote the number of colored extended Dynkin diagrams  $\Gamma$  of  $SU(n + 1)$  such that  $\Gamma \in \mathbb{D}_i$ ,  $v$  is white-colored and  $|W_\Gamma|$  is prime to  $p$ . Let

$$\chi(n, p) = \sum_{i=0}^n (-1)^i f_i(n, p).$$

**Example 7.6** All colored extended Dynkin diagrams  $\Gamma$  of  $SU(4)$  satisfying that a fixed vertex  $v$  is white-colored and  $|W_\Gamma|$  is prime to  $p = 3$  are as below. Then  $f_0 = 0, f_1 = 1, f_2 = 3, f_3 = 1$ , implying  $\chi(3, 3) = 1$ .

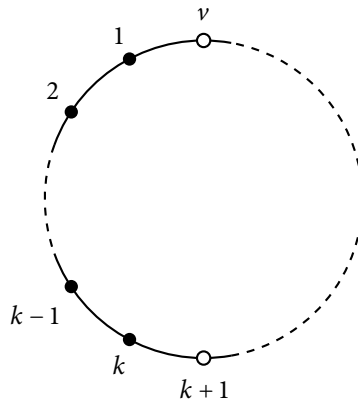


We compute  $\chi(n, p)$  for general  $n$  and  $p$ .

**Lemma 7.7** We have

$$\chi(n, p) = \begin{cases} -1, & n \equiv -1 \pmod p, \\ 1, & n \equiv 0 \pmod p, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof** It is easy to see the statement holds for  $n \leq p - 1$ . Let  $\Gamma$  be a colored extended Dynkin diagram of  $SU(n + 1)$  such that  $v$  is white-colored and  $|W_\Gamma|$  is prime to  $p$ . Then for  $n \geq p$ , it follows from Lemma 7.2 that  $\Gamma$  is given as below, where  $0 \leq k \leq p - 2$ .

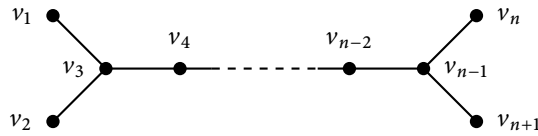


Thus, we get

$$\chi(n, p) = \sum_{i=1}^{p-2} -\chi(n - i, p),$$

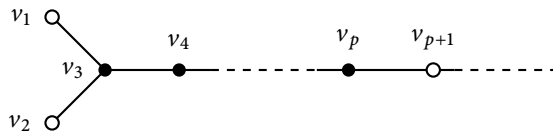
and so the proof is done by induction on  $n$ . ■

We name the vertices of the extended Dynkin diagram of  $Spin(2n)$  as follows.



Let  $\Gamma_1$  and  $\Gamma_2$  be colored extended Dynkin diagrams of  $Spin(2n)$ . Suppose all but one vertices of  $\Gamma_1$  and  $\Gamma_2$  have the same colors. If  $\Gamma_1$  and  $\Gamma_2$  correspond to faces of  $\Delta_p(k)$ , then we can cancel out these faces in the computation of  $\chi(\Delta_p(k))$ . Thus, we specify the case  $\Gamma_1$  is a face of  $\Delta_p(k)$  but  $\Gamma_2$  is not.

**Lemma 7.8** *Let  $\Gamma_1$  and  $\Gamma_2$  be colored extended Dynkin diagrams of  $Spin(2n)$  such that all vertices but the vertex  $v_1$  have the same color. If  $\Gamma_1$  corresponds to a face of  $\Delta_p(r - 1)$  and  $\Gamma_2$  is not, then  $\Gamma_2$  is of the following form.*



**Proof** The statement follows from Lemma 7.2. ■

**Theorem 7.9** *If  $p \leq n$  and  $n \equiv 0, 1 \pmod p$ , then for  $m \geq 2$ ,  $\text{Hom}(\mathbb{Z}^m, Spin(2n))_1$  has  $p$ -torsion in homology.*

**Proof** As in the proof of Theorem 7.5, it suffices to show  $\chi(\Delta_p(r - 1)) \neq 1$  for  $p \leq n$  and  $n \equiv 0, 1 \pmod p$ . Let

$$\tilde{\chi}(n, p) = \sum_{i=0}^n (-1)^i (|P_i(\Delta)| - |P_i(\Delta_p(r - 1))|).$$

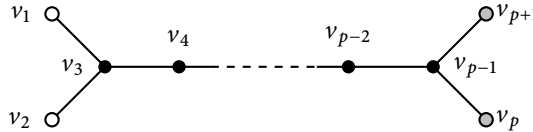
Then, since  $\chi(\Delta) = 1$ , we have  $\chi(\Delta_p(r - 1)) = 1 - \tilde{\chi}(n, p)$ , and so  $\chi(\Delta_p(r - 1)) \neq 1$  if and only if  $\tilde{\chi}(n, p) \neq 0$ .

First, we consider the  $n \geq 2p + 2$  case. Note that Lemma 7.8 holds if we replace  $v_1$  with  $v_n$ . Then, since  $n \geq 2p + 2$ , we only need to count colored extended Dynkin diagrams of  $Spin(2n)$  such that vertices  $v_1, v_2, \dots, v_{p+1}, v_{n-p+1}, v_{n-p+2}, \dots, v_{n+1}$  are colored as in Lemma 7.8. If we delete  $v_1, v_2, \dots, v_{p+1}, v_{n-p+1}, v_{n-p+2}, \dots, v_{n+1}$  and add a white vertex  $v$  together with edges  $vv_{p+2}$  and  $vv_{n-p}$ , then we get a colored extended Dynkin diagram of  $SU(n - 2p)$ . Through this operation, there is a one-to-one correspondence between colored extended Dynkin diagrams of  $Spin(2n)$  whose left and right ends are as in Lemma 7.8 and colored extended Dynkin diagrams of  $SU(n - 2p)$  such that a fixed vertex  $v$  is white-colored. Then we get

$$\tilde{\chi}(n, p) = -\chi(n - 2p - 1, p).$$

Therefore, for  $n \geq 2p + 2$ , the proof is finished by Lemma 7.7.

Next, we consider the  $p \leq n < 2p + 2$  case. We only need to count colored extended Dynkin diagrams such that vertices  $v_1, v_2, \dots, v_{p+1}$  are colored as in Lemma 7.8. Except for  $n = p, p + 1, 2p, 2p + 1$ , vertices  $v_{p+2}, v_{p+3}, \dots, v_{n+1}$  can have arbitrarily color, and so  $\tilde{\chi}(n, p) = 0$ . For  $n = p$ , we only need to count the graphs whose vertices  $v_1, \dots, v_{p-1}$  are colored as in Lemma 7.8. Then there are only three graphs to be counted as follows, where  $v_p$  and  $v_{p+1}$  are either black or white such that they cannot be white at the same time.



In each case  $n = p + 1, 2p, 2p + 1$ , it follows from Lemma 7.8 that there is only one graph to be counted. For  $n = p + 1$ , the end vertices  $v_1, v_2, v_{p+1}, v_{p+2}$  are white and the remaining vertices are black. For  $n = 2p$ , the end vertices  $v_1, v_2, v_{2p}, v_{2p+1}$  and the middle vertex  $v_{p+1}$  are white and the remaining vertices are black. For  $n = 2p + 1$ , the end vertices  $v_1, v_2, v_{2p+1}, v_{2p+2}$  and the middle vertices  $v_{p+1}, v_{p+2}$  are white and the remaining vertices are black. Summarizing,  $\tilde{\chi}(n, p) \neq 0$  for  $n = p, p + 1, 2p, 2p + 1$ . Therefore the statement is proved. ■

### 7.3 The exceptional case

We continue to compute torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  when  $G$  is the exceptional Lie group. Let  $G$  be exceptional, and let  $p$  be a prime dividing  $|W|$ , where  $|W|$  is given as in Table 1. Then  $G$  has  $p$ -torsion in homology except for

$$(7.1) \quad (G, p) = (G_2, 3), (E_6, 5), (E_7, 5), (E_7, 7), (E_8, 7)$$

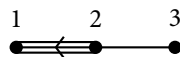
(see [23, Theorem 5.11, Chapter 7]). Since  $G$  is a retract of  $\text{Hom}(\mathbb{Z}^m, G)_1$ ,  $\text{Hom}(\mathbb{Z}^m, G)_1$  has  $p$ -torsion in homology except possibly for the cases (7.1). On the other hand,  $\text{Hom}(\mathbb{Z}^m, G)_1$  has no  $p$ -torsion in homology when  $p$  does not divide  $|W|$ . Then we only need to consider the cases (7.1).

Now, we prove the following.

**Theorem 7.10** *Let  $G$  be exceptional. Then  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$  has  $p$ -torsion in homology if and only if  $p$  divides  $|W|$ , except possibly for  $(G, p) = (E_7, 5), (E_7, 7), (E_8, 7)$ .*

**Proof** As observed above, it follows from Corollary 6.4 that we only need to show that  $\Delta_p(0)$  has nontrivial mod  $p$  homology for  $(G, p)$  in (7.1) for  $(G, p) = (G_2, 3), (E_6, 5)$ .

Let  $G = G_2$ . Then the extended Dynkin diagram is given as below. So, by Lemma 7.2,  $\Delta_3(0)$  consists only of two vertices 1 and 3, implying it has nontrivial mod  $p$  homology.

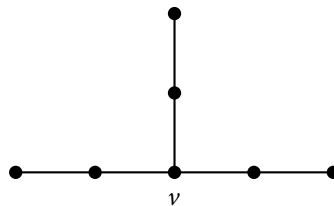




Type	Lie group	Rank	$W$	$ W $
$A_n$	$SU(n + 1)$	$n$	$\Sigma_{n+1}$	$(n + 1)!$
$B_n$	$Spin(2n + 1)$	$n$	$B_n$	$2^n n!$
$C_n$	$Sp(n)$	$n$	$B_n$	$2^n n!$
$D_n$	$Spin(2n)$	$n$	$B_n^+$	$2^{n-1} n!$
$G_2$	$G_2$	2	---	$2^2 \cdot 3$
$F_4$	$F_4$	4	---	$2^7 \cdot 3^2$
$E_6$	$E_6$	6	---	$2^7 \cdot 3^4 \cdot 5$
$E_7$	$E_7$	7	---	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
$E_8$	$E_8$	8	---	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

Table 1: Simple Lie groups and their Weyl groups.

Let  $G = E_6$ . Then the extended Dynkin diagram is given as below.



It has symmetry of rotation around the vertex  $v$ . By Lemma 7.2, we can see that if a colored extended Dynkin diagram is symmetric with respect to the rotation around  $v$ , then its corresponding face of  $\Delta$  does not lie in  $\Delta_5(0)$ . Then the number of  $i$ -faces of  $\Delta_5(0)$  is divisible by 3 for each  $i$ , implying  $\chi(\Delta_5(0))$  is divisible by 3. Thus,  $\Delta_5(0)$  has nontrivial mod 5 homology. ■

By Theorems 7.5, 7.9, and 7.10, we dare to pose the following.

**Conjecture 7.11** *The homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$  has  $p$ -torsion if and only if  $p$  divides the order of  $W$ .*

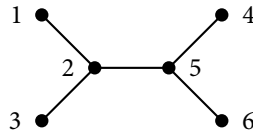
### 8 Negative results

Although we have pose Conjecture 7.11, the complex  $\Delta_p(k)$  does not work in the remaining cases, unfortunately. To be fair, we prove this, but we have to notice that those negative results do not imply the nonexistence of torsion in homology because the non-triviality of the homology of  $\Delta_p(k)$  is only a sufficient condition for the existence of  $p$ -torsion in the homology of  $\text{Hom}(\mathbb{Z}^m, G)_1$  for  $m \geq 2$ .

First, we consider the  $Spin(2n)$  case by examining  $(G, p) = (Spin(10), 3)$  which is not included in Theorem 7.9. Since  $|W| = 2^4 \cdot 5!$  for  $G = Spin(10)$ , we only need to consider  $\Delta_3(0)$ .

**Proposition 8.1** *The complex  $\Delta_3(0)$  of  $Spin(10)$  is contractible.*

**Proof** We prove the statement by applying discrete Morse theory. We refer to [22] for materials of discrete Morse theory. We name the vertices of the extended Dynkin diagram of  $Spin(10)$  as follows.



By Lemma 7.2, it is straightforward to see that facets of  $\Delta_3(0)$  are

$$1234, \quad 1236, \quad 1346, \quad 1456, \quad 3456.$$

Then we have the following acyclic partial matching.

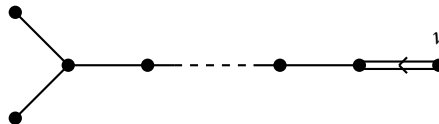
(1234,234)	(1236,236)	(1346,134)	(1456,145)	(3456,345)	(123,12)	(124,24)
(126,26)	(136,13)	(146,14)	(156,15)	(346,34)	(356,35)	(456,45)
(16,1)	(23,2)	(36,3)	(46,4)	(56,5)		

Since all faces of  $\Delta_3(0)$  but the vertex 6 appear in the acyclic partial matching above, it follows from the fundamental theorem of discrete Morse theory that  $\Delta_3(0)$  collapses onto the vertex 6, implying  $\Delta_3(0)$  is contractible. ■

Next, we consider the  $G = Spin(2n + 1), Sp(n)$  case.

**Proposition 8.2** *Let  $G$  be  $Spin(2n + 1)$  for  $n \geq 3$  or  $Sp(n)$ , and let  $p$  be an odd prime dividing  $|W|$ . Then for each  $k, \Delta_p(k)$  is contractible.*

**Proof** We only prove the case  $G = Spin(2n + 1)$  because the case  $G = Sp(n)$  is quite similarly proved. The extended Dynkin diagram of  $Spin(2n + 1)$  is given as follows.

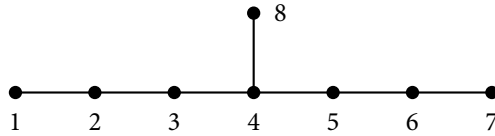


Let  $\widehat{\Delta}_p(k)$  be the subcomplex of  $\Delta_p(k)$  consisting of faces  $\sigma$  such that in the corresponding colored extended Dynkin diagram, the vertex  $\nu$  is black. Let  $\Gamma = \Psi_i(\sigma)$  for an  $i$ -face  $\sigma$  of  $\widehat{\Delta}_p(k)$ . Let  $\Gamma'$  be a colored extended diagram whose vertices have the same color as  $\Gamma$  except for the vertex  $\nu$ . So the vertex  $\nu$  of  $\Gamma'$  is white. By Lemma 7.2,  $\Gamma'$  corresponds to the join  $\nu * \sigma$ . Since  $|W_\Gamma|/|W_{\Gamma'}|$  is a power of 2, the join  $\nu * \sigma$  is an  $(i + 1)$ -face of  $\Delta_p(k)$ . Thus,  $\Delta_p(k)$  is the join  $\nu * \widehat{\Delta}_p(k)$ , completing the proof. ■

Finally, we consider the exceptional case. The only cases that are not included in Theorem 7.10 are  $(G, p) = (E_7, 5), (E_7, 7), (E_8, 7)$ . Since  $|W(E_7)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$ , we only need to consider  $\Delta_p(0)$  for  $G = E_7$  and  $p = 5, 7$ .

**Proposition 8.3** For  $G = E_7$  and  $p = 5, 7$ ,  $\Delta_p(0)$  is contractible.

**Proof** Let  $G = E_7$ . Then its extended Dynkin diagram is given as follows.



Then the facets of  $\Delta_5(0)$  are

$$1237, \quad 1238, \quad 1278, \quad 1567, \quad 1678, \quad 5678.$$

So we have the following acyclic partial matching.

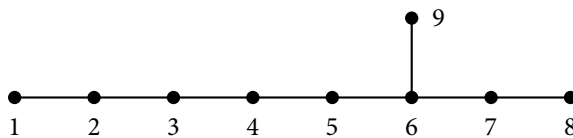
(1237,137)	(1238,138)	(1278,278)	(1567,157)	(1678,168)	(5678,578)
(123,13)	(127,17)	(128,18)	(156,15)	(167,67)	(178,78)
(237,37)	(238,38)	(567,57)	(568,58)	(678,68)	(12,2)
(16,6)	(23,3)	(27,7)	(28,8)	(56,5)	

Note that all faces of  $\Delta_5(0)$  but the vertex 1 appear in the acyclic partial matching above. Thus, by the fundamental theorem of discrete Morse theory,  $\Delta_5(0)$  collapses onto the vertex 1, implying  $\Delta_5(0)$  is contractible. By Lemma 7.2, the facets of  $\Delta_7(0)$  are 18 and 78. Then  $\Delta_7(0)$  is a path graph of length 2, implying it is contractible. ■

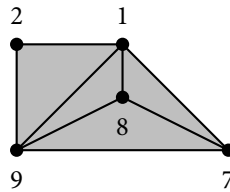
Since  $|W(E_8)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ , we only need to consider  $\Delta_7(0)$  for  $(G, p) = (E_8, 7)$ .

**Proposition 8.4** For  $G = E_8$ ,  $\Delta_7(0)$  is contractible.

**Proof** The extended Dynkin diagram of  $E_8$  is given as below.



Then, by Lemma 7.2,  $\Delta_7(0)$  is the following two-dimensional simplicial complex.



Thus  $\Delta_7(0)$  is contractible. ■

## References

- [1] A. Adem and F. R. Cohen, *Commuting elements and spaces of homomorphisms*. Math. Ann. 338(2007), 587–626.
- [2] A. Adem, F. R. Cohen, and E. Torres-Giese, *Commuting elements, simplicial spaces, and filtrations of classifying spaces*. Math. Proc. Cambridge Philos. Soc. 152(2012), no. 1, 91–114.
- [3] A. Adem, J. M. Gómez, and S. Gritschacher, *On the second homotopy group of spaces of commuting elements in Lie groups*. Int. Math. Res. Not. 2022(2022), 1–73.
- [4] M. F. Atiyah and R. Bott, *The Yang–Mills equations over Riemann surfaces*. Philos. Trans. R. Soc. Lond. Ser. A 308(1983), no. 1505, 523–615.
- [5] T. Baird, L. C. Jeffrey, and P. Selick, *The space of commuting  $n$ -tuples in  $SU(2)$* . Illinois J. Math. 55(2011), no. 3, 805–813.
- [6] T. J. Baird, *Cohomology of the space of commuting  $n$ -tuples in a compact Lie group*. Algebr. Geom. Topol. 7(2007), 737–754.
- [7] M. Bergeron, *The topology of nilpotent representations in reductive groups and their maximal compact subgroups*. Geome. Topol. 19(2015), no. 3, 1383–1407.
- [8] M. Bergeron and L. Silberman, *A note on nilpotent representations*. J. Group Theory 19(2016), no. 1, 125–135.
- [9] A. Borel, *Sous-groupes commutatifs et torsion des groupes de Lie compacts*. Tôhoku Math. J. 13(1962), 216–240.
- [10] A. Borel, R. Friedman, and J. Morgan, *Almost commuting elements in compact Lie groups*. Mem. Amer. Math. Soc. 157(2002), 747.
- [11] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, 304, Springer, Berlin, 1972.
- [12] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, 46, Academic Press, New York, 1972.
- [13] M. C. Crabb, *Spaces of commuting elements in  $SU(2)$* , Proc. Edinb. Math. Soc. 54(2011), 67–75.
- [14] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of 3-manifolds*. Ann. of Math. (2) 117(1983), no. 1, 109–146.
- [15] W. M. Goldman, *Topological components of spaces of representations*. Invent. Math. 93(1988), no. 3, 557–607.
- [16] J. M. Gómez, A. Pettet, and J. Souto, *On the fundamental group of  $\text{Hom}(\mathbb{Z}^k, G)$* . Math. Z. 271(2012), 33–44.
- [17] S. Hasui, D. Kishimoto, M. Takeda, and M. Tsutaya, *Tverberg’s theorem for cell complexes*. accepted by Bull. Lond. Math. Soc. (2023).
- [18] N. J. Hitchin, *The self-duality equations on a Riemann surface*. Proc. Lond. Math. Soc. (3) 55(1987), no. 1, 59–126.
- [19] J. E. Humphreys, *Reflection groups and coxeter groups*, Cambridge Studies in Advanced Mathematics, 29, Cambridge University Press, Cambridge, 1990.
- [20] V. G. Kac and A. V. Smilga, *Vacuum structure in supersymmetric Yang–Mills theories with any gauge group*. In: The many faces of the superworld, edited by M. Shifman (University of Minnesota) World Scientific Publishing, River Edge, NJ, 1999, pp. 185–234.
- [21] D. Kishimoto and M. Takeda, *Spaces of commuting elements in the classical groups*. Adv. Math. 386(2021), 107809.
- [22] D. Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, 21, Springer, Berlin, 2008.

- [23] M. Mimura and H. Toda, *Topology of Lie groups I, II*, Translations of Mathematical Monographs, 91, American Mathematical Society, Providence, RI, 1991.
- [24] D. A. Ramras and M. Stafa, *Hilbert–Poincaré series for spaces of commuting elements in Lie groups*. *Math. Z.* 292(2019), 591–610.
- [25] D. A. Ramras and M. Stafa, *Homological stability for spaces of commuting elements in Lie groups*. *Int. Math. Res. Not.* 2021(2021), no. 5, 3927–4002.
- [26] M. Takeda, *Cohomology of the spaces of commuting elements in Lie groups of rank two*. *Topology Appl.* 303(2021), 107851.
- [27] K. Wehrheim, *Uhlenbeck compactness*, EMS Series of Lectures in Mathematics, European Mathematical Society, Zürich, 2004.
- [28] E. Witten, *Constraints on supersymmetry breaking*. *Nuclear Phys. B* 202(1982), 253–316.
- [29] E. Witten, *Toroidal compactification without vector structure*. *J. High Energy Phys.* 6(1998), 43 pp.

*Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan*  
*e-mail:* kishimoto@math.kyushu-u.ac.jp

*Institute for Liberal Arts and Sciences, Kyoto University, Kyoto 606-8316, Japan*  
*e-mail:* takeda.masahiro.87u@st.kyoto-u.ac.jp