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On a class of power-associative periodic rings

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A power-associative ring A is called a p-ring provided there exists a prime p so that for every x in A, $x^p = x$ and $px = 0$. It is shown that if A is such a ring with $p \neq 2$, then A is isomorphic to a subdirect sum of copies of $\mathrm{GF}(p)$, the Galois field with p elements.

1. Introduction

A power-associative ring A is called a p-ring provided there exists a prime p so that for every x in A, $x^p = x$ and $px = 0$. It is well known that any associative p-ring is commutative and is isomorphic to a subdirect sum of copies of $\mathrm{GF}(p)$ (see for example, [3, p. 144]). In this paper we will extend this result to power-associative p-rings with $p \neq 2$. Stated formally we have:

THEOREM. *Let A be a power-associative p-ring with $p \neq 2$, then A is associative and commutative. Thus, A is a subdirect sum of copies of* $\mathrm{GF}(p)$.

Before proceeding we need the following terminology. Let A be an algebra over a field F not of characteristic 2. Then A^+ will denote the algebra which is the same set as A with addition and scalar multiplication defined as in A and multiplication defined by $x.y = \frac{1}{2}(xy+yx)$, where juxtaposition denotes the product in A. Hence A^+ is a commutative algebra which is power-associative if A is

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power-associative.

The proof of the theorem will proceed as follows. We begin by letting A be a power-associative p-ring of characteristic not 2 and show that A^+ is a Jordan ring. The argument used here is similar to the one employed by Albert which showed that for any finite strictly power-associative division ring D, D^+ is a Jordan algebra [5, p. 133]. Next we shall show that A^+ is associative. For this purpose we first apply the Shirshov-Cohn Theorem and then the Vandermonde determinant argument employed by Forsythe and McCoy which showed that associative p-rings are commutative [3, p. 144]. The proof is completed by showing that if A is a power-associative algebra of characteristic not 2 such that for every a in A there exists an integer $n(a) > 1$, depending on a, with $a^{n(a)} = a$, then A is commutative and associative provided A^+ is associative.

2. Proof

In order to show that A^+ is a Jordan algebra, it is necessary to show that (x, y, x^2), the associator in x, y, and x^2, is zero for every x and y in A^+. For this purpose we look at the ring $\langle x \rangle$ generated by x. Clearly we may suppose that $x \neq 0$. Since A is power-associative, then $\langle x \rangle$ is associative. Also since A is a p-ring, it follows that $\langle x \rangle$ is a finite, semi-simple algebra over $\mathrm{GF}(p)$. Thus, $\langle x \rangle$ is equal to the direct sum of a finite number of copies of $\mathrm{GF}(p)$. Hence if $\langle x \rangle = \sum_{i=1}^{n} F_i$, with $F_i = \mathrm{GF}(p)$ for every $i = 1, \ldots, n$, then there are elements λ_i and $\mu_j \in \mathrm{GF}(p)$, $i, j = 1, \ldots, n$, with

$$x = \sum_{i=1}^{n} \lambda_i e_i \quad \text{and} \quad x^2 = \sum_{j=1}^{n} \mu_j e_j ,$$

where e_i is the identity of F_i.

Also it is clear that $\{e_i\}_{i=1}^{n}$ is a set of orthogonal idempotents in A^+. Now with this representation for x and x^2 it follows that

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$$(x, y, x^2) = \left(\sum_{i=1}^{n} \lambda_i e_i, y, \sum_{j=1}^{n} \mu_j e_j\right)$$
$$= \sum_{i,j=1}^{n} \lambda_i \mu_j (e_i, y, e_j)$$
$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \lambda_i \mu_j (e_i, y, e_j) ,$$

since A^+ is commutative and hence flexible. However, Albert has shown that in a commutative, power-associative algebra R, $(e, r, e') = 0$ for every $r \in R$ and every pair of orthogonal idempotents e, e' [5, Lemma 5.2, p. 133]. Thus, indeed $(x, y, x^2) = 0$, and A^+ is Jordan.

Next we claim that the Jordan algebra A^+ is associative. First of all note that because A^+ has no nonzero nilpotent elements, it suffices to show that A^+ is alternative. This is the case due to the well-known fact that any commutative, alternative ring without nonzero nilpotent elements is associative [4, Lemma 3, p. 1175]. Hence, we only need show that if A^+ is generated by two elements, then it is associative. Therefore, we can suppose by the Shirshov-Cohn Theorem that A^+ is a special Jordan algebra.

Let $\left(S(A^+), \sigma_u\right)$ be the special universal envelope of A^+. Then we have that σ_u is an injection mapping and A^+ can be assumed to be contained in $S(A^+)^+$. Now $S(A^+)$ has the following properties. It is an associative algebra with identity generated by $\{x : x \in A^+ \cup \mathrm{GF}(p)\}$. Also if $x, y \in A^+$ with $x.y$ their product in A^+ and xy their product in $S(A^+)$, then $x.y = \frac{1}{2}(xy+yx)$. Therefore, to show that A^+ is associative it suffices to show that $S(A^+)$ is commutative, since in this case A^+ is a subalgebra of $S(A^+)^+ = S(A^+)$. Clearly $S(A^+)$ is commutative if and only if $xy = yx$ for every $x, y \in A^+$.

Now with x and y in A^+, we have in $S(A^+)$

(1) $$(x+y)^p = x^p + y^p .$$

Therefore in $S(A^+)$,

$$A_1 + A_2 + \dots + A_{p-1} = 0 ,$$

where A_i is the sum of all words in the expansion of $(x+y)^p$ in which y appears i times and x appears $(p-i)$ times. Then substituting λy for y in (1) for any $\lambda \in \mathrm{GF}(p)$ we have

(2) $$\lambda A_1 + \lambda^2 A_2 + \dots + \lambda^{p-1} A_{p-1} = 0 .$$

Thus, if m denotes the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{p-1} \\ \vdots & \vdots & & \vdots \\ (p-1) & (p-1)^2 & \dots & (p-1)^{p-1} \end{bmatrix}$$

and if $m_1, \dots, m_{p-1}$ denote the co-factors of the elements of the first column of the above matrix, then by multiplying the λ-th equation in (2) by m_λ and adding we have from an elementary property of determinants that $mA_1 = 0$. But since the above determinant is a Vandermonde determinant it follows that m and p are relatively prime. Hence $A_1 = 0$. Now by an easy calculation we have

$$0 = xA_1 - A_1 x = x^p y - yx^p = xy - yx .$$

Therefore $S(A^+)$ is commutative, and hence A^+ is associative.

It remains to show that A is commutative, since then $A = A^+$ and the result will follow. For this purpose we have the following lemma.

LEMMA. *Let R be a power-associative algebra of characteristic not 2 such that for every a in R there exists a positive integer $n(a) > 1$, depending on a, with $a^{n(a)} = a$. Then R is commutative and* *associative if* R^+ *is associative.*

Proof. As above, it suffices to show that R is commutative. Let x and y be in R and look at the ring B generated by x and y. Since B^+ is a finitely generated, associative, commutative ring, without nonzero nilpotent elements and satisfying the hypothesis of the lemma, then B^+ is the direct sum of a finite number of Galois fields, that is, $B^+ = \sum_{i=1}^{n} F_i$ where each F_i is a finite field. Hence to show that A is commutative it suffices to show that $st = ts$ for every $s \in F_i$ and $t \in F_j$ for some choice of i and j. Also, we can suppose that neither s nor t is zero. If $i = j$, then there is a $z \in B$ and positive integers α and β with $z^\alpha = s$ and $z^\beta = t$. So by the power-associativity of B, $st = ts$. If $i \neq j$, then we look at the following identity which holds in any power-associative algebra not of characteristic 2

$$[a.b, c] + [a.c, b] + [b.c, a] = 0 \tag{3}$$

for every $a, b, c \in B$ [5, p. 129]. (Here $a.b$ denotes the product in B^+ and $[u, v]$ is the commutator in B of u and v.) Since B^+ is the direct sum of the fields F_i, $i = 1, \ldots, n$, then $u.v = 0$ for every $u \in F_i$, $v \in F_j$, $i \neq j$. Then by setting $a = s$, $b = s^{n(s)-1}$, $c = t$ in (3) it follows that the last two commutators in (3) are zero. Hence

$$0 = [s.s^{n(s)-1}, t] = [s, t] .$$

Therefore, B is indeed commutative, and since x and y were chosen arbitrarily, R is also commutative. This completes the proof of the lemma and also the proof of the theorem.

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