Canad. Math. Bull. Vol. 64 (4), 2021, pp. 970–983
http://dx.doi.org/10.4153/S0008439520000983

On Moser's regularization of the Kepler system: Positive and negative energies

Sebastián Ferrer and Francisco Crespo

Abstract. We present a generalization of Moser's theorem on the regularization of Keplerian systems that include positive and negative energies. Our approach does not consider the geodesics of the hyperboloid embedded in a Lorentz space for the unbounded orbits, as it is previously done in the literature. Instead, we connect the Keplerian positive and negative energy orbits with the harmonic oscillator with negative and positive frequencies. The connection is established through the canonical extension of the stereographic projection, as it is done in Moser's original paper. How we base our study reveals that Kustaanheimo–Stiefel map KS and Moser regularizations are alternative ways of showing the spatial Kepler system as a subdynamics of the 4D harmonic oscillator.

1 Introduction and main result

The spatial Kepler system describes the motion of a particle in a central potential and has the following energy function:

$$\mathcal{K}_\mu = \frac{1}{2}|\mathbf{y}|^2 - \frac{\mu}{|\mathbf{x}|}, \tag{1.1}$$

where μ is the positive gravitational constant and $\mathbf{x}, \mathbf{y} \in T^*\mathbb{R}^3_0$, with $\mathbb{R}^n_0 = \mathbb{R}^n - \{0\}$. In this paper, we consider the Kepler system's regularization, which has been done in various forms by several authors. After reparametrization of the independent variable and imposing several constraints, Moser [13, 15] and Kustaanheimo and Stiefel [9, 10] linked the Keplerian flow at each energy level with well-known linear systems. Each procedure has pros and cons; Moser's technique is easily stated for arbitrary dimension, but it is not suitable for positive energy in its original formulation. The Kustaanheimo–Stiefel map (KS) regularization is not restricted to bounded orbits; however, it works only for the spatial case. In this work, we focus on Moser's procedure.

The main result of this paper is the extension of the Moser theorem for the case of unbounded Keplerian orbits. This is achieved by employing a canonical stereographic-type transformation $\mathcal{ST}$, the result of considering the stereographic projection in the simplectic context, which will be detailed in Section 4. Furthermore,

Received by the editors September 3, 2020; revised November 19, 2020.
Published online on Cambridge Core December 18, 2020.
AMS subject classification: 70F15, 70F16, 70H05.
Keywords: Stereographic-type transformation, Kepler system, harmonic oscillator.

in our proof, the Keplerian orbits are embedded in the Hamiltonian flow given by

$$\mathcal{H}_\omega = \frac{1}{2}\left(|\mathbf{p}|^2 + \omega\,|\mathbf{q}|^2\right), \tag{1.2}$$

with $\mathbf{q}, \mathbf{p} \in T^*\mathbb{R}^4_0$. The frequency ω is assumed to be an arbitrary real number, which will allow to handle the unbounded orbits.

Theorem 1.1 (Extended Moser) *The inverse stereographic projection $\mathcal{ST}^{-1}$ connects positive and negative Keplerian energy orbits on $\mathcal{K}_\mu = k$ with the Hamiltonian flow given by (1.2) on the manifold $\mathcal{H}_\omega = h$, when properly constrained and reparametrized. Moreover, we have that*

(1) *Bounded orbits are linked to the geodesic flow in $T\mathcal{S}^3(r)$, with $r = |\mathbf{q}|$, which is an invariant manifold of the flow associated to (1.2) in the energy level $h = \omega r^2$. This connection yields a bijection between oriented great circles on $\mathcal{S}^3(r)$ and the hodographs of the bounded Keplerian orbits with energy $k = -\sqrt{\omega}/(2r^2)$ and $\omega > 0$.*
(2) *The map $\mathcal{ST}^{-1}$ connects the unbounded Keplerian orbits, with energy $k = 1/(2h^2)$ and the Hamiltonian flow given by (1.2) with negative frequency $\omega < 0$ and energy $\mathcal{H}_\omega = h$.*

The above theorem is proved in Section 5; it includes unbounded Keplerian orbits as a remarkable novelty. To the best of our knowledge, Belbruno [1] was the first one to deal with the case of positive energies. He replaced Moser's transformation and established the Kepler motion equivalence with the geodesic flow on a two-sheeted hyperboloid embedded in a Lorentz space (see also [16]). In their treatment, the authors prioritize the Kepler system's connection with the geodesic flow as Moser does it. Later, Kummer [8] gave a geometric explanation of Moser's procedure in terms of the hidden special orthogonal $(SO)(4)$ symmetry of the Kepler system and showed that it is intimately related to the KS-regularized Kepler system. Precisely, in [6, 8] the Kepler system is obtained due to the reduction of the action of a 1D subgroup of $SO(4)$, which in a more physical context is dubbed as a gauge group. Section 2 provides a set of symplectic coordinates especially well suited for this geometric setting. Moreover, we shall prove that time reparametrizations play a key role in leading straightforwardly to the KS or Moser regularization. Our methodology detaches from that of Belbruno since the connection with the geodesic flow is no longer the priority, which will only be maintained for bounded motions. Instead, our approach extends Moser's procedure not by force to maintain the geodesic flow but by fixing the stereographic-type map connecting the Kepler system and (1.2). As a reward, this strategy allows straightforwardly handling the positive energy case without introducing the mentioned elements and maintaining the same methodology as in the negative energies.

The symplectic stereographic-type transformation $\mathcal{ST}$ defined in Section 4 plays a key role in our approach. In configuration space, it is the same as the one given by Moser in [15, Section 1.6]. However, there is a slight difference in the canonical extension allowing us to consider the positive energy case. Precisely, while Moser in

[13, 15] was focused on the restriction to the tangent bundle of the sphere, we define a canonical extension valid in an open set of $T^*\mathbb{R}^4_0$.

The use of the Hamiltonian (1.2) for the regularization process is one of the possible choices. Indeed, in [15], Moser changed the Hamiltonian function employed previously by himself in [13]. In this regard, how different Hamiltonians could be chosen for regularization purposes is discussed in Section 3. The reason why we employ (1.2) is justified in Section 2. Precisely, by using an angle-based chart, dubbed as the Projective Euler variables, it is shown that one of the Hamiltonian flows used by Moser for the Kepler regularization is already obtained as a subdynamic of the harmonic oscillator. This fact suggests that the Hamiltonian function (1.2) is also suitable for Moser's regularization process. Moreover, with the aid of the Projective Euler chart, in Section 2, we show that KS and Moser regularizations are alternative ways of embedding the Keplerian flow in a 4-degrees of freedom (DOF) oscillator. Each of these regularizations corresponds with a different reparametrization of the Hamiltonian system (1.2). The relation between KS and Moser's methods was first detailed by Kummer in [8].

A shared disadvantage of KS and Moser's treatments is that they manage each energy level independently. In this respect, Ligon and Schaaf [11] provided an almost global treatment considering all positive energies and all negative energies separately. However, the process they gave asks for a considerable effort, and simplified versions were provided in [3, 4]. Additionally, in a recent publication, van der Meer [17] gives a full explanation of the relation between KS, Moser, and Ligon and Schaaf regularizations in terms of constructive geometric reduction. Still, another drawback of the Ligon and Schaaf regularization is pointed out in [4], where Cushman and Duistermaat claimed that, given a certain function f in the negative energy manifold, the composition with the Ligon-Schaaf map might not possess the required smoothness for the application of the averaging method. To the best of our knowledge, this question remains open.

Throughout this work, we will use several sets of variables. With a slight abuse of notation, we shall use the same symbol to denote functions expressed in different charts. The context will make clear which one we are referring to.

2 Projective Euler variables and the 4D oscillator

In [8], it is shown that the Moser regularization exposes the hidden $SO(4)$ symmetry of the Kepler system, and it is intimately related to the KS-regularized Kepler system. Precisely, in [6, 8] the Kepler system is obtained due to the reduction of the action of a 1D subgroup of $SO(4)$, which in a more physical context is dubbed as a gauge group. This section provides a set of symplectic coordinates especially well suited for this geometric setting. Moreover, we shall prove that time reparametrizations play a key role in this system, leading straightforwardly to the KS or Moser regularization.

The Projective Euler variables are based on Euler parameters and have been used in several works (see, for example, [5, 6] and the references therein). This set of variables has proved to be very useful to describe different dynamical aspects of the 4D oscillator. Notably, in [6], the Projective Euler variables establish the connection with the Kepler system and the spherical rotor straightforwardly. For the benefit of the reader, we briefly recall here the Projective Euler chart, which is defined through the transformation

$$\mathcal{PE} : (\rho, \phi, \theta, \psi, P, \Phi, \Theta, \Psi) \to (\mathbf{q}, \mathbf{p}),$$

given by

$$\begin{aligned} q_1 &= \sqrt{\rho}\, c_\alpha s_\beta, \quad p_1 = \frac{2}{\sqrt{\rho}} s_\beta \left(c_\alpha P\rho - s_\alpha \Theta\right) + \frac{\Phi + \Psi}{\sqrt{\rho}\, c_\alpha} c_\beta, \\ q_2 &= \sqrt{\rho}\, s_\alpha s_\gamma, \quad p_2 = \frac{2}{\sqrt{\rho}} c_\gamma \left(s_\alpha P\rho + c_\alpha \Theta\right) - \frac{\Phi - \Psi}{\sqrt{\rho}\, s_\alpha} s_\gamma, \\ q_3 &= \sqrt{\rho}\, s_\alpha s_\gamma, \quad p_3 = \frac{2}{\sqrt{\rho}} s_\gamma \left(s_\alpha P\rho + c_\alpha \Theta\right) + \frac{\Phi - \Psi}{\sqrt{\rho}\, s_\alpha} c_\gamma, \\ q_4 &= \sqrt{\rho}\, c_\alpha c_\beta, \quad p_4 = \frac{2}{\sqrt{\rho}} c_\beta \left(c_\alpha P\rho - s_\alpha \Theta\right) - \frac{\Phi + \Psi}{\sqrt{\rho}\, c_\alpha} s_\beta, \end{aligned} \tag{2.1}$$

where we have introduced the following notation with the aim of getting compact expressions $c_x = \cos x$ and $s_x = \sin x$. Moreover, $\alpha = \theta/2$, $\beta = (\phi + \psi)/2$, $\gamma = (\phi - \psi)/2$, and $(\rho, \phi, \theta, \psi) \in \Lambda = R^+ \times (0, 2\pi) \times (0, \pi) \times (0, 2\pi)$. Excluding the manifolds $\mathcal{M}_1 = \{(q, Q) | q_1 = q_4 = 0\}$ and $\mathcal{M}_2 = \{(q, Q) | q_2 = q_3 = 0\}$, the Hamiltonian (1.2) in the new variables reads as follows:

$$\mathcal{H}_\omega = \frac{\omega}{2}\rho + 2\rho P^2 + \frac{1}{\rho}\, \mathcal{Z}(\theta, \_, \_, \Theta, \Phi, \Psi), \tag{2.2}$$

where

$$\mathcal{Z} = 2\left(\Theta^2 + \frac{\Phi^2 + \Psi^2 - 2\,\Phi\Psi \cos\theta}{\sin^2\theta}\right). \tag{2.3}$$

In the above expression, the variables ϕ and ψ are cyclic, with Φ and Ψ as the corresponding first integrals. In addition, the computation of the Poisson brackets shows that the function $\mathcal{Z}$ is an integral for $\mathcal{H}_\omega$. This function will play a key role in what follows and defines a well-known Hamiltonian dubbed as the spherical rotor, which is obtained by making equal all the principal moments of inertia of the free rigid body system in Euler angles (see [12]). Mainly, the equations of motion associated with $\mathcal{Z}$ describe the dynamics of a full symmetric rigid body, with uniform distribution of mass and freely rotating without external interactions. That is to say, the flow generated by $\mathcal{Z}$, when restricted to a sphere of arbitrary radius, corresponds to uniform rotations.

2.1 The role of time reparametrizations

Now, we explore two reparametrizations of the independent variable for the 4D oscillator system. Each of them will reveal a particular 3D subdynamics of the original system. The following theorem was first given in [6] .

Theorem 2.1 *Considering the time reparametrization $d\tau = (4\rho)^{-1} ds$ on system (1.2), and fixing the energy level $\mathcal{H}_\omega = h$, the flow is described by the Hamiltonian $\tilde{\mathcal{H}}_\omega = \frac{1}{4\rho}(\mathcal{H}_\omega - h)$ on the manifold $\tilde{\mathcal{H}}_\omega = 0$, which in the Projective Euler chart is given by*

$$\tilde{\mathcal{H}}_\omega = \tilde{\mathcal{K}}_\mu + \frac{\Psi^2 - 2\Phi\Psi\cos\theta}{2\rho^2\sin^2\theta}, \tag{2.4}$$

where

$$\tilde{\mathcal{K}}_\mu = \frac{1}{2}\left(P^2 + \frac{\Theta^2}{\rho^2} + \frac{\Phi^2}{\rho^2\sin^2\theta}\right) - \frac{\mu}{\rho}, \tag{2.5}$$

and $\mu = h/4$.

Proof It is a straightforward computation using the Projective Euler chart (see [6, Theorem 1]). ∎

Remark 2.2 Notice that by restricting to $\Phi = 0$ or $\Psi = 0$, we are led to the 3D Kepler system in spherical coordinates (see [7]). Indeed, the expressions for these momenta in Cartesian coordinates are given by

$$\begin{aligned} \Phi(q,p) &= \frac{1}{2}(p_1q_4 - p_4q_1 - p_2q_3 + p_3q_2), \\ \Psi(q,p) &= \frac{1}{2}(p_1q_4 - p_4q_1 + p_2q_3 - p_3q_2). \end{aligned} \tag{2.6}$$

Namely, each of them is one of the possible choices for the bilinear relation constraint imposed for the KS transformation. Moreover, in [6], we also showed that the KS map connects both systems in rectangular coordinates, just by inverting the spherical coordinates transformation

$$\mathcal{S} : (\rho, \phi, \theta, R, \Phi, \Theta) \to (x_1, x_2, x_3, y_1, y_2, y_3),$$

$$\begin{aligned} x_1 &= \rho\, s_\theta\, c_\phi, & y_1 &= 1/2\,\rho\, s_\theta\left[c_\phi\left(\Theta s_{2\theta} + 2P\rho s_\theta^2\right) - 2\Phi s_\phi\right], \\ x_2 &= \rho\, s_\theta\, s_\phi, & y_2 &= 1/2\,\rho\, s_\theta\left[s_\phi\left(\Theta s_{2\theta} + 2P\rho s_\theta^2\right) + 2\Phi c_\phi\right], \\ x_3 &= \rho\, c_\theta, & y_3 &= Pc_\theta - \Theta s_\theta/\rho, \end{aligned} \tag{2.7}$$

with $(\rho, \phi, \theta) \in \Lambda_4 = R^+ \times (0, 2\pi) \times (0, \pi)$. In other words, the composition of the previous transformations $\mathcal{S} \circ P_4 \circ \mathcal{PE}^{-1}$

$$(q,p) \to (\rho, \phi, \theta, \psi, P, \Phi, \Theta, \Psi) \to (\rho, \phi, \theta, P, \Phi, \Theta) \to (x_1, x_2, x_3, y_1, y_2, y_3),$$

where P_4 is the projection over $(\rho, \phi, \theta, P, \Phi, \Theta)$, is given by

$$\begin{aligned} x_1 &= 2(q_2q_4 - q_1q_3), & y_1 &= 1/2\rho\,(p_4q_2 + p_2q_4 - p_1q_3 - p_3q_1), \\ x_2 &= 2(q_1q_2 + q_3q_4), & y_2 &= 1/2\rho\,(p_2q_1 + p_1q_2 + p_4q_3 + p_3q_4), \\ x_3 &= q_1^2 - q_2^2 - q_3^2 + q_4^2, & y_3 &= 1/2\rho\,(p_1q_1 - p_2q_2 - p_3q_3 + p_4q_4), \end{aligned} \tag{2.8}$$

https://doi.org/10.4153/S0008439520000983 Published online by Cambridge University Press

which is the KS map. This fact shows that the following diagram is commutative:

$$
\begin{tikzcd}
\Psi^{-1}(0) \subset T^*\mathbb{R}^4_0 \arrow[r, "\mathcal{P}\mathcal{E}^{-1}"] \arrow[d, "KS"'] & T^*\Lambda \arrow[d, "p_4"] \\
T^*\mathbb{R}^3_0 & T^*\Lambda_4, \arrow[l, "\mathcal{S}"]
\end{tikzcd}
\tag{2.9}
$$

where $\Psi^{-1}(0) = \{(q,p) \in T^*\mathbb{R}^4_0 : \ \Psi(q,p) = 0\}$.

Note that this operation is equivalent to reducing the S^1 symmetry given by the Hamiltonian action associated with $\Phi(q,p)$ or $\Psi(q,p)$. A detailed treatment of this process is given in [2, 6].

The following result explores an alternative time reparametrization. It shows a connection between the 4D oscillator and another remarkable 3D physical system. Namely, employing a suitable reparametrization, the oscillator may be separated into two 1-DOF subsystems being one of them the spherical rotor.

Theorem 2.3 *Let us consider the 4D harmonic oscillator expressed in the Projective Euler chart. Then, by fixing the energy level $\mathcal{H}_\omega = h$ and carrying out the time reparametrization*

$$d\tau = \rho\, ds, \tag{2.10}$$

the flow is described by the Hamiltonian $\tilde{\tilde{\mathcal{H}}}_\omega = \rho\,(\mathcal{H}_\omega - h)$ on the manifold $\tilde{\tilde{\mathcal{H}}}_\omega = 0$. After this manipulation, the 4D harmonic oscillator becomes separable and includes the dynamics of the spherical rotor.

Proof According to the Poincaré technique with $d\tau = \rho\, ds$, we obtain the following expression for $\tilde{\tilde{\mathcal{H}}}_\omega$ in the Projective Euler chart:

$$\tilde{\tilde{\mathcal{H}}}_\omega = \rho\,(\mathcal{H}_\omega - h) = \mathcal{K}_\rho + \mathcal{K}_\theta, \tag{2.11}$$

where

$$\mathcal{K}_\rho = \frac{\omega\rho^2}{2} + 2\rho^2 P^2 - h\,\rho, \qquad \mathcal{K}_\theta = 2\left(\Theta^2 + \frac{\Phi^2 + \Psi^2 - 2\,\Phi\Psi\,\cos\theta}{\sin^2\theta}\right). \tag{2.12}$$

That is to say, $\mathcal{K}$ has been separated to two 1-DOF subsystems being the function $\mathcal{K}_\theta = \mathcal{Z}$, the Hamiltonian defining the system associated to the spherical rotor. ∎

The connection of the 4D oscillator and the spherical rotor may appear less important than the previous one with the Kepler system. However, considering the expression of $\mathcal{K}_\theta$ in rectangular coordinates, the link with the Moser regularization is established. Precisely, we have

$$
\begin{aligned}
\mathcal{K}_\theta(\mathbf{q},\mathbf{p}) &= \mathcal{Z} = \frac{1}{2}\left(|\mathbf{q}|^2|\mathbf{p}|^2 - \langle\mathbf{q},\mathbf{p}\rangle^2\right), \\
\mathcal{K}_\rho(\mathbf{q},\mathbf{p}) &= \frac{1}{2}\left(\omega|\mathbf{q}|^4 + \langle\mathbf{q},\mathbf{p}\rangle^2 - 2h|\mathbf{q}|^2\right).
\end{aligned}
\tag{2.13}
$$

https://doi.org/10.4153/S0008439520000983 Published online by Cambridge University Press

The Hamiltonian function $\mathcal{Z}$ is precisely the one considered by Moser and Zehnder in [15] to regularize the Keplerian flow, which after constraining to $|\mathbf{q}| = 1$ and $\langle \mathbf{q}, \mathbf{p} \rangle = 0$, leads to the geodesic flow on the sphere S^3. This fact suggests that the harmonic oscillator may also lead to the Kepler system through the Moser procedure. The next section investigates this issue.

Indeed, constraints $|\mathbf{q}| = 1$ and $\langle \mathbf{q}, \mathbf{p} \rangle = 0$ are equivalent to $\rho = 1$ and $\rho P = 0$. Moreover, for the case $\omega = h = 1$, and taking into account expression (2.12), the constrained manifold obtained from imposing $\rho = 1$ and $\rho P = 0$ is an invariant manifold of the Hamiltonian system given by (2.11). This can be checked by noting that ρ and ρP commute with $\mathcal{K}_\theta$, and considering the equations of motion associated to $\mathcal{K}_\rho$ given by

$$\dot{\rho} = 4\rho^2 P, \quad \dot{P} = \omega\rho + 4\rho P^2 - h = \rho + 4\rho P^2 - 1.$$

3 Alternative Moser regularizations

In this section, we investigate a broad family of Hamiltonian functions giving the geodesic flow when restricted to TS^3. Note that Moser used two different Hamiltonians to regularize the Keplerian flow. Indeed, in [13, 15], the Kepler system connection was made through the Hamiltonian system defined by the following functions:

$$\mathcal{Z} = \frac{1}{2}\left(|\mathbf{q}|^2|\mathbf{p}|^2 - \langle \mathbf{q}, \mathbf{p} \rangle^2\right), \quad \mathcal{M} = \frac{1}{2}\left(|\mathbf{q}|^2|\mathbf{p}|^2\right). \tag{3.1}$$

Of course, both systems define the same vector field on the tangent bundle of the sphere TS^3, the Hamiltonian version of the geodesic flow. That is to say, given a Riemannian manifold (M, g), the geodesic flow on M may be given by the restriction to the configuration space of any of the Hamiltonian flows on TM or T^*M defined, respectively, by the Hamiltonian functions

$$H(x, y) = \frac{1}{2} g_x(y), \quad \tilde{H}(x, p_y) = \frac{1}{2}\tilde{g}_x(p_y),$$

where $\tilde{g}$ is the inverse of the metric tensor. Considering $M = \mathbb{R}^4$, with the metric tensor given by the usual Euclidean inner product, we have that the geodesic flow is defined in the submanifold $|\mathbf{q}| = r$ and $\langle \mathbf{q}, \mathbf{p} \rangle = 0$, together with the energy surface $\tilde{H}(\mathbf{q}, \mathbf{p}) = |\mathbf{p}|^2 = p^2$. We are using the terminology of geodesic flow in a broad sense, allowing the flow to traverse geodesics at constant arbitrary velocity, rather that imposing $p = 1$.

The reparametrization and splitting of the 4-DOF harmonic oscillator given in Theorem 2.3 suggest that the regularization procedure given by Moser in [13] may also work for the oscillator. In this regard, we shall characterize a wide family of Hamiltonians allowing the Kepler regularization through the Moser procedure.

Any Hamiltonian flow in $T^*\mathbb{R}^4$ containing the geodesic flow of $S^3(r)$ (at constant velocity) must satisfy that the submanifold

$$\Omega_{r,p} = \{(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^4_0 : \ |\mathbf{q}| = r, \ |\mathbf{p}| = p, \ \langle \mathbf{q}, \mathbf{p} \rangle = 0\}, \tag{3.2}$$

https://doi.org/10.4153/S0008439520000983 Published online by Cambridge University Press

is invariant, where $\Omega_{r,p}$ is a submanifold of $TS^3(r)$ containing the geodesics travelled at constant rate p. Moreover, the functions $|\mathbf{q}|^2$, $|\mathbf{p}|^2$, and $\langle \mathbf{q}, \mathbf{p}\rangle$ span the following Lie algebra.

Proposition 3.1 *In $T^*\mathbb{R}^4 \equiv T\mathbb{R}^4 \equiv \mathbb{R}^8$ with the standard Poisson bracket $\{,\}$, the set of functions $G = \{G_1 = |\mathbf{q}|^2/2,\ G_2 = |\mathbf{p}|^2/2,$ and $G_3 = \langle \mathbf{q}, \mathbf{p}\rangle\}$ span a Lie algebra in $C^\infty(\mathbb{R}^8)$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Therefore, $G \equiv SL(2,\mathbb{R})$, where G denotes the group generated by the flows of G_i. Moreover, the function $\mathcal{Z}$ given in (3.1) is the generator of the center of G.*

Proof The computation of the Poisson bracket for G_1, G_2, and G_3 yields $\{G_1, G_2\} = G_3$, $\{G_1, G_3\} = 2G_1$, and $\{G_2, G_3\} = 2G_2$, that is, those functions span a Lie algebra isomorphic to $\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R})$ and $G \equiv SL(2,\mathbb{R})$. Finally, just a direct computation is needed to show that $\{\mathcal{Z}, F\} = 0$ for any F in G, and consequently, $\mathcal{Z}$ is the generator of the center of G. ■

Proposition 3.1 shows the key role that the function $\mathcal{Z}$ plays. It is the Hamiltonian function of the spheric rotor and the generator of the center of G. Thus, every submanifold given by the combination of any of the constraints $|\mathbf{q}|^2 = k_1$, $|\mathbf{p}|^2 = k_2$, or $\langle \mathbf{q}, \mathbf{p}\rangle = k_3$ is an invariant manifold for the Hamiltonian system associated to $\mathcal{Z}$. In this regard, it is no wonder that Moser chose it to define the geodesic flow after the corresponding constraints are imposed. However, there are many functions $f(x,y,z)$, evaluated in $x = |\mathbf{q}|^2$, $y = |\mathbf{p}|^2$, and $z = \langle \mathbf{q}, \mathbf{p}\rangle$, which may also be suitable for this purpose. To be precise, such a function must satisfy that when constrained to $\Omega_{r,p}$, the following brackets vanish:

$$\{F(\mathbf{q},\mathbf{p}), |\mathbf{q}|^2\} = \{F(\mathbf{q},\mathbf{p}), |\mathbf{p}|^2\} = \{F(\mathbf{q},\mathbf{p}), \langle \mathbf{q}, \mathbf{p}\rangle\} = 0, \quad \mathbf{q}, \mathbf{p} \in \Omega_{r,p},$$

for some r and p, and $F(\mathbf{q},\mathbf{p}) = f(|\mathbf{q}|^2, |\mathbf{p}|^2, \langle \mathbf{q}, \mathbf{p}\rangle)$. These conditions impose the following relations involving partial derivatives:

$$|\mathbf{q}|^2 f_x = |\mathbf{p}|^2 f_y, \quad |\mathbf{p}|^2 f_z + 2\langle \mathbf{q}, \mathbf{p}\rangle f_x = 0, \quad |\mathbf{q}|^2 f_z + 2\langle \mathbf{q}, \mathbf{p}\rangle f_y = 0. \tag{3.3}$$

Note that any f in the center of G satisfies the above conditions. Nevertheless, there are functions, not in the center of G, for which (3.3) holds. For instance, the harmonic oscillator $\mathcal{H}_\omega$, associated to $f(x,y) = 1/2(y^2 + \omega x^2)$ and $\omega > 0$, is a possible choice. Precisely, for $r = p/\sqrt{\omega}$ and $\langle \mathbf{q}, \mathbf{p}\rangle = 0$, we have that

$$\{\mathcal{H}_\omega, |\mathbf{q}|^2\} = -2\langle \mathbf{q}, \mathbf{p}\rangle = 0, \quad \{\mathcal{H}_\omega, |\mathbf{p}|^2\} = 2\omega\langle \mathbf{q}, \mathbf{p}\rangle = 0,$$

and

$$\{\mathcal{H}_\omega, \langle \mathbf{q}, \mathbf{p}\rangle\} = \omega|\mathbf{q}|^2 - |\mathbf{p}|^2 = \omega r^2 - p^2 = 0.$$

Therefore, among all the submanifolds $\Omega_{r,p}$ in $T^*\mathbb{R}^4$, the harmonic oscillator only leaves invariant that one given by $|\mathbf{q}| = r = p/\sqrt{\omega}$, $|\mathbf{p}| = p$ and $\langle \mathbf{q}, \mathbf{p}\rangle = 0$. According to Theorem 2.3, the Hamiltonian vector fields generated by the functions $\mathcal{H}_\omega$ and $\mathcal{Z}$ agree along this manifold. Furthermore, as it was showed by Moser in [15], the constrained Hamiltonian $\mathcal{Z}$ leads to the geodesic flow, hence so does $\mathcal{H}_\omega$.

https://doi.org/10.4153/S0008439520000983 Published online by Cambridge University Press

4 Symplectic stereographic-type transformation

This section is devoted to the transformation connecting the dynamics of the Kepler and oscillator flows. We define our transformation based on Moser's work, but it is not the same as we are going to see. Our variation allows us to extend Moser's regularization to the positive energy case. In our view, the reason this work was not finished in [13] is that Moser was not concerned about parabolic or hyperbolic motions, as it is indicated in the title of the paper [13]. On the contrary, he was more focused on small perturbations of bounded Keplerian orbits related to big questions as the stability of the solar system [14].

First, we define the transformation in the configuration space. For this purpose, we consider the following stereographic-type transformation, which was dubbed by Moser in [15] as the "homogeneous" version of the stereographic mapping:

$$\Sigma\Pi : \mathbb{R}^4_0 - L_0 \longrightarrow \Delta \subset \mathbb{R}^4_0, \qquad \mathbf{q} \to \mathbf{x},$$

and defined as follows

$$x_0 = |\mathbf{q}|, \quad x_j = \frac{q_j}{|\mathbf{q}| - q_0}, \tag{4.1}$$

where L_0 is the closed positive q_0-axis and Δ is the open set of $\mathbb{R}^4_0$ defined by $x_0 > 0$. Notice that this map is related to the stereographic projection, but it is not the same. Indeed, the following geometrical interpretation was given in [15, Section 1.6]. *"For $x_0 = 1$, the transformation (4.1) is the stereographic projection, while for arbitrary x_0, it maps rays through the origin into vertical half lines on $\mathbb{R}^4_0$ such that the heights x_0 agree with the distance on the ray."* Now, we carry out the canonical extension by considering the corresponding generating function

$$W(\mathbf{x},\mathbf{p}) = p_0 x_0 \frac{|\tilde{\mathbf{x}}|^2 - 1}{|\tilde{\mathbf{x}}|^2 + 1} + \frac{2x_0}{|\tilde{\mathbf{x}}|^2 + 1}(p_1x_1 + p_2x_2 + p_3x_3).$$

Thus, the canonical extension is given by

$$\mathcal{ST} : T^*(\mathbb{R}^4_0 - L_0) \longrightarrow \Delta \times \mathbb{R}^4 \subset T^*\mathbb{R}^4_0, \quad (\mathbf{q},\mathbf{p}) \to (\mathbf{x},\mathbf{y}),$$

$$\begin{aligned} x_0 &= |\mathbf{q}|, & x_j &= \frac{q_j}{|\mathbf{q}| - q_0}, \\ y_0 &= \frac{\langle \mathbf{q},\mathbf{p}\rangle}{|\mathbf{q}|}, & y_i &= (|\mathbf{q}| - q_0)p_i - \left(\frac{\langle \mathbf{q},\mathbf{p}\rangle}{|\mathbf{q}|} - p_0\right)q_i. \end{aligned} \tag{4.2}$$

This transformation has been named symplectic stereographic-type transformation due to the geometrical relation with the stereographic projection in configuration space. Next, for the sake of completeness, we also provide the inverse transformation of the stereographic-type map given above

$$\mathcal{ST}^{-1} : \Delta \times \mathbb{R}^4 \subset T^*\mathbb{R}^4_0 \longrightarrow T^*(\mathbb{R}^4_0 - L_0), \quad (\mathbf{x},\mathbf{y}) \to (\mathbf{q},\mathbf{p}).$$

The explicit expression of the inverse is obtained by considering the inverse of the configuration space transformation given in (4.1). Then, by using the generating function

$$\mathcal{W}(\mathbf{q},\mathbf{y}) = y_0|\mathbf{q}| + \frac{1}{|\mathbf{q}| - q_0}\,(y_1q_1 + y_2q_2 + y_3q_3),$$

we arrive to the inverse stereographic-type transformation

$$\begin{aligned} q_0 &= x_0\,\frac{|\tilde{\mathbf{x}}|^2 - 1}{|\tilde{\mathbf{x}}|^2 + 1}, & q_j &= x_0\,\frac{2x_j}{|\tilde{\mathbf{x}}|^2 + 1}, \\ p_0 &= \frac{\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle}{x_0} + y_0\frac{|\tilde{\mathbf{x}}|^2 - 1}{|\tilde{\mathbf{x}}|^2 + 1}, & p_j &= \frac{|\tilde{\mathbf{x}}|^2 + 1}{2\,x_0}\,y_j - \langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle\,\frac{x_j}{x_0} + \frac{2y_0x_j}{|\tilde{\mathbf{x}}|^2 + 1}, \end{aligned} \tag{4.3}$$

where $\mathbf{z} = (z_0, z_1, z_2, z_3)$, $\tilde{\mathbf{z}} = (z_1, z_2, z_3)$, and $j = 1, 2, 3$.

The direct stereographic-type transformation is the same as the one given by Moser in [15, Section 1.6]. However, by comparison of (4.3) with the formulas (1.82) given in [15], the reader may observe a slight difference in the inverse stereographic-type transformation, allowing us to consider the positive energy case. Precisely, while Moser in [13, 15] was focused on the restriction to the tangent bundle of the sphere, we define a canonical extension between open sets in $T^*\mathbb{R}^4_0$.

5 Proof of Theorem 1.1

This section demonstrates Theorem 1.1 by introducing a suitable change of variables and distinguishing the cases of positive and negative energies.

5.1 Differential system in stereographic variables

First, we consider the inverse symplectic stereographic-type transformation given in (4.3). After applying this change of variables, we can replace the Hamiltonian $\mathcal{H}_\omega(\mathbf{q},\mathbf{p})$ given in (1.2) by

$$\mathcal{G}_\omega(\mathbf{x},\mathbf{y}) = \frac{\omega x_0^2 + y_0^2}{2} + \frac{1}{2x_0^2}\frac{1}{4}\left(|\tilde{\mathbf{x}}|^2 + 1\right)^2|\tilde{\mathbf{y}}|^2. \tag{5.1}$$

Previous to the regularization, we prepare the Hamiltonian system associated to (5.1). For this purpose, we consider a function $\mathcal{F}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ satisfying

$$\mathcal{F}^2(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \frac{1}{4}\left(|\tilde{\mathbf{x}}|^2 + 1\right)^2|\tilde{\mathbf{y}}|^2. \tag{5.2}$$

Of course, we have two options, which will be used for the cases of positive and negative energies. Thus, $\mathcal{G}_\omega$ is given by

$$\mathcal{G}_\omega(\mathbf{x},\mathbf{y}) = \frac{\omega x_0^2 + y_0^2}{2} + \frac{1}{2x_0^2}\mathcal{F}^2. \tag{5.3}$$

Moreover, the function $\mathcal{F}$ is an integral of the Hamiltonian $\mathcal{G}_\omega$, which associated equations of motion are given by

$$\begin{aligned} &\dot{x}_0(t) = y_0(t), \qquad \dot{y}_0(t) = -\omega x_0(t) + \frac{1}{x_0(t)^3}\mathcal{F}^2, \\ &\dot{\mathbf{x}}(t) = \frac{1}{x_0^2(t)}\mathcal{F}\mathcal{F}_{\tilde{\mathbf{y}}}, \quad \dot{\mathbf{y}}(t) = -\frac{1}{x_0^2(t)}\mathcal{F}\mathcal{F}_{\tilde{\mathbf{x}}}. \end{aligned} \tag{5.4}$$

5.2 Part 1. Regularization of bounded orbits

For the regularization of the bounded orbits, we consider the case of $\omega > 0$. Then, we restrict the 4D oscillator to the invariant submanifold $\Omega_r \subset T^*(\mathbb{R}_0^4 - L_0)$ given by

$$\Omega_r = \{(\mathbf{q},\mathbf{p}) \in TS^3(r) \subset T^*\mathbb{R}_0^4 : \ |\mathbf{q}| = r, \ |\mathbf{p}| = p = r\sqrt{\omega}, \ \langle \mathbf{q},\mathbf{p}\rangle = 0, \ q_0 \neq r\}. \tag{5.5}$$

We already proved that the flow of the harmonic oscillator and $\mathcal{Z}$ agree in Ω_r, and hence they both agree with the geodesic flow. Thus, we only need to prove that the transformation (4.3) connects the negative energy orbits of the Kepler system with $\mathcal{H}_\omega$.

The invariant manifold Ω_r corresponds in the $\mathbf{xy}$-space to

$$\Omega_r^* = \{(\mathbf{x},\mathbf{y}) \in T^*\mathbb{R}_0^4 : \ x_0 = r, \ y_0 = 0, \ \left(|\tilde{\mathbf{x}}|^2 + 1\right)^2 |\tilde{\mathbf{y}}|^2 = 4r^4\omega\}, \tag{5.6}$$

where r is a positive real constant. By imposing this restriction to the invariant manifold Ω_r^*, and taking into account that $x_0 = r$ and $\mathcal{F} = r^2\sqrt{\omega}$, the equations of motion (5.4) are given as follows:

$$\begin{aligned} &\dot{x}_0 = 0, \qquad \dot{y}_0 = 0, \\ &\dot{\mathbf{x}} = \tilde{\mathcal{F}}_{\tilde{\mathbf{y}}}, \quad \dot{\mathbf{y}} = -\tilde{\mathcal{F}}_{\tilde{\mathbf{x}}}, \end{aligned} \tag{5.7}$$

where $\tilde{\mathcal{F}}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \sqrt{\omega}/2\left(|\tilde{\mathbf{x}}|^2 + 1\right)|\tilde{\mathbf{y}}|$, and we are restricted to the energy manifold $\tilde{\mathcal{F}}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \omega\, r^2$.

Now, we perform another change of variables, and by abuse of notation, we use the symbols $(\mathbf{x},\mathbf{y})$ in the following way:

$$\tilde{\mathbf{x}} = r\,\mathbf{y}, \quad \tilde{\mathbf{y}} = -\frac{1}{r}\,\mathbf{x}, \tag{5.8}$$

which leads to the Hamiltonian

$$\tilde{\tilde{\mathcal{F}}}(\mathbf{x},\mathbf{y}) = \frac{\sqrt{\omega}}{2}\left(r^2|\mathbf{y}|^2 + 1\right)\frac{1}{r}\,|\mathbf{x}| - \omega\, r^2.$$

However, the context makes clear what objects are we referring to in each case. Then, after this symplectic change of variables, the introduction of a new independent variable

$$dt = \left(\sqrt{\omega}\, r\,|\mathbf{x}|\right)^{-1} d\tau$$

and the elimination of constant terms lead to the system given by the Hamiltonian function

$$\mathcal{F}_\mu(\mathbf{x}, \mathbf{y}) = \frac{1}{2}|\mathbf{y}|^2 - \frac{\mu}{|\mathbf{x}|}, \tag{5.9}$$

in the manifold $\mathcal{F}_\mu = -1/(2r^2)$, where $\mu = \sqrt{\omega}\, r$. That is to say, the Kepler system at the level energy $-\sqrt{\omega}/(2r^2)$ is connected with the 4D isotropic oscillator at the level energy $\omega\, r^2$.

5.3 Part 2. Regularization of unbounded orbits

The regularization of the unbounded orbits is related to the case $\omega < 0$ and positive energy $\mathcal{H}_\omega = h > 0$. The assumption of negative frequencies implies that Ω_r^* is no longer an invariant manifold. Moreover, we choose the integral function $\mathcal{F}$ in (5.2) to be

$$\mathcal{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = -\frac{1}{2}\left(|\tilde{\mathbf{x}}|^2 + 1\right)|\tilde{\mathbf{y}}|, \tag{5.10}$$

with fixed value $\mathcal{F} = f < 0$.

Equations of system (5.4) can be separated in two parts. Variables (x_0, y_0) define a 1-DOF subsystem, and by using the Hamiltonian invariance $\mathcal{G}_\omega(\mathbf{x}, \mathbf{y}) = h$, the variable y_0 is expressed as follows:

$$y_0^2 = 2h + |\omega| x_0^2 - f^2/x_0^2.$$

Therefore, the differential equation for $\dot{x}_0$ is integrated by rewriting it as

$$\dot{x}_0 = \pm\sqrt{2h + |\omega| x_0^2 - f^2/x_0^2}. \tag{5.11}$$

Once the subsystem $(x_0(t), y_0(t))$ is solved, we introduce the following new independent variable:

$$ds = f^2/x_0^2(t)\, dt.$$

Then, subsystem $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ becomes

$$\dot{\tilde{\mathbf{x}}}(t) = \mathcal{F}_{\tilde{\mathbf{y}}}, \quad \dot{\tilde{\mathbf{y}}}(t) = -\mathcal{F}_{\tilde{\mathbf{x}}}. \tag{5.12}$$

By restricting to the energy level $\mathcal{F} = f$ and introducing the following symplectic change of coordinates

$$\tilde{\mathbf{x}} = h\,\mathbf{y}, \quad \tilde{\mathbf{y}} = -\frac{1}{h}\,\mathbf{x}, \tag{5.13}$$

we obtain that system (5.12) is given by the Hamiltonian

$$\hat{\mathcal{F}}(\mathbf{x}, \mathbf{y}) = -\frac{1}{2}\left(h^2|\mathbf{y}|^2 + 1\right)|\mathbf{x}|/h - f. \tag{5.14}$$

https://doi.org/10.4153/S0008439520000983 Published online by Cambridge University Press

Finally, we introduce a new independent variable τ,

$$ds = -(h\,|\mathbf{x}|)^{-1}\,d\tau,$$

which leads to the system defined by the Hamiltonian function

$$\mathcal{F}_\mu(\mathbf{x},\mathbf{y}) = \frac{1}{2}|\mathbf{y}|^2 - \frac{\mu}{|\mathbf{x}|}, \tag{5.15}$$

in the manifold $\mathcal{F}_\mu = 1/(2h^2)$, where $\mu = -f/h > 0$. That is to say, the Kepler system at the positive energy level $1/(2h^2)$ is connected with the system given by $\mathcal{H}_\omega$ at the level energy h.

Acknowledgment Support from the Universidad del Bío-Bío and the Research Agencies of Spain and Chile is acknowledged. They came in the form of research project ESP2017-87271-P of the Ministry of Science of Spain, the project 11160224 of the Chilean national agency FONDECYT, and the UBB project 2020157 IF/R. We deeply appreciate comments and suggestions of the referee who contributed to the improvement and clarity of the paper.

References

[1] E. A. Belbruno, *Two-body motion under the inverse square central force and equivalent geodesic flows*. Celest. Mech. **15**(1977), no. 4, 467–476.

[2] F. Crespo and S. Ferrer, *Alternative reduction by stages of Keplerian systems. Positive, negative, and zero energy*. SIAM J. Appl. Dyn. Syst. **19**(2020), no. 2, 1525–1539. http://dx.doi.org/10.1137/19M1264060.

[3] R. H. Cushman and L. M. Bates, *Global aspects of classical integrable systems*. 2nd ed., Birkhäuser Verlag, Basel, 2015.

[4] R. H. Cushman and J. J. Duistermaat, *A characterization of the Ligon-Schaaf regularization map*. Comm. Pure Appl. Math. **50**(1997), no. 8, 773–787.

[5] S. Ferrer and F. Crespo, *Parametric quartic Hamiltonian model. A unified treatment of classic integrable systems*. J. Geom. Mech. **6**(2014), no. 4, 479–502.

[6] S. Ferrer and F. Crespo, *Alternative angle-based approach to the $\mathcal{KS}$-map. An interpretation through symmetry and reduction*. J. Geom. Mech. **10**(2018), no. 3, 359–372. http://dx.doi.org/10.3934/jgm.2018013.

[7] H. Goldstein, C. Poole, and J. Safko, *Classical mechanics*. 3rd ed., Addison Wesley, New York, 2002.

[8] M. Kummer, *On the regularization of the Kepler problem*. Commun. Math. Phys. **84**(1982), no. 1, 133–152.

[9] P. Kustaanheimo, *Spinor regularization of the Kepler motion*. Annales Universitatis Turkuensis 73(1964), no. 3, 3–7.

[10] P. Kustaanheimo and E. Stiefel, *Perturbation theory of Kepler motion based on spinor regularization*. J. Reine Angew. Math. **218**(1965), 204–219.

[11] T. Ligon and M. Schaaf, *On the global symmetry of the classical Kepler problem*. Rep. Math. Phys. **9**(1976), no. 3, 281–300.

[12] J. E. Marsden and T. S. Ratiu, *Introduction to mechanics and symmetry*. 2nd ed., Springer-Verlag, Inc., New York, 1999.

[13] J. Moser, *Regularization of Kepler's problem and the averaging method on a manifold*. Commun. Pure Appl. Math. **XXIII**(1970), 609–636.

[14] J. Moser, *Is the solar system stable*? Math. Intell. **1**(1978), no. 2, 65–71.

[15] J. Moser and E. J. Zehnder, *Notes on dynamical systems*. AMS and the Courant Institute of Mathematical Sciences at New York University, Providence, 2005.

[16] Y. S. Osipov, *The Kepler problem and geodesic flows in spaces of constant curvature*. Celest. Mech. **16**(1977), no. 2, 191–208.

[17] J. C. van der Meer, *Reduction of the harmonic oscillator and regularization of the Kepler problem*. CASA-Rep. **20**(2020), no. 1, 23.

Grupo de Dinámica Espacial, Departamento de Ingeniería y Tecnología de Computadores, Facultad de Informática, Universidad Murcia, Murcia 30100, Spain
e-mail: sferrer@um.es

Grupo de Investigación en Sistemas Dinámicos y Aplicaciones, Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Collao 1202, Concepción, Casilla 5-C, Chile
e-mail: fcrespo@ubiobio.cl

https://doi.org/10.4153/S0008439520000983 Published online by Cambridge University Press