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On the energy equality for very weak solutions to 3D MHD equations

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In this paper, we consider the energy equality of the 3D Cauchy problem for the magneto-hydrodynamics (MHD) equations. We show that if a very weak solution of MHD equations belongs to $L^4(0, T; L^4(\mathbb{R}^3))$, then it is actually in the Leray–Hopf class and therefore must satisfy the energy equality in the time interval $[0, T]$.

Keywords: MHD equations; energy equality; very weak solutions

1. Introduction

This paper is concerned with energy equality of the weak solution to the standard magneto-hydrodynamics (MHD) equations, for a velocity field u, a magnetic field B and a pressure field p, as follows

$$\left.\begin{aligned} \partial_t u + (u\cdot\nabla)u - \nu_1\Delta u + \nabla P_* &= (B\cdot\nabla)B \\ \partial_t B + (u\cdot\nabla)B - \nu_2\Delta B &= (B\cdot\nabla)u \\ \operatorname{div} u = \operatorname{div} B &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\times[0,T) \tag{1.1}$$

for any $T > 0$ and the initial conditions

$$u(\cdot,0) = u_0, \quad B(\cdot,0) = B_0 \text{ on } \mathbb{R}^3 \tag{1.2}$$

where $P_* = p + \frac{1}{2}|B|^2$ is the total pressure, and $\nu_1, \nu_2 > 0$ are coefficients of viscosity and coefficient of magnetic resistivity, respectively. The MHD equations are generally derived by coupling the Navier–Stokes equations for the velocity field of a fluid to Maxwell's equations governing the electric and magnetic fields (see, e.g., Duvaut–Lions [3]). Existence and uniqueness theory for MHD is closely related to that of the fundamental models of fluid mechanics, the Navier–Stokes equations

$$\left.\begin{aligned} \partial_t u + (u\cdot\nabla)u - \nu_1\Delta u + \nabla p &= 0 \\ \div u &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\times[0,T) \tag{1.3}$$

with the initial conditions

$$u(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}^3, \tag{1.4}$$

for which the global-in-time existence and uniqueness of smooth solutions is still a famous open problem. Similar to the the Navier–Stokes equations, a global weak solution (in the sense of definition 2.2) and local strong solution to (1.1) with the initial boundary value condition were constructed by Duvaut and Lions [3]. Later, these results were extended to the Cauchy problem (1.1) and (1.2) by Sermange and Teman [11]. Here, their main tools are regularity theory of the Stokes operator and the energy method.

It is well known that the concept of kinetic energy becomes particularly important in existence and uniqueness theory of models of fluid mechanics, starting with the fact that Leray [13] and Hopf [10] prove the existence of the weak solution to the Navier–Stokes equations (1.3) and (1.4). Because the boundedness of kinetic energy provides a primary *a priori* estimate, it allows us to construct a weak solution (called nowadays Leray–Hopf weak solutions when we are restricted under Navier–Stokes equations), and such weak solutions satisfy energy inequality. The energy inequality can be regarded as weak solutions lacking sufficient regularity, actually, if weak solution has sufficient regularity, it can satisfy the equal sign in the energy inequality, i.e., energy equality. The classical result in this direct go back to the Lions [15], which shows that if u is a weak solution to the Navier–Stokes equations (1.3) and (1.4) in addition to satisfying

$$u \in L^4(0,T;L^4(\mathbb{R}^3)) \tag{1.5}$$

then necessarily u obeys

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 \,\mathrm{d}\tau = \|u_0\|_2^2, \quad \text{for } t \in [0,T]. \tag{1.6}$$

Subsequently, Lions's result was extended to the general case by Shinbrot [20], precisely, they proved if

$$u \in L^r(0,T;L^s(\mathbb{R}^3)) \text{ with } \frac{2}{r} + \frac{2}{s} = 1 \text{ for } s \geqslant 4, \tag{1.7}$$

then (1.6) is still valid, where their argument relies on the regularity result of Serrin [19], which sates that a Leray–Hopf weak solution u is regular, furnished the following criterion holds

$$v \in L^r(0,T;L^s(\mathbb{R}^3)) \text{ with } \frac{2}{r} + \frac{3}{s} = 1 \text{ for } s \geqslant 3. \tag{1.8}$$

Recently, Galdi [6] improved Lion's result by a mollifying procedure and a duality argument. Precisely, he showed that if $u \in L_{\text{loc}},\, \sigma^2(\mathbb{R}^3 \times (0,\, T))$ is a very weak solution (for details, please see definition 2.3), and satisfies (1.5), then necessarily u obeys the energy equality (1.6). Later, Galdi's result was extended to the general case (1.7) by Berselli–Chiodaroli [1].

Turning to the standard 3D MHD equations, there is a large body of work on various regularity criteria. For example, He-Xin[9] proved that if $(u,\, B)$ is a weak

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solution of (1.1), and u satisfies (1.8), then (u, B) is smooth in $\mathbb{R}^3 \times (0, T]$, and of course it satisfies the related energy equality:

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_0^t (\|\nabla u(\tau)\|_2^2 + \|\nabla B(\tau)\|_2^2)\,\mathrm{d}\tau = \|u_0\|_2^2 + \|B_0\|_2^2 \qquad (1.9)$$

for all $0 < t \leqslant T$. For more results in this field, please see [**2**, **8**, **16**, **17**, **21**]. However, there is a little literature on energy conservation criteria for the standard MHD equations, the only reference, to our knowledge, is [**12**] where the author proved if the pair (u, B) is a weak solution of the MHD equations and (u, B) fulfills energy conservation criteria (1.7), then the energy equality (1.9) holds.

In the present paper, we, inspired by the argument of Galdi [**6**], will extend energy conservation criteria of Galdi to MHD equations, and prove that if $(u, B) \in L^4(0, T; L^4(\mathbb{R}^3))$ is a very weak solution to (1.1), then it is actually in the Leray–Hopf class. Thus the energy equality (1.9) holds. The main result of the present paper is stated as

Theorem 1.1. *Let $u_0, B_0 \in L^2_\sigma(\mathbb{R}^3)$, and the pair $(u, B) \in L^2_{loc}(\mathbb{R}^3 \times (0, T))$ be a very weak solution of (1.1) in the sense of definition 2.3. If u, B are in $L^4(0, T; L^4(\mathbb{R}^3))$, then the very weak solution pair (u, B) is, actually, in the Leray–Hopf class. Thus it obeys the energy equality (1.9).*

The rest paper of this paper is organized as follow. In § 2, we give some preliminaries. Section 3 presents the detailed proof of our main results. Finally, the energy conservation theorem of weak solution to the standard MHD equation (1.1) is provided in appendix A.

2. Preliminaries

2.1. Functional spaces

Let $T > 0$ and let $\mathbf{X}$ be a Banach space. We shall consider $L^p(0, T; \mathbf{X})$, $1 \leqslant p \leqslant \infty$, which is the space of functions from $[0, T]$ into $\mathbf{X}$, which are L^p for the Lebesǵue measure $\mathrm{d}t$. This is a Banach space endowed with the norm

$$\left(\int_0^T \|u(t)\|_{\mathbf{X}}^p\,\mathrm{d}t\right)^{\frac{1}{p}} \quad \text{if} \quad 1 \leqslant p < \infty, \ \operatorname{esssup}_{0\leqslant t\leqslant T}\|u(t)\|_{\mathbf{X}}, \quad \text{for } p = \infty.$$

For simplicity, we write $\|\cdot\|_{L^p(\mathbb{R}^3)} = \|\cdot\|_p$ when $\Omega = \mathbb{R}^3$. And the space $W^{k,p}(\Omega)$ is the usual Sobolev space, if $u \in W^{k,p}(\Omega)$ we define its norm by

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\alpha|\leqslant k}\int_\Omega |D^\alpha u|^p\,\mathrm{d}x\right)^{\dfrac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \displaystyle\sum_{|\alpha|\leqslant k} \operatorname*{ess\,sup}_{\Omega} |D^\alpha u| & \text{if } p = \infty. \end{cases}$$

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As usual, we write $W^{1,2}(\Omega) = H^1(\Omega)$, $W_0^{1,2}(\Omega) = H_0^1(\Omega)$. Besides, we give the following spaces which are usually used when one investigates the mathematical theory of Navier–Stokes equations:

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &:= \{\varphi \in C_c^\infty(\Omega); \mathrm{div}\varphi = 0\}; \\ L_\sigma^q(\Omega) &:= \text{the completion of } C_{0,\sigma}^\infty(\Omega) \ \text{ under the norm of } L^q; \\ H_{0,\sigma}^1(\Omega) &:= \text{the completion of } C_{0,\sigma}^\infty(\Omega) \ \text{ under the norm of } W^{1,2}; \\ \mathcal{D}_T &:= \{\varphi \in C_0^\infty(\mathbb{R}^3 \times [0,T)) : \mathrm{div}\, \varphi = 0\}. \end{aligned}$$

Finally, we define t-anisotropic Sobolev spaces

$$W_{q,T}^{2,1} := \{u \in L_{\mathrm{loc}}^1(\mathbb{R}^3 \times (0,T)) : u \in W^{1,q}(0,T; L_\sigma^q(\mathbb{R}^3)) \cap L^q(0,T; W^{2,q}(\mathbb{R}^3))\}.$$

Usually, one denotes by

$$A := -\mathbb{P}\Delta_D \tag{2.1}$$

the Stokes operator in Ω, where

$$\mathbb{P} : L^p(\Omega) \to L_\sigma^p(\Omega)\, (1 < p < \infty)$$

is the Helmholtz–Leray projection given by

$$(\mathbb{P}u)_i = u_i + \partial_i(-\Delta)^{-1}\nabla \cdot u$$

when $\Omega = \mathbb{R}^n$, and Δ_D is the Laplace operator under the Dirichlet boundary condition. The domain of Stokes operator is $D(A) = H_{0,\sigma}^1(\Omega) \cap W^{2,2}(\Omega)$.

Next we give a basic theorem, which will be used in our proof.

THEOREM 2.1 [**18**]. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Then there exists a family of functions $\mathcal{N} = \{a_1, a_2, a_3, \cdots\}$ such that*

(i) *$\mathcal{N}$ is an orthogonal basis in $L_\sigma^2(\Omega)$;*

(ii) *$a_j \in D(A) \cap C^\infty(\overline{\Omega})$ are eigenfunctions of the Stokes operator, that is $Aa_j = \lambda_j a_j$ for all $j \in \mathbb{N}$ with*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \quad \text{and} \quad \lambda_j \to \infty;$$

(iii) *$\mathcal{N}$ is an orthogonal basis in $H_{0,\sigma}^1(\Omega)$.*

2.2. The definition of weak solutions, the form $\mathfrak{B}_0$ and the operator $\mathfrak{B}$

First, we present the notion of energy weak solution of (1.1).

DEFINITION 2.2. The pair $(u, B) \in L^\infty(0, T; L_\sigma^2(\mathbb{R}^3)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^3))$ is called as energy weak solution in $\mathbb{R}^3 \times (0, T)$ if

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(1) The pair (u, B) solves (1.1) in the distribution sense

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \varphi - \nabla u \cdot \nabla \varphi - u \cdot \nabla u \cdot \varphi + B \cdot \nabla B \cdot \varphi) \,\mathrm{d}x\mathrm{d}t \\ &\quad = - \int_{\mathbb{R}^3} u_0 \cdot \varphi(\cdot, 0) \,\mathrm{d}x \\ &\int_0^T \int_{\mathbb{R}^3} (B \cdot \partial_t \phi - \nabla B \cdot \nabla \phi - u \cdot \nabla B \cdot \phi + B \cdot \nabla u \cdot \phi) \,\mathrm{d}x\mathrm{d}t \\ &\quad = - \int_{\mathbb{R}^3} B_0 \cdot \phi(\cdot, 0) \,\mathrm{d}x \end{aligned} \tag{2.2}$$

for all $\varphi,\, \phi \in \mathcal{D}_T$ and $u_0,\, B_0 \in L^2_\sigma(\mathbb{R}^3)$.

(2) Such solution pair (u, B) satisfies the energy inequality

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_0^t (\|\nabla u(\tau)\|_2^2 + \|\nabla B(\tau)\|_2^2) \,\mathrm{d}\tau \leqslant (\|u_0\|_2^2 + \|B_0\|_2^2) \tag{2.3}$$

for all $0 < t < T$.

In the present paper, the main goal is to consider the energy equality of very weak solutions to (1.1). To this end, we give the definition of very weak solutions of (1.1) as follows

DEFINITION 2.3. We say that the pair $(u, B) \in L^2_{loc,\sigma}(\mathbb{R}^3 \times (0, T))$ is a very weak solution of equation (1.1), if

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \varphi + u \cdot \Delta \varphi + u \cdot \nabla \varphi \cdot u - B \cdot \nabla \varphi \cdot B) \,\mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}^3} u_0 \cdot \varphi(0) \,\mathrm{d}x, \\ &\int_0^T \int_{\mathbb{R}^3} (B \cdot \partial_t \phi + B \cdot \Delta \phi + u \cdot \nabla \phi \cdot B - B \cdot \nabla \phi \cdot u) \,\mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}^3} B_0 \cdot \phi(0) \,\mathrm{d}x \end{aligned} \tag{2.4}$$

for some $u_0,\, B_0 \in L^2_\sigma(\mathbb{R}^3)$ and all $\varphi,\, \phi \in \mathcal{D}_T$.

REMARK 2.4. In fact, denote by $\Gamma = (u, B)$, $\Gamma_1 = (B, u)$ and $\Psi = (\varphi, \phi)$, we can rewrite (2.4) as

$$\int_0^T \int_{\mathbb{R}^3} \Gamma \cdot \partial_t \Psi + \Gamma \cdot \Delta \Psi + u \cdot \nabla \Psi \cdot \Gamma - B \cdot \nabla \Psi \cdot \Gamma_1 \,\mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}^3} \Gamma_0 \cdot \Psi_0 \,\mathrm{d}x \tag{2.5}$$

with $\Gamma_0 = (u_0, B_0)$, $\Psi_0 = (\varphi(x, 0), \phi(x, 0))$.

In order to deal with nonlinear term conveniently, we introduce a trilinear form $\mathfrak{B}_0$. First, we define now a trilinear form on $(H_0^1(\Omega))^3$ by setting

$$b(u,v,w)=\sum_{i,j=1}^{3}\int_\Omega u_i\partial_i v_j w_j\,\mathrm{d}x=\int_\Omega u\cdot\nabla v\cdot w\,\mathrm{d}x$$

whenever the integral makes sense. Actually, from Hölder inequality and embedding theorem,

$$\int_\Omega u\cdot\nabla v\cdot w\,\mathrm{d}x\leqslant\|u\|_{L^4(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}\leqslant\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}\|w\|_{H_0^1(\Omega)}$$

so b is a trilinear continuous form on $(H_0^1(\Omega))^3$. In particular, if $u\in H^1_{0,\sigma}(\Omega)$, we can easily get by a direct calculation

$$b(u,v,v)=0,\quad\text{for all } v\in H_0^1(\Omega) \tag{2.6}$$

and

$$b(u,v,w)=-b(u,w,v),\quad\text{for all } v,w\in H_0^1(\Omega). \tag{2.7}$$

In order to write (2.5) as a simpler form, we define a trilinear form $\mathfrak{B}_o$ on $(H^1_{0,\sigma}(\Omega))^3$ as

$$\mathfrak{B}_o(\Phi^1,\Phi^2,\Phi^3)=b(u,v,w)-b(U,V,w)+b(u,V,W)-b(U,v,W)$$

for all Φ^1, Φ^2, $\Phi^3\in H^1_{0,\sigma}(\Omega)$. Here

$$\Phi^1=(u,U),\quad\Phi^2=(v,V),\ \Phi^3=(w,W).$$

Due to the continuous b, one derives that $\mathfrak{B}_o$ is trilinearly continuous on $(H^1_{0,\sigma}(\Omega))^3$. This let us give a continuous bilinear operator $\mathfrak{B}$ from $H^1_{0,\sigma}(\Omega)\times H^1_{0,\sigma}(\Omega)$ into $(H^1_{0,\sigma}(\Omega))'$ as

$$\langle\mathfrak{B}(\Phi^1,\Phi^2),\Phi^3\rangle=\mathfrak{B}_0(\Phi^1,\Phi^2,\Phi^3)\text{ for all }\Phi^i\in H^1_{0,\sigma}(\Omega). \tag{2.8}$$

The definition of $\mathfrak{B}_0$, along with (2.6) and (2.7), gives

$$\left.\begin{aligned}\mathfrak{B}_0(\Phi^1,\Phi^2,\Phi^2)&=0\\ \mathfrak{B}_0(\Phi^1,\Phi^2,\Phi^3)&=-\mathfrak{B}_0(\Phi^1,\Phi^3,\Phi^2)\end{aligned}\right\}\text{ for all }\Phi^i\in H^1_{0,\sigma}(\Omega). \tag{2.9}$$

This yields an alternative form of weak formulation (2.5) as follows, which will be used in the proof of the lemma 3.1. Precisely, if we choose φ, $\phi\in C^\infty_{0,\sigma}(\mathbb{R}^3)$, one

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can rewrite (2.2) as

$$\partial_t(u,\varphi)+(\nabla u,\nabla\varphi)+b(u,u,\varphi)-b(B,B,\varphi)=0,$$
$$\partial_t(B,\phi)+(\nabla B,\nabla\phi)+b(u,B,\phi)-b(B,u,\phi)=0,$$

with $(f,\, g)=\int_{\mathbb{R}^3} f\cdot g\,\mathrm{d}x$, then this system can be equivalently rewritten as

$$\partial_t(\Gamma,\Psi)+(\nabla\Gamma,\nabla\Psi)+\mathfrak{B}_0(\Gamma,\Gamma,\Psi)=0. \tag{2.10}$$

Using the operators A and $\mathfrak{B}$ perviously defined, (2.10) is equivalent to the following formula

$$\partial_t\Gamma+A\Gamma+\mathfrak{B}(\Gamma,\Gamma)=0, \tag{2.11}$$

i.e., (2.10) is a weak formulation of the problem (2.11).

Next we will introduce the standard mollifying techniques which will be used in the proof of our main results. Let $\epsilon>0$ be a sufficiently small parameter, we define space and space-time mollifiers of h with $h\in L^1_{\rm loc}(\mathbb{R}^3\times[0,\,T))$, respectively, by

$$h^{(\epsilon)}(x,\cdot)=\int_{\mathbb{R}^3}k_\epsilon(x-y)h(y,\cdot)\,dy,\quad h_{(\epsilon)}(x,t)=\int_0^T j_\epsilon(t-s)h^{(\varepsilon)}(x,s)\,\mathrm{d}s,$$

where

$$j_\epsilon(\tau):=\epsilon^{-1}j(\tau/\epsilon),\quad k_\epsilon(\xi):=\eta^{-1}k(\xi/\epsilon),\quad (\tau,\xi)\in\mathbb{R}\times\mathbb{R}^3$$

with $j\in C_0^\infty(-\epsilon,\,\epsilon)$ and $k\in C_0^\infty(\mathbb{R}^3)$.

We conclude this section by introducing an essential lemma, which ensures that the divergence of the approximate sequence is also zero when we approximate the sequence to the initial condition of the divergence of zero.

LEMMA 2.5. *For any $v\in H^1(\mathbb{R}^3)\cap L^2_\sigma(\mathbb{R}^3)$, one can find an approximation sequence $v_R\in H^1_0(\mathbb{B}_R)\cap L^2_\sigma(\mathbb{B}_R)$, where $\mathbb{B}_R$ is the ball with the origin as the centre and R as the radius, and*

$$\lim_{R\to\infty}\|v-v_R\|_{H^1(\mathbb{R}^3)}=0.$$

Proof. Let $\psi\in C^1(\mathbb{R})$ be a cut-off function with $\psi(\xi)=1$ if $|\xi|\leqslant 1$, $\psi(\xi)=0$ if $|\xi|\geqslant 2$ and set $\psi^R(x)=\psi(\frac{|x|}{R})$. From [5], we can get an unique solution $w_R\in H^1_0(\mathbb{B}_{R,2R})$ (where $\mathbb{B}_{R,2R}=\{x|R<|x|<2R\}$) to the problem

$$\nabla\cdot w_R=-v\cdot\nabla\psi^R$$

and w_R satisfies

$$\|\nabla w_R\|_{L^2(\mathbb{B}_{R,2R})}\leqslant c\|v\cdot\nabla\psi^R\|_{L^2(\mathbb{B}_{R,2R})} \tag{2.12}$$

with the constant c independent of R. Moreover, since $\nabla\psi^R=\mathcal{O}(\frac{1}{R})$ uniformly in x, using Poincaré inequality and (2.12), one can get

$$\|w_R\|_{L^2(\mathbb{B}_{R,2R})}\leqslant c_1R\|\nabla w_R\|_{L^2(\mathbb{B}_{R,2R})}\leqslant c_2\|v\|_{L^2(\mathbb{B}_{R,2R})}. \tag{2.13}$$

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Then we set $w_R \equiv 0$ in the complement of $\mathbb{B}_{R,2R}$ and define

$$\overline{v} = \psi^R v + w_R$$

it is easy to check $\overline{v} \in H^1_{0,\sigma}(\mathbb{B}_{2R})$. Next, for any given $\epsilon \geqslant 0$, we can acquire a sequence $\overline{v}^{(\epsilon)} \in C^\infty_{0,\sigma}(\Omega_{2R})$ due to the mollifying techniques such that

$$\|\overline{v} - \overline{v}^{(\epsilon)}\|_{H^1_0(\mathbb{B}_{2R})} < \epsilon.$$

Therefore,

$$\begin{aligned}\|v - \overline{v}^{(\epsilon)}\|_2 &\leqslant \|\overline{v}^{(\epsilon)} - \overline{v}\|_2 + \|v - \overline{v}\|_2 \\ &< \epsilon + \|(1-\psi^R)v\|_2 + \|w_R\|_{L^2(\mathbb{B}_{R,2R})}.\end{aligned}$$

Because of (2.13) and the properties of ψ^R, when R is sufficiently large and ϵ is sufficiently small, we can make the right side of this inequality as small as we please, namely,

$$\|v - \overline{v}^{(\epsilon)}\|_2 \to 0, \text{as } \epsilon \to 0, R \to \infty.$$

Similarly, by (2.12) and the properties of ψ^R, when R sufficiently large and ϵ sufficiently small, one can obtain

$$\begin{aligned}\|\nabla v - \nabla\overline{v}^{(\epsilon)}\|_2 &\leqslant \|\nabla\overline{v}^{(\epsilon)} - \nabla\overline{v}\|_2 + \|\nabla v - \nabla\overline{v}\|_2 \\ &< \epsilon + \|(1-\psi^R)\nabla v\|_2 + \|\nabla w_R\|_{L^2(\mathbb{B}_{R,2R})} + \|\nabla\psi^R v\|_2 \\ &\to 0.\end{aligned}$$

So, we can choose $v_R = \overline{v}^{(\epsilon)}$, which completes the proof of this lemma. □

3. Proof of theorem 1.1

To prove theorem 1.1, we first consider the following regularize problem related to equation (1.1)

$$\left.\begin{aligned}\partial_t w + \alpha\cdot\nabla w - \beta\cdot\nabla E = \Delta w - \nabla p + f_1, \\ \partial_t E + \alpha\cdot\nabla E - \beta\cdot\nabla w = \Delta E + f_2, \\ \operatorname{div} w = \operatorname{div} E = 0,\end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0,T), \\ w(\cdot,0) = w_0, \quad E(\cdot,0) = E_0 \; x \in \mathbb{R}^3, \tag{3.1}$$

where α, $\beta \in C^\infty_{0,\sigma}([0,T)\times\mathbb{R}^3)$, w_0, $E_0 \in L^2_\sigma(\mathbb{R}^3)$ and f_1, $f_2 \in C^\infty_0((0,T)\times\mathbb{R}^3)$ with some $T > 0$. Note that using the same method as in remark 2.4, we can rewrite equation (3.1) as

$$\left.\begin{aligned}\partial_t\Phi + A\Phi + \mathfrak{B}(\Theta,\Phi) = f \\ \operatorname{div}\Phi = 0,\end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0,T) \\ \Phi(\cdot,0) = (w(\cdot,0), E(\cdot,0)) = \Phi_0 \quad x \in \mathbb{R}^3 \tag{3.2}$$

where $\Phi = (w, E)$, $f = (f_1, f_2)$, $\Theta = (\alpha, \beta)$ and A is the Stokes operator defined in (2.1).

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By a standard Galerkin technique, we can construct a energy weak solution of (3.1). If Φ is a weak solution in a bounded domain $\Omega \subset \mathbb{R}^3$ guarantees that at each time $t > 0$ the function $\Phi(t)$ is an element of the infinite-dimensional space L^2_σ spanned by the eigenfunctions of the Stokes operator. The Galerkin method allows us to construct a weak solution Φ as the limit of approximate solutions Φ_n that each time $t > 0$ belongs to the finite-dimensional space $P_n L^2_\sigma$ spanned by the first n eigenfunctions,

$$P_n L^2_\sigma := \text{span}\{a_1, a_2, \cdots, a_n\} \quad a_i \in \mathcal{N}.$$

Here $\mathcal{N}$ is the basis given in theorem 2.1 and by P_n we denote the projection operator $P_n : L^2 \to L^2_\sigma$ defined by

$$P_n \Phi = \sum_{i=1}^{n} \langle \Phi, a_i \rangle a_i, \text{where } a_i \in \mathcal{N}. \tag{3.3}$$

More precisely, we have the following technical lemma which plays a key role in our proof of theorem 1.1.

Lemma 3.1. (i) *If $(w_0, E_0) \in L^2_\sigma(\mathbb{R}^3)$, the Cauchy problem (3.1) exists a unique solution pair (w, E) such that*

$$(w, E) \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^3)), \tag{3.4}$$

moreover

$$\begin{aligned} &\max_{t\in[0,T]} (\|w(t)\|_2^2 + \|E(t)\|_2^2) + \int_0^T (\|\nabla w(t)\|_2^2 + \|\nabla E(t)\|_2^2)\, dt \\ &\leqslant \|w_0\|_2^2 + \|E_0\|_2^2 + c_0 \int_0^T (\|f_1(t)\|_{\frac{6}{5}}^2 + \|f_2(t)\|_{\frac{6}{5}}^2)\, dt \end{aligned} \tag{3.5}$$

with some positive constant c_0.

(ii) *(Improved regularity.) If $(w_0, E_0) \in H^1(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$, then*

$$w, E \in W^{2,1}_{2,T} \subset C(0, T; H^1(\mathbb{R}^3)). \tag{3.6}$$

In addition, if $w_0, E_0 \equiv 0$, we have also $(w, E) \in W^{2,1}_{4/3,T} \times W^{2,1}_{4/3,T} \times L^{\frac{4}{3}}(0, T; L^{4/3}(\mathbb{R}^3))$.

Proof. (i) We will prove the related result by using the standard Galerkin technique. Let $\mathbb{B}_R \subset \mathbb{R}^3$ be the ball of radius R centred at the origin, we consider the following

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problem

$$\left.\begin{aligned}\partial_t w+\alpha\cdot\nabla w-\beta\cdot\nabla E&=\Delta w-\nabla p+f_1\\ \partial_t E+\alpha\cdot\nabla E-\beta\cdot\nabla w&=\Delta E+f_2\\ \operatorname{div} w=\operatorname{div} E&=0\end{aligned}\right\}\text{in}\quad \mathbb{B}_R\times(0,T) \tag{3.7}$$

endowed with the initial-boundary value condition

$$\begin{aligned}&w=E=0,\quad \text{on } \partial\mathbb{B}_R\times(0,T);\\ &w(\cdot,0)=w_{0_R},\quad E(\cdot,0)=E_{0_R}\text{in}\quad \mathbb{B}_R\end{aligned} \tag{3.8}$$

where w_{0_R}, $E_{0_R}\in L^2_\sigma(\mathbb{B}_R)$ obey

$$\lim_{R\to\infty}\|w_0-w_{0_R}\|_2=0,\quad \lim_{R\to\infty}\|E_0-E_{0_R}\|_2=0. \tag{3.9}$$

Here the existence of w_{0_R}, E_{0_R} can be ensured by lemma 2.5. Similarly, (3.7) can be equivalently written as

$$\left.\begin{aligned}\partial_t\Phi+A\Phi+\mathfrak{B}(\Theta,\Phi)&=f\\ \operatorname{div}\Phi&=0\end{aligned}\right\}\quad\text{in } \mathbb{B}_R\times(0,T) \tag{3.10}$$

with

$$\Phi\big|_{\partial\mathbb{B}_R\times(0,T)}=0,\quad\text{and}\quad \Phi(x,0)=\Phi_{0_R}\qquad x\in\mathbb{B}_R.$$

According to (3.9) we also have

$$\lim_{R\to\infty}\|\Phi_0-\Phi_{0_R}\|_2=0. \tag{3.11}$$

In the following, we will invoke the classical Galerkin method, coupled with 'invading domains' technique to explore the existence of solutions for (3.10) in the class of (3.4).

Step 1. In this step, we will prove that the system (3.10) admits a solution at least locally in time. To this end we consider the problem

$$\partial_t\Phi_n+A\Phi_n+P_n\mathfrak{B}(\Theta_n,\Phi_n)=P_nf \tag{3.12}$$

$$\Phi_n(0)=P_n\Phi_{0_R} \tag{3.13}$$

for the form of functions Φ_n and Θ_n

$$\Phi_n(x,t)=\sum_{k=1}^n c_k^n(t)a_k(x),\quad \Theta_n(x,t)=\sum_{k=1}^n \widehat{c}_k^n(t)a_k(x)$$

where $a_k(x)\in\mathcal{N}$ is the basis of $L^2(\mathbb{B}_R)$. To determine Φ_n we need to find the functions $c_k^n(t)$. To get the desired result, we take the inner product of (3.12) in

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L^2 with a_k, $k = 1, 2, \cdots n$. Since $(P_n v,\, a_k) = (v,\, a_k)$ for every $v \in L^2(\mathbb{R}^3)$ and $1 \leqslant k \leqslant n$, one derives

$$\sum_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}c_j^n(t)a_j, a_k\right) + \sum_{j=1}^{n}(c_j^n(t)Aa_j, a_k) + \sum_{i,j=1}^{n}\langle \mathfrak{B}(\widehat{c}_i^n(t)a_i, c_j^n(t)a_j), a_k\rangle = (f, a_k).$$

Using the fact that the a_k are eigenfunctions of the Stokes operator and are orthonormal in $L^2(\mathbb{B}_R)$ we therefore obtain a system of ODEs,

$$\frac{\mathrm{d}}{\mathrm{d}t}c_k^n(t) + \lambda_k c_k^n(t) + \sum_{i,j=1}^{n} D_{ijk}\widehat{c}_i^n(t)c_j^n(t) = f_k, \quad k = 1, 2, \cdots, n \tag{3.14}$$

where

$$D_{ijk} = \langle \mathfrak{B}(a_i, a_j), a_k\rangle = \mathfrak{B}_0(a_i, a_j, a_k), \quad f_k(t) = (f(\cdot, t), a_k)$$

and $\widehat{c}_i^n(t) = (\Theta_n,\, a_i)$ is the coefficients of Θ_n. To find initial conditions for c_k^n we take the inner product of (3.13) with a_k, which yields

$$c_k^n(0) = \langle \Phi_{0_R}, a_k\rangle.$$

From the classical theory of ODEs, one immediately obtains (3.14) admits a unique solution $(c_1^n, \cdots, c_n^n)$ on some time interval $[0,\, T_n)$, from which we obtain the corresponding solution Φ_n of (3.12).

Step 2. *We obtain uniform estimates on the solutions Φ_n and hence show that $T_n = T$.* We already know that Φ_n exists at least on some time interval $[0,\, T_n)$. Let $s \in (0,\, T_n)$, we take the inner product of (3.12) with $\Phi_n(s)$ to get

$$(\partial_s\Phi_n(s), \Phi_n(s)) + (A\Phi_n(s), \Phi_n(s)) + \langle P_n\mathfrak{B}(\Theta_n(s), \Phi_n(s)), \Phi_n(s)\rangle = (f, \Phi_n(s)).$$

On the one hand,

$$(\partial_s\Phi_n(s), \Phi_n(s)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\Phi_n(s)\|^2_{L^2(\mathbb{B}_R)}$$

and

$$\begin{aligned}(A\Phi_n(s), \Phi_n(s)) &= (-\mathbb{P}\Delta\Phi_n, \Phi_n) = (-\Delta\Phi_n, \mathbb{P}\Phi_n) = (-\Delta\Phi_n, \Phi_n)\\ &= \|\nabla\Phi_n(s)\|^2_{L^2(\mathbb{B}_\mathbb{R})}.\end{aligned}$$

On the other hand, the nonlinear term vanishes due to the definition of operator $\mathfrak{B}$ and $\operatorname{div}\Theta_n = 0$, i.e.,

$$\begin{aligned}\langle P_n\mathfrak{B}(\Theta_n, \Phi_n), \Phi_n\rangle &= \langle \mathfrak{B}(\Theta_n, \Phi_n), P_n\Phi_n\rangle\\ &= \langle \mathfrak{B}(\Theta_n, \Phi_n), \Phi_n\rangle = \mathfrak{B}_0(\Theta_n, \Phi_n, \Phi_n) = 0.\end{aligned}$$

Therefore for all $s > 0$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\Phi_n(s)\|_2^2 + \|\nabla\Phi_n(s)\|_2^2 = (f, \Phi_n(s)). \tag{3.15}$$

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Now, integrating (3.15) from 0 to t, for all $t \in [0, T_n)$, along with (3.11), Sobolev inequality and Cauchy–Schwartz inequality, yields

$$\|\Phi_n(t)\|^2_{L^2(\mathbb{B}_\mathbb{R})} + \int_0^t \|\nabla \Phi_n(s)\|^2_{L^2(\mathbb{B}_\mathbb{R})} \,\mathrm{d}\tau \leqslant \|\Phi_0\|_2^2 + c_0 \int_0^t \|f(s)\|^2_{\frac{6}{5}} \,\mathrm{d}\tau \leqslant C \quad (3.16)$$

with C independent of t and n. In particular, (3.16) shows that $|c_k^n(t)| \leqslant C^{\frac{1}{2}}$ for all $k = 1, \cdots, n$ and $t \in [0, T_n)$ which in turn, by a standard technique on ordinary differential equations, implies that the c_k^n do not blow up at $t = T_n$ and hence $T_n = T$. In addition, from (3.16) we get

$$\sup_{t\in[0,T]} \|\Phi_n(t)\|^2_{L^2(\mathbb{B}_\mathbb{R})} \leqslant \|\Phi_0\|_2^2 + c_0 \int_0^T \|f(s)\|^2_{\frac{6}{5}} \,\mathrm{d}\tau$$

and hence

$$\int_0^T \|\nabla \Phi_n(s)\|^2_{L^2(\mathbb{B}_\mathbb{R})} \,\mathrm{d}\tau \leqslant \|\Phi_0\|_2^2 + c_0 \int_0^T \|f(s)\|^2_{\frac{6}{5}} \,\mathrm{d}\tau. \quad (3.17)$$

Therefore we have shown that the approximate solution sequence Φ_n is bounded uniformly in $L^\infty(0, T; L^2(\mathbb{B}_R)) \cap L^2(0, T; H^1_{0,\sigma}(\mathbb{B}_R))$.

Step 3. *We establish the uniform bounds on $\partial_t \Phi_n$ in $L^2(0, T; (H^1_{0,\sigma}(\mathbb{B}_R))')$.* For any $\varphi \in H^1_{0,\sigma}(\mathbb{B}_R)$, we take the L^2 inner product of the Galerkin equation (3.12) with φ to obtain

$$\begin{aligned}(\partial_t \Phi_n, \varphi) &= -(A\Phi_n, \varphi) - \langle P_n \mathfrak{B}(\Theta_n, w_n), \varphi\rangle + (f, \varphi)\\ &= -(A\Phi_n, \varphi) - \langle \mathfrak{B}(\Theta_n, \Phi_n), P_n\varphi\rangle + (f, \varphi).\end{aligned}$$

To estimate the norm $\|\partial_t \Phi_n\|_{(H^1_{0,\sigma}(\mathbb{B}_R))'}$, we need to estimate each term of the right-hand side of the above equality. It is clear that

$$\begin{aligned}|(A\Phi_n, \varphi)| &= |(-\mathbb{P}\Delta\Phi_n, \varphi)| = |(-\Delta\Phi_n + \nabla\tilde{\varphi}, \varphi)| = |(-\Delta\Phi_n, \varphi)|\\ &= |(\nabla\Phi_n, \nabla\varphi)| \leqslant \|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)} \|\varphi\|_{H^1_0(\mathbb{B}_R)}\end{aligned}$$

for some smooth $\tilde{\varphi}$. On the other hand, by Hölder inequality and the Sobolev embedding theorem

$$\begin{aligned}|\langle \mathfrak{B}(\Theta_n, \Phi_n), P_n\varphi\rangle| = |\mathfrak{B}_0(\Theta_n, \Phi_n, P_n\varphi)| &\leqslant \|\Theta_n\|_{L^3(\mathbb{B}_R)} \|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)} \|P_n\varphi\|_{L^6(\mathbb{B}_R)}\\ &\leqslant c\|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)} \|P_n\varphi\|_{H^1_{0,\sigma}(\mathbb{B}_R)}\\ &\leqslant c\|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)} \|\varphi\|_{H^1_{0,\sigma}(\mathbb{B}_R)}\end{aligned}$$

and

$$|\langle f, \varphi\rangle| \leqslant \|f\|_{L^{\frac{6}{5}}(\mathbb{B}_R)} \|\varphi\|_{L^6(\mathbb{B}_R)} \leqslant c\|f\|_{L^{\frac{6}{5}}(\mathbb{B}_R)} \|\varphi\|_{H^1_{0,\sigma}(\mathbb{B}_R)}.$$

Therefore,

$$\|\partial_t\Phi_n\|_{(H^1_{0,\sigma}(\mathbb{B}_R))'} \leqslant c(\|\nabla\Phi_n\|_{L^2(\mathbb{B}_R)} + \|f\|_{L^{\frac{6}{5}}(\mathbb{B}_R)})$$

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with the constant c independent of n. From this, one immediately derives

$$\begin{aligned}\int_0^T \|\partial_t\Phi_n\|^2_{(H^1_{0,\sigma}(\mathbb{B}_R))'}\,\mathrm{d}s &\leqslant c\left(\int_0^T \|\nabla\Phi_n(s)\|^2_{L^2(\mathbb{B}_R)}\,\mathrm{d}s + \int_0^T \|f(s)\|^2_{L^{\frac{6}{5}}(\mathbb{B}_R)}\,\mathrm{d}s\right)\\ &\leqslant c\left(\|\Phi_0\|_2^2 + \int_0^T \|f(s)\|^2_{\frac{6}{5}}\,\mathrm{d}s\right)\end{aligned}$$

where in the last inequality we have used the inequality (3.16).

Step 4. *We extract a convergent subsequence and pass to the limit in the equation.* From step 2, there exists an absolute constant C such that

$$\|\Phi_n\|_{L^\infty(0,T;L^2(\mathbb{B}_R))\cap L^2(0,T;H^1_{0,\sigma}(\mathbb{B}_R))} \leqslant C;\ \ \|\partial_t\Phi_n\|_{L^2(0,T;(H^1_{0,\sigma}(\mathbb{B}_R))')} \leqslant C.$$

Therefore, by Aubin–Lions Lemma [**18**, theorem 4.3] one can find a subsequence of $\{\Phi_n\}$ (we still denote by $\{\Phi_n\}$) such that

$$\Phi_n \to \Phi_R, \quad \text{strongly in } L^2(0,T;L^2_\sigma(\mathbb{B}_R)) \tag{3.18}$$

$$\Phi_n \to \Phi_R, \text{weakly star in } L^\infty(0,T;L^2_\sigma(\mathbb{B}_R))$$

$$\nabla\Phi_n \to \nabla\Phi_R, \quad \text{weakly in } L^2(0,T;L^2(\mathbb{B}_R)) \tag{3.19}$$

We now show that Φ_R is a weak solution of equation (3.10). It is enough to check that for any fixed $\varphi\in\mathcal{D}_T$ we have

$$\begin{aligned}&-\int_0^T(\Phi,\partial_t\varphi)\,\mathrm{d}t + \int_0^T(\nabla\Phi,\nabla\varphi)\,\mathrm{d}t \quad + \int_0^T\langle\mathfrak{B}(\Theta,\Phi),\varphi\rangle\,\mathrm{d}t\\ &= (\Phi_{0_R},\varphi(0)) + \int_0^T(f,\varphi)\,\mathrm{d}t.\end{aligned} \tag{3.20}$$

If we take the dot product of (3.12) with φ, and integrate in space, then integrating the second term by part yield, we get

$$(\partial_t\Phi_n,\varphi) + (\nabla\Phi_n,\nabla\varphi) + \langle\mathfrak{B}(\Theta_n,\Phi_n),\varphi\rangle = (f,\varphi).$$

Integrating in $[0,\,T)$ and using integrating by parts in the first term we obtain that

$$\begin{aligned}&-\int_0^T(\Phi_n,\partial_t\varphi)\,\mathrm{d}t + \int_0^T(\nabla\Phi_n,\nabla\varphi)\,\mathrm{d}t + \int_0^T(\mathfrak{B}(\Theta_n,\Phi_n),\varphi)\,\mathrm{d}t\\ &= (\Phi_{0_R},\varphi(0)) + \int_0^T(f,\varphi)\,\mathrm{d}t.\end{aligned} \tag{3.21}$$

We pass to limit in (3.21), as $n\to\infty$. From the convergence (3.18) and (3.19) we have

$$\int_0^T(\Phi_n,\partial_t\varphi)\,\mathrm{d}t \to \int_0^T(\Phi_R,\partial_t\varphi)\,\mathrm{d}t$$

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and

$$\int_0^T (\nabla\Phi_n, \nabla\varphi)\,\mathrm{d}t \to \int_0^T (\nabla\Phi_R, \nabla\varphi)\,\mathrm{d}t.$$

To prove that

$$\int_0^T \langle \mathfrak{B}(\Theta_n, \Phi_n), \varphi\rangle\,\mathrm{d}t \to \int_0^T \langle \mathfrak{B}(\Theta, \Phi_R), \varphi\rangle\,\mathrm{d}t$$

we notice that

$$\mathfrak{B}(\Theta_n, \Phi_n) - \mathfrak{B}(\Theta, \Phi_R) = \mathfrak{B}(\Theta_n - \Theta, \Phi_n) + \mathfrak{B}(\Theta, \Phi_n - \Phi_R)$$

Hence we need to show that

$$\int_0^T \langle \mathfrak{B}(\Theta_n - \Theta, \Phi_n), \varphi\rangle\,\mathrm{d}t \to 0$$

and

$$\int_0^T \langle \mathfrak{B}(\Theta, \Phi_n - \Phi_R), \varphi\rangle\,\mathrm{d}t \to 0.$$

From the bound (3.17) it follows that

$$\begin{aligned}|\int_0^T \langle \mathfrak{B}(\Theta_n - \Theta, \Phi_n), \varphi\rangle\,\mathrm{d}t| &\leqslant C\int_0^T \|\Theta_n - \Theta\|_4\|\nabla\Phi_n\|_2\|\varphi\|_4\,\mathrm{d}t \\ &\leqslant C_\varphi\left(\int_0^T \|\Theta_n - \Theta\|_4^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|\nabla\Phi_n\|_2^2\,\mathrm{d}t\right)^{\frac{1}{2}} \to 0.\end{aligned}$$

Moreover, from the weak convergence of gradients in (3.19) it follows that

$$\int_0^T \langle \partial_j(\Phi_n - \Phi_R)_i, \Theta_j\varphi_i\rangle\,\mathrm{d}t \to 0$$

as $n \to \infty$ for all $1 \leqslant i,\ j \leqslant 3$, as $\Theta_j\varphi_i \in L^2$. Hence

$$\int_0^T \langle \mathfrak{B}(\Theta, \Phi_n - \Phi_R), \varphi\rangle\,\mathrm{d}t \to 0.$$

Therefore we obtain (3.20).

Step 5. *We prove that (3.1) exists at least one weak solution.*

From steps 1–4, we can find a weak solution Φ_R of the equations (3.10). We extend the functions Φ_R by zero outside $\mathbb{B}_R$ and still denote such extended functions by Φ_R. Note that due to the zero boundary conditions these extended functions belong to $H^1_{0,\sigma}(\mathbb{R}^3)$ for almost every time t. The sequence $\{\Phi_R\}$ of weak solutions shares many properties with the sequence of Galerkin approximations considered in the

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step 1–4. In particular, it follows from our method of construction that we have a uniform bound

$$\frac{1}{2}\|\Phi_R(t)\|_2^2 + \int_0^t \|\nabla\Phi_R\|_2^2\,\mathrm{d}s \leqslant C$$

with some constant independent of R. From the above bound we conclude that for some subsequence of $\{\Phi_R\}$ (which we relabel)

$$\Phi_R \rightharpoonup \Phi \text{weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3))$$

for some $\Phi \in L^2(0, T; W^{1,2}(\mathbb{R}^3))$. Furthermore, for all $R > 1$ the estimates on the time derivative in step 3 shows

$$\int_0^T \|\partial_t\Phi_R\|^2_{(H^1_\sigma(\mathbb{R}^3))'}\,\mathrm{d}s \leqslant C\left(\|\Phi_0\|_2^2 + \int_0^T \|f(s)\|^2_{\frac{6}{5}}\,\mathrm{d}s\right)$$

for some $C > 0$ independent of R. Since $(H^1_{0,\sigma})'(\mathbb{B}_R) \subset (H^1_{0,\sigma})'(\mathbb{B}_M)$ for all $R \geqslant M$ with

$$\|\cdot\|_{(H^1_{0,\sigma})'(\mathbb{B}_M)} \leqslant \|\cdot\|_{(H^1_{0,\sigma})'(\mathbb{B}_R)},$$

then we have for all $R \geqslant M$

$$\|\partial_t\Phi_R\|_{L^2(0,T;(H^1_{0,\sigma})'(\mathbb{B}_M)} + \|\Phi_R\|_{L^2(0,T;H^1_0(\mathbb{B}_M))} \leqslant C.$$

Thus, by Aubin–Lions Lemma, for every $M \in \mathbb{N}$ we can find a subsequence of $\{\Phi_R\}$ which converges strongly in $L^2(0, T; L^2_\sigma(\mathbb{B}_M))$. Using the standard diagonal argument we can choose a subsequence of $\{\Phi_R\}$, still denoted by $\{\Phi_R\}$, such that

$$\Phi_R \to \Phi \text{strongly in } L^2(0,T;L^2(\mathbb{B}_M))$$

for every $M = 1, 2, 3\cdots$. It remains to show that the limit function Φ is a weak solution of the equation (3.1). To do this, take any test function $\phi \in \mathcal{D}_T$ where the support of ϕ is contained in $\mathbb{B}_M \times [0, T)$ for a large enough M. Then for all $R > M$ we have

$$\begin{aligned}
-\int_0^T (\Phi_R, \partial_t\phi)\,\mathrm{d}t &+ \int_0^T (\nabla\Phi_R, \nabla\phi)\,\mathrm{d}t + \int_0^T \langle\mathfrak{B}(\Theta, \Phi_R), \phi\rangle\,\mathrm{d}t \\
&= (\Phi_{0_R}, \phi(0)) + \int_0^T (f, \phi)\,\mathrm{d}t
\end{aligned}$$

where we have used the fact that Φ_R is a weak solution of the equation (3.7) in $\mathbb{B}_R \times [0, T)$ and $\phi \in \mathcal{D}_T(\mathbb{B}_R)$. We pass to the limit as $R \to \infty$ using the weak convergence of $\nabla\Phi_R$ and strong convergence of Φ_R in $L^2(0, T; L^2(\mathbb{B}_M))$, and prove that Φ is a weak solution of the equation (3.1) with the initial Φ_0.

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(ii) To improve the regularity of the solution, multiplying both sides of (3.12) by $\mathbb{P}\Delta\Phi_n$, and integrating by parts the resulting relations over $\mathbb{B}_R$, one derives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi_n\|_2^2 + \|\mathbb{P}\Delta\Phi_n\|_2^2 &= (P_n\mathfrak{B}(\Theta_n,\Phi_n),\mathbb{P}\Delta\Phi_n) - (f,\mathbb{P}\Delta\Phi_n)\\ &\leqslant (\|\Theta\||_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla\Phi_n\|_2 + \|f\|_2)\|\mathbb{P}\Delta\Phi_n\|_2\\ &\leqslant \frac{1}{2}(\|\Theta\||_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla\Phi_n\|_2 + \|f\|_2)^2 + \frac{1}{2}\|\mathbb{P}\Delta\Phi_n\|_2^2\\ &\leqslant \|\Theta\||^2_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla\Phi_n\|_2^2 + \|f\|_2^2 + \frac{1}{2}\|\mathbb{P}\Delta\Phi_n\|_2^2\end{aligned}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi_n\|_2^2 + \|\mathbb{P}\Delta\Phi_n\|_2^2 \leqslant 2(\|\Theta\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla\Phi_n\|_2^2 + \|f\|_2^2), \tag{3.22}$$

where we have used the fact $\Theta \in L^\infty(0,\,\infty,\,L^\infty(\mathbb{R}^3))$. From this, one immediately obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi_n\|_2^2 \leqslant 2\|\Theta\||^2_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla\Phi_n\|_2^2 + 2\|f\|_2^2.$$

This, along with Gronwall's inequality, yields for all $t \in [0,\,T]$

$$\|\nabla\Phi_n\|_2^2 \leqslant Ce^{T\|\Theta(s)\||^2_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}}\left(\|\nabla\Phi_n(0)\|_2^2 + \int_0^t \|f(s)\|_2^2\,\mathrm{d}s\right) \leqslant C.$$

Here, we have used the fact that $\|\nabla\Phi_n(0)\|_2^2 < \|\Phi_{0_R}\|_{H^1(\mathbb{B}_R)} < \infty$. Next, we integrate both sides of (3.22) from 0 to t, to gain

$$\int_0^t \|\mathbb{P}\Delta\Phi_n\|_2^2\,\mathrm{d}\tau \leqslant C.$$

Similar as above, dot-multiplying both sides of (3.12) by $\partial_t\Phi_n$, one can gain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi_n\|_2^2 + \|\partial_t\Phi_n\|_2^2 &= -(P_n\mathfrak{B}(\Theta_n,w_n),\partial_t\Phi_n) + (f,\partial_t\Phi_n)\\ &\leqslant (\|\Theta\||_{L_t^\infty L_x^\infty}\|\nabla\Phi_n\|_2 + \|f\|_2)\|\partial_t\Phi_n\|_2.\end{aligned} \tag{3.23}$$

Using Gronwall's inequality again, we get

$$\int_0^t \|\partial_\tau\Phi_n\|_2^2\,\mathrm{d}\tau \leqslant C.$$

This, together with (3.11), (3.16), and the well-known estimate

$$\|D^2\Phi\|_2 \leqslant C(\|\mathbb{P}\Delta\Phi\|_2 + \|\nabla\Phi\|_2)$$

with a constant C independent of R (for details, please see [7]), implies

$$\int_0^T (\|\partial_\tau\Phi_n(\tau)\|_2^2 + \|\Phi_n(\tau)\|^2_{W^{2,2}})\,\mathrm{d}\tau \leqslant C, \tag{3.24}$$

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where the constant C depends only on T, f, $\|\Theta\|_{L_t^\infty L_x^\infty}$, and $\|\Phi_0\|_2$ and is therefore independent of n. Therefore, (3.24) implies $\Phi \in W_{2,T}^{2,1}$, and from the classical interpolation result in [**14**] we can get $W_{2,T}^{2,1} \subset C([0, T]; H^1(\mathbb{R}^3))$.

Now, we consider the case $\Phi_0 \equiv 0$. By Hölder inequality, (3.16) and $\Theta \in L^4(0, T; L^4(\mathbb{R}^3))$, we have

$$\begin{aligned}\|\mathfrak{B}(\Theta, \Phi)\|_{L^{4/3}(0,T;L^{4/3}(\mathbb{R}^3))} &= \sup_{\varphi\neq 0} \frac{|\mathfrak{B}_0(\Theta, \Phi, \varphi)|}{\|\varphi\|_{L^4(0,T;L^4(\mathbb{R}^3))}} \\ &\leqslant c\|\Theta\|_{L^4(0,T;L^4(\mathbb{R}^3))}\|\nabla\Phi\|_{L^2(0,T;L^2(\mathbb{R}^3))} \\ &\leqslant c_1\|\Theta\|_{L^4(0,T;L^4(\mathbb{R}^3))}\|f\|_{L^2(0,T;L^2(\mathbb{R}^3))}\end{aligned}$$

which implies, in particular, that

$$\mathfrak{B}(\Theta, \Phi) \in L^{4/3}(0, T; L^{4/3}(\mathbb{R}^3)).$$

Therefore, from classical results of [**5**, theorem VIII.4.1], the problem

$$\begin{aligned}&\partial_t\Phi = \Delta\Phi - \nabla\chi + F, \quad \text{div } \Phi = 0 \text{in } \mathbb{R}^3 \times (0, T),\\ &\Phi(x, 0) = 0, \quad x \in \mathbb{R}^3,\end{aligned} \tag{3.25}$$

with $F = \mathfrak{B}(\Theta, \Phi) + f$, has at least one solution $\overline{\Phi}$ such that

$$(\overline{\Phi}, \nabla\chi) \in W_{4/3,T}^{2,1} \times L^{4/3}(0, T; L^{4/3}(\mathbb{R}^3)).$$

However, by uniqueness of Stokes operator, see, for example, [**5**, lemma VIII.4.2], we must have $\Phi \equiv \overline{\Phi}$, which completes the proof of this lemma. □

We are now in a position to show theorem 1.1.

Proof of theorem 1.1. To obtain the desired result, we, in equations (3.2), first take

$$\Theta(x, t) = (\alpha, \beta) \equiv (u_{(\epsilon)}, B_{(\epsilon)}) = \Gamma_{(\epsilon)}(x, t), \quad f \equiv (f_1, f_2) \equiv 0$$

and

$$\Phi_0 = (u_0^{(\epsilon)}, B_0^{(\epsilon)}) = \Gamma_0^\epsilon$$

where u, B, u_0, B_0 are defined in *Theorem* 1.1. By lemma 3.1, equations (3.2) admits a solution $\Phi_\epsilon = (w_\epsilon, E_\epsilon) \in W_{2,T}^{2,1}$ which fulfills for any pair $(\varphi, \phi) \in \mathcal{D}_T$

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^3} (w_\epsilon \cdot \partial_t\varphi + w_\epsilon \cdot \Delta\varphi + u_{(\epsilon)} \cdot \nabla\varphi \cdot w_\epsilon - B_{(\epsilon)} \cdot \nabla\varphi \cdot E_\epsilon)\, \mathrm{d}x\mathrm{d}t \\ &\quad = -\int_{\mathbb{R}^3} u_0^{(\epsilon)} \cdot \varphi(0)\, \mathrm{d}x, \\ &\int_0^T \int_{\mathbb{R}^3} (E_\epsilon \cdot \partial_t\phi + E_\epsilon \cdot \Delta\phi + u_{(\epsilon)} \cdot \nabla\phi \cdot E_\epsilon - B_{(\epsilon)} \cdot \nabla\phi \cdot w_\epsilon)\, \mathrm{d}x\mathrm{d}t \\ &\quad = -\int_{\mathbb{R}^3} B_0^{(\epsilon)} \cdot \phi(0)\, \mathrm{d}x.\end{aligned} \tag{3.26}$$

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From (2.4) and (3.26) we infer that

$$\int_0^T (u - w_\epsilon, \partial_t \varphi + \Delta\varphi)\,\mathrm{d}t + \int_0^T u\cdot\nabla\varphi\cdot u\,\mathrm{d}t - \int_0^T B\cdot\nabla\varphi\cdot B\,\mathrm{d}t$$
$$- \int_0^T u_{(\epsilon)}\cdot\nabla\varphi\cdot w_\epsilon \mathrm{d}t + \int_0^T B_{(\epsilon)}\cdot\nabla\varphi\cdot E_\epsilon\,\mathrm{d}t$$
$$= (u_0^{(\epsilon)} - u_0, \varphi(0))$$
$$\int_0^T (B - E_\epsilon, \partial_t \phi + \Delta\phi)\,\mathrm{d}t + \int_0^T u\cdot\nabla\phi\cdot B\,\mathrm{d}t - \int_0^T B\cdot\nabla\phi\cdot u\,\mathrm{d}t$$
$$- \int_0^T u_{(\epsilon)}\cdot\nabla\phi\cdot E_\epsilon\,\mathrm{d}t + \int_0^T B_{(\epsilon)}\cdot\nabla\phi\cdot w_\epsilon\,\mathrm{d}t$$
$$= (B_0^{(\epsilon)} - B_0, \phi(0)).$$

Using the same method as in remark 2.4, for any $\Psi = (\varphi, \phi) \in \mathcal{D}_T$, we can write the above two equalities as

$$\int_0^T (\Gamma - \Phi_\epsilon, \partial_t\Psi + \Delta\Psi + \mathfrak{B}(\Gamma_{(\epsilon)}, \Psi))\,\mathrm{d}t = -\int_0^T (\mathfrak{B}(\Gamma - \Gamma_{(\epsilon)}, \Psi), \Gamma)\,\mathrm{d}t - (\Gamma_0 - \Gamma_0^{(\epsilon)}, \Psi(0)). \tag{3.27}$$

On the other hand, according to lemma 3.1, the duality equation

$$\left.\begin{aligned} \partial_t\Phi + A\Phi + \mathfrak{B}(\Theta, \Phi) &= f^T \\ \operatorname{div}\Phi &= 0, \end{aligned}\right\}\text{in } \mathbb{R}^3\times(0,\infty)$$
$$\Phi(x,0) = 0\text{in } \mathbb{R}^3$$

with

$$\Theta(x,t) = \Gamma^T_{(\epsilon)}(x,t) = -\Gamma_{(\epsilon)}(x, T-t), \quad f^T(x,t) = -f(x, T-t),$$

admits a solution $\overline{\Phi}_\epsilon \in W^{2,1}_{4/3,T} \cap W^{2,1}_{2,T}$. Now let $\Lambda_\epsilon(x,\, t) = \overline{\Phi}_\epsilon(T-t,\, x)$, then it solves the final-value problem

$$\begin{cases} \partial_t\Lambda + \Delta\Lambda + \mathfrak{B}(\Gamma_{(\epsilon)}, \Lambda) = \nabla\Xi - f, \quad \operatorname{div}\Lambda = 0 \quad \text{in} \quad \mathbb{R}^3\times(0,T), \\ \Lambda(x,T) = 0, \quad \text{in } \mathbb{R}^3. \end{cases} \tag{3.28}$$

On the other hand, from the fact that $\mathcal{D}_T$ is dense in $\dot{W}^{1,2}_{q,T} := \{u\in W^{1,2}_{q,T},\, u(\cdot,\, T)=0\}$, for details please see [**6**, lemma A.1], we can take $\Psi = \Lambda_\epsilon$ in (3.27) and use $(3.28)_1$ to deduce

$$\int_0^T (\Gamma - \Phi_\epsilon, f)\,\mathrm{d}t = -\int_0^T (\mathfrak{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon), \Gamma)\,\mathrm{d}t - (\Gamma_0 - \Gamma_0^{(\epsilon)}, \Lambda_\epsilon(0)). \tag{3.29}$$

Here we have used the fact $\int_0^T ((\Gamma - \Phi_\epsilon),\, \nabla\Xi_\epsilon)\,\mathrm{d}t = 0$ due to $\operatorname{div}\Gamma = \operatorname{div}\Phi_\epsilon = 0$.

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In what follows, we need to pass to the limit $\epsilon \to 0$ in (3.29) to finish our proof. Indeed, by (3.5),

$$\|\Lambda_\epsilon(0)\|_2 \leqslant C \int_0^T \|f(t)\|_{\frac{6}{5}}^2 \,\mathrm{d}t$$

with some constant C independent of ϵ. From this, we immediately derive

$$\lim_{\epsilon \to 0} |(\Gamma_0 - \Gamma_0^{(\epsilon)}, \Lambda_\epsilon(0))| \leqslant \lim_{\epsilon \to 0} \|\Gamma_0 - \Gamma_0^{(\epsilon)}\|_2 \|\Lambda_\epsilon(0)\|_2 \to 0. \tag{3.30}$$

Secondly,

$$\begin{aligned} &\left| \int_0^T (\mathfrak{B}(\Gamma - \Gamma_{(\epsilon)}, \Lambda_\epsilon), \Gamma) \,\mathrm{d}t \right| \\ &\quad \leqslant \|\Gamma - \Gamma_{(\epsilon)}\|_{L^4(0,T;L^4(\mathbb{R}^3))} \|\nabla \Lambda_\epsilon\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|\Gamma\|_{L^4(0,T;L^4(\mathbb{R}^3))} \\ &\quad \to 0, \quad \text{as } \epsilon \to 0. \end{aligned} \tag{3.31}$$

On the other hand, by (3.5), we can find a subsequence of $\{\Phi_\epsilon\}$, still denoted by $\{\Phi_\epsilon\}$, such that as $\epsilon \to 0$

$$\Phi_\epsilon \to \Phi, \quad \text{weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3))$$

for some $\Phi \in L^\infty(0,\, T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,\, T; W^{1,2}(\mathbb{R}^3))$. From this, we immediately gain

$$\lim_{\epsilon \to 0} \int_0^T (\Gamma - \Phi_\epsilon, f) \,\mathrm{d}t = \int_0^T (\Gamma - \Phi, f) \,\mathrm{d}t \tag{3.32}$$

for any $f \in C_0^\infty(\mathbb{R}^3 \times (0,\, T))$. Finally, from (3.29)–(3.32) we conclude that

$$\int_0^T (\Gamma - \Phi, f) \,\mathrm{d}t = 0$$

which in turn, by the arbitrariness of f, implies $\Gamma = \Phi$. Therefore,

$$\Gamma \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\mathbb{R}^3)),$$

then by theorem A.2, $\Gamma = (u,\, B)$ obeys the energy equality (1.9) in the time interval $[0,\, T]$. □

Appendix A. Energy equality of weak solutions of the MHD equations

In this part, we will give a proof of the energy equality of weak solution to the MHD equations, for reader's convenience. Before starting the proof, we give a technical lemma as follows.

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Lemma A.1. *Assume Φ be a weak solution of (1.1) with the initial value $\Phi_0 \in L^2(\mathbb{R}^3)$. Then Φ can be redefined on a set of zero Lebesgue measure in such a way that $\Phi(\cdot,\, t) \in L^2(\mathbb{R}^3)$ for all $t \in [0,\, T)$ and satisfies the identity*

$$(\Phi(t), \Psi(t)) - (\Phi_0, \Psi(0)) = \int_0^t \left\{ \left(\Phi(\tau), \frac{\partial \Psi(\tau)}{\partial t} \right) - (\nabla\Phi(\tau), \nabla\Psi(\tau)) - \langle \mathfrak{B}(\Phi(\tau), \Phi(\tau)), \Psi(\tau) \rangle \right\} d\tau \tag{A.1}$$

for all $\Psi \in \mathcal{D}_T$.

The proof of this lemma can be found in, such as, [**4**, **20**].

Now we present the following theorem, which is about the energy equality of weak solutions of the MHD equations. Although its proof is classical, we can not find it in the literature. I will present its proof for reader's convenience.

Theorem A.2. *Let $\Phi_0 = (u_0,\, B_0) \in L^2_\sigma$, and let $\Phi = (u,\, B)$ be a weak solution of (1.1) and (1.2). In addition,*

$$\Phi \in L^r(0, T; L^s(\mathbb{R}^3)) \text{ for any } \frac{2}{r} + \frac{2}{s} = 1, \quad s \geqslant 4 \tag{A.2}$$

then Φ satisfies the energy equality

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_0^t (\|\nabla u(\tau)\|_2^2 + \|\nabla B(\tau)\|_2^2)\, d\tau = \|u_0\|_2^2 + \|B_0\|_2^2 \tag{A.3}$$

for all $t \in [0,\, T]$.

Proof. Let $\Phi^i = (u^i,\, B^i) \subset \mathcal{D}_T$ be a sequence such that

$$\Phi^i \to \Phi, \quad \text{strongly in } L^\infty(0, T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^3)) \cap L^r(0, T; L^s(\mathbb{R}^3)).$$

Thus, we can choose $\Psi(t) = (\Phi^i)_\epsilon(t)$ in (A.1), where $(\cdot)_\epsilon$ is the standard time mollifying operator, i.e. $(\Phi^i)_\epsilon(t) = \int_0^T j_\epsilon(t-s)\Phi^i(s)\mathrm{d}s$. Then, from lemma A.1 one has

$$(\Phi(t), (\Phi^i)_\epsilon(t)) - (\Phi_0, (\Phi^i)_\epsilon(0)) = \int_0^t \left\{ \left(\Phi(\tau), \frac{\partial (\Phi^i)_\epsilon(\tau)}{\partial t} \right) - (\nabla\Phi(\tau), \nabla(\Phi^i)_\epsilon(\tau)) - \langle \mathfrak{B}(\Phi(\tau), \Phi(\tau)), (\Phi^i)_\epsilon(\tau) \rangle \right\} \mathrm{d}\tau,$$

i.e.,

$$(u, (u^i)_\epsilon) - (u_0, (u^i)_\epsilon(0)) = \int_0^t \left\{ \left(u, \frac{\partial (u^i)_\epsilon}{\partial t} \right) - (\nabla u, \nabla (u^i)_\epsilon) - b(u, u, (u^i)_\epsilon) + b(B, B, (u^i)_\epsilon) \right\} \mathrm{d}\tau \tag{A.4}$$

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and

$$(B,(B^i)_\epsilon)-(B_0,(B^i)_\epsilon(0))=\int_0^t\left\{\left(B,\frac{\partial(B^i)_\epsilon}{\partial t}\right)-(\nabla B,\nabla(B^i)_\epsilon)\right. \\ \left. -b(u,B,(B^i)_\epsilon)+b(B,u,(B^i)_\epsilon)\right\}\mathrm{d}\tau. \tag{A.5}$$

Note that, (2.7), along with the Hölder inequality and the interpolation inequality, shows that when $i\to\infty$

$$\begin{aligned}\left|\int_0^t b(u,u,(u^i)_\epsilon-u_\epsilon)\,\mathrm{d}\tau\right|&=\left|\int_0^t b(u,(u^i)_\epsilon-u_\epsilon,u)\,\mathrm{d}\tau\right|\\&\leqslant\int_0^t\|u\|_s\|\nabla\big((u^i)_\epsilon-u_\epsilon\big)\|_2\|u\|_r\,\mathrm{d}\tau\\&\leqslant\int_0^t\|u\|_s\|\nabla\big((u^i)_\epsilon-u_\epsilon\big)\|_2\|u\|_2^{2-\frac{r}{2}}\|u\|_s^{\frac{r}{2}-1}\,\mathrm{d}\tau\\&\leqslant C\|u\|^{\frac{r}{2}}_{L^r(0,T;L^s(\mathbb{R}^3))}\|(u^i)_\epsilon-u_\epsilon\|_{L^2(0,T;W^{1,2}(\mathbb{R}^3))}\\&\to 0,\end{aligned}$$

where we have used the fact $u\in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3))\cap L^2(0,T;W^{1,2}(\mathbb{R}^3))\cap L^r(0,T;L^s(\mathbb{R}^3))$. Similarly, we can also get when $i\to\infty$

$$\left|\int_0^t b(B,B,(u^i)_\epsilon-u_\epsilon)\,\mathrm{d}\tau\right|+\left|\int_0^t b(u,B,(B^i)_\epsilon-B_\epsilon)\,\mathrm{d}\tau\right|\to 0$$

and $|\int_0^t b(B,u,(B^i)_\epsilon-B_\epsilon)\,\mathrm{d}\tau|\to 0$. Therefore, let i tend to infinity in (A.4) and (A.5), one has

$$\begin{aligned}&(u,u_\epsilon)-(u_0,(u_0)_\epsilon)\\&\quad=\int_0^t\left\{\left(u,\frac{\partial u_\epsilon}{\partial t}\right)-(\nabla u,\nabla u_\epsilon)-b(u,u,u_\epsilon)+b(B,B,u_\epsilon)\right\}\mathrm{d}\tau,\\&(B,B_\epsilon)-(B_0,(B_0)_\epsilon)\\&\quad=\int_0^t\left\{\left(B,\frac{\partial B_\epsilon}{\partial t}\right)-(\nabla B,\nabla B_\epsilon)-b(u,B,B_\epsilon)+b(B,u,B_\epsilon)\right\}\mathrm{d}\tau.\end{aligned}\tag{A.6}$$

In what follows, we need to pass to the limit $\epsilon\to0$ in (A.6) to finish our proof. In fact, from the fact j_ϵ is even in $(-\epsilon,\epsilon)$ and the basic properties of mollifiers,

$$\begin{aligned}\int_0^t\left(u,\frac{\partial u_\epsilon}{\partial t}\right)\mathrm{d}\tau&=\int_0^t\int_0^t\frac{dj_\epsilon(t-\tau)}{\mathrm{d}t}(u(t),u(\tau))\,\mathrm{d}t\mathrm{d}\tau\\&=-\int_0^t\left(u,\frac{\partial u_\epsilon}{\partial t}\right)\mathrm{d}\tau=0\end{aligned}$$

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$$\int_0^t \left(B, \frac{\partial B_\epsilon}{\partial t}\right) \mathrm{d}\tau = \int_0^t \int_0^t \frac{dj_\epsilon(t-\tau)}{\mathrm{d}t} (B(t), B(\tau))\, \mathrm{d}t\mathrm{d}\tau$$
$$= -\int_0^t \left(B, \frac{\partial B_\epsilon}{\partial t}\right) \mathrm{d}\tau = 0,$$

and

$$\lim_{\epsilon\to 0} \int_0^t (\nabla u, \nabla u_\epsilon)\, \mathrm{d}\tau = \int_0^t (\nabla u, \nabla u)\mathrm{d}\tau,$$
$$\lim_{\epsilon\to 0} \int_0^t (\nabla B, \nabla B_\epsilon)\, \mathrm{d}\tau = \int_0^t (\nabla B, \nabla B)\, \mathrm{d}\tau.$$

Besides, the weak L^2 continuity of weak solution, along with the fact $\int_0^\epsilon j_\epsilon s)\mathrm{d}s = \frac{1}{2}$, implies

$$(u(t), u_\epsilon(t)) = \int_0^\epsilon j_\epsilon(s)(u(t), u(t-s))\, \mathrm{d}s$$
$$= \int_0^\epsilon j_\epsilon(s)\big((u(t), u(t)) + (u(t), u(t-s) - u(t))\big)\, \mathrm{d}s$$
$$= \frac{1}{2}\|u(t)\|_2^2 + O(\epsilon) \to \frac{1}{2}\|u(t)\|_2^2 \text{ when } \epsilon \to 0.$$

Similarly, we also have when $\epsilon \to 0$

$$(u_0, (u_0)_\epsilon) + (B_0, (B_0)_\epsilon) = \frac{1}{2}\|u_0\|_2^2 + \frac{1}{2}\|B_0\|_2^2,$$
$$(B(t), B_\epsilon(t)) \to \frac{1}{2}\|B(t)\|_2^2.$$

Now we turn to the nonlinear term in (A.6). Indeed, a simple calculation shows as $\epsilon \to 0$

$$\left| \int_0^t b(u, u, u_\epsilon - u)\mathrm{d}\tau \right| \leqslant \int_0^t \|u\|_s \|\nabla u\|_2 \|u_\epsilon - u\|_r \mathrm{d}\tau$$
$$\leqslant \int_0^t \|u\|_s \|\nabla u\|_2 \|u_\epsilon - u\|_2^{2-\frac{r}{2}} \|u_\epsilon - u\|_s^{\frac{r}{2}-1} \mathrm{d}\tau$$
$$\leqslant \|u\|_{L^r(0,T;L^s(\mathbb{R}^3))} \|\nabla u\|_{L^2(0,T;L^2(\mathbb{R}^3))}$$
$$\times \|u_\epsilon - u\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}^{2-\frac{r}{2}} \left(\int_0^t \|u_\epsilon - u\|_s^{(\frac{r}{2}-1)s} \mathrm{d}\tau \right)^{\frac{1}{s}}$$
$$\leqslant \|u\|_{L^r(0,T;L^s(\mathbb{R}^3))} \|\nabla u\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|u_\epsilon$$
$$- u\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}^{2-\frac{r}{2}} \|u_\epsilon - u\|_{L^r(0,T;L^s(\mathbb{R}^3))}^{\frac{r}{2}-1}$$
$$\to 0.$$

Here, we have used the fact

$$u \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^3)) \cap L^r(0, T; L^s(\mathbb{R}^3)).$$

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Similarly, as $\epsilon \to 0$

$$\left| \int_0^t b(B, B, u_\epsilon - u)\, \mathrm{d}\tau \right| + \left| \int_0^t b(u, B, B_\epsilon - B)\, \mathrm{d}\tau \right| + \left| \int_0^t b(B, u, B_\epsilon - B)\, \mathrm{d}\tau \right| \to 0.$$

Thus, let $\epsilon \to 0$ in (A.6), we deduce

$$\begin{aligned} \|u(t)\|_2^2 - \|u_0\|_2^2 &= 2\int_0^t -\|\nabla u(\tau)\|_2^2 - b(u, u, u) + b(B, B, u)\, \mathrm{d}\tau \\ \|B(t)\|_2^2 - \|B_0\|_2^2 &= 2\int_0^t -\|\nabla B(\tau)\|_2^2 - b(u, B, B) + b(B, u, B)\, \mathrm{d}\tau \end{aligned} \tag{A.7}$$

This, along with the definition of $\mathfrak{B}_0$, yields

$$\begin{aligned} &\|u(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_0^t (\|\nabla u(\tau)\|_2^2 + \|B(t)\|_2^2)\mathrm{d}\tau \\ &= -\int_0^t \mathfrak{B}_0(\Phi, \Phi, \Phi)\mathrm{d}\tau + \|u_0\|_2^2 + \|B_0\|_2^2. \end{aligned}$$

Since $\Phi(\cdot, t) = (u(\cdot, t), B(\cdot, t)) \in H_\sigma^1(\mathbb{R}^3)$ for a.e t, we get by an approximation technique

$$\int_0^t \mathfrak{B}_0(\Phi(\tau), \Phi(\tau), \Phi(\tau))\mathrm{d}\tau = 0.$$

From which the desired result is obtained. □

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