



COMPOSITIO MATHEMATICA

Higher index Fano varieties with finitely many birational automorphisms

Nathan Chen and David Stapleton

Compositio Math. **158** (2022), 2033–2045.

[doi:10.1112/S0010437X22007771](https://doi.org/10.1112/S0010437X22007771)



FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865





Higher index Fano varieties with finitely many birational automorphisms

Nathan Chen and David Stapleton

ABSTRACT

A famous problem in birational geometry is to determine when the birational automorphism group of a Fano variety is finite. The Noether–Fano method has been the main approach to this problem. The purpose of this paper is to give a new approach to the problem by showing that in every positive characteristic, there are Fano varieties of arbitrarily large index with finite (or even trivial) birational automorphism group. To do this, we prove that these varieties admit ample and birationally equivariant line bundles. Our result applies the differential forms that Kollár produces on p -cyclic covers in characteristic $p > 0$.

Introduction

Recall that a *Fano variety* X is a variety with mild (at worst klt) singularities such that $-K_X$ is ample. The *index* of X is the largest number r such that $-K_X \equiv rH$ for an ample Weil divisor H . Iskovskikh and Manin [IM71] showed that the birational automorphism group, $\text{Bir}(X)$, of a smooth quartic threefold is finite, which implied that a smooth quartic threefold is not rational. Given a rational map of Fano varieties, their approach relied on a detailed study of singularities of divisors in the corresponding linear series. This approach is now referred to as the *Noether–Fano* method (using ideas of Fano [Fan08, Fan15]). An immense amount of work has been devoted to proving the birational rigidity, and thus finiteness, of $\text{Bir}(X)$ for other Fano varieties of index one. From the contributions of many authors (cf. [Che00, Cor95, dF13, dF16, dFEM03, Puk87, Puk02]) we now know that in characteristic zero any smooth Fano hypersurface $X \subset \mathbb{C}P^{n+1}$ of degree $n + 1$ is superrigid. As a consequence, these index-one Fano hypersurfaces have finite birational automorphism groups. For index-two Fano varieties, Pukhlikov has a number of results on finiteness of birational automorphisms [Puk10, Puk16, Puk20, Puk21]. These results classify rationally connected fibrations of these varieties with base of dimension one. Little is known about $\text{Bir}(X)$ for higher index Fano varieties. Here we give the first examples of higher index Fano varieties with finite birational automorphism groups.¹

Received 18 October 2021, accepted in final form 29 August 2022, published online 2 December 2022.

2020 Mathematics Subject Classification 14E07 (primary), 14E05, 14M20 (secondary).

Keywords: Fano varieties, hypersurfaces, birational automorphisms, rigidity.

The first author's research was partially supported by an NSF postdoctoral fellowship, DMS-2103099. The second author's research is partially supported by the NSF FRG grant number 1952399.

¹ One could alternatively define the Fano index by only considering Cartier divisors. The varieties that we consider here are factorial because they have isolated hypersurface singularities in codimension at least four. Thus, the two definitions are equivalent in all of our examples.

© 2022 The Author(s). The publishing rights in this article are licensed to Foundation Compositio Mathematica under an exclusive licence.

THEOREM A. *For every characteristic $p > 0$ there are singular Fano varieties of arbitrarily large index over a field of characteristic p with trivial birational automorphism groups.*

In positive characteristic, Kollár [Kol95] observed that certain Fano varieties that are p -cyclic covers in characteristic p carry global differential forms, and used these to deduce that a very general hypersurface $X \subset \mathbb{CP}^{n+1}$ of degree at least $2\lceil(n+3)/3\rceil$ is not rational. These forms have also been used to show that Fano hypersurfaces of high degree are far from being rational in other ways. For example, Totaro [Tot16] used these forms to prove that hypersurfaces in a slightly larger range are not even stably rational. Using unramified cohomology as an obstruction, Schreieder [Sch19] improved these results and showed that a very general hypersurface of degree $d \geq \log_2(n) + 2$ is not stably rational. The arguments of Totaro and Schreieder both involve the specialization property of decomposition of the diagonal, which was developed by Voisin [Voi13] and expanded upon in work of Colliot-Thélène and Pirutka [CP16]. In other degree ranges, by studying the positivity properties of these forms in more detail, the authors demonstrated that the degrees of irrationality of complex Fano hypersurfaces can be arbitrarily large and, in a different range, the degrees of possible rational endomorphisms on complex Fano hypersurfaces must satisfy certain congruence conditions (see [CS20, CS21]).

We work with the p -cyclic covers that Kollár used:

$$\nu: Y \rightarrow X.$$

They have mild (terminal) isolated singularities and admit a straightforward resolution of singularities:

$$\sigma: Z \rightarrow Y.$$

An important step in proving Theorem A is the computation of the space of global $(n - 1)$ -forms on Z . In doing so, we show that the only global $(n - 1)$ -forms are the forms that Kollár found.

THEOREM B. *Let k be an algebraically closed field of characteristic $p > 0$. Let $n \geq 3$ (if $p = 2$, then assume n is even). Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth degree e hypersurface, fix an integer $d > 0$, and let Y be a p -cyclic cover branched over a general section of $\mathcal{O}_X(pd)$. There exists a resolution of singularities Z of Y . Moreover, if*

$$(p - 1)d \leq n - e \leq pd - 3,$$

then $H^0(Z, \wedge^{n-1}\Omega_Z) \cong H^0(X, \omega_X(pd))$.

The first inequality implies that Y is Fano of index at least two. The second inequality implies that $\omega_X(pd)$ is very ample.

We introduce the notion of *birational equivariance* for line bundles, which arises naturally in this setting. We show that the existence of a nontrivial birationally equivariant line bundle is a strong condition. In particular, the global sections of a birationally equivariant line bundle \mathcal{L} on Y are naturally a representation of $\text{Bir}(Y)$. This allows us to show the following result.

COROLLARY C. *In the setting of Theorem B, let $\nu: Y \rightarrow X$ denote the p -cyclic cover. Then $\nu^*(\omega_X(pd))$ is an ample and birationally equivariant line bundle on Y , and there is an injection $\text{Bir}(Y) \cong \text{Aut}(Y) \hookrightarrow \text{Aut}(X)$.*

For an alternative perspective on these results, in [Kol96, V.5.20] Kollár views the map $Y \rightarrow X$ as a birational invariant of Y .

These theorems are proved in slightly greater generality and apply to other p -cyclic covers with appropriate hypotheses. The results lead us to ask the following question:

QUESTION. Can finiteness of birational automorphisms of Fano varieties in characteristic p be used to prove that complex Fano varieties have finitely many birational automorphisms?

Outline and conventions

We begin in § 1 by defining birational equivariance for line bundles \mathcal{L} and giving some properties and relevant examples. In § 2, we describe the resolutions of certain p -cyclic covers and check that they have terminal singularities. Finally, in § 3 we prove Theorem B by computing the $(n - 1)$ -forms on the cyclic covers, which leads to a proof of Theorem A and Corollary C.

Throughout we work over an algebraically closed field k . A variety is an integral k -scheme of finite type. We do not give the birational automorphism group any scheme structure.

1. Birationally equivariant line bundles

The goal of this section is to introduce the notion of a birationally equivariant line bundle, to give some examples, to state some basic properties, and explain how they can be used to study the birational automorphism group. We consider $\text{Bir}(X)$ as an abstract group, without any scheme structure.

Let k be an algebraically closed field, and let X be a normal projective algebraic variety over k . By variety, we mean an integral scheme of finite type over k . Let

$$f: X \dashrightarrow X$$

be a rational endomorphism. The map f is defined on some open set $U \hookrightarrow X$ such that $X \setminus U$ has codimension two in X . To start we define the pullback of a line bundle along f .

DEFINITION 1.1. Let \mathcal{L} be a line bundle on X . The *pullback of \mathcal{L} along f* is defined by (1) first pulling back to a line bundle $f^*(\mathcal{L}) \in \text{Pic}(U)$, and then (2) pushing forward $i_*(f^*(\mathcal{L}))$ to get a reflexive rank-one sheaf. This gives a group homomorphism:

$$f^*: \text{Pic}(X) \rightarrow \text{Cl}(X),$$

(where we identify the divisor class group with the group of reflexive rank-one sheaves with reflexive tensor product).

DEFINITION 1.2. We say that \mathcal{L} is equipped with a *birationally equivariant structure* (or simply \mathcal{L} is *birationally equivariant*) if for every $g \in \text{Bir}(X)$ there is a choice of an isomorphism

$$\phi_g: g^*\mathcal{L} \rightarrow \mathcal{L}$$

subject to the following compatibility condition: for all $g_1, g_2 \in \text{Bir}(X)$, there is the following commutative diagram.

$$\begin{array}{ccc} (g_1 \cdot g_2)^*(\mathcal{L}) = g_2^*(g_1^*\mathcal{L}) & \xrightarrow{g_2^*(\phi_{g_1})} & g_2^*(\mathcal{L}) \\ & \searrow \phi_{g_1 \cdot g_2} & \downarrow \phi_{g_2} \\ & & \mathcal{L} \end{array}$$

Remark 1.3. It also makes sense to talk about G -birationally equivariant line bundles for any subgroup $G \leq \text{Bir}(X)$ and any group homomorphism $G \rightarrow \text{Bir}(X)$, as well as birationally equivariant vector bundles on X .

Example 1.4. For $n \geq 2$, the *Cremona involution* $\tau: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is defined by

$$\begin{aligned} \tau([x_0 : \cdots : x_n]) &= [1/x_0 : \cdots : 1/x_n] \\ &= [x_1 \cdots x_n : x_0 x_2 \cdots x_n : \cdots : x_0 \cdots x_{n-1}]. \end{aligned}$$

The pullback $\tau^*: \text{Pic}(\mathbb{P}^n) \rightarrow \text{Pic}(\mathbb{P}^n)$ is multiplication by n . However, $\tau^{o2} = \text{id} \in \text{Bir}(\mathbb{P}^n)$. Thus, even if $\text{Pic}(X) = \text{Cl}(X)$, the map

$$\text{Bir}(X) \rightarrow \text{Hom}(\text{Pic}(X), \text{Pic}(X))$$

is only a map of sets: it does not always respect composition. This also shows that \mathbb{P}^n does not admit any nontrivial birationally equivariant line bundles when $n \geq 2$.

THEOREM 1.5 (Basic properties of birationally equivariant line bundles).

- (i) If \mathcal{L}_1 and \mathcal{L}_2 are birationally equivariant line bundles on X , then so is $\mathcal{L}_1 \otimes \mathcal{L}_2$.
- (ii) Likewise, the inverse of a birationally equivariant line bundle is naturally birationally equivariant. In particular,

$$\text{Pic}_{\text{Bir}(X)}(X) := \{\text{line bundles with birational equivariant structure}\}$$

is a group under tensor product and the forgetful map

$$\text{Pic}_{\text{Bir}(X)}(X) \rightarrow \text{Pic}(X)$$

is a group homomorphism with kernel equal to the group of one-dimensional k -representations of $\text{Bir}(X)$ under tensor product.

- (iii) Let $\mu: \tilde{X} \rightarrow X$ be a proper birational morphism. If \mathcal{L} is a line bundle on X and $\mu^*\mathcal{L}$ has a birationally equivariant structure, then \mathcal{L} is naturally birationally equivariant.
- (iv) If \mathcal{L} is a birationally equivariant line bundle on X and $H^0(X, \mathcal{L}) \neq 0$, then there is a representation $\rho: \text{Bir}(X) \rightarrow \text{GL}(H^0(X, \mathcal{L})^\vee)$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & X \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}(H^0(X, \mathcal{L})^\vee) & \xrightarrow{\quad \rho(g) \quad} & \mathbb{P}(H^0(X, \mathcal{L})^\vee) \end{array}$$

- (v) In the setting of property (iv), let $X' \subset \text{PGL}(H^0(\mathcal{L})^\vee)$ be the closure of the image of X . For all $g \in \text{Bir}(X)$, $\rho(g)$ restricts to an automorphism of X' , which induces a group homomorphism:

$$\text{Bir}(X) \rightarrow \text{Aut}(X'),$$

and the kernel consists of $g \in \text{Bir}(X)$ such that $\pi \circ g = \pi$.

- (vi) In the setting of property (iv), if there is a nonempty open set $U \subset X$ such that $\pi|_U$ is injective on the k -points of U (e.g. if π is birational or generically finite and purely inseparable), then the homomorphism $\text{Bir}(X) \rightarrow \text{Aut}(X')$ is injective.
- (vii) If X has an ample birationally equivariant line bundle, then $\text{Bir}(X) \cong \text{Aut}(X)$.

Proof. For property (i), to give the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ a birationally equivariant structure, one may assign

$$\phi_g: g^*(\mathcal{L}_1 \otimes \mathcal{L}_2) \cong g^*\mathcal{L}_1 \otimes g^*\mathcal{L}_2 \xrightarrow{(\phi_1)_g \otimes (\phi_2)_g} \mathcal{L}_1 \otimes \mathcal{L}_2.$$

Note that the first isomorphism is canonical. The compatibility condition is easy to check.

In property (ii), if \mathcal{L} is birationally equivariant with isomorphisms ϕ_g , then there are isomorphisms

$$\phi'_g = (\phi_g^\vee)^{-1}: g^*(\mathcal{L}^\vee) \rightarrow \mathcal{L}^\vee.$$

Compatibility is easy to check. It is clear that the map

$$\text{Pic}_{\text{Bir}(X)}(X) \rightarrow \text{Pic}(X)$$

is a group homomorphism. The kernel is given by equivariant structures on the trivial line bundle. These give rise to one-dimensional representations on $H^0(X, \mathcal{O}_X)$ which determine the birationally equivariant bundle up to isomorphism.

To prove property (iii), let $g \in \text{Bir}(X)$ let $\tilde{g} \in \text{Bir}(\tilde{X})$ be the corresponding birational automorphism. Assume that both g and \tilde{g} are defined away from codimension two. Let $U \subset X$ be an open set so that (a) $X \setminus U$ has codimension at least two in X , (b) π^{-1} is defined on U , (c) g is defined on U , and (d) \tilde{g} is defined on $\pi^{-1}(U)$. By assumption, there is an isomorphism

$$\phi_{\tilde{g}}|_{\mu^{-1}(U)} : (\tilde{g}^* \mu^* \mathcal{L})|_{\mu^{-1}(U)} \rightarrow (\mu^* \mathcal{L})|_{\mu^{-1}(U)}.$$

This gives an isomorphism

$$\phi_g|_U : g^*(\mathcal{L})|_U \rightarrow \mathcal{L}|_U$$

which uniquely extends to an isomorphism

$$\phi_g : g^*(\mathcal{L}) \rightarrow \mathcal{L}.$$

Lastly, compatibility follows as it can be checked on any nonempty open set (such as U).

For property (iv), the isomorphisms ϕ_g give rise to isomorphisms of global sections:

$$H^0(X, L) \xrightarrow{g^*} H^0(X, g^* \mathcal{L}) \xrightarrow{\phi_g} H^0(X, L).$$

Let $\rho(g)^\vee$ denote the composition. The compatibility implies that the dual isomorphisms satisfy

$$\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2) \in \text{GL}(H^0(X, L)^\vee).$$

Commutativity of the diagram follows from the fact that for a general $x \in X$, the map $\rho(g)^\vee$ gives an isomorphism between sections of $H^0(X, \mathcal{L})$ vanishing at x and those vanishing at $g(x)$.

To prove property (v), it suffices to observe that the matrix $\rho(g)$ preserves the closure of the image $\pi(X)$, which is clear from commutativity.

For property (iv), by the Nullstellensatz any birational automorphism $g \in \text{Bir}(X)$ which is equal to the identity on the k -points of some nonempty open subset $U \subset X$ must be equal to the identity on U . Therefore, $g = \text{id} \in \text{Bir}(X)$.

Part (vii) is proved by taking a tensor power of \mathcal{L} that is very ample and applying property (vi). □

Now we shift our focus to giving examples of birationally equivariant line bundles.

PROPOSITION 1.6. *Let X be a smooth projective variety.*

- (i) *If ω_X is a globally generated line bundle, then it is birationally equivariant.*
- (ii) *More generally, if the image of the evaluation map*

$$H^0(X, \wedge^i \Omega_X) \otimes_k \mathcal{O}_X \rightarrow \wedge^i \Omega_X$$

is a line bundle $\mathcal{L} \subset \wedge^i \Omega_X$ (which is necessarily globally generated), then \mathcal{L} is birationally equivariant.

Proof. Part (ii) implies part (i), so we just prove part (ii). Let $g \in \text{Bir}(X)$ and let $i_g : U_g \hookrightarrow X$ denote the inclusion of the open set on which g is defined (so the complement $X \setminus U_g$ has codimension ≥ 2).

The derivative map

$$\wedge^i dg: g^*(\wedge^i \Omega_X) \rightarrow \wedge^i \Omega_{U_g}$$

pushes forward to an inclusion

$$i_{g*}(\wedge^i dg): i_{g*}g^*(\wedge^i \Omega_X) \rightarrow \wedge^i \Omega_X.$$

This gives an injection on global sections:

$$H^0(X, i_{g*}g^*(\wedge^i \Omega_X)) \hookrightarrow H^0(X, \wedge^i \Omega_X).$$

Now $i_{g*}g^*(\wedge^i \Omega_X)$ contains the line bundle $g^*\mathcal{L}$ (here we use smoothness of X to say that Weil divisors are Cartier). Moreover, every global section of \mathcal{L} pulls back to a global section of $g^*\mathcal{L}$. Thus, we have a commuting diagram of inclusions as follows.

$$\begin{array}{ccccc} H^0(X, \mathcal{L}) & \xleftarrow{g^*} & H^0(X, g^*\mathcal{L}) & & H^0(X, \mathcal{L}) \\ \parallel & & \downarrow & & \parallel \\ H^0(X, \wedge^i \Omega_X) & \xleftarrow{g^*} & H^0(X, i_{g*}g^*(\wedge^i \Omega_X)) & \xrightarrow{\wedge^i dg} & H^0(X, \wedge^i \Omega_X) \end{array}$$

As the spaces on the left and the right are of the same dimension and the maps are all inclusions, it follows that every map is an isomorphism. Lastly, the commutative diagram of evaluation maps

$$\begin{array}{ccc} H^0(X, g^*\mathcal{L}) \otimes_k \mathcal{O}_X & \xrightarrow{\cong} & H^0(X, \wedge^i \Omega_X) \otimes_k \mathcal{O}_X \\ \downarrow & & \downarrow \\ g^*\mathcal{L} & \xrightarrow{\phi_g} & \wedge^i \Omega_X \end{array}$$

shows that the image of $g^*\mathcal{L}$ in $\wedge^i \Omega_X$ contains \mathcal{L} . It remains to show that the natural inclusion $\mathcal{L} \hookrightarrow \phi_g(g^*\mathcal{L}) \cong g^*\mathcal{L}$ is an isomorphism.

Suppose for contradiction that $g^*\mathcal{L} \cong \mathcal{L}(\Delta)$ for some effective divisor Δ . As $H^0(X, \mathcal{L})$ and $H^0(X, g^*\mathcal{L})$ have the same dimension it follows that the fixed component of the linear series $|g^*\mathcal{L}|$ equals Δ . However, the sections in $H^0(X, g^*\mathcal{L})$ globally generate $g^*\mathcal{L}$ on U_g , which has a complement of codimension at least two in X , so there is no fixed component. Therefore, ϕ_g defines an isomorphism between $g^*\mathcal{L}$ and \mathcal{L} .

To check equality of isomorphisms

$$\phi_{g_2} \circ g_2^*(\phi_{g_1}) = \phi_{g_1 \cdot g_2},$$

it suffices to check on any nonempty open set (as global automorphisms of a line bundle on a projective variety are constant). This reduces to the chain rule:

$$\wedge^i dg_2 \circ g_2^*(\wedge^i dg_1) = \wedge^i d(g_1 \cdot g_2)$$

on an open set where everything is defined. □

2. p -cyclic covers and their resolutions

The goal of this section is to define p -cyclic covers in characteristic p , present Kollár’s resolution [Kol95, § 21], and check that they have terminal singularities (by further passing to a log resolution, and computing discrepancies). Throughout we work over an algebraically closed field k of characteristic $p > 0$.

First we define cyclic covers. Fix a k -scheme X together with a line bundle \mathcal{L} on X . Let

$$\mathbb{L} = \text{Spec}_{\mathcal{O}_X} \left(\bigoplus_{i \geq 0} \mathcal{L}^{-i} \cdot y^i \right)$$

(respectively, $\mathbb{L}^{\otimes m}$) be the total space of the line bundle \mathcal{L} (respectively, $\mathcal{L}^{\otimes m}$). Let $s \in H^0(X, \mathcal{L}^{\otimes m})$ be a section, which corresponds to the following map.

$$\begin{array}{ccc} \mathbb{L}^{\otimes m} & \longrightarrow & X \\ & \searrow s & \end{array}$$

There is also an m th power map, $\mathbb{L} \xrightarrow{p_m} \mathbb{L}^{\otimes m}$, which is a μ_m -quotient.

DEFINITION 2.1. The m -cyclic cover branched over s is $Y := p_m^{-1}(s(X))$ with the map $\nu: Y \rightarrow X$. We say that the cyclic cover Y has *branch divisor* $(s = 0) \subset X$.

It follows that $Y \cong \text{Spec}_{\mathcal{O}_X} \left(\bigoplus_{i \geq 0} \mathcal{L}^{-i} \cdot y^i / (y^m - s) \right)$.

Let X be a smooth projective k -variety with a line bundle \mathcal{L} and let $s \in H^0(X, \mathcal{L}^{\otimes p})$ be a section. The p -cyclic cover Y branched along s is inseparable and typically singular. However, if s is general then Kollár shows how to resolve these singularities. We say s has *non-degenerate critical points* [Kol96, 17.3] if when we locally describe s as a function, any critical point of s has a non-degenerate Hessian matrix (when the characteristic is two this forces the dimension to be even, we leave out the odd-dimensional case here). In this case, any critical point of s gives rise to an isolated hypersurface singularity on Y of the form

$$y^p = f_2(x_1, \dots, x_n) + f_3,$$

where f_2 and f_3 are functions on X , f_3 vanishes to order three, and $f_2(x_1, \dots, x_n)$ is a quadratic polynomial with non-degenerate Hessian. Kollár shows that these isolated singularities can be resolved by a sequence of blow-ups at points.

If $p = 2$ then Y is resolved after one blow-up of each singular point, and this is a log-resolution (the exceptional divisor over each point is given by the quadric:

$$y^2 - f_2(x_1, \dots, x_n) = 0 \subset \mathbb{P}^n,$$

which can be checked to be smooth).

If $p > 2$, then Kollár shows that a sequence of $(p - 1)/2$ blow-ups of isolated double points resolves the singularities of Y . At the i th step, the new exceptional divisor over Y is a quadric in the new exceptional divisor over \mathbb{L} whose equation is given by

$$f_2(x_1, \dots, x_n) = 0 \subset \mathbb{P}^n,$$

where \mathbb{P}^n has coordinates $[x_1 : \dots : x_n : y]$. This exceptional divisor is smooth away from the point $[0 : \dots : 0 : 1]$. The only exceptional divisor it intersects is the one from the step before, and the intersection is given by $(y = 0) \cap (f_2 = 0) \subset \mathbb{P}^n$, which is smooth. Here the strict transform of Y has the new local equation:

$$y^{p-2i} = f_2(x_1, \dots, x_n) + f_3.$$

Thus, it is resolved after $(p - 1)/2$ steps. To give a log resolution, that is, to resolve the singularity of the $(p - 1)/2$ th exceptional divisor, we must blow-up one more time at the point $[0 : \dots : 0 : 1]$. The last exceptional divisor over Y is a smooth projective space \mathbb{P}^{n-1} with coordinates $[x_1 : \dots : x_n]$, and the intersection of the last two exceptional divisors is again the quadric $f_2(x_1, \dots, x_n) = 0 \subset \mathbb{P}^{n-1}$. This shows that the total exceptional divisor is simple normal crossing.

Call this log-resolution Z . This gives a log resolution of Y which fits into the following diagram.

$$\begin{array}{ccc} Z \subset \tilde{\mathbb{L}} & \xrightarrow{\tilde{\mu}} & \\ \sigma \downarrow & \pi \downarrow & \\ Y \subset \mathbb{L} & \xrightarrow{\mu} & X \end{array}$$

PROPOSITION 2.2. *Let X be a smooth k -variety of dimension $n \geq 3$ with a line bundle \mathcal{L} . If $s \in H^0(X, \mathcal{L}^{\otimes p})$*

is a section with non-degenerate critical points then the p -cyclic cover branched over s has terminal singularities.

Proof. This can be checked locally at each singularity of the form

$$y^p + f_2(x_1, \dots, x_n) + f_3.$$

First, when the characteristic of k is two (with n even), if $\sigma: Z \rightarrow Y$ is the resolution of singularities and E is the unique exceptional divisor it suffices to compute the coefficient α of E in the equation

$$K_Z = \sigma^*(K_Y) + \alpha E = \pi^*(K_{\mathbb{L}} + Y)|_Z + (n - 2)E.$$

If $n \geq 3$, then $\alpha > 0$.

When p is odd, let $E_i \subset Z$ denote the strict transform of the exceptional divisor of the i th blow-up of T . Let $r = (p - 1)/2$. Then it suffices to compute the coefficients α_i of E_i :

$$\begin{aligned} K_Z &= \sigma^*(K_Y) + \alpha_1 E_1 + \dots + \alpha_r E_r + \alpha_{r+1} E_{r+1} \\ &= \pi^*(K_{\mathbb{L}} + Y)|_Z + (n - 2)E_1 + (2n - 4)E_2 + \dots + (rn - 2r)E_r + ((r + 1)n - p)E_{r+1}. \end{aligned}$$

This gives $\alpha_i = i(n - 2)$ for $0 \leq i \leq r$ and $\alpha_{r+1} = (r + 1)n - p$. These are all positive for $n \geq 3$. □

3. Computing the space of $(n - 1)$ -forms

Again, assume k has characteristic $p > 0$. In this section we prove a slightly more general version of Theorem B. Specifically we consider the following situation:

- (i) X is a smooth projective k -variety of dimension $n \geq 3$;
- (ii) \mathcal{L} is an effective line bundle on X with total space \mathbb{L} ;
- (iii) $s \in H^0(X, \mathcal{L}^{\otimes p})$ is a global section with non-degenerate critical points;
- (iv) $\nu: Y \rightarrow X$ is the p -cyclic cover branched over s ;
- (v) and assume that $-K_Y$ is ample (i.e. Y is Fano).

In Proposition 3.2, assuming that

$$H^0(X, T_X \otimes \omega_X \otimes \mathcal{L}^{p-1}) = 0,$$

we show

$$H^0(Y, \wedge^{n-1} \Omega_Z) = H^0(X, \omega_X \otimes \mathcal{L}^p).$$

As a consequence, we show the line bundle $\mu^*(\omega_X \otimes \mathcal{L}^p)|_Y$ is birationally equivariant on Y . Theorems A and B and Corollary C follow from results in §1.

Let $Z \subset \tilde{\mathbb{L}}$ be the log-resolution of the cyclic cover as in §2. Following Kollár, consider the relative cotangent sequence for $\tilde{\mathbb{L}}/X$ restricted to Z and the cotangent sequence for $Z \subset \tilde{\mathbb{L}}$. This gives rise to the following diagram.

$$\begin{array}{ccccccc}
 & & & I/I^2 & \longrightarrow & \tau & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \tilde{\mu}^*\Omega_X|_Z & \longrightarrow & \Omega_{\tilde{\mathbb{L}}}|_Z & \xrightarrow{\rho_1} & \Omega_{\tilde{\mathbb{L}}/X}|_Z \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Q & \longrightarrow & \Omega_Z & \xrightarrow{\rho_2} & B \longrightarrow 0
 \end{array}$$

Here, τ (respectively, B) is the torsion (respectively, torsion-free) part of $\Omega_{\tilde{\mathbb{L}}/X}|_Z$. To check that there is a map ρ_2 that makes the diagram commute, it suffices to check that I/I^2 maps to 0 in B which can be done generically as these are both torsion-free. Generically, this follows from the fact that the y -derivative of the equation of the cyclic cover

$$y^p - s(x_1, \dots, x_n) = 0$$

vanishes (as we are in characteristic p). Here Q is defined to be the kernel of ρ_2 .

PROPOSITION 3.1. *In the setting described previously:*

(i) *the natural map*

$$\wedge^{n-1}\Omega_Z \cong \Omega_Z^\vee \otimes \omega_Z \rightarrow Q^\vee \otimes \omega_Z$$

is surjective outside of codimension two;

- (ii) *the kernel is isomorphic to $\det(Q)$;*
- (iii) *if $H^0(X, T_X \otimes \omega_X \otimes \mathcal{L}^{p-1}) = 0$ then $H^0(Z, \wedge^{n-1}\Omega_Z) \cong H^0(Z, \det(Q))$.*

Proof. For part (i), the map

$$\Omega_Z^\vee \otimes \omega_Z \rightarrow Q^\vee \otimes \omega_Z$$

only fails to be surjective on the locus where B is not locally free (which has codimension at least two as B is torsion-free). Letting A denote the kernel of the map above, we observe that A reflexive as it is the kernel of a map of reflexive sheaves. A is therefore a line bundle as it is rank one.

Now that we know the kernel is a line bundle, part (ii) can be verified outside codimension two where the sequence

$$0 \rightarrow A \rightarrow \Omega_Z^\vee \otimes \omega_Z \rightarrow Q^\vee \otimes \omega_Z \tag{1}$$

becomes exact. Thus, we have

$$c_1(A) + c_1(Q^\vee \otimes \omega_Z) = c_1(\wedge^{n-1}\Omega_Z),$$

which gives $c_1(A) = c_1(Q)$, that is, $A \cong \det Q$.

It remains to check part (iii). By (1) and part (ii) it suffices to show that $Q^\vee \otimes \omega_Z$ has no global sections. There is an inclusion

$$Q^\vee \otimes \omega_Z \subset (\tilde{\mu}^*T_X)|_Z \otimes \omega_Z,$$

so it suffices to show

$$H^0(Z, (\tilde{\mu}^*T_X)|_Z \otimes \omega_Z) = 0.$$

As Y has terminal singularities there is an exact sequence

$$0 \rightarrow \sigma^* \omega_Y \rightarrow \omega_Z \rightarrow \mathcal{O}_\Delta(\Delta) \rightarrow 0$$

(where Δ is an effective exceptional divisor in Z). Pushing forward to Y shows $\omega_Y \cong \sigma_* \omega_Z$ (as they are isomorphic outside of points, torsion-free, and the first is a line bundle). By the projection formula,

$$\mu^* T_X|_Y \otimes \omega_Y \rightarrow \mu^* T_X|_Y \otimes \sigma_* \omega_Z$$

is an isomorphism. Therefore, as push-forward preserves global sections it suffices to show that

$$H^0(Y, \mu^* T_X|_Y \otimes \omega_Y) = 0.$$

Now $\omega_Y = \nu^*(\omega_X \otimes \mathcal{L}^{p-1})$. As Y is a p -cyclic cover,

$$\nu_*(\mathcal{O}_Y) \cong \bigoplus_{i=0}^{p-1} \mathcal{L}^{-i}.$$

Pushing forward $\nu^*(T_X) \otimes \omega_Y$ gives

$$H^0(Y, \nu^*(T_X) \otimes \omega_Y) = \bigoplus_{i=0}^{p-1} H^0(X, T_X \otimes \omega_X \otimes \mathcal{L}^i).$$

This vanishes by the assumptions that $H^0(X, T_X \otimes \omega_X \otimes \mathcal{L}^{p-1}) = 0$ (here we use that \mathcal{L} is effective to show the other summands vanish). □

PROPOSITION 3.2. *Assume:*

- (i) $H^0(X, T_X \otimes \omega_X \otimes \mathcal{L}^{p-1}) = 0$;
- (ii) $\omega_X \otimes \mathcal{L}^p$ is globally generated; and
- (iii) Y is Fano (or that $H^0(Y, \omega_Y) = 0$).

Then

$$\det(Q) \cong \tilde{\mu}^*(\omega_X \otimes \mathcal{L}^p)|_Z(\Delta)$$

for some effective divisor Δ that is exceptional for the birational map σ , and

$$H^0(Z, \wedge^{n-1} \Omega_Z) \cong H^0(Z, \det(Q)) \cong H^0(Z, \tilde{\mu}^*(\omega_X \otimes \mathcal{L}^p)|_Z) \cong H^0(X, \omega_X \otimes \mathcal{L}^p). \tag{2}$$

Proof. By Kollár’s work ([Kol95, § 23]) there is an injection:

$$\tilde{\mu}^*(\omega_X \otimes \mathcal{L}^p)|_Z \hookrightarrow \wedge^{n-1} \Omega_Z. \tag{3}$$

The line bundle $\tilde{\mu}^*(\omega_X \otimes \mathcal{L}^p)|_Z$ is globally generated, so it must land inside of $\det(Q)$. Hence, $\det(Q) = \tilde{\mu}^*(\omega_X \otimes \mathcal{L}^p)|_Z(\Delta)$ for some effective divisor Δ . On the other hand, away from the exceptional divisors of σ , the vector bundle Q is the pull-back of a vector bundle on the complement of the singular locus of Y with determinant $\nu^*(\omega_X \otimes \mathcal{L}^p)$. Thus, the line bundles are isomorphic away from the exceptional divisors. Thus, Δ is exceptional for σ .

Pushing forward along σ gives a map on Y :

$$\nu^*(\omega_X \otimes \mathcal{L}^p) \rightarrow \sigma_*(\det(Q)),$$

which is necessarily an isomorphism, as $\sigma_*(\det(Q))$ is torsion-free and they are isomorphic away from points. It follows that

$$H^0(Z, \det(Q)) \cong H^0(Z, \tilde{\mu}^*(\omega_X \otimes \mathcal{L}^p)|_Z) \cong H^0(Y, \nu^*(\omega_X \otimes \mathcal{L}^p)).$$

Lastly, we have

$$\nu_*(\nu^*(\omega_X \otimes \mathcal{L}^p)) \cong \bigoplus_{i=0}^{p-1} (\omega_X \otimes \mathcal{L}^{p-i}). \tag{4}$$

By the Fano assumption, $H^0(Y, \omega_Y) = 0$. We also have $\omega_Y = \nu^*(\omega_X \otimes \mathcal{L}^{p-1})$. Thus,

$$\nu_*(\omega_Y) = \bigoplus_{i=0}^{p-1} (\omega_X \otimes \mathcal{L}^{p-1-i})$$

has no global sections. It follows that the only global sections on the right-hand side of (4) come from $\omega_X \otimes \mathcal{L}^p$, giving:

$$H^0(Y, \nu^*(\omega_X \otimes \mathcal{L}^p)) \cong H^0(X, \omega_X \otimes \mathcal{L}^p),$$

which completes the proof. □

Proof of Theorem B. We check that in the setting of Theorem B, the assumptions of Proposition 3.2 are satisfied. Let e be a positive integer such that

$$e + (p - 1)d \leq n \leq e + pd - 3.$$

Consider a hypersurface $X \subset \mathbb{P}_k^{n+1}$ of degree e and let $\mathcal{L} = \mathcal{O}_X(d)$ for some $d \geq 1$. We claim that

$$H^0(X, T_X \otimes \omega_X \otimes \mathcal{L}^{p-1}) = H^0(X, T_X(e + (p - 1)d - n - 2)) = 0.$$

The Euler sequence restricted to X

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(e + (p - 1)d - n - 2) \rightarrow \mathcal{O}_X^{\oplus(n+2)}(e + (p - 1)d - n - 1) \\ \rightarrow T_{\mathbb{P}^{n+1}}(e + (p - 1)d - n - 2)|_X \rightarrow 0 \end{aligned}$$

can be used to show that $T_{\mathbb{P}^{n+1}}(e + (p - 1)d - n - 2)|_X$ has no global sections. Thus, the above vanishing follows from taking global sections for the inclusion of tangent bundles:

$$T_X(e + (p - 1)d - n - 2) \hookrightarrow T_{\mathbb{P}^{n+1}}(e + (p - 1)d - n - 2)|_X.$$

Next, observe that

$$\omega_X \otimes \mathcal{L}^p \cong \mathcal{O}_X(e + pd - n - 2),$$

so the inequality $e + pd - 3 \geq n$ implies that $\omega_X \otimes \mathcal{L}^p$ is globally generated.

Now for any point $x \in X$ the restriction map $H^0(X, \mathcal{L}^p) \rightarrow \mathcal{L}^p/\mathfrak{m}_x^3$ is surjective as $p \geq 2$ and $\mathcal{L}^p = \mathcal{O}_X(pd)$ (this follows from the analogous result for \mathbb{P}^{n+1} by restricting sections). Therefore, by [Kol96, V.5.7.1], a general section $s \in H^0(X, \mathcal{O}_X(pd))$ has non-degenerate critical points. Y is Fano as $\omega_Y = \nu^*(\mathcal{O}_X(e + (p - 1)d - n - 2))$. By Proposition 3.2, it follows that

$$H^0(Z, \wedge^{n-1}\Omega_Z) \cong H^0(X, \omega_X(pd)). \tag{5}$$

Proof of Corollary C. By §2, Y admits a log resolution $\sigma: Z \rightarrow Y$ and by Theorem B there is an injection

$$\tilde{\mu}^*(\omega_X(pd))|_Z \hookrightarrow \wedge^{n-1}\Omega_Z,$$

which induces the isomorphism on global sections in (2). In particular, the image of the evaluation map

$$H^0(Z, \wedge^{n-1}\Omega_Z) \otimes \mathcal{O}_Z \rightarrow \wedge^{n-1}\Omega_Z$$

is precisely the line bundle

$$\mathcal{L} := \tilde{\mu}^*(\omega_X(pd))|_Z,$$

so by Proposition 1.6(ii) it is birationally equivariant on Z . Theorem 1.5(iii) shows that $\mu^*(\omega_X(pd))|_Y$ is birationally equivariant on Y . The map ν is purely inseparable and the sections of $\nu^*(\omega_X(pd))$ define a map

$$Y \rightarrow X \subset \mathbb{P}(H^0(Y, \nu^*(\omega_X(pd)))^\vee).$$

Therefore, Theorem 1.5(vi) and (vii) imply that $\text{Bir}(Y) \cong \text{Aut}(Y) \hookrightarrow \text{Aut}(X)$. □

We are now ready to give the following proof.

Proof of Theorem A. Let X be a general degree $e \geq 3$ hypersurface over an algebraically closed field of characteristic $p > 0$ and fix the line bundle $\mathcal{L} := \mathcal{O}_X(d)$. Assume (as in Corollary C) that $n \geq 3$ and

$$e + (p - 1)d \leq n \leq e + pd - 3.$$

Let Y be a cyclic cover branched over a general section of $H^0(X, \mathcal{O}_X(pd))$. By Corollary C,

$$\text{Bir}(Y) \hookrightarrow \text{Aut}(X).$$

By the work of Matsumura and Monsky [MM63] (see [Poo05, Corollary 1.9] for a more modern treatment), we may assume that $\text{Aut}(X) = \{1\}$. The index of such a Y is $n + 2 - e - (p - 1)d$. For appropriate choices of e and n this can be made arbitrarily large. For example, $p = 2$, $e = 3$, $d = 3$, and $n = 6$ give index-two examples. When $p = 2$, $e = 3$, $d = 4$, and $n = 8$, there are index-three examples. □

Remark 3.3. Fixing a prime number p . There are examples of Fano varieties of dimension n , index equal to $i \geq 2$, and trivial birational automorphisms once

$$n \geq p(i + 1).$$

Indeed, setting $d = i + 1$, there are always solutions to $n + 3 - e = pd$ with $e \geq 3$ once n satisfies the above inequality. For such values of n , e , and d , the index is

$$n + 2 - e - (p - 1)d = pd - 1 - (p - 1)d = d - 1 = i.$$

ACKNOWLEDGEMENTS

We would like to thank J. Kollár, M. Mustařă, A. Perry, A. Pukhlikov, and B. Totaro for valuable comments and discussions. We especially thank the referees for a thorough report and helpful suggestions.

REFERENCES

Che00 I. Cheltsov, *On a smooth four-dimensional quintic*, Mat. Sb. **191** (2000), 139–160.
 CS20 N. Chen and D. Stapleton, *Fano hypersurfaces with arbitrarily large degrees of irrationality*, Forum Math. Sigma **8** (2020), e24.
 CS21 N. Chen and D. Stapleton, *Rational endomorphisms of Fano hypersurfaces*, Preprint (2021). [arXiv:2103.12207](https://arxiv.org/abs/2103.12207).
 CP16 J.-L. Colliot-Thélène and A. Pirutka, *Hypersurfaces quartiques de dimension 3: non rationalité stable*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), 371–397.
 Cor95 A. Corti, *Factoring birational maps of 3-folds after Sarkisov*, J. Algebraic Geom. **4** (1995), 223–254.
 dF13 T. de Fernex, *Birationally rigid hypersurfaces*, Invent. Math. **192** (2013), 533–566.
 dF16 T. de Fernex, *Erratum to: Birationally rigid hypersurfaces*, Invent. Math. **203** (2016), 675–680.

- dFEM03 T. de Fernex, L. Ein and M. Mustață, *Bounds for log canonical thresholds with applications to birational rigidity*, Math. Res. Lett. **10** (2003), 219–236.
- Fan08 G. Fano, *Sopra alcune varietà algebriche a tre dimensioni aventi tutti i generi nulli*, Atti. Ac. Torino **43** (1908), 973–977.
- Fan15 G. Fano, *Osservazioni sopra alcune varietà non razionali aventi tutti i generi nulli*, Atti. Ac. Torino **50** (1915), 1067–1072.
- IM71 V. A. Iskovskikh and J. I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. **86** (1971), 140–166.
- Kol95 J. Kollár, *Nonrational hypersurfaces*, J. Amer. Math. Soc. **8** (1995), 241–249.
- Kol96 J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32 (Springer, Berlin, 1996).
- MM63 H. Matsumura and P. Monsky, *On the automorphisms of hypersurfaces*, J. Math. Kyoto Univ. **3** (1963), 347–361.
- Poo05 B. Poonen, *Varieties without extra automorphisms III: hypersurfaces*, Finite Fields Appl. **11** (2005), 230–268.
- Puk87 A. V. Pukhlikov, *Birational isomorphisms of four-dimensional quintics*, Invent. Math. **87** (1987), 303–329.
- Puk02 A. V. Pukhlikov, *Birationally rigid Fano hypersurfaces*, Izv. Math. **66** (2002), 1243.
- Puk10 A. V. Pukhlikov, *Birational geometry of Fano double spaces of index two*, Izv. Math. **74** (2010), 925–991.
- Puk16 A. V. Pukhlikov, *Birational geometry of Fano hypersurfaces of index two*, Math. Ann. **366** (2016), 721–782.
- Puk20 A. V. Pukhlikov, *Birational geometry of singular Fano hypersurfaces of index two*, Manuscripta Math. **161** (2020), 161–203.
- Puk21 A. V. Pukhlikov, *Birational geometry of singular Fano double spaces of index two*, Sb. Math. **212** (2021), 551–566.
- Sch19 S. Schreieder, *Stably irrational hypersurfaces of small slopes*, J. Amer. Math. Soc. **32** (2019), 1171–1199.
- Tot16 B. Totaro, *Hypersurfaces that are not stably rational*, J. Amer. Math. Soc. **29** (2016), 883–891.
- Voi13 C. Voisin, *Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal*, J. Algebraic Geom. **22** (2013), 141–174.

Nathan Chen nathanchen@math.harvard.edu

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

David Stapleton dajost@umich.edu

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA