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COMPLETIONS OF RANK RINGS

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In this note, we prove three results on regular rings possessing a rank function: (a) the completion of a *-regular rank ring is a regular Baer *-ring; (b) (a) is used to construct regular Baer * factors of type II_f with centre any complex subfield closed under conjugation; (c) the units of a unit-regular rank ring form a dense topological subgroup of the units of the completion. We also outline a proof that for suitable simple regular rings, all proper normal subgroups of the commutator subgroup of the group of units are central.

Definitions. All rings are associative, with 1 and are usually denoted R. A ring R is *regular* if for all r in R, there exists t in R (called a quasi-inverse for r) such that $rtr = r$. A *rank function* N on the regular ring R is a function $N: R \to [0, 1]$ satisfying:

(i) $N(r) = 0$ if and only if $r = 0$
(ii) $N(1) = 1$
(iii) $N(rs) \leq N(r), N(s)$
(iv) if $e = e^2$, $f = f^2$, $ef = fe = 0$ then $N(e+f) = N(e) + N(f)$.

For more details, see [4, 11, 5, 2, 3].

A regular ring possessing a rank function is called a rank ring. It follows from the above definitions that $N(r+s) \leq N(r) + N(s)$; if $rR \cap sR = (0)$, then $N(r) + N(s) = N(e)$ where $eR = rR + sR$. In particular the function d_N defined by $d_N(x, y) = N(x-y)$ is a metric on R, called the *rank-metric* associated with N. In the d_N-metric topology, R becomes a topological ring, and N is uniformly continuous ([11; Cap. 18]).

A ring is **-regular* if it possesses an involution * such that every principal right ideal is generated by a projection (an element $p = p^* = p^2$). From ([10; Ex. p. 38]) a ring is *-regular if and only if

(A) it is regular and possesses an involution * such that $rr^* = 0$ implies $r = 0$.

A Baer (*) ring is a ring (with involution *) such that the right annihilator of any set is generated by an idempotent (projection). By ([10; Ex. p. 39]), a

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regular ring is a Baer *-ring if and only if

(B) it is a *-regular Baer ring.

Following the theory of types in [10], we say a Baer *-ring R is a factor of type II_f if $xx^*=1$ implies $x^*x=1$, R has no minimal right annihilator ideals, and the centre of R is a domain. With regularity added, the collection of principal right ideals of R becomes an orthocomplemented modular irreducible complete lattice, hence ([9]) a continuous geometry, so by [11], R is a non-artinian right and left self-injective simple regular ring.

PROPOSITION 1. *If R is a *-regular rank ring, then the completion of R with respect to any rank-metric is a Baer *-ring.*

Proof. Let $\bar{R}$ denote the completion of R with respect to d_N. $\bar{R}$ is right and left self-injective regular (e.g. [2; Theorem 14]), so $\bar{R}$ is a regular Baer ring. Choose r non-zero in R. If $r^*rt=0$ for some t in R, then $0=t^*r^*rt=(rt)^*rt$, whence $rt=0$. Thus the right annihilator of r^*r equals that of r, so $r^*rR \simeq rR$ (as right R-modules; in fact one has $Rr^*r=Rr$), so that $N(r^*r)=N(r)$. But $N(r^*r)\le N(r^*)$, so $N(r^*)\ge N(r)$. By symmetry $N(r)\ge N(r^*)$, and thus $N(r)=N(r^*)=N(r^*r)$. Hence if (r_i) is a Cauchy sequence (with respect to d_N), then (r_i^*) is also Cauchy and this defines the extension of * to $\bar{R}$. Suppose $(r_i^*)(r_i)$ is zero in $\bar{R}$, then $\lim_{i\to\infty} N(r_i^*r_i)=0$, so $\lim N(r_i)=0$ and hence (r_i) is a null sequence (i.e. equals zero in $\bar{R}$). Thus $\bar{R}$ satisfies (A), so is a *-regular Baer ring and from (B), it is a Baer *-ring.

M_nR will denote the ring of $n\times n$ matrices with entries from R.

Let F be a field. Consider the maps $M_{2^n}F \to M_{2^{n+1}}F$ given by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

Suppose F has an involution #. Then # induces an involution * on the ring $R=\lim M_{2^n}F$; set $A^*=A^{\#t}$ (^{t} of course indicates transpose) if A is a matrix. * is easily seen to be invariant under the diagonal maps, so R becomes a *-ring. R, being a union of regular rings is regular, but need not be *-regular.

LEMMA 2 ([9]). *Let F be a field with an involution # such that for all integers n, for all subsets $\{s_i\}_{i=1}^n$ of F,*

$$\sum s_is_i^{\#}=0 \quad \text{implies all the } s_i \text{ are } 0.$$

*Then $R=\lim M_{2^n}F$ is a *-regular ring.*

Proof. The * is as defined above. Since R is regular, we need only verify condition (A). If $B=(b_{ij})$ is a non-zero matrix in $M_{2^n}F$, then trace $(BB^*)=\sum b_{ij}b_{ij}^{\#}\neq 0$, so BB^* is not zero.

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Now $R = \lim M_{2^n}F$ is a rank ring ($N(x) = \text{rank } x/2^n$ if x belongs to $M_{2^n}F$). According to [5; p. 718], Alexander has shown that the centre of the completion $\bar{R}$, is F. We prove this result for all fields of characteristic 0, and thereby obtain type II_f Baer *-regular factors with centre the rationals or the rationals with $\sqrt{-1}$ adjoined.

PROPOSITION 3. *Let F be a field of characteristic* 0, *and let A be an $n \times n$ matrix with entries in F, satisfying*

$$\inf_{\beta \in F} \text{rank}(A - \beta I) \geq n/2. \tag{1}$$

Then there exists B in M_nF such that

$$\text{rank}(AB - BA) \geq n/4. \tag{2}$$

Proof. The n^2 entries of A generate a finitely generated field over $\mathbb{Q}$, so we may assume F is a subfield of the complex numbers, $\mathbb{C}$.

Case 1. $F = \mathbb{C}$. Obviously, (1) and (2) are invariant under change of base, so we may assume A is in Jordan Normal form,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1 is diagonal of size n_1, and A_2 is of size n_2 and a (matrix) direct sum of blocks of the form (α):

$$(\alpha): \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$$

For each block of the form (α), of size m, pick $m-1$ distinct, non-zero complex numbers $a_1, a_2, \ldots, a_{m-1}$. Denoting $CD - DC$ by $|C, D|$,

$$E = \left| \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & 0 & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & 0 & & & \lambda \end{bmatrix}, \begin{bmatrix} 0 & & & & & \\ a_1 & 0 & & & & 0 \\ & a_2 & 0 & & & \\ & & a_3 & 0 & & \\ & & & \ddots & 0 & \\ & 0 & & & \ddots & 0 \end{bmatrix} \right|$$

is diagonal and all its non-zero entries are a_1, $a_2 - a_1$, $a_3 - a_2$, etc., so the commutant E is invertible. Taking a direct sum of suitable size matrices with $\{a_i\}$ below the diagonal, we obtain a matrix B_2 of size n_2 such that $\text{rank}(A_2B_2 - B_2A_2) = n_2$.

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Now by (1), the multiplicity of any eigen value of A_1 must be not greater than $n/2$. Hence there exist at least $t=(\frac{1}{2})(n_1-(n/2))$ pairs of eigen values of A_1, each pair consisting of two *distinct* eigen values. (If $n_1<n/2$, the statement is meaningless but true). Now consider, if $\lambda\neq\mu$, $\lambda, \mu\in F$,

$$\left|\left[\begin{matrix}\lambda & 0\\ 0 & \mu\end{matrix}\right],\left[\begin{matrix}0 & 1\\ 1 & 0\end{matrix}\right]\right|=\left[\begin{matrix}0 & \lambda-\mu\\ \mu-\lambda & 0\end{matrix}\right].$$

By rearranging the eigen values of A_1 into pairs of distinct elements, and taking B_1 to be a suitable direct sum of copies of $\left[\begin{matrix}0 & 1\\ 1 & 0\end{matrix}\right]$ and [0], we obtain a $B_1\in M_{n_1}F$ such that

$$\operatorname{rank}(A_1B_1-B_1A_1)=\max\{0, 2((\tfrac{1}{2})(n_1-(n/2)))\}.$$

If $B=\left[\begin{matrix}B_1 & 0\\ 0 & B_2\end{matrix}\right]$, we see

$$\begin{aligned}\operatorname{rank}(AB-BA)&=\max\{0, n_1-(n/2)\}+n_2\\ &=\max\{n_2, n-(n/2)\}\quad (\text{as } n_1+n_2=n)\\ &=\max\{n_2, (n/2)\}\geq n/2.\end{aligned}$$

Now let F be any subfield of $\mathbb{C}$. We show A satisfies (1) as an element of $M_n\mathbb{C}$. Let $\bar{F}$ denote the algebraic closure of F in $\mathbb{C}$. If $\beta\in\mathbb{C}-\bar{F}$, $\operatorname{rank}(A-\beta I)=n$, since the eigen values of A lie in $\bar{F}$. Suppose $\beta\in\bar{F}-F$; then $F[\beta]$ is finite-dimensional over F, and there exists an automorphism of $\bar{F}$ fixing F and sending β to a distinct root of the same polynomial that β satisfies, γ. Obviously $\operatorname{rank}(A-\beta I)=\operatorname{rank}(A-\gamma I)$. If $\operatorname{rank}(A-\beta I)<n/2$, then β and thus γ are both eigen values of A of multiplicity greater than $n/2$; so A would have too many eigen values! So A satisfies (1) as an element of $M_n\mathbb{C}$. Similarly, if $F\subset\mathbb{R}$, then A satisfies (1) as a matrix in $M_n\mathbb{R}$.

Case 2. $F=\mathbb{R}$. As A satisfies (1) as an element of $M_n\mathbb{C}$ (by the preceding paragraph), there exists $B=C+iD$, $C, D\in M_n\mathbb{R}$ such that $AB-BA=AC-CA+i(AD-DA)$ has rank not less than $n/2$, hence either $AC-CA$, or $AD-DA$ has rank not less than $n/4$.

Case 3. Any $F\subseteq\mathbb{C}$. Either F is dense in $\mathbb{R}$ or $\mathbb{C}$, so with the Euclidean metric, M_nF is dense in the corresponding matrix ring. By the paragraph preceding Case 2, and Case 1 or Case 2, there exists B in $M_n\mathbb{C}$ or $M_n\mathbb{R}$ such that $t=\operatorname{rank}(AB-BA)\geq n/4$. Let J denote the matrix with 1's in the (i, i) position for $i\leq n-t$, and 0's elsewhere. There exist matrices $U, V\in M_n\,\mathbb{C}$ or $M_n\mathbb{R}$ such that $U(AB-BA)V+J=I$. By the density, there exist sequences $\{U_p\}$, $\{B_p\}$, $\{V_p\}$ of matrices over F converging to U, B, V respectively, so $\{U_p(AB_p-B_pA)V_p+J\}$ converges to I. As the determinant is continuous, there

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exists an integer q such that $U_q(AB_q - B_qA)V_q + J$ is invertible. Then

$$\operatorname{rank}(AB_q - B_qA) \geq \operatorname{rank}(U_q(AB_q - B_qA)V_q) \geq n\text{-rank } J = n-(n-t) = t \geq n/4.$$

So $B_q \in M_nF$ is the desired B.

A slight adjustment to the proof of Case 1 above shows, if $F \subseteq \mathbb{C}$ but $F \not\subset \mathbb{R}$, then there exists B in M_nF such that $AB - BA$ is invertible.

THEOREM 4. *Let F be a field of characteristic* 0, *and define* $R = \operatorname{Lim} M_{2^n}F$; *then the centre of the completion of R in its rank metric is F.*

Proof. The completion, $\bar{R}$, is simple [3; Theorem 4.5], so the centre of $\bar{R}$ is a field containing F. Let t be central, but not in F. We may find r in R such that $N(t-r) < \frac{1}{8}$. For all β in F, $t - \beta$ is central and non-zero, whence is a unit, so $N(t-\beta) = 1$. Now $t - \beta = (t-r) + (r-\beta)$. We thus have $N(r-\beta) > \frac{7}{8}$ for all β in F. Regarded as a matrix in some M_nF, r satisfies (1) of the preceding proposition, so there exists s in R with $N(rs - sr) \geq \frac{1}{4}$. But

$$(t-r)s - s(t-r) = -(rs - sr)$$

as t is central, so

$$N(rs - sr) = N((t-r)s - s(t-r)) \leq 2N(t-r) < \tfrac{1}{4},$$

a contradiction.

COROLLARY 5. *Let F be a field that is either formally real or is a subfield of the complex numbers closed under complex conjugation. Then the completion of* $\operatorname{Lim} M_{2^n}F$ *is a regular Baer * factor of type II_f with centre F.*

The method of proof of Theorem 4 suggests the following invariant for $\bar{R}$.

Let R be a regular ring with a rank function N, whose centre is a field F. R satisfies property $p_{\delta,k}$ $(0 < \delta \leq \frac{1}{2};\ 0 < k \leq 1)$ if

(a) $\bar{R}$ is simple

(b) for all $\varepsilon > 0$, for all r in R satisfying

(1) $$\operatorname*{Inf}_{\beta \in F} N(r-\beta) \geq 1 - \delta$$

there exists a in R, depending on ε such that

(2) $$N(ar - ra) > k - \varepsilon.$$

PROPOSITION 6. *Let R be a regular ring with a rank function N, and centre F, a field. If R satisfies $p_{\delta,k}$ for some (δ, k), then the centre of $\bar{R}$ is F and $\bar{R}$ satisfies $p_{\delta,k}$. Conversely, if the centre of $\bar{R}$ is F and $\bar{R}$ satisfies $p_{\delta,k}$, then R satisfies $p_{\delta,k}$.*

Proof. Suppose R satisfies $p_{\delta,k}$. To prove the centre of $\bar{R}$ is F, simply mimic the proof of Theorem 4. Given $\bar{r}$ in $\bar{R}$ with

$$\operatorname*{Inf}_{\lambda \in F} \bar{N}(\bar{r} - \lambda) > 1 - \delta,$$

we may find r in R with $N(r-\bar{r})<\varepsilon/3$ and $\operatorname{Inf}_{\lambda\in F} N(r-\lambda)>1-\delta$. There exists a in R such that $N(ar-ra)>k-\varepsilon/3$. Then

$$\bar{N}(a\bar{r}-\bar{r}a)>k-\varepsilon.$$

Conversely, if the centre of $\bar{R}$ is F and in $\bar{R}$, r satisfies (1), then in satisfies it in R, and we may approximate the 'a' in (2) by an element of R.

We obtain a curious result on the density of the units of R in those of $\bar{R}$.

A ring is *unit-regular* ([1]) if for all x in R there exists a unit u such that $xux=x$. Unit-regularity is equivalent to the cancellation law for finitely generated projective modules, over regular rings ([6]). It is conjectured that all rank rings are unit-regular. If F is a field, lim $M_{2^n}F$ is easily seen to be unit-regular.

PROPOSITION 7 ([8; Proposition 8]). *A regular ring R is unit-regular if and only if*

for all a, b in R satisfying $aR+bR=R$,
there exists t in R such that $a+bt$ in a unit.

For the ring R, we denote the group of units by $R^{\cdot}$.

PROPOSITION 8. *Let R be a rank ring. Then the units of R form a topological group (in the relative rank-metric topology), and if R is unit-regular, the group of units of R is dense in that of $\bar{R}$.*

Proof. If u, v are units, then $u^{-1}-v^{-1}=u^{-1}(v-u)v^{-1}$, so $N(u^{-1}-v^{-1})=N(u-v)$ (since $N(u)=N(v)=1$). Thus $u\mapsto u^{-1}$ is continuous. Now R is a topological ring, so multiplication is continuous; thus $R^{\cdot}$ is a topological group.

Now suppose R is unit-regular. Choose a unit u in $\bar{R}$. Given $\varepsilon>0$, there exists r in R such that $\bar{N}(r-u)<\varepsilon$. Let s be any element such that $rR\oplus sR=R$. Then $N(r)+N(s)=1$, so $N(s)<\varepsilon$. By unit-regularity, there exists t in R such that $r+st$ is a unit; as $N(r+st-r)\leq N(s)<\varepsilon$, we have $N(r+st-u)<2\varepsilon$. Thus any unit in $\bar{R}$ can be approximated by units of R.

If $R=\text{Lim } M_{2^n}F$ or its completion, the commutator subgroup is dense in $R^{\cdot}$. Presumably, this phenomenon holds for any simple right and left self-injective ring.

For $R=\text{Lim } M_{2^n}F$, the only closed normal subgroups of $R^{\cdot}$ and the only normal subgroups of the commutator are central. These results can be extended by following the first part of the proof of Theorem D of [10; p. 137]:

THEOREM. *Let R be a simple regular ring satisfying*

(i) *the comparability axiom* [3]
(ii) *there exist integers $n\geq 2m\geq 6$ such that $R\simeq M_mS\simeq M_nT$ for some rings S and T*
(iii) $xy=1$ *implies* $yx=1$.

Then all proper normal subgroups of the commutator subgroup of $R^{\cdot}$ are central.

Outline of proof. By [3, Corollary 3.15], (i) and (iii) guarantee the existence of a unique rank function on R and (ii) is used as in [10; p. 137–140] to show the prospective normal subgroup contains all transvections with respect to a specific set of n^2 matrix units. Then [7, Theorem II.4] guarantees the normal subgroup contains all of the commutator.

In particular, this applies to all simple right and left self-injective rings that are not artinian.

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