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# ON THE TEMPERED FUNDAMENTAL GROUPS OF HYPERBOLIC CURVES OF GENUS 0 OVER $\overline{\mathbb{Q}}_p$

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*Abstract* Let $p$ be a prime number. In the present paper, we prove that the moduli of hyperbolic curves of genus 0 over an algebraic closure of the field of $p$-adic numbers may be completely determined by their tempered fundamental groups.

*Key words*: anabelian geometry; hyperbolic curve; genus 0; tempered fundamental group; Grothendieck Conjecture

*2020 Mathematics subject classification*: Primary 14H30

## Contents



## Introduction

Let $p$ be a prime number. For any perfect field $F$, we shall write $\overline{F}$ for the algebraic closure (determined up to isomorphisms) of $F$. We shall write $\mathbb{Q}_p$ for the field of $p$-adic numbers; $\mathbb{C}_p$ for the $p$-adic completion of $\overline{\mathbb{Q}}_p$. One of the central subjects/results in anabelian geometry is the Grothendieck Conjecture (cf. [7], Theorem A; [12], Theorem 0.4). The Grothendieck Conjecture-type results assert that 'anabelian' varieties over 'sufficiently arithmetic' fields (for instance, hyperbolic curves over number fields, $p$-adic local fields or finite fields) may be reconstructed from their étale fundamental groups. On the other hand, we note that the group structure of the étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic 0 (i.e. fields far from 'sufficiently arithmetic') may be completely determined by the genus and the number of cusps of the hyperbolic curves. In particular, the moduli of hyperbolic curves over algebraically closed fields of characteristic 0 may not be determined by their étale fundamental groups.

Next, let us recall the tempered fundamental groups of smooth algebraic varieties (i.e. smooth, separated, of finite type and geometrically integral schemes) over non-Archimedean complete valuation fields introduced by André, which may be regarded as a $p$-adic analogue of the usual topological fundamental groups of complex manifolds (cf. [1], [2]). Let $Z$ be a smooth algebraic variety over a non-Archimedean complete valuation field; $\widetilde{Z} \to Z$ a pro-universal étale covering (determined up to isomorphisms). Then the tempered fundamental group $\Pi_Z^{\mathrm{tp}}$ of $Z$ (relative to a suitable choice of basepoint) may be defined as

$$\Pi_Z^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \varprojlim_{Z' \to Z} \mathrm{Aut}((Z'^{\mathrm{an}})^{\mathrm{top}}/Z^{\mathrm{an}}),$$

where $Z' \to Z$ ranges over the finite étale Galois subcoverings of the fixed pro-universal étale covering $\widetilde{Z} \to Z$; $(-)^{\mathrm{an}}$ denotes the Berkovich analytification of $(-)$; $(-)^{\mathrm{top}}$ denotes the topological universal covering of $(-)$. Here, each group $\mathrm{Aut}((Z'^{\mathrm{an}})^{\mathrm{top}}/Z^{\mathrm{an}})$ may be regarded as a topological group endowed with the discrete topology and $\Pi_Z^{\mathrm{tp}}$ may be regarded as a topological group endowed with the subspace topology of the product topology on $\prod_{Z' \to Z} \mathrm{Aut}((Z'^{\mathrm{an}})^{\mathrm{top}}/Z^{\mathrm{an}})$. Note that the calculation of topological fundamental groups of the Berkovich spaces associated to smooth algebraic varieties is already difficult in general. Thus, the determination of the topological group structure of the tempered fundamental group $\Pi_Z^{\mathrm{tp}}$ may be a highly nontrivial problem. Moreover, one may expect that, in general, the topological group structure of the tempered fundamental group $\Pi_Z^{\mathrm{tp}}$ tends to become so complicated and depends heavily on the geometric structure of $Z$, even if the base fields are algebraically closed fields of characteristic 0. So, it is natural to pose the following anabelian geometric question:

**Question 1.** *What geometric information does the tempered fundamental group carry?*

In the remainder, for a smooth algebraic variety $S$ over $\overline{\mathbb{Q}}_p$, we shall write $\Pi_S^{\mathrm{tp}}$ for the tempered fundamental group of $S \times_{\overline{\mathbb{Q}}_p} \mathbb{C}_p$, relative to a suitable choice of basepoint. With regard to Question 1, the following theorems have been obtained by Mochizuki and Lepage so far:

**Theorem 0.1** ([9], Corollary 3.11). *Let $X$, $Y$ be hyperbolic curves over $\overline{\mathbb{Q}}_p$;*

$$\alpha : \Pi_X^{\mathrm{tp}} \xrightarrow{\sim} \Pi_Y^{\mathrm{tp}},$$

*an isomorphism of topological groups. Write $\mathcal{G}_X$, $\mathcal{G}_Y$ for the semi-graphs of anabelioids associated to the special fibres of the stable models of $X$, $Y$, respectively. Then $\alpha$ induces an isomorphism of semi-graphs of anabelioids*

$$\mathcal{G}_X \xrightarrow{\sim} \mathcal{G}_Y$$

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*in a fashion that is functorial with respect to* $\alpha$*. In particular, the following hold:*

- *The isomorphism* $\alpha$ *maps the cuspidal inertia subgroups of* $\Pi_X^{\mathrm{tp}}$ *to the cuspidal inertia subgroups of* $\Pi_Y^{\mathrm{tp}}$ *(cf. Notations and conventions, Fundamental groups).*
- *Write* $\Gamma_X$*,* $\Gamma_Y$ *for the underlying semi-graphs of* $\mathcal{G}_X$*,* $\mathcal{G}_Y$ *(i.e. dual semi-graphs associated to the special fibres of the stable models of* $X$*,* $Y$*), respectively. Then* $\alpha$ *induces an isomorphism of semi-graphs*

$$\alpha_\Gamma : \Gamma_X \xrightarrow{\sim} \Gamma_Y$$

*in a fashion that is functorial with respect to* $\alpha$*.*

**Theorem 0.2** ([4], Theorem 4.13; [5], Theorem 0.2). *In the notation of Theorem 0.1, suppose that* $X$ *and* $Y$ *are hyperbolic Mumford curves over* $\overline{\mathbb{Q}}_p$*. Then the following hold:*

(i) *The isomorphism* $\alpha_\Gamma$ *(cf. Theorem 0.1) is an isomorphism of metric semi-graphs.*

(ii) *There exists a canonical homeomorphism between the underlying topological spaces of the Berkovich spaces* $(X \times_{\overline{\mathbb{Q}}_p} \mathbb{C}_p)^{\mathrm{an}}$ *and* $(Y \times_{\overline{\mathbb{Q}}_p} \mathbb{C}_p)^{\mathrm{an}}$*.*

**Theorem 0.3** ([5], Theorem 0.3). *Let* $E_1$*,* $E_2$ *be once-punctured Tate elliptic curves over* $\overline{\mathbb{Q}}_p$*. Write* $q_1$*,* $q_2$ *for the* $q$*-parameters of* $E_1$*,* $E_2$*, respectively. Suppose that there exists an isomorphism of topological groups*

$$\Pi_{E_1}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{E_2}^{\mathrm{tp}}.$$

*Then there exists an element* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *such that* $q_2 = \sigma(q_1)$*.*

In particular, the above theorems imply that the tempered fundamental groups of hyperbolic curves carry sufficiently rich scheme-theoretic (or, geometric) information even if the base fields are algebraically closed fields of characteristic 0.

In the present paper, inspired by the above theorems, we consider Question 1 for hyperbolic curves of genus 0 over $\overline{\mathbb{Q}}_p$ and prove that their tempered fundamental groups completely determine their moduli:

**Theorem A.** *Let* $n$ *be an integer such that* $n \geq 3$*. Suppose that there exists an isomorphism of topological groups*

$$\alpha : \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1, x_2, \ldots, x_n\}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1, x'_2, \ldots, x'_n\}}.$$

*Note that* $\alpha$ *induces a bijection*

$$\alpha_{\mathrm{cusp}} : \{x_1, x_2, \ldots, x_n\} \xrightarrow{\sim} \{x'_1, x'_2, \ldots, x'_n\}$$

*(cf. Theorem 0.1; Notations and conventions, Fundamental groups). Then there exists an isomorphism of schemes*

$$\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1, x_2, \ldots, x_n\} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1, x'_2, \ldots, x'_n\},$$

*such that the bijection* $\{x_1, x_2, \ldots, x_n\} \xrightarrow{\sim} \{x'_1, x'_2, \ldots, x'_n\}$ *induced by the isomorphism* $\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1, x_2, \ldots, x_n\} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1, x'_2, \ldots, x'_n\}$ *coincides with the bijection* $\alpha_{\mathrm{cusp}}$*.*

We will apply Theorems 0.1 and 0.3, together with some complicated calculations concerning certain Belyi maps (cf. Lemma 1.1, (i), (ii)), to prove Theorem A. Note that Theorem A is related with the partial reconstruction result of hyperbolic curves obtained in an author's previous work (cf. [17], Theorem C), whose proof is a direct application of Theorem 0.1 (cf. [9], Corollary 3.11), together with the theory of resolution of nonsingularities (cf. [5], [15]). On the other hand, in light of Theorems 0.3; A, it is natural to pose the following question:

**Question 2.** *Let $E_1$, $E_2$ be hyperbolic curves of genus* 1 *over $\overline{\mathbb{Q}}_p$. Suppose that there exists an isomorphism of topological groups*

$$\Pi^{\mathrm{tp}}_{E_1} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{E_2}.$$

*Then does there exist an isomorphism of schemes $E_1 \xrightarrow{\sim} E_2$?*

However, at the time of writing of the present paper, the author does not know whether Question 2 is affirmative or not (even if we assume that $E_1$ and $E_2$ are once-punctured). Furthermore, interestingly, one may regard Theorem A as an analogous result in characteristic 0 of the corresponding result for hyperbolic curves of genus 0 over $\overline{\mathbb{F}}_p$ proved by Tamagawa (cf. [13], Theorem 0.2). So, it would also be interesting to investigate the extent to which the analogous results in characteristic 0 of the various results for hyperbolic/stable curves over $\overline{\mathbb{F}}_p$ (cf. for instance, [11], [14], [16], [18]) hold.

The present paper is organised as follows. In §1, we observe that the open subgroups of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ associated to certain Belyi maps are preserved (up to composition with an inner automorphism) via any automorphism of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$. In §2, we execute some elementary computations concerning the Belyi maps that appear in §1. In §3, we apply the results obtained in the previous sections, together with Lepage's reconstruction result for the once-punctured Tate elliptic curves over $\overline{\mathbb{Q}}_p$, to prove Theorem A.

## Notations and conventions

**Numbers:** The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. If $p$ is a prime number, then the notation $\mathbb{Q}_p$ will be used to denote the $p$-adic completion of $\mathbb{Q}$. The notation $\overline{\mathbb{Q}}_p$ will be used to denote an algebraic closure of $\mathbb{Q}_p$. For each positive integer $r$, we fix a primitive $p^r$-th root of unity $\zeta_{p^r} \in \overline{\mathbb{Q}}_p$. The notation $\mathbb{C}_p$ will be used to denote the $p$-adic completion of $\overline{\mathbb{Q}}_p$. It is well-known that $\mathbb{C}_p$ is an algebraically closed field.

**Valuations:** We shall write $v_p$ for the additive valuation on $\overline{\mathbb{Q}}_p$ normalised by $v_p(p) = 1$.

**Topological groups:** Let $G$ be a topological group. Then we shall write $\mathrm{Aut}(G)$ for the group of continuous automorphisms of $G$.

**Curves:** Let $k$ be an algebraically closed field; $X$ a 1-dimensional, connected, smooth, separated, of finite type scheme over $k$. Then we shall write $X(k)$ for the set of $k$-valued points of $X$; $\overline{X}$ for the smooth compactification of $X$ over $k$. We shall refer to an element $\in \overline{X} \setminus X$ as a *cusp* of $X$. Let $(g,n)$ be a pair of nonnegative integers. Then we shall say that $X$ is *of type* $(g,n)$ if $X$ has genus $g$ and the cardinality of the set of cusps of $X$ is $n$. Suppose that $X$ is of type $(g,n)$. Then we shall say that $X$ is a *hyperbolic curve* if $2g-2+n>0$ (so if $g=0$, then $n\geq 3$). We shall write $\mathbb{P}^1_{\overline{\mathbb{Q}}_p}$ for the projective line over $\overline{\mathbb{Q}}_p$. We shall use $t$ for the standard coordinate of $\mathbb{P}^1_{\overline{\mathbb{Q}}_p}$. We shall identify $\overline{\mathbb{Q}}_p$ with $\mathbb{P}^1_{\overline{\mathbb{Q}}_p}(\overline{\mathbb{Q}}_p)\setminus\{\infty\}$.

**Fundamental groups:** Let $X$ be a hyperbolic curve over $\overline{\mathbb{Q}}_p$. Then we shall write $\Pi^{\mathrm{tp}}_X$ for the *tempered fundamental group* of $X\times_{\overline{\mathbb{Q}}_p}\mathbb{C}_p$, relative to a suitable choice of basepoint (cf. [1], [2]). Note that the projection morphism $X\times_{\overline{\mathbb{Q}}_p}\mathbb{C}_p\to X$ induces a bijection between the respective sets of cusps. Let $x$ be a cusp of $X$ (so $x$ determines a cusp $x_{\mathbb{C}_p}$ of $X\times_{\overline{\mathbb{Q}}_p}\mathbb{C}_p$). Then we shall refer to the stabiliser subgroup of $\Pi^{\mathrm{tp}}_X$ associated to some pro-cusp of the pro-universal tempered covering of $X\times_{\overline{\mathbb{Q}}_p}\mathbb{C}_p$ that lies over $x_{\mathbb{C}_p}$ as a *cuspidal inertia subgroup* of $\Pi^{\mathrm{tp}}_X$ associated to $x$. Note that it follows immediately from the various definitions involved that the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_X$ associated to $x$ are *conjugate*. Note also that, if we write $I_X$ for the set of the conjugacy classes of cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_X$, then the natural map $\overline{X}\setminus X\to I_X$ is *bijective*. (Indeed, the surjectivity follows immediately from the various definitions involved, and the injectivity follows immediately from the well-known structure of (the abelianisations of) the étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic 0, together with [2], Proposition 4.4.1.). We shall identify $\overline{X}\setminus X$ with $I_X$ via this natural bijection.

## 1. Numerical characterisations of certain Belyi maps

In the present section, we observe that the open subgroups associated to certain Belyi maps (which will be of use in the proof of our main theorem in §3) are preserved (up to composition with an inner automorphism) via any automorphism of the geometric tempered fundamental group of projective line minus three points (cf. Lemma 1.3).

Let $p$ be a prime number.

**Lemma 1.1.** *The following hold:*

(i) *Let $r$ be a positive integer. Write*

$$\phi_{p^r}:\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\left\{0,\zeta^i_{p^r}\ (0\leq i\leq p^r-1),\infty\right\}\longrightarrow\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}$$

*for the Belyi map determined by the assignment*

$$t\mapsto t^{p^r}.$$

*Then the connected finite étale covering $\phi_{p^r}$ may be uniquely characterised (up to isomorphisms of connected finite étale coverings) as the connected finite étale covering*

$$g:X\longrightarrow\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\},$$

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*satisfying the following conditions:*

- $\deg(g) = p^r$;
- $g$ *is unramified over* $1$;
- $g$ *is totally ramified over* $0$ *and* $\infty$.

(ii) *Let* $(m,n)$ *be a pair of positive integers. Write*

$$\psi_{m,n} : \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0, 1, \frac{m}{m+n}, \ldots, \infty\right\} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty\}$$

*for the Belyi map determined by the assignment*

$$t \mapsto \frac{(m+n)^{m+n}}{m^m n^n} t^m (1-t)^n.$$

*Then the connected finite étale covering* $\psi_{m,n}$ *may be uniquely characterised (up to isomorphisms of connected finite étale coverings) as the connected finite étale covering*

$$g : X \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty\},$$

*satisfying the following conditions:*

- $\deg(g) = m+n$.
- *The genus of* $X$ *is* $0$.
- *Write* $\overline{g} : \overline{X} \to \mathbb{P}^1_{\overline{\mathbb{Q}}_p}$ *for the finite morphism induced by* $g$ *(cf. Notations and conventions, Curves). Then* $\overline{g}^{-1}(0)$ *consists of two closed points of* $\overline{X}$, *and* $\overline{g}^{-1}(1)$ *consists of* $m+n-1$ *closed points of* $\overline{X}$.
- *The ramification index of* $\overline{g}$ *at a closed point over* $0$ *coincides with* $m$, *and the ramification index of* $\overline{g}$ *at another closed point over* $0$ *coincides with* $n$.
- $g$ *is totally ramified over* $\infty$.

**Proof.** Assertion (i) follows immediately from the well-known calculation of the étale fundamental group of the multiplicative group $\mathbb{G}_m$. Next, we verify assertion (ii). Write $\overline{g}^{-1}(0) \stackrel{\text{def}}{=} \{a,b\}$. Then it follows immediately from the various definitions involved that we may assume without loss of generality that the ramification index of $\overline{g}$ at $a$ coincides with $m$, and the ramification index of $\overline{g}$ at $b$ coincides with $n$. Note that since the genus of $X$ is 0, there exists a(n) (unique) isomorphism

$$\overline{X} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p}$$

over $\overline{\mathbb{Q}}_p$ that map $a$, $b$, the unique point $\in \overline{g}^{-1}(\infty)$ to 0, 1, $\infty$, respectively. In particular, we may also assume without loss of generality that

- $X$ is an open subscheme of $\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty\}$;
- $\overline{X} = \mathbb{P}^1_{\overline{\mathbb{Q}}_p}$;
- $a = 0$, $b = 1$, and $\overline{g}(\infty) = \infty$.

Next, since $g$ is totally ramified over $\infty$, it holds that $g$ is defined by a polynomial $h(t) \in \overline{\mathbb{Q}}_p[t]$. Observe that 0, 1 are roots of $h(t)$ with multiplicity $m$, $n$, respectively. Thus, since $\deg(h(t)) = \deg(g) = m+n$, there exists an element $c \in \overline{\mathbb{Q}}_p$ such that

$$h(t) = c \cdot t^m (1-t)^n.$$

Then it holds that

$$h'(t) = c \cdot t^{m-1}(1-t)^{n-1}(m-(m+n)t).$$

On the other hand, since $\overline{g}^{-1}(1)$ consists of $m+n-1$ closed points, and $0 < \frac{m}{m+n} < 1$, it holds that $\overline{g}$ ramifies over 1. Thus, we conclude that $h(\frac{m}{m+n}) = 1$, hence that $c = \frac{(m+n)^{m+n}}{m^m n^n}$. This completes the proof of Lemma 1.1. □

**Remark 1.1.1.** In the remainder of the present paper, for each positive integer $m$, we shall write $\psi_m$ for $\psi_{m,1}$.

**Remark 1.1.2.** Note that $\psi_{m,n}$ is a connected finite étale covering that appears in the proof of the well-known Belyi's theorem (cf. [3], [8]).

**Definition 1.2.** We shall write

$$\Pi^{\mathrm{tp}}_{\phi_{p^r}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}, \quad \Pi^{\mathrm{tp}}_{\psi_{m,n}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$$

for the open subgroups (determined up to $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$-conjugate) of finite index determined by the connected finite étale coverings $\phi_{p^r}$, $\psi_{m,n}$, respectively (cf. Lemma 1.1, (i), (ii)).

**Lemma 1.3.** *Let $\alpha \in \mathrm{Aut}(\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}})$ be an automorphism of topological groups. Recall that $\alpha$ induces a bijection on the set of the conjugacy classes of cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ (cf. [9], Corollary 3.11) that determines a bijection $\alpha_{\mathrm{cusp}} : \{0,1,\infty\} \xrightarrow{\sim} \{0,1,\infty\}$. Suppose that*

$$\alpha_{\mathrm{cusp}} \quad \textit{is the identity automorphism.}$$

*Then there exists an inner automorphism $\iota$ of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$, such that the composite $\alpha \circ \iota \in \mathrm{Aut}(\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}})$ induces an automorphism of $\Pi^{\mathrm{tp}}_{\phi_{p^r}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}}$) via the inclusion $\Pi^{\mathrm{tp}}_{\phi_{p^r}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$) (cf. Definition 1.2) that maps the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\phi_{p^r}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}}$) associated to $*$ to the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\phi_{p^r}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}}$) associated to $*$, where $* \in \{0,1,\infty\}$.*

**Proof.** Note that since $\Pi^{\mathrm{tp}}_{\phi_{p^r}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$) is an open subgroup of finite index, it holds that $\alpha(\Pi^{\mathrm{tp}}_{\phi_{p^r}}) \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ (respectively, $\alpha(\Pi^{\mathrm{tp}}_{\psi_{m,n}}) \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$) is also an open subgroup of finite index. Thus, the inclusion

$\alpha(\Pi^{\mathrm{tp}}_{\phi_{p^r}}) \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ (respectively, $\alpha(\Pi^{\mathrm{tp}}_{\psi_{m,n}}) \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$) determines a connected finite étale covering

$$g_1 : X_1 \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\} \text{ (respectively, } g_2 : X_2 \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}).$$

Next, observe that the numerical information that appears in the conditions in Lemma 1.1, (i) (respectively, Lemma 1.1, (ii)) may be reconstructed from the set of cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ (cf. Notations and conventions, Fundamental groups). In particular, since $\alpha_{\mathrm{cusp}}$ is the identity automorphism, it follows immediately from [9], Corollary 3.11, that $g_1$ (respectively, $g_2$) satisfies the conditions in Lemma 1.1, (i) (respectively, Lemma 1.1, (ii)). Therefore, by replacing $\alpha$ by the composite of $\alpha$ with a suitable inner automorphism of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$, we may assume without loss of generality that

$$\alpha(\Pi^{\mathrm{tp}}_{\phi_{p^r}}) = \Pi^{\mathrm{tp}}_{\phi_{p^r}} \text{ (respectively, } \alpha(\Pi^{\mathrm{tp}}_{\psi_{m,n}}) = \Pi^{\mathrm{tp}}_{\psi_{m,n}})$$

(cf. Lemma 1.1, (i) (respectively, Lemma 1.1, (ii))). Write $\alpha_{p^r}$ (respectively, $\alpha_{m,n}$) for the automorphism of $\Pi^{\mathrm{tp}}_{\phi_{p^r}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}}$) induced by $\alpha$ via the inclusion $\Pi^{\mathrm{tp}}_{\phi_{p^r}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$). Recall that $\alpha_{\mathrm{cusp}}$ is the identity automorphism. Thus, by replacing $\alpha$ by the composite of $\alpha$ with a suitable inner automorphism of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty\}}$, again, if necessary, we conclude from the conditions in Lemma 1.1, (i) (respectively, Lemma 1.1, (ii)) that $\alpha_{p^r}$ (respectively, $\alpha_{m,n}$) maps the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\phi_{p^r}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}}$) associated to $*$ to the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\phi_{p^r}}$ (respectively, $\Pi^{\mathrm{tp}}_{\psi_{m,n}}$) associated to $*$, where $* \in \{0,1,\infty\}$. This completes the proof of Lemma 1.3. □

## 2. Elementary lemmas

Let $p$ be a prime number. In the present section, we discuss elementary calculations concerning the Belyi maps that appear in §1 and the $p$-adic valuation $v_p$ on $\overline{\mathbb{Q}}_p$, which will be of use in the proof of our main theorem in the next section.

**Lemma 2.1.** *In the notation of Lemma 1.1, (i), let $x,y \in \overline{\mathbb{Q}}_p$ be such that $v_p(y) = 0$. Then it holds that*

$$\max_{x_r \in \phi_{p^r}^{-1}(x),\ y_r \in \phi_{p^r}^{-1}(y)} v_p(x_r - y_r) \ \leq\ \max\left\{\frac{1}{p-1},\ v_p(x-y) - r\right\}.$$

**Proof.** Fix elements $x_r \in \phi_{p^r}^{-1}(x)$, $y_r \in \phi_{p^r}^{-1}(y)$. Note that it follows immediately from the definition of $\phi_{p^r}$ that, for each element $y'_r \in \phi_{p^r}^{-1}(y)$, there exists a nonnegative integer $j$ such that $y'_r = \zeta^j_{p^r} \cdot y_r$. Next, observe that

$$x - y = \prod_{0 \leq j \leq p^r - 1} (x_r - \zeta^j_{p^r} \cdot y_r).$$

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Suppose that there exists an integer $i$ such that

$$0 \le i \le p^r - 1, \quad v_p(x_r - \zeta_{p^r}^i \cdot y_r) > \ v_p(1-\zeta_p) = \frac{1}{p-1}$$

(cf. [10], Chapter I, Lemma 10.1). Then it follows immediately from our assumption that $v_p(y) = 0$ [so $v_p(y_r) = 0$] that, for each $j = 0, \dots, i-1, i+1, \dots, p^r - 1$, it holds that

$$v_p(x_r - \zeta_{p^r}^j \cdot y_r) = v_p(x_r - \zeta_{p^r}^i \cdot y_r + \zeta_{p^r}^i \cdot y_r(1-\zeta_{p^r}^{j-i})) = v_p(1-\zeta_{p^r}^{j-i}).$$

On the other hand, observe (cf., e.g. the second display in the proof of [10], Chapter I, Lemma 10.1) that

$$\sum_{\substack{0 \le j \le p^r - 1 \\ j \ne i}} v_p(1-\zeta_{p^r}^{j-i}) = v_p(p^r) = r.$$

Thus, since

$$v_p(x-y) = \sum_{0 \le j \le p^r - 1} v_p(x_r - \zeta_{p^r}^j \cdot y_r),$$

we conclude that

$$v_p(x_r - \zeta_{p^r}^i \cdot y_r) = v_p(x-y) - r.$$

This completes the proof of Lemma 2.1. □

**Lemma 2.2.** *Let $x \in \overline{\mathbb{Q}}_p \setminus \{0,1\}$ be such that $v_p(x) > -p$; $r$ a positive integer. Then, in the notation of Lemma 1.1, (ii) (cf. Remark 1.1.1), there exists a(n) [unique] element $x_1 \in \psi_{p^r}^{-1}(x)$ such that*

- $v_p(1-x_1) = rp^r + v_p(x)$ $(>0)$, *and*
- *for each $y \in \psi_{p^r}^{-1}(x) \setminus \{x_1\}$, it holds that* $v_p(y) = \frac{rp^r + v_p(x)}{p^r}$ $(>0)$.

**Proof.** For each $y \in \psi_{p^r}^{-1}(x)$, it holds that

$$(p^r+1)^{p^r+1}(y^{p^r} - y^{p^r+1}) - (p^r)^{p^r} x = 0.$$

Thus, by using the Newton polygon (cf. [10], Chapter II, Proposition 6.3), we obtain the desired conclusion. This completes the proof of Lemma 2.2. □

**Lemma 2.3.** *In the notation of Lemma 2.2, suppose that*

$$r = 2, \quad v_p(x) = 0, \quad v_p(1-x) \le 1.$$

*Let*

$$s \in \psi_{p^2}^{-1}(x) \ (\subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty\}(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p \setminus \{0,1\})$$

*be such that $v_p(s) = 2$. Write $\mathcal{C}_s \subseteq \overline{\mathbb{Q}}_p$, $\mathcal{C}_x \subseteq \overline{\mathbb{Q}}_p$ for the subsets of the Galois-conjugates of $s \in \overline{\mathbb{Q}}_p$, $x \in \overline{\mathbb{Q}}_p$, respectively. Suppose, moreover, that*

$$\max_{w_x \in \mathcal{C}_x \setminus \{x\}} v_p(x - w_x) \le 1.$$

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*Then it holds that*

$$\max_{w\in\mathcal{C}_s\setminus\{s\}} v_p(s-w) < 4.$$

**Proof.** First, it follows immediately from the definition of $\psi_{p^2}$ that

$$\frac{(p^2+1)^{p^2+1}}{(p^2)^{p^2}}(s^{p^2}-s^{p^2+1}) = x.$$

Next, let $w\in\mathcal{C}_s\setminus\{s\}$ be an element. Then since $w$ is a *Galois-conjugate* of $s$, there exists a *Galois-conjugate* $w_x$ of $x$ such that

$$\frac{(p^2+1)^{p^2+1}}{(p^2)^{p^2}}(w^{p^2}-w^{p^2+1}) = w_x.$$

Thus, by taking the difference of the above equalities, we obtain an equality

$$\frac{(p^2+1)^{p^2+1}}{(p^2)^{p^2}}(s-w)\Bigg(\Bigg(\sum_{0\le l\le p^2-1} s^l w^{p^2-1-l}\Bigg)-\Bigg(\sum_{0\le l\le p^2} s^l w^{p^2-l}\Bigg)\Bigg) = x-w_x.$$

On the other hand, since $v_p(s)=v_p(w)=2$, it holds that

$$v_p\Bigg(\Bigg(\sum_{0\le l\le p^2-1} s^l w^{p^2-1-l}\Bigg)-\Bigg(\sum_{0\le l\le p^2} s^l w^{p^2-l}\Bigg)\Bigg) \ge 2p^2-2.$$

Thus, in the case where $x\ne w_x$, it follows immediately from our assumption that $v_p(x-w_x)\le 1$ that $v_p(s-w)\le 3<4$. In particular, we may assume without loss of generality that

$$x = w_x.$$

Then since $s-w\ne 0$, it holds that

$$\Bigg(\sum_{0\le l\le p^2-1} s^l(s+(w-s))^{p^2-1-l} =\Bigg) \quad \sum_{0\le l\le p^2-1} s^l w^{p^2-1-l}$$

$$= \sum_{0\le l\le p^2} s^l w^{p^2-l} \quad \Bigg(= \sum_{0\le l\le p^2} s^l(s+(w-s))^{p^2-l}\Bigg).$$

Observe that

$$\sum_{0\le l\le p^2-1} s^l(s+(w-s))^{p^2-1-l} = \sum_{0\le h\le p^2-1} c_h\cdot s^{p^2-1-h}(w-s)^h;$$

$$\sum_{0\le l\le p^2} s^l(s+(w-s))^{p^2-l} = \sum_{0\le h\le p^2} d_h\cdot s^{p^2-h}(w-s)^h,$$

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where

$$c_h \stackrel{\text{def}}{=} \sum_{0\le l\le p^2-1-h} \binom{p^2-1-l}{h}, \quad d_h \stackrel{\text{def}}{=} \sum_{0\le l\le p^2-h} \binom{p^2-l}{h}.$$

Here, we note that

$$c_0 = p^2, \quad c_1 = \frac{p^2(p^2-1)}{2}, \quad v_p(c_1) \ge 1, \quad d_0 = p^2+1.$$

Next, suppose that $v_p(s-w) \ge 4$. Then since $v_p(s) = 2$, it holds that, for each $h \ge 1$,

$$v_p(c_h \cdot s^{p^2-1-h}(w-s)^h) \ge 2p^2+1, \quad v_p(d_h \cdot s^{p^2-h}(w-s)^h) \ge 2p^2+1.$$

Thus, in summary, we conclude from the above discussion that

$$v_p(p^2 \cdot s^{p^2-1} - (p^2+1)\cdot s^{p^2}) \ge 2p^2+1,$$

hence that

$$v_p\left(\frac{p^2}{p^2+1} - s\right) \ge 3.$$

Write

$$s' \stackrel{\text{def}}{=} s - \frac{p^2}{p^2+1}, \quad a_0 \stackrel{\text{def}}{=} (p^2)^{p^2}(1-x), \quad a_{p^2+1} \stackrel{\text{def}}{=} -(p^2+1)^{p^2+1}.$$

For each $l = 1, \ldots, p^2$, write

$$a_l = (p^2)^{p^2-l}\cdot(p^2+1)^l \cdot \left(\binom{p^2}{l} - p^2\cdot\binom{p^2}{l-1}\right).$$

Then it follows immediately from the equality in the first display of the present proof that

$$\sum_{0\le l\le p^2+1} a_l\cdot(s')^l = (p^2+1)^{p^2+1}\left(s' + \frac{p^2}{p^2+1}\right)^{p^2}\left(\frac{1}{p^2+1} - s'\right) - (p^2)^{p^2}x = 0.$$

On the other hand, since $0 \le v_p(1-x) \le 1$, and $a_1 = 0$, it follows immediately from the various definitions involved that

$$2p^2 \le v_p(a_0) \le 2p^2+1, \quad v_p(a_{p^2+1}) = 0, \quad v_p(a_1) = \infty,$$
$$v_p(a_l) = 2p^2 - 2l + v_p\left(\binom{p^2}{l}\right) \quad (l = 2, \ldots, p^2).$$

Moreover, for each $l = 2, \ldots, p^2$, it holds that

$$1 - \frac{l}{p^2} \le v_p\left(\binom{p^2}{l}\right),$$

hence that

$$-\frac{v_p(a_0)}{p^2}l + v_p(a_0) \le -\frac{2p^2+1}{p^2}l + 2p^2 + 1 \le 2p^2 - 2l + v_p\left(\binom{p^2}{l}\right) = v_p(a_l).$$

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Then, by using the Newton polygon, we observe that

$$\left(v_p\left(s - \frac{p^2}{p^2+1}\right) = \right) \quad v_p(s') = \frac{v_p(a_0)}{p^2} \quad \left(\leq \frac{2p^2+1}{p^2} < 3\right).$$

This contradicts the inequality $v_p(\frac{p^2}{p^2+1} - s) \geq 3$. Thus, we conclude that $v_p(s-w) < 4$. This completes the proof of Lemma 2.3. □

## 3. Reconstruction of moduli of hyperbolic curves of genus 0 from their geometric tempered fundamental groups

Let $p$ be a prime number. In the present section, we apply the results obtained in the previous sections, together with Lepage's reconstruction result for the Tate elliptic curves, to prove that the tempered fundamental groups of hyperbolic curves of genus 0 over $\overline{\mathbb{Q}}_p$ completely determine their moduli.

First, we begin by recalling Lepage's result:

**Theorem 3.1** ([5], Theorem 4.1). *Let $q_1, q_2 \in \overline{\mathbb{Q}}_p$ be such that $v_p(q_1) > 0$, and $v_p(q_2) > 0$. Write*

$$E_{q_1} \stackrel{\text{def}}{=} \mathbb{G}_m^{\text{an}}/q_1^{\mathbb{Z}}, \quad E_{q_2} \stackrel{\text{def}}{=} \mathbb{G}_m^{\text{an}}/q_2^{\mathbb{Z}}$$

*(i.e. Tate elliptic curves). Suppose that there exists an isomorphism of topological groups*

$$\Pi^{\text{tp}}_{E_{q_1}\setminus\{1\}} \stackrel{\sim}{\to} \Pi^{\text{tp}}_{E_{q_2}\setminus\{1\}}.$$

*Then there exists an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that $q_2 = \sigma(q_1)$.*

**Remark 3.1.1.** In the notation of Theorem 3.1, write $j_1$, $j_2$ for the *j-invariants* of the Tate elliptic curves $E_{q_1}$, $E_{q_2}$, respectively. Then it follows immediately from [6], Theorem 2.1.1, that

$$j_2 = \sigma(j_1).$$

Next, we apply Lemmas 1.3, 2.2; Theorem 3.1, to prove that the moduli of hyperbolic curves of type $(0,4)$ over $\overline{\mathbb{Q}}_p$ may be completely determined by their tempered fundamental groups.

**Proposition 3.2.** *Let $x, x' \in \overline{\mathbb{Q}}_p \setminus \{0,1\}$ be elements. Suppose that there exists an isomorphism of topological groups*

$$\alpha : \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}} \stackrel{\sim}{\to} \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'\}}.$$

*Note that $\alpha$ induces a bijection*

$$\alpha_{\text{cusp}} : \{0,1,\infty,x\} \stackrel{\sim}{\to} \{0,1,\infty,x'\}$$

*(cf. [9], Corollary 3.11). Suppose that*

$$\alpha_{\text{cusp}}(0) = 0, \quad \alpha_{\text{cusp}}(1) = 1, \quad \alpha_{\text{cusp}}(\infty) = \infty.$$

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*Then there exists an element* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *such that* $x' = \sigma(x)$*. In particular, there exists an isomorphism of schemes*

$$\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\},$$

*such that the bijection* $\{0,1,\infty,x\} \xrightarrow{\sim} \{0,1,\infty,x'\}$ *induced by the isomorphism* $\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}$ *coincides with the bijection* $\alpha_{\mathrm{cusp}}$*.*

**Proof.** First, we verify the following assertion:

Claim 3.2.A: We may assume without loss of generality that $v_p(1-x) > 0$.

Indeed, suppose that Proposition 3.2 in the case where $v_p(1-x) > 0$ holds. First, by using suitable geometric automorphisms of $\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty\}$ over $\overline{\mathbb{Q}}_p$, we may assume without loss of generality that $v_p(x) = 0$. Next, write

$$\psi_{p,x} : \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0,1,\frac{p}{p+1},\ldots,\infty\right\} \cup \psi_p^{-1}(x) \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\},$$

$$\psi_{p,x'} : \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0,1,\frac{p}{p+1},\ldots,\infty\right\} \cup \psi_p^{-1}(x') \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}$$

for the connected finite étale coverings induced by $\psi_p$ (cf. Lemma 1.1, (ii); Remark 1.1.1);

$$\Pi^{\mathrm{tp}}_{\psi_{p,x}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\}}, \quad \Pi^{\mathrm{tp}}_{\psi_{p,x'}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}}$$

for the open subgroups (determined up to $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\}}$-conjugate, $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}}$-conjugate, respectively) determined by $\psi_{p,x}$, $\psi_{p,x'}$, respectively. Then since $\alpha_{\mathrm{cusp}}(0) = 0$, $\alpha_{\mathrm{cusp}}(1) = 1$, and $\alpha_{\mathrm{cusp}}(\infty) = \infty$, it follows immediately from Lemma 1.3 that there exists an inner automorphism $\iota$ of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}}$ satisfying the following conditions:

- The composite $\iota \circ \alpha$ induces an isomorphism of topological groups

  $$\beta : \Pi^{\mathrm{tp}}_{\psi_{p,x}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\psi_{p,x'}}$$

  via the inclusions $\Pi^{\mathrm{tp}}_{\psi_{p,x}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\}}$ and $\Pi^{\mathrm{tp}}_{\psi_{p,x'}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}}$.
- Write

  $$\beta_{\mathrm{cusp}} : \left\{0,1,\frac{p}{p+1},\ldots,\infty\right\} \cup \psi_p^{-1}(x) \xrightarrow{\sim} \left\{0,1,\frac{p}{p+1},\ldots,\infty\right\} \cup \psi_p^{-1}(x')$$

  for the bijection induced by $\beta$ (cf. [9], Corollary 3.11). Then it holds that

  $$\beta_{\mathrm{cusp}}(0) = 0, \quad \beta_{\mathrm{cusp}}(1) = 1, \quad \beta_{\mathrm{cusp}}(\infty) = \infty.$$

Let $x_1 \in \psi_p^{-1}(x)$ be such that $v_p(1-x_1) > 0$ (cf. Lemma 2.2). Write $x'_1 \stackrel{\mathrm{def}}{=} \beta_{\mathrm{cusp}}(x_1)$. Note that the kernels of the natural surjections

$$\Pi^{\mathrm{tp}}_{\psi_{p,x}} \twoheadrightarrow \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_1\}}, \quad \Pi^{\mathrm{tp}}_{\psi_{p,x'}} \twoheadrightarrow \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'_1\}}$$

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(induced by the natural open immersions of hyperbolic curves over $\overline{\mathbb{Q}}_p$) are topologically generated by cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\psi_p,x}$, $\Pi^{\mathrm{tp}}_{\psi_p,x'}$ associated to the cusps $\notin \{0,1,\infty,x_1\}$, the cusps $\notin \{0,1,\infty,x'_1\}$, respectively. Then $\beta : \Pi^{\mathrm{tp}}_{\psi_p,x} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\psi_p,x'}$ induces an isomorphism of topological groups

$$\alpha_1 : \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x_1\}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'_1\}}$$

via the above surjections. Moreover, for each $* \in \{0,1,\infty\}$, it holds that $\alpha_1$ maps the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x_1\}}$ associated to $*$ to the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'_1\}}$ associated to $*$. Then since $v_p(1-x_1) > 0$, it follows from our assumption (that Proposition 3.2 in the case where $v_p(1-x) > 0$ holds) that there exists an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that $x'_1 = \sigma(x_1)$. Thus, since $\psi_p$ is defined over $\mathbb{Q}_p$, it holds that $x' = \sigma(x)$. This completes the proof of Claim 3.2.A.

Next, we verify the following assertion:

Claim 3.2.B: Suppose that $v_p(1-x) > 0$. Then it holds that

$$x' = \sigma(x), \quad \text{or} \quad (x')^{-1} = \sigma(x).$$

Indeed, it follows immediately from [9], Corollary 3.11, that $v_p(1-x') > 0$. Write

$$E \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}, \quad E' \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'\}$$

for the finite étale Galois coverings of degree 2 that ramify over every cusp of $\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}$, $\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'\}$, respectively;

$$\Pi^{\mathrm{tp}}_E \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}}, \quad \Pi^{\mathrm{tp}}_{E'} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'\}}$$

for the normal open subgroups of index 2 determined by the above finite étale Galois coverings of degree 2. Observe that the normal open subgroup $\Pi^{\mathrm{tp}}_E \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}}$ coincides with the kernel of the unique surjection

$$q : \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$$

such that the image of every cuspidal inertia subgroup of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}}$ is nontrivial. The normal open subgroup $\Pi^{\mathrm{tp}}_{E'} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'\}}$ admits a similar characterisation. Thus, since $\alpha$ maps the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}}$ to the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'\}}$, the isomorphism $\alpha$ induces an isomorphism of topological groups

$$\Pi^{\mathrm{tp}}_E \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{E'}$$

via the inclusions $\Pi^{\mathrm{tp}}_E \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x\}}$ and $\Pi^{\mathrm{tp}}_{E'} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,x'\}}$. On the other hand, since $v_p(1-x) > 0$, and $v_p(1-x') > 0$, the hyperbolic curves $E$, $E'$ (of type (1,4)) may be regarded as open subschemes of once-punctured Tate elliptic curves $E_1$, $E'_1$ over $\overline{\mathbb{Q}}_p$, where the cusps of $E_1$, $E_1'$ are the origins and correspond to the cusps of $E$, $E'$ that lie over $\infty$ via the finite étale Galois coverings $E \to \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\}$, $E' \to \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}$, respectively. In particular, since $\alpha_{\mathrm{cusp}}(\infty) = \infty$, the isomorphism $\Pi^{\mathrm{tp}}_E \overset{\sim}{\to} \Pi^{\mathrm{tp}}_{E'}$ induces an isomorphism

$$\Pi^{\mathrm{tp}}_{E_1} \overset{\sim}{\to} \Pi^{\mathrm{tp}}_{E_1'}.$$

Write $j(E_1), j(E_1')$ for the $j$-invariants of $E_1$, $E_1'$, respectively. Then it follows immediately from Theorem 3.1, together with Remark 3.1.1, that there exists an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that

$$j(E_1') = \sigma(j(E_1)).$$

Therefore, it holds that

$$\sigma(x) \in \left\{x', \frac{1}{1-x'}, \frac{x'-1}{x'}, \frac{1}{x'}, \frac{x'}{x'-1}, 1-x'\right\}.$$

Moreover, since $v_p(1-x) > 0$, and $v_p(1-x') > 0$, we conclude that

$$x' = \sigma(x), \quad \text{or} \quad (x')^{-1} = \sigma(x).$$

This completes the proof of Claim 3.2.B.

To complete the proof of Proposition 3.2, by applying Claims 3.2.A and 3.2.B, we may assume without loss of generality that

$$v_p(1-x) > 0, \quad (x')^{-1} = \sigma(x).$$

Write

$$X_x \overset{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0, 1, \frac{1}{2}, \infty, \frac{1+\sqrt{1-x}}{2}, \frac{1-\sqrt{1-x}}{2}\right\};$$

$$X_{x'} \overset{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0, 1, \frac{1}{2}, \infty, \frac{1+\sqrt{1-x'}}{2}, \frac{1-\sqrt{1-x'}}{2}\right\};$$

$$\psi_{1,x} : X_x \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\}, \quad \psi_{1,x'} : X_{x'} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}$$

for the finite étale Galois coverings of degree 2 defined by the assignment $t \mapsto 4t(1-t)$. Then $\alpha$ induces an isomorphism of topological groups

$$\gamma : \Pi^{\mathrm{tp}}_{X_x} \overset{\sim}{\to} \Pi^{\mathrm{tp}}_{X_{x'}}$$

(cf. Lemma 1.3). By replacing $\alpha$ by a suitable composite of $\alpha$ with an inner automorphism of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}}$, we may assume without loss of generality that $\gamma$ maps the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{X_x}$ associated to 1 to the cuspidal inertia subgroups of $\Pi^{\mathrm{tp}}_{X_{x'}}$ associated to 1. Write

$$Y_x \overset{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, 1, \infty, 1+\sqrt{1-x}, 1-\sqrt{1-x}\};$$

$$Y_{x'} \overset{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, 1, \infty, 1+\sqrt{1-x'}, 1-\sqrt{1-x'}\}.$$

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Then $\gamma$ induces, via the respective quotients of $\Pi^{\mathrm{tp}}_{X_x}$, $\Pi^{\mathrm{tp}}_{X_{x'}}$ by the normal closed subgroups topologically generated by the cuspidal inertia subgroups associated to 1, an isomorphism of topological groups

$$\delta : \Pi^{\mathrm{tp}}_{Y_x} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{Y_{x'}}.$$

Write

$$\delta_{\mathrm{cusp}} : \{0, 1, \infty, 1+\sqrt{1-x}, 1-\sqrt{1-x}\} \xrightarrow{\sim} \{0, 1, \infty, 1+\sqrt{1-x'}, 1-\sqrt{1-x'}\}$$

for the bijection induced by $\delta$ (cf. [9], Corollary 3.11);

$$x_1 \overset{\mathrm{def}}{=} 1+\sqrt{1-x}; \quad x_1' \overset{\mathrm{def}}{=} \delta_{\mathrm{cusp}}(x_1).$$

Observe that

$$\delta_{\mathrm{cusp}}(0) = 0, \quad \delta_{\mathrm{cusp}}(1) = 1, \quad \delta_{\mathrm{cusp}}(\infty) = \infty, \quad x_1' \in \{1+\sqrt{1-x'}, 1-\sqrt{1-x'}\}.$$

Then since $v_p(1-x_1) = v_p(\sqrt{1-x}) = \frac{v_p(1-x)}{2} > 0$, it follows from Claim 3.2.B that there exists an element $\sigma_1 \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that

$$x_1' = \sigma_1(x_1), \text{ or } (x_1')^{-1} = \sigma_1(x_1).$$

Note that $\psi_{1,x}$ and $\psi_{1,x'}$ are defined over $\mathbb{Q}_p$. Therefore, if $x_1' = \sigma_1(x_1)$, then $x' = \sigma_1(x)$. In particular, it suffices to consider the case where

$$(x')^{-1} = \sigma(x), \quad (x_1')^{-1} = \sigma_1(x_1).$$

In this case, it holds that

$$P((x_1')^{-1}) = \sigma_1(P(x_1)) = \sigma_1(x) = \sigma_1\sigma^{-1}((x')^{-1}),$$

where $P(t) \overset{\mathrm{def}}{=} t(2-t) \in \overline{\mathbb{Q}}_p[t]$. Moreover, it holds that

$$(2-x_1')(2-(x_1')^{-1}) = P(x_1')P((x_1')^{-1}) = x'\sigma_1\sigma^{-1}((x')^{-1}).$$

This implies that one of the following assertions holds:

(a) $x_1'$ and $(x_1')^{-1}$ are *Galois-conjugate*.

(b) $x_1'$ is contained in the Galois closure of $\mathbb{Q}_p(x')$ over $\mathbb{Q}_p$.

Note that since $x_1' \in \{1+\sqrt{1-x'}, 1-\sqrt{1-x'}\}$, it holds that

$$v_p(1-x_1') = \frac{v_p(1-x')}{2}.$$

Note also that the $p$-adic valuation on the Galois closure of $\mathbb{Q}_p(x')$ over $\mathbb{Q}_p$ is *discrete*. Then, by applying the above discussion repeatedly, we may assume without loss of generality that assertion (b) does not hold. In particular, assertion (a) holds. Thus, since $(x_1')^{-1} = \sigma_1(x_1)$, we conclude that $x_1$ and $x_1'$ are *Galois-conjugate*, hence that $x$ and $x'$ are *Galois-conjugate*. This completes the proof of Proposition 3.2. □

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**Remark 3.2.1.** At the time of writing of the present paper, the author does not know

> *whether the given isomorphism* $\alpha$ *arises from some isomorphism of schemes* $\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}$ *or not.*

Finally, we apply Lemmas 2.1, 2.3, and Proposition 3.2, to prove our main theorem (i.e. the tempered fundamental groups of hyperbolic curves of genus 0 over $\overline{\mathbb{Q}}_p$ completely determine their moduli):

**Theorem 3.3.** *Let* $n$ *be an integer such that* $n \geq 3$. *Suppose that there exists an isomorphism of topological groups*

$$\alpha : \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1,x_2,\ldots,x_n\}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1,x'_2,\ldots,x'_n\}}.$$

*Note that* $\alpha$ *induces a bijection*

$$\alpha_{\mathrm{cusp}} : \{x_1,x_2,\ldots,x_n\} \xrightarrow{\sim} \{x'_1,x'_2,\ldots,x'_n\}$$

*(cf. [9], Corollary 3.11). Then there exists an isomorphism of schemes*

$$\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1,x_2,\ldots,x_n\} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1,x'_2,\ldots,x'_n\},$$

*such that the bijection* $\{x_1,x_2,\ldots,x_n\} \xrightarrow{\sim} \{x'_1,x'_2,\ldots,x'_n\}$ *induced by the isomorphism* $\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1,x_2,\ldots,x_n\} \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1,x'_2,\ldots,x'_n\}$ *coincides with the bijection* $\alpha_{\mathrm{cusp}}$.

**Proof.** First, if $n = 3$, then the desired assertion follows immediately from the well-known structure of the automorphism group of $\mathbb{P}^1_{\overline{\mathbb{Q}}_p}$. Thus, we may assume without loss of generality that

$$n \geq 4.$$

Moreover, by replacing $\alpha$ by the composite of $\alpha$ with the outer isomorphisms arising from suitable geometric automorphisms of $\mathbb{P}^1_{\overline{\mathbb{Q}}_p}$, together with the various definitions involved, we may also assume without loss of generality that

- $x_1 = x'_1 = 0$; $x_2 = x'_2 = 1$; $x_3 = x'_3 = \infty$;
- $\alpha_{\mathrm{cusp}}(x_i) = x'_i$, for each $i = 1,\ldots,n$.

Then our goal is to prove that

($*_n$) there exists an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that $x'_i = \sigma(x_i)$, for each $i = 4,\ldots,n$.

Next, we verify the following assertion:

Claim 3.3.A: We may assume without loss of generality that

$$v_p(x_i) = 0$$

for each $i = 4,\ldots,n$.

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Indeed, let $r$ be a positive integer such that, for each $i = 4,\dots,m$, it holds that $v_p(y_i) > -p$, where $y_i$ denotes a $p^r$-th root of $x_i$. Write

$$Y \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, \zeta^j_{p^r} y_i \ (i = 2,4,\dots,n,\ j = 0,\dots,p^r - 1), \infty\};$$

$$Y' \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, \zeta^j_{p^r} y'_i \ (i = 2,4,\dots,n,\ j = 0,\dots,p^r - 1), \infty\};$$

$$\phi_{p^r,x} : Y \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x\}, \quad \phi_{p^r,x'} : Y' \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}$$

for the finite étale Galois coverings of degree $p^r$ determined by the assignment $t \mapsto t^{p^r}$ (cf. Lemma 1.1, (i)), where $y_2 \stackrel{\text{def}}{=} 1$; $y'_2 \stackrel{\text{def}}{=} 1$; $y'_i$ denotes a $p^r$-th root of $x'_i$. Then $\alpha$ induces an isomorphism of topological groups

$$\epsilon : \Pi^{\text{tp}}_Y \xrightarrow{\sim} \Pi^{\text{tp}}_{Y'}$$

(cf. Lemma 1.3). By replacing $\alpha$ by the composite of $\alpha$ with a suitable inner automorphism of $\Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x'\}}$, we may assume without loss of generality that $\epsilon$ maps the cuspidal inertia subgroups of $\Pi^{\text{tp}}_Y$ associated to $y_2 = 1$ to the cuspidal inertia subgroups of $\Pi^{\text{tp}}_{Y'}$ associated to $y'_2 = 1$. Moreover, by replacing $y'_i$ by a suitable $p^r$-th root of $x'_i$, if necessary, we may assume without loss of generality that $\epsilon$ maps the cuspidal inertia subgroups associated to $y_i$ to the cuspidal inertia subgroups associated to $y'_i$ for each $i = 4,\dots,n$. Then $\epsilon$ induces, via the quotients of $\Pi^{\text{tp}}_Y$, $\Pi^{\text{tp}}_{Y'}$ by the normal closed subgroups topologically generated by cuspidal inertia subgroups associated to the cusps $\notin \{0,1,\infty,y_i \ (i = 4,\dots,n)\}$, the cusps $\notin \{0,1,\infty,y'_i \ (i = 4,\dots,n)\}$, respectively, an isomorphism of topological groups

$$\Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,y_4,\dots,y_n\}} \xrightarrow{\sim} \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,y'_4,\dots,y'_n\}}.$$

Since $\phi_{p^r,x}$ and $\phi_{p^r,x'}$ are defined over $\mathbb{Q}_p$, by replacing $y_i$, $y'_i$ by $x_i$, $x'_i$, respectively, we may assume without loss of generality that $v_p(x_i) > -p$.

Next, we consider the connected finite étale covering

$$\psi_p : \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0, 1, \frac{p}{p+1}, \dots, \infty\right\} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty\}$$

(cf. Lemma 1.1, (ii)). For each $i = 4,\dots,n$, let $z_i \in \psi_p^{-1}(x_i)$ be such that $v_p(1 - z_i) > 0$ (cf. Lemma 2.2). Recall that $\psi_p$ is also defined over $\mathbb{Q}_p$. Thus, in light of Lemmas 1.3 and 2.2, it follows from a similar argument to the above argument that, by replacing $z_i$ by $x_i$, we may assume without loss of generality that $v_p(x_i) = 0$ for each $i = 4,\dots,n$. This completes the proof of Claim 3.3.A.

Write

$$\mathcal{C}_i \subseteq \overline{\mathbb{Q}}_p$$

for the set of the *Galois-conjugates* of $x_i$. Next, we verify the following assertion:

Claim 3.3.B: We may assume without loss of generality that

$$v_p(x_i) = 0, \quad v_p(1 - x_i) \leq 1, \quad \max_{w \in \mathcal{C}_i \setminus \{x_i\}} v_p(x_i - w) \leq 1,$$

for each $i = 4,\dots,n$.

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Indeed, by applying Claim 3.3.A, we may assume without loss of generality that $v_p(x_i) = 0$ for each $i = 4,\dots,m$. Then, in light of Lemma 2.1, one may apply a similar argument to the argument applied in the proof of Claim 3.3.A, together with the use of the connected finite étale covering

$$\phi_{p^r} : \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, \zeta^i_{p^r}\ (0 \le i \le p^r - 1), \infty\} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty\}$$

(cf. Lemma 1.1, (i)) for sufficiently large $r$, to obtain the desired conclusion. This completes the proof of Claim 3.3.B.

In the remainder, we prove $(*_n)$ by induction on $n$. We already observed that $(*_4)$ holds (cf. Proposition 3.2). Let $m$ be a positive integer such that $m \ge 4$. Suppose that $(*_n)$ in the case where $n \le m$ holds. Then our goal is to prove that $(*_{m+1})$ holds.

Next, we verify the following assertion:

Claim 3.3.C: Suppose that

- $v_p(x_4) \ge v_p(x_{m+1})$;
- $\max_{w \in \mathcal{C}_i \setminus \{x_i\}} v_p(x_i - w) < 2$, for each $i = 4,\dots,m$;
- $v_p(1 - x_{m+1}) \ge 2$.

Then $(*_{m+1})$ holds.

First, we note that since $\alpha_{\text{cusp}}(x_{m+1}) = x'_{m+1}$, the isomorphism $\alpha$ induces an isomorphism of topological groups

$$\Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,x_m\}} \xrightarrow{\sim} \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,x'_m\}}.$$

Then, by applying the induction hypothesis, we may assume without loss of generality that

$$x_i = x'_i,$$

for each $i = 4,\dots,m$. Next, write

$$f : \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,x_{m+1}\}} \xrightarrow{\sim} \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,\frac{x_4}{x_{m+1}}\}}$$

$$f' : \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,x'_{m+1}\}} \xrightarrow{\sim} \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,\frac{x_4}{x'_{m+1}}\}}$$

for the isomorphisms of topological groups (determined up to $\Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,\frac{x_4}{x_{m+1}}\}}$-conjugate, $\Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,\frac{x_4}{x'_{m+1}}\}}$-conjugate, respectively) induced by the isomorphism of schemes defined by the assignment $t \mapsto \frac{x_4}{t}$. Then we obtain an isomorphism of topological groups

$$\eta \overset{\text{def}}{=} f' \circ \alpha \circ f^{-1} : \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,\frac{x_4}{x_{m+1}}\}} \xrightarrow{\sim} \Pi^{\text{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0,1,\infty,x_4,\dots,\frac{x_4}{x'_{m+1}}\}}.$$

Next, write

$$\eta_{\text{cusp}} : \left\{0,1,\infty,x_4,\dots,\frac{x_4}{x_{m+1}}\right\} \xrightarrow{\sim} \left\{0,1,\infty,x_4,\dots,\frac{x_4}{x'_{m+1}}\right\}$$

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for the bijection induced by $\eta$. Recall that $\alpha_{\mathrm{cusp}}(x_i) = x'_i$, for each $i = 1, \ldots, n$. Then it follows immediately from the definition of $\eta_{\mathrm{cusp}}$ that

$$\eta_{\mathrm{cusp}}(0) = 0, \quad \eta_{\mathrm{cusp}}(1) = 1, \quad \eta_{\mathrm{cusp}}(\infty) = \infty, \quad \eta_{\mathrm{cusp}}(x_4) = x_4,$$

$$\eta_{\mathrm{cusp}}\left(\frac{x_4}{x_j}\right) = \frac{x_4}{x'_j},$$

for each $j = 5, \ldots, m+1$. In particular, $\eta$ induces an isomorphism of topological groups

$$\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\mathbb{Q}_p} \setminus \{0,1,\infty, \frac{x_4}{x_5}, \ldots, \frac{x_4}{x_{m+1}}\}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\mathbb{Q}_p} \setminus \{0,1,\infty, \frac{x_4}{x'_5}, \ldots, \frac{x_4}{x'_{m+1}}\}}.$$

Thus, by applying the induction hypothesis, we obtain an element $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that

$$\frac{x_4}{x'_j} = \tau\left(\frac{x_4}{x_j}\right),$$

for each $j = 5, \ldots, m+1$. Next, observe that

- $v_p(x_{m+1}) = v_p(x'_{m+1})$ (cf. [4], Theorem 4.6);
- $v_p(1 - x_{m+1}) = v_p(1 - x'_{m+1})$ (cf. [4], Theorem 4.6);
- $v_p(x_4 - \frac{x_4}{x'_{m+1}}) = v_p(x_4 - \tau(x_4) + \tau(x_4) - \tau(\frac{x_4}{x_{m+1}}))$;
- $v_p(x_4 - \frac{x_4}{x_{m+1}}) = v_p(\tau(x_4) - \tau(\frac{x_4}{x_{m+1}}))$.

Note that the first, second and forth equalities imply that

$$v_p\left(x_4 - \frac{x_4}{x'_{m+1}}\right) = v_p\left(\tau(x_4) - \tau\left(\frac{x_4}{x_{m+1}}\right)\right).$$

Thus, it follows immediately from the third equality that

$$v_p(x_4 - \tau(x_4)) \geq v_p\left(x_4 - \frac{x_4}{x'_{m+1}}\right).$$

On the other hand, since

$$v_p(x_4) \geq v_p(x_{m+1}) = v_p(x'_{m+1}), \quad v_p(1 - x'_{m+1}) = v_p(1 - x_{m+1}) \geq 2,$$

it holds that

$$v_p\left(x_4 - \frac{x_4}{x'_{m+1}}\right) \geq 2.$$

Then we obtain an inequality

$$v_p(x_4 - \tau(x_4)) \geq 2.$$

Thus, by applying our assumption that $\max_{w \in \mathcal{C}_4 \setminus \{x_4\}} v_p(x_4 - w) < 2$, we conclude that

$$x_4 = \tau(x_4).$$

Therefore, by combining with the equality $\frac{x_4}{x'_j} = \tau(\frac{x_4}{x_j})$, we also conclude that

$$x'_j = \tau(x_j),$$

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for each $j = 5, \ldots, m+1$. This completes the proof of Claim 3.3.C.

Next, we verify the following assertion:

Claim 3.3.D: Suppose that, for each $i = 4, \ldots, m+1$, it holds that

- $v_p(x_i) = 0$,
- $v_p(1-x_i) \le 1$, and
- $\max_{w \in \mathcal{C}_i \setminus \{x_i\}} v_p(x_i - w) \le 1$.

For each $i = 4, \ldots, m$, let $s_i \in \psi_{p^2}^{-1}(x_i)$ be, such that $v_p(s_i) = 2$ (cf. Lemma 2.2). Let $s_{m+1} \in \psi_{p^2}^{-1}(x_{m+1})$ be such that $v_p(1 - s_{m+1}) > 0$ (cf. Lemma 2.2). For each $i = 4, \ldots, m+1$, write $u_i \overset{\text{def}}{=} 1 - \frac{p^2}{(p^2+1)s_i}$; $\mathcal{C}_{u_i}$ for the set of the *Galois-conjugates* of $u_i$. Then it holds that

$$v_p(u_4) \ge 0 = v_p(u_{m+1}), \quad v_p(1-u_{m+1}) = 2,$$

$$\max_{c_i \in \mathcal{C}_{u_i} \setminus \{u_i\}} v_p(u_i - c_i) < 2,$$

for each $i = 4, \ldots, m$.

The assertions in the first display follow immediately from the facts that $v_p(s_4) = 2$, and $v_p(s_{m+1}) = 0$. Next, we verify the assertion in the second display. For each $i = 4, \ldots, m$, write $\mathcal{C}_{s_i}$ for the set of the *Galois-conjugates* of $s_i$. Let $w_i \in \mathcal{C}_{s_i} \setminus \{s_i\}$ be an element. Then since $v_p(s_i) = v_p(w_i) = 2$, it suffices to prove that

$$v_p(s_i - w_i) < 4.$$

However, this inequality follows from Lemma 2.3. This completes the proof of Claim 3.3.D.

Finally, we complete the proof of the assertion $(*_{m+1})$. By applying Claim 3.3.B, we may assume without loss of generality that

$$v_p(x_i) = 0, \quad v_p(1-x_i) \le 1, \quad \max_{w \in \mathcal{C}_i \setminus \{x_i\}} v_p(x_i - w) \le 1,$$

for each $i = 4, \ldots, m+1$. Write

$$T \overset{\text{def}}{=} \{x_4, \ldots, x_{m+1}\}; \quad T' \overset{\text{def}}{=} \{x'_4, \ldots, x'_{m+1}\};$$

$$\psi_{p^2,x} : \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0, 1, \frac{p^2}{p^2+1}, \ldots, \infty\right\} \cup \psi_{p^2}^{-1}(T) \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1, x_2, \ldots, x_n\},$$

$$\psi_{p^2,x'} : \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \left\{0, 1, \frac{p^2}{p^2+1}, \ldots, \infty\right\} \cup \psi_{p^2}^{-1}(T') \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1, x'_2, \ldots, x'_n\}$$

for the connected finite étale coverings induced by $\psi_{p^2}$ (cf. Lemma 1.1, (ii); Remark 1.1.1);

$$\Pi^{\mathrm{tp}}_{\psi_{p^2,x}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1, x_2, \ldots, x_n\}}, \quad \Pi^{\mathrm{tp}}_{\psi_{p^2,x'}} \subseteq \Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1, x'_2, \ldots, x'_n\}}$$

for the open subgroups of finite index (determined up to $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x_1, x_2, \ldots, x_n\}}$-conjugate, $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{x'_1, x'_2, \ldots, x'_n\}}$-conjugate, respectively) determined by $\psi_{p^2,x}$, $\psi_{p^2,x'}$, respectively. Here,

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we note that $\frac{p^2}{p^2+1}$ is a unique cusp $*$ such that $\psi_{p^2}$ ramifies at $*$, and $*$ lies over 1 via $\psi_{p^2}$ (cf. Lemma 1.1, (ii)). Then since $\alpha_{\mathrm{cusp}}(0)=0$, $\alpha_{\mathrm{cusp}}(1)=1$, $\alpha_{\mathrm{cusp}}(\infty)=\infty$, and $\alpha_{\mathrm{cusp}}(x_i)=x_i'$, for each $i=4,\dots,n$, it follows immediately from Lemma 1.3 that there exists an inner automorphism $\iota$ of $\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{x_1',x_2',\dots,x_n'\}}$ satisfying the following conditions:

- The composite morphism $\iota\circ\alpha$ induces an isomorphism of topological groups

$$\theta:\Pi^{\mathrm{tp}}_{\psi_{p^2,x}}\xrightarrow{\sim}\Pi^{\mathrm{tp}}_{\psi_{p^2,x'}}$$

  via the inclusions $\Pi^{\mathrm{tp}}_{\psi_{p^2,x}}\subseteq\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{x_1,x_2,\dots,x_n\}}$ and $\Pi^{\mathrm{tp}}_{\psi_{p^2,x'}}\subseteq\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{x_1',x_2',\dots,x_n'\}}$.
- Write

$$\theta_{\mathrm{cusp}}:\left\{0,1,\frac{p^2}{p^2+1},\dots,\infty\right\}\cup\psi_{p^2}^{-1}(T)\xrightarrow{\sim}\left\{0,1,\frac{p^2}{p^2+1},\dots,\infty\right\}\cup\psi_{p^2}^{-1}(T')$$

  for the bijection induced by $\theta$ (cf. [9], Corollary 3.11). Then it holds that

$$\theta_{\mathrm{cusp}}(0)=0,\quad \theta_{\mathrm{cusp}}(1)=1,\quad \theta_{\mathrm{cusp}}(\infty)=\infty,\quad \theta_{\mathrm{cusp}}\left(\frac{p^2}{p^2+1}\right)=\frac{p^2}{p^2+1}.$$

For each $i=4,\dots,m+1$, write $s_i'\overset{\mathrm{def}}{=}\theta_{\mathrm{cusp}}(s_i)$ (cf. Claim 3.3.D). Then the isomorphism $\theta$ induces an isomorphism of topological groups

$$\xi:\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,\frac{p^2}{p^2+1},s_4,\dots,s_{m+1}\}}\xrightarrow{\sim}\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,\frac{p^2}{p^2+1},s_4',\dots,s_{m+1}'\}}.$$

Write

$$\omega:\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\left\{\infty,\frac{1}{p^2+1},0,\dots,1\right\}\xrightarrow{\sim}\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\left\{0,1,\frac{p^2}{p^2+1},\dots,\infty\right\}$$

for the inverse of the isomorphism determined by the assignment $t\mapsto 1-\frac{p^2}{(p^2+1)t}$. For each $i=4,\dots,m+1$, write $u_i'\overset{\mathrm{def}}{=}1-\frac{p^2}{(p^2+1)s_i'}$. Then $\xi$ and $\omega$ induce, in a similar way to the construction of $\eta$, an isomorphism of topological groups

$$\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,u_4,\dots,u_{m+1}\}}\xrightarrow{\sim}\Pi^{\mathrm{tp}}_{\mathbb{P}^1_{\overline{\mathbb{Q}}_p}\setminus\{0,1,\infty,u_4',\dots,u_{m+1}'\}}.$$

Note that, if we write

$$h:\{0,1,\infty,u_4,\dots,u_{m+1}\}\xrightarrow{\sim}\{0,1,\infty,u_4',\dots,u_{m+1}'\}$$

for the bijection induced by the above isomorphism, then it holds that

$$h(0)=0,\quad h(1)=1,\quad h(\infty)=\infty,\quad h(u_i)=u_i',$$

for each $i=4,\dots,m+1$. On the other hand, observe that the composite morphism $\psi_{p^2}\circ\omega$ is defined over $\mathbb{Q}_p$. Then, by replacing $u_i$ by $x_i$, together with Claim 3.3.D, we may assume without loss of generality that

$$v_p(x_4)\geq v_p(x_{m+1}),\qquad \max_{w\in\mathcal{C}_i\setminus\{x_i\}}v_p(x_i-w)<2\quad(4\leq i\leq m),\quad v_p(1-x_{m+1})\geq 2.$$

Thus, we conclude from Claim 3.3.C that $(*_{m+1})$ holds. This completes the proof of Theorem 3.3. □

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**Conflicts of Interest.** The author declares none.

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