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WEAK REFLECTION PRINCIPLE, SATURATION OF THE NONSTATIONARY IDEAL ON ω_1 AND DIAMONDS

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Abstract. We prove that WRP and saturation of the ideal NS_{ω_1} together imply $\diamondsuit\{a \in [\lambda]^{\omega_1} : \operatorname{cof}(\sup(a)) = \omega_1\}$, for every cardinal λ with $\operatorname{cof}(\lambda) \geq \omega_2$.

§1. **Introduction.** The reflection of stationary sets has been one of the most studied themes in modern Set Theory, as it shows up in basically all of its parts such as the large-cardinal theory, forcing and combinatorial Set Theory, inner model theory and determinacy.

We recall the especially fruitful study of reflection principles for stationary subsets of sets of the form $[\kappa]^\omega$. For example, one of the crucial observations of the ground-breaking work of Foreman, Magidor and Shelah [3] is that the reflection of stationary subsets of $[\kappa]^\omega$ reduces the class of posets that preserve stationary subsets of ω_1 to the class of semi-proper posets. This result leads to the consistency of Martin's Maximum, arguably the strongest forcing axiom for $\aleph_1$ dense sets. Around the same time, motivated also by some work of Baumgartner and Taylor, Todorčević showed that the reflection of stationary subsets of $[\omega_2]^\omega$ implies that the cardinality of the continuum is bounded by $\aleph_2$ [22], or written in modern notation, that

$$\mathrm{WRP}(\omega_2) \longrightarrow 2^{\aleph_0} \leq \aleph_2.$$

Following some work of Veličković [26] and Foreman-Todorčević [4] that generalize this to higher cardinals, Shelah has shown that reflection of stationary subsets of arbitrary $[\kappa]^\omega$ implies that $\lambda^{\aleph_0} = \lambda$ for all regular cardinals $\lambda \geq \omega_2$, or in short that

$$\mathrm{WRP} \longrightarrow (\forall \lambda = \mathrm{cf}(\lambda) \geq \aleph_2)\ \lambda^{\aleph_0} = \lambda.$$

So, in particular, WRP implies the Singular Cardinal Hypothesis [19].

In [3, Theorem 10], Foreman–Magidor–Shelah showed that MM implies $\lambda^{\omega_1} = \lambda$ for every regular cardinal $\lambda \geq \omega_2$. However, it has been noted by Woodin in his well known monograph [27] that WRP, which is a consequence of MM, is not sufficient for giving us this stronger conclusion $\lambda^{\aleph_1} = \lambda$ even for $\lambda = \aleph_2$.

In this paper we present some results appeared in [25]. We observe that the same weak reflection principle for stationary sets, WRP, used by Shelah to obtain $\lambda^{\aleph_0} = \lambda$, will give us the stronger cardinal arithmetic $\lambda^{\aleph_1} = \lambda$ for all regular $\lambda \geq \omega_2$ as long

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as we add to it the assumption that the ideal NS_{ω_1} of nonstationary subsets of ω_1 is saturated (see Section 4). This saturation, which is also a consequence of MM ([3, Theorem 12]), is in the sense that every family $\mathcal{A}$ of stationary subsets of ω_1 such that $A \cap B$ is not stationary for every $A \neq B$ in $\mathcal{A}$ must have cardinality smaller than $\aleph_2$. In short,

$$\mathrm{WRP} + \mathrm{sat}(\mathrm{NS}_{\omega_1}) = \aleph_2 \longrightarrow (\forall \lambda = \mathrm{cf}(\lambda) \geq \aleph_2)\ \lambda^{\aleph_1} = \lambda.$$

The main result of these notes is involving a two-cardinal version of Jensen's Diamond, introduced by Jech. We show that this assumption will give us a stronger result,

$$\mathrm{WRP} + \mathrm{sat}(\mathrm{NS}_{\omega_1}) = \aleph_2 \longrightarrow (\forall \lambda = \mathrm{cf}(\lambda) \geq \aleph_2)\ \Diamond_{\omega_2,\lambda}.$$

Moreover, supposing WRP and $\mathrm{sat}(\mathrm{NS}) = \omega_2$, we get

$$\Diamond_{\omega_2,\lambda}\left(\{a \in [\lambda]^{\omega_1} : \mathrm{cof}\left(\sup(a)\right) = \omega_1\}\right).$$

We remark that by Matet (see Lemma 10.3 in [13]), if $\kappa < \lambda$ with λ regular and $2^{<\kappa} \leq \lambda^{\aleph_0}$, then $\Diamond_{\kappa,\lambda}$ holds. So by Lemma 4.1, taking $\kappa = \omega_2$, we have indeed $2^{<\kappa} \leq \lambda^{\aleph_0}$ for every regular $\lambda \geq \omega_2$. See also [1]. So in these notes we give an alternative direct proof to obtain $\Diamond_{\omega_2,\lambda}$, but which is besides concentrated on the set $\{a \in [\lambda]^{\omega_1} : \mathrm{cof}(\sup(a)) = \omega_1\}$. Compare it also, for example, with Shelah's result in [20], where he showed that in particular, if $2^{\omega_1} = \omega_2$, then $\Diamond_{\omega_2}(E^{\omega_2}_{\omega})$ holds. Also in [20], Claim 3.2, GCH does not imply $\Diamond(E^{\omega_2}_{\omega_1})$.

On the other hand, the referee pointed out that in the proof of [16], it is implicit the following: Assume $\Diamond(E^{\omega_2}_{\omega})$, for every stationary set $S \subseteq E^{\omega_2}_{\omega}$ there is $\alpha \in E^{\omega_2}_{\omega_1}$ such that $S \cap \alpha$ is stationary, and Saturation of NS. Then $\Diamond(E^{\omega_2}_{\omega_1})$ holds.

§2. Notations, basic definitions and some of its properties. For a given set A and a cardinal μ such that $|A| \geq \mu$, we will denote $[A]^{\mu} = \{a \subseteq A : |a| = \mu\}$, $[A]^{<\mu} = \{a \subseteq A : |a| < \mu\}$. For an ordinal λ, let $\lambda^{\uparrow<\mu}$ be the set of increasing sequences of λ of cardinality $< \mu$.

Let λ, μ be two cardinals such that $\lambda \geq \mu \geq \omega$ with μ uncountable and regular.

Definition 2.1. We say that a subset $C \subseteq [\lambda]^{<\mu}$ is *closed* if for every $\gamma < \mu$ and for every chain $x_0 \subseteq x_1 \subseteq \cdots \subseteq x_\xi \subseteq \cdots$ of elements of C with $\xi < \gamma$, the union $\bigcup_{\xi<\gamma} x_\xi$ is also a member of C. C is *unbounded* if for every $b \in [\lambda]^{<\mu}$, there is $a \in C$ with $a \supseteq b$. A closed and unbounded set we will often call it just a *club* set. Observe for example that $[\lambda]^{\kappa}$ is a club in $[\lambda]^{<\kappa^+}$ for any infinite regular cardinal κ.

We call $S \subseteq [\lambda]^{<\mu}$ *stationary in* $[\lambda]^{<\mu}$ if for every club $C \subseteq [\lambda]^{<\mu}$, $S \cap C \neq \emptyset$.

We state a two cardinal version of Jensen's diamond principle due to Jech [6] and its equivalence with a (apparently) weaker form.

Let $\lambda > \mu \geq \omega$ with μ a regular cardinal.

Definition 2.2. Let $S \subseteq [\lambda]^{<\mu}$. Let $\langle A_a \rangle_{a\in S}$ be a sequence such that $A_a \subseteq a$ for all $a \in [\lambda]^{<\mu}$. Then $\langle A_a \rangle_{a\in S}$ is a $\Diamond_{\mu,\lambda}(S)$-sequence if for all $A \subseteq \lambda$, the set

$$\{a \in S : A \cap a = A_a\}$$

is stationary. So, the principle $\Diamond_{\mu,\lambda}(S)$ states that there is a $\Diamond_{\mu,\lambda}(S)$-sequence. We write $\Diamond_{\mu,\lambda}$ when $S = [\lambda]^{<\mu}$.

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Definition 2.3. Let $S \subseteq [\lambda]^{<\mu}$. Let $\langle \mathscr{A}_a \rangle_{a \in S}$ be a sequence such that $\mathscr{A}_a \subseteq P(a)$ for all $a \in [\lambda]^{<\mu}$, and $|\mathscr{A}_a| < \mu$. Then $\langle \mathscr{A}_a \rangle_{a \in S}$ is a $\diamondsuit^-_{\mu,\lambda}(S)$-sequence if for all $A \subseteq \lambda$, the set

$$\{a \in S : A \cap a \in \mathscr{A}_a\}$$

is stationary. So, the principle $\diamondsuit^-_{\mu,\lambda}(S)$ states that there is a $\diamondsuit^-_{\mu,\lambda}(S)$-sequence. We write $\diamondsuit^-_{\mu,\lambda}$ when $S = [\lambda]^{<\mu}$.

It turns out, similarly to Jensen's original diamond principle, that these two versions are equivalent.

Lemma 2.4. *Let μ be an infinite regular cardinal. Then for every $S \subseteq [\lambda]^{\mu}$, $\diamondsuit^-_{\mu^+,\lambda}(S) \leftrightarrow \diamondsuit_{\mu^+,\lambda}(S)$.*

Proof. We follow very closely the proof of Theorem 7.14 in Chapter II from Kunen's book ([11]). $\diamondsuit_{\mu^+,\lambda}(S) \rightarrow \diamondsuit^-_{\mu^+,\lambda}(S)$ is straightforward. So assume $\diamondsuit^-_{\mu^+,\lambda}(S)$ and we will conclude $\diamondsuit_{\mu^+,\lambda}(S)$.

Let $\langle \mathcal{A}_a \rangle_{a \in S}$ be a $\diamondsuit^-_{\mu^+,\lambda}(S)$-sequence. Since $|\mathcal{A}_a| \leq \mu$, let

$$\mathcal{A}_a = \{a_\xi : \xi < \mu\}.$$

We fix a bijection

$$f : \lambda \rightarrow \mu \times \lambda.$$

Also for $A \subseteq \mu \times \lambda$ and for every $\xi < \mu$, we define

$$\text{proj}_\xi A = \{\alpha \in \lambda : \langle \xi, \alpha \rangle \in A\}.$$

It is enough to prove the following

Claim 2.5. *There is $\xi < \mu$ such that for every set $X \subseteq \lambda$ the set*

$$\{a \in S : X \cap a = \text{proj}_\xi f\left[a_\xi\right]\}$$

is stationary.

In order to have $A_a \subseteq a$, remark that $\{a \in [\lambda]^\mu : f[a] \subseteq \mu \times a\}$ is a club. If we define $A_a = \text{proj}_\xi f[a_\xi]$ if $f[a] \subseteq \mu \times a$, and $A_a = \emptyset$ otherwise, $\langle A_a \rangle_{a \in S}$ will be our desired $\diamondsuit_{\mu^+,\lambda}(S)$-sequence.

Before proving Claim 2.5, we will need next Subclaim, which is the place where we actually make use of the $\diamondsuit^-_{\mu^+,\lambda}(S)$-sequence.

Subclaim 2.6. *For every $X \subseteq \mu \times \lambda$, the set*

$$E_X = \{a \in S : \textit{there is } \xi \in \mu \textit{ such that } X \cap (\mu \times a) = f\left[a_\xi\right]\}$$

is stationary in $[\lambda]^\mu$.

Proof of Subclaim 2.6. Let $X \subseteq \mu \times \lambda$. Since $\langle \mathcal{A}_a = \{a_\xi : \xi < \mu\} \rangle_{a \in S}$ is a $\diamondsuit^-_{\mu^+,\lambda}(S)$ sequence, the set

$$S_X = \{a \in S : \exists \xi < \mu (f^{-1}[X] \cap a = a_\xi)\}$$

is stationary in $[\lambda]^\mu$. The set $C = \{a \in [\lambda]^\mu : f[a] \subseteq \mu \times a\}$ is a club, and so $S_X \cap C$ is stationary. Since any set containing a stationary set is stationary, then it is enough to show

$$S_X \cap C \subseteq E_X.$$

Let $a \in C$ and $\xi < \mu$ be such that

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$$f^{-1}[X] \cap a = a_\xi. \tag{1}$$

We have to show that

$$X \cap (\mu \times a) = f\left[a_\xi\right].$$

Let $\langle \eta, \beta \rangle \in X \cap (\mu \times a)$. It is sufficient to see that $f^{-1}\langle \eta, \beta \rangle \in a_\xi$. We are assuming (1), so it is enough to verify that $f^{-1}[\mu \times a] \subseteq a$. But this is true, since $a \in C$.

In order to prove the other inclusion, take $\langle \eta, \beta \rangle \in f\left[a_\xi\right]$. So we need to prove that $\langle \eta, \beta \rangle \in X \cap (\mu \times a)$, i.e., we must check that $\langle \eta, \beta \rangle \in X$ and $\beta \in a$. By our supposition, we know that $f^{-1}\langle \eta, \beta \rangle \in a_\xi = f^{-1}[X] \cap a$, in particular $\langle \eta, \beta \rangle \in X$. Again, since $a \in C$ it follows that $f[a] \subseteq \mu \times a$ and $\beta \in a$. ⊣

Now suppose Claim 2.5 does not hold. Then for every $\xi < \mu$, there exists $X_\xi \subseteq \lambda$ such that the set

$$\{a \in S : X_\xi \cap a = \operatorname{proj}_\xi f\left[a_\xi\right]\} \tag{2}$$

is nonstationary.

We will try to get a contradiction with Subclaim 2.6 considering the set $\bigcup_{\eta<\mu} (\{\eta\} \times X_\eta) \subseteq \mu \times \lambda$. We remark that for $a \in [\lambda]^\mu$ and $\xi < \mu$, if

$$\left(\bigcup_{\eta<\mu} \{\eta\} \times X_\eta\right) \cap (\mu \times a) = f\left[a_\xi\right],$$

then

$$\alpha \in X_\xi \cap a \iff \langle \xi, \alpha \rangle \in f\left[a_\xi\right] \iff \alpha \in \operatorname{proj}_\xi f\left[a_\xi\right].$$

Therefore $\{a \in [\lambda]^\mu : (\bigcup_{\eta<\mu} \{\eta\} \times X_\eta) \cap (\mu \times a) = f\left[a_\xi\right]\} \subseteq \{a \in [\lambda]^\mu : X_\xi \cap a = \operatorname{proj}_\xi f\left[a_\xi\right]\}$, and so the set

$$G_\xi = \{a \in S : \left(\bigcup_{\eta<\mu} \{\eta\} \times X_\eta\right) \cap (\mu \times a) = f\left[a_\xi\right]\}$$

is also nonstationary as a subset of (2). The ideal of nonstationary sets of $[\lambda]^\mu$ is μ-closed (Theorem 3.2 in [6]), so

$$\bigcup_{\xi<\mu} G_\xi = \{a \in S : \exists \xi \in \mu \left(\bigcup_{\eta<\mu} \{\eta\} \times X_\eta\right) \cap (\mu \times a) = f\left[a_\xi\right]\}$$

is nonstationary, contradicting Subclaim 2.6. ⊣

We state the Weak Reflection Principle (WRP):

Definition 2.7 (WRP). Let $\lambda \geq \aleph_2$ be an arbitrary ordinal. We say that WRP(λ) holds if for every stationary set $S \subseteq [\lambda]^\omega$, the set

$$\{x \in [\lambda]^{\omega_1} : x \supseteq \omega_1 \text{ and } S \cap [x]^\omega \text{ is stationary in } [x]^\omega\}$$

is stationary in $[\lambda]^{\omega_1}$. Then WRP states that WRP(λ) holds for every $\lambda \geq \omega_2$.

The Weak Reflection Principle was introduced in [3]. We are using the equivalent version provided by Feng and Jech ([2]). See also comments below Proposition 6.2, page 119 in [8].

We will denote the ideal of nonstationary sets of ω_1 by NS_{ω_1} or even just by NS.

Definition 2.8. NS is saturated if every collection W of stationary sets in ω_1 such that for every S and T in W, $S \cap T$ is nonstationary, has cardinality $\leq \aleph_1$.

In general, for an ideal $I \subseteq P(A)$, we denote by $\mathrm{sat}(I)$ the minimal cardinal κ such that if W is a an almost-disjoint family of subsets of A (i.e., for every $X, Y \in W$, if $X \neq Y$ then $X \cap Y \in I$), then $|W| < \kappa$. All along this paper we will suppose $\mathrm{sat}(\mathrm{NS}) = \aleph_2$.

§3. Saturation of NS_{ω_1}**, WRP, and diamonds.** In this section we prove our principal theorem:

Theorem 3.1. *The Weak Reflection Principle and the saturation of the ideal* NS_{ω_1} *imply* $\diamondsuit_{\omega_2,\lambda}\left(\{a \in [\lambda]^{\omega_1} : \mathrm{cof}\left(\sup(a)\right) = \omega_1\}\right)$ *for every cardinal* λ *such that* $\mathrm{cof}(\lambda) \geq \omega_2$.

Proof. We shall also rely on the following result from Todorčević. He showed $\diamondsuit_{\omega_1,\omega_2}$ holds without any further assumption over ZFC. Its proof is sketched in a more general form in the Appendix for the convenience of the reader as it is written in ([12]). See also [23], Lemma 43. This result has also been proved independently by Shelah (see [21]). In addition, we show that we can obtain the extra condition (b) here below.

Theorem 3.2 (Shelah, Todorčević (independently)). *For every* λ *with* $\mathrm{cof}(\lambda) \geq \omega_2$, *there exists a sequence* $\langle \theta_a : a \in [\lambda]^\omega \rangle$ *such that*:

(a) *for every* $W \subseteq \lambda$, $\{a \in [\lambda]^\omega : a \cap W = \theta_a\}$ *is stationary*,

(b) *for every* $\delta < \lambda$ *of countable cofinality*, $\{a \in [\delta]^\omega : \theta_a = \emptyset\}$ *is a club in* $[\delta]^\omega$.

We continue with the proof of Theorem 3.1.

From now on let $\langle \theta_a \rangle_{a \in [\lambda]^\omega}$ be a fixed $\diamondsuit_{\omega_1,\lambda}$-sequence with the properties of Theorem 3.2. We also fix for each $x \in [\lambda]^{\omega_1}$ a $\subseteq$-continuous increasing chain $\langle a^x_\xi \rangle_{\xi < \omega_1}$ of countable sets such that $x = \bigcup\limits_{\xi < \omega_1} a^x_\xi$.

Let $T_x = \langle \{a^x_\xi\}_{\xi < \omega_1}, <_x \rangle$ be the associated tree where the ordering is as follows: $a^x_\xi <_x a^x_{\xi'}$ iff $\xi < \xi'$ and $\theta_{a^x_{\xi'}} \cap a^x_\xi = \theta_{a^x_\xi}$.

Take $S \subseteq \omega_1$ with the property that

$$\{a^x_\xi : \xi \in S\}$$

is a chain in the tree order.

For each S with this property, let

$$F^x_S = \bigcup_{\xi \in S} \theta_{a^x_\xi}.$$

Now, consider

$$\mathscr{S}_x = \{F^x_S : S \text{ is stationary in } \omega_1 \text{ and } \langle a^x_\xi \rangle_{\xi \in S} \text{ is a } <_x\text{-chain}\}.$$

Claim 3.3. $F^x_S \neq F^x_T$ *implies that* $|S \cap T| \leq \aleph_0$, *so in particular* $S \cap T$ *is nonstationary.*

Proof. We are going to prove it using the contrapositive: Suppose $|S \cap T| = \aleph_1$. We will prove that $F^x_S = F^x_T$. So, we are going to show that $F^x_S \subseteq F^x_T$ (the proof of $F^x_T \subseteq F^x_S$ is similar). It is enough to verify that for every $\xi \in S$, there is $\eta \in T$ such that $\theta_{a^x_\xi} \subseteq \theta_{a^x_\eta}$. So, take $\xi \in S$. Since $|S \cap T| = \aleph_1$, there is $\eta \in S \cap T$ such that

$\xi > \eta$. Then, by definition of $<_x$, $\theta_{a^x_\xi} = a^x_\xi \cap \theta_{a^x_\eta}$. In particular, $\theta_{a^x_\xi} \subseteq \theta_{a^x_\eta}$, and we are done. ⊣

Corollary 3.4. *Saturation of* NS_{ω_1} *implies that* $|\mathscr{S}_x| \le \omega_1$.

Proof. Every $\mathscr{S}_x$ witnesses the existence of an antichain modulo the ideal NS_{ω_1}, and then apply NS_{ω_1}-saturation. ⊣

Claim 3.5. $\langle \mathscr{S}_x \rangle_{x \in [\lambda]^{\omega_1}}$ *is a* $\diamondsuit_{\omega_2,\lambda}$*-sequence.*

Proof. Let $W \subseteq \lambda$. We need to prove that :

$$G_W = \{x \in [\lambda]^{\omega_1} : W \cap x \in \mathscr{S}_x\}$$

is stationary in $[\lambda]^{\omega_1}$.

Considering

$$E_W = \{a \in [\lambda]^{\omega} : W \cap a = \theta_a\},$$

which is stationary (remember that $\langle \theta_a \rangle_{a \in [\lambda]^\omega}$ is a $\diamondsuit_{\omega_1,\lambda}$-sequence), we can apply $\mathrm{WRP}(\lambda)$. Then the set

$$E'_W = \{x \in [\lambda]^{\omega_1} : E_W \cap [x]^\omega \text{ is stationary in } [x]^\omega \text{ and } x \supseteq \omega_1\} \tag{3}$$

is stationary in $[\lambda]^{\omega_1}$.

Subclaim 3.6. *If* $x \in E'_W$*, then the set* $S = \{\xi \in \omega_1 : a^x_\xi \in E_W\}$ *is stationary and* $W \cap x \in \mathscr{S}_x$.

Proof. First: $\langle a^x_\xi \rangle_{\xi \in S}$ is a $<_x$-chain. Let $\xi, \xi' \in S$ such that $\xi < \xi'$. We need to prove that $\theta_{a^x_\xi} = a^x_\xi \cap \theta_{a^x_{\xi'}}$. Since $\xi, \xi' \in S$, we get:

$$\theta_{a^x_\xi} = a^x_\xi \cap W$$

and

$$\theta_{a^x_{\xi'}} = a^x_{\xi'} \cap W.$$

Therefore

$$a^x_\xi \cap \theta_{a^x_{\xi'}} = a^x_\xi \cap (a^x_{\xi'} \cap W) = (a^x_\xi \cap a^x_{\xi'}) \cap W = a^x_\xi \cap W.$$

Second: S is stationary in ω_1. Let $C \subseteq \omega_1$ be a club. So, we would like to find $\xi \in C$ such that $W \cap a^x_\xi = \theta_{a^x_\xi}$. By our hypothesis, $\{a \in [x]^\omega : W \cap a = \theta_a\}$ is stationary in $[x]^\omega$. So, it is enough to show that $\langle a^x_\xi \rangle_{\xi \in C}$ is a club in $[x]^\omega$. Since $\langle a^x_\xi \rangle_{\xi \in \omega_1}$ is already continuous, it is enough to prove that $\langle a^x_\xi \rangle_{\xi \in C}$ is unbounded $[x]^\omega$. So, let $b \in [x]^\omega$. Let $b = \{\alpha_n\}_{n \in \omega}$. Then, for every $n \in \omega$, there is $\xi_n \in \omega_1$ such that $\alpha_n \in a^x_{\xi_n}$. Since ω_1 is regular, then $\sup_{n \in \omega} \xi_n < \omega_1$. Then, since C is a club, there is $\xi \ge \sup_{n \in \omega} \xi_n$ such that $\xi \in C$. Therefore, $b \subseteq a^x_\xi$, and $\langle a^x_\xi \rangle_{\xi \in C}$ is a club in $[x]^\omega$. So, there is $\xi \in C$ such that $W \cap a^x_\xi = \theta_{a^x_\xi}$.

Last: $W \cap x = \bigcup_{\xi \in S} \theta_{a^x_\xi}$. Two remarks:

Remark 1: Note that if a set $A \subseteq \omega_1$ is unbounded in ω_1, $\bigcup_{\xi \in A} a^x_\xi = x$, since for every a^x_ξ, there is $\xi' \in A \cap (\xi, \omega_1)$. So, $a^x_\xi \subseteq a^x_{\xi'}$, and $x = \bigcup_{\xi \in A} a^x_\xi$ (it is clear that $\bigcup_{\xi \in A} a^x_\xi \subseteq x = \bigcup_{\xi \in \omega_1} a^x_\xi$).

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Remark 2: Every stationary set in particular is unbounded, since for every $\xi \in \omega_1$, the interval (ξ, ω_1) is a club in ω_1.

So, coming back to our proof:

$$\bigcup_{\xi \in S} \theta_{a_\xi^x} = \bigcup_{\xi \in S} (W \cap a_\xi^x) = W \cap \left(\bigcup_{\xi \in S} a_\xi^x \right) = W \cap x. \quad \dashv$$

$\dashv$

We have then obtained $\diamondsuit_{\omega_2,\lambda}$, but we can say something more. We make some remarks about the stationary set $E'_W = \{x \in [\lambda]^{\omega_1} : \{a \in [x]^\omega : W \cap a = \theta_a\}$ is stationary in $[x]^\omega$ and $x \supseteq \omega_1\}$ described in (3).

We are going to show that there is a stationary set $E \subseteq E'_W \cap \{a \in [\lambda]^{\omega_1} : \mathrm{cof}\,(\sup(a)) = \omega_1\}$ and so $\diamondsuit\{a \in [\lambda]^{\omega_1} : \mathrm{cof}\,(\sup(a)) = \omega_1\}$ holds.

Without loss of generality we can suppose $W \neq \emptyset$, since we can assume that $\emptyset \in \mathscr{S}_a$ for all $a \in [\lambda]^{\omega_1}$. Observe that the set $D = \{a \in [\lambda]^{\omega_1} : a \cap W \neq \emptyset\}$ is a club.

Then the set $E = E'_W \cap D$ is stationary. We claim that $\mathrm{cof}\,(\sup(a)) = \omega_1$ for every $a \in E$.

Suppose it is not the case and take $x \in E$ such that if $\delta = \sup(x)$ then $\mathrm{cof}(\delta) = \omega$. Also, since $\delta \cap W \neq \emptyset$, the set $\{a \in [\delta]^\omega : a \cap W \neq \emptyset\}$ is a club in $[\delta]^\omega$. Since $\delta \in E'_W$, there is $a \in [\delta]^\omega$ such that $a \cap W = \theta_a$ and $a \cap W \neq \emptyset$, contradicting (b) of Theorem 3.2. $\dashv$

Therefore, we have the following Corollary, which was found, as mentioned in the Introduction, by Shelah in [16].

Corollary 3.7. *The Weak Reflection Principle and the saturation of the ideal* NS_{ω_1} *imply* $\diamondsuit_{\omega_2}\{\delta < \omega_2 : \mathrm{cof}\,\delta = \omega_1\}$.

Actually, and thanks to the referee's observations, we can get a stronger and more general result:

Theorem 3.8. *Suppose that* NS_{ω_1} *is saturated and* λ *is a cardinal greater than or equal to* ω_2 *such that* $\mathrm{WRP}(\lambda)$ *holds. Then for any finite set of regular cardinals* $r \subseteq \{\mu : \omega_2 \leq \mu \leq \lambda\}$, $\diamondsuit_{\omega_2,\lambda}(X_r)$ *holds, where* $X_r = \{x : \forall \mu \in r(\mathrm{cof}(\sup(x \cap \mu)) = \omega_1)\}$.

Proof. Fix a finite set of regular cardinals $r \subseteq \{\mu : \omega_2 \leq \mu \leq \lambda\}$. For $\mu \in r$, let $\langle H^\mu_\eta : \eta < \omega_1 \rangle$ be a partition of E^μ_ω into $\aleph_1$ disjoint stationary sets. Then by (the proof of) Lemma 10.4 in [13], we may find a $\diamondsuit_{\omega_1,\lambda}(K_r)$-sequence, where K_r consists of all a such that $\sup(a \cap \mu) \in H^\mu_{\sup(a \cap \omega_1)}$ for all $\mu \in r$.

Let $\langle \theta_a \rangle_{a \in [\lambda]^\omega}$ be a $\diamondsuit_{\omega_1,\lambda}(K_r)$-sequence, and like in the proof of Theorem 3.1, use it to define in the same way the sequence $\langle \mathscr{S}_x \rangle_{x \in [\lambda]^{\omega_1}}$.

Let $W \subseteq \lambda$. It is only left to show that the set

$$G_W = \{x \in X_r : W \cap x \in \mathscr{S}_x\}$$

is stationary in $[\lambda]^{\omega_1}$ (the rest of the proof is similar to the one of Theorem 3.1).

Considering

$$E_W = \{a \in K_r : W \cap a = \theta_a\},$$

which is stationary in $[\lambda]^\omega$, we can apply $\mathrm{WRP}(\lambda)$. Then the set

$$E'_W = \{x \in [\lambda]^{\omega_1} : E_W \cap [x]^\omega \text{ is stationary in } [x]^\omega \text{ and } x \supseteq \omega_1\} \tag{4}$$

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is stationary in $[\lambda]^{\omega_1}$. It is enough to show $E'_W \subseteq G_W$. Take $x \in E'_W$. To show $W \cap x \in \mathscr{S}_x$ is similar as in the proof of Theorem 3.1.

Now suppose that $\omega_1 \subseteq x = \bigcup_{i<\omega_1} x_i$ and each x_i is countable, and $S = \{i \in \omega_1 : x_i \in K_r\}$ is stationary. Then clearly for each $i \in S$, there is $j \in S$ with $i < j$ and $\sup(x_i \cap \omega_1) < \sup(x_j \cap \omega_1)$ (and therefore $\sup(x_i \cap \mu) < \sup(x_j \cap \mu)$ for all μ in r). Hence $x \in X_r$. Note that x has the extra (not needed but remarkable) property that for each $\mu \in r$, it contains a club subset of $\sup(x \cap \mu)$. ⊣

§4. Some last remarks on cardinal arithmetic.

We remark some consequences of WRP + sat(NS) $= \aleph_2$ on cardinal arithmetic. We have the following Proposition:

PROPOSITION 4.1. WRP *and saturation of* NS_{ω_1} *imply*

$$\lambda^{\omega_1} = \begin{cases} \lambda & \text{if } \operatorname{cof}\lambda > \omega_1, \\ \lambda^+ & \text{if } \operatorname{cof}\lambda \le \omega_1. \end{cases}$$

PROOF. By Conclusion 1.4(1)(a) of [19], WRP implies the *Strong Hypothesis*. By Theorem 6.3(1) of [17], this yields that for every cardinals $\lambda > \kappa$,

$$\operatorname{cof}([\lambda]^{\omega_1}) = \begin{cases} \lambda & \text{if } \operatorname{cof}\lambda > \omega_1, \\ \lambda^+ & \text{if } \operatorname{cof}\lambda \le \omega_1. \end{cases}$$

We cite without proof a Lemma by Jech and Prikry ([9, 10]), and also can be found in [7] (Theorem 22.16):

LEMMA 4.2. *Let I be a σ-complete ideal on ω_1. If $2^{\aleph_0} < \aleph_{\omega_1}$, then*

$$2^{\aleph_1} \le \max\{2^{\aleph_0}, \operatorname{sat}(I)\}.$$

We cite also a result from Todorčević .

THEOREM 4.3 (Todorčević). WRP(ω_2) *implies* $2^{\aleph_0} \le \aleph_2$.

PROOF. See, for example Theorem 37.18 in [7]. ⊣

We know that $2^{\aleph_1} = {\aleph_2}^{\aleph_1}$. Then assuming WRP$(\omega_2)$ + sat(NS) $= \aleph_2$ and using previous Theorem and Lemma, we have

$$2^{\aleph_1} = \aleph_2^{\aleph_1} \le \max\{2^{\aleph_0}, \aleph_2\} = \aleph_2.$$

Therefore

$$\aleph_2^{\aleph_1} = \aleph_2.$$

Since $\lambda^\kappa = 2^\kappa \cdot \operatorname{cof}([\lambda]^\kappa)$ for every cardinal $\lambda > \kappa$, we have:

$$\operatorname{cof}([\lambda]^{\omega_1}) = \begin{cases} \lambda & \text{if } \operatorname{cof}\lambda > \omega_1, \\ \lambda^+ & \text{if } \operatorname{cof}\lambda \le \omega_1. \end{cases} \quad ⊣$$

We thank the referee for the suggestions for this proof, which is a lot shorter than our original one.

§5. Appendix.

PROOF OF THEOREM 3.2. Let $\langle C_\delta\rangle_{\delta \in E^\lambda_\omega}$ be a club guessing sequence of λ (see [18]); i.e., for every $\delta \in E^\lambda_\omega$

(1) o.t.$C_\delta = \omega$,
(2) $\sup C_\delta = \delta$,
(3) for every club $C \subseteq \lambda$, there is $\delta \in E^\lambda_\omega$ such that $C_\delta \subseteq C$.

We denote by $C_\delta(n)$ the n-th element of C_δ.

Consider x a countable subset of λ.

We define the *oscillation* of x

$$\mathrm{osc_x} : \omega \to 2$$

as follows: $\mathrm{osc}_x(n) = 1$ iff $x \cap [C_{\sup x}(n), C_{\sup x}(n+1)) \neq \emptyset$.

For every real $r \in 2^\omega$, we define its *section in* n, $r_n \in 2^\omega$ as

$$r_n(k) = r(2^n(2k+1)).$$

REMARK 5.1. Given a countable collection of reals $\langle r_n \rangle_{n\in\mathbb{N}}$ and a fixed value $a_0 \in \{0,1\}$, we can obtain a unique real $r \in 2^\omega$ whose sections are the respective $\{r_n\}_{n\in\omega}$ and $r(0) = a_0$.

We recall that for every infinite cardinal κ, we have $2^\kappa = (\kappa^+)^\kappa$: On one hand $2 \leq \kappa^+ \to 2^\kappa \leq (\kappa^+)^\kappa$. On the other hand, by Cantor's Theorem, $2^\kappa \geq \kappa^+$. Therefore, $(\kappa^+)^\kappa \leq 2^\kappa = (2^\kappa)^\kappa \leq (\kappa^+)^\kappa$.

Then we can fix a bijection

$$\varphi : 2^\omega \to [\omega_1]^\omega \cup \{\emptyset\}$$

such that the characteristic function of ω is sent to $\emptyset$, i.e., for the constant function $\chi_\omega : \omega \to 2$ such that $\chi_\omega(n) = 1$ for all $n \in \omega$, we have $\varphi(\chi_\omega) = \emptyset$.

We define the function $^* : 2^\omega \to [\omega_1]^\omega$ as

$$r^* = \bigcup_{n\in\omega} \varphi(r_n).$$

For $x \in [\lambda]^\omega$, let $\pi_x : x \to \mathrm{o.t.}(x)$ be the canonical bijection. So, let

$$\theta_x = \pi_x^{-1}[\mathrm{osc}_x^*].$$

We are going to show that $\langle \theta_x \rangle_{x\in[\lambda]^\omega}$ is a $\diamondsuit_{\omega_1,\lambda}$-sequence as appeared in [12].

Let $f : \lambda^{<\omega} \to \lambda$ be a function, and take $W \subseteq \lambda$. So, we have to find $x \in [\lambda]^\omega$ such that

(1) $f[x^{<\omega}] \subseteq x$, and
(2) $W \cap x = \theta_x$.

Let $\lambda^{\uparrow<\omega}$ be the collection of finite strictly increasing sequences of ordinals smaller than λ. In $\lambda^{\uparrow<\omega}$ we define a partial order: for every $s, t \in \lambda^{\uparrow\omega}$, we say that $s < t$ if $s = t \cap (\max s + 1)$.

DEFINITION 5.2. We call a tree $T \subseteq \lambda^{\uparrow<\omega}$ a Namba tree if for every node $s \in T$, the set

$$\{\alpha < \lambda : s^\frown\alpha \in T\}$$

is unbounded in λ.

We shall need the following consequence of the standard Namba lemma (see [4, 12, 14, 15]). A proof can be found after Lemma 3.2 in [24].

LEMMA 5.3. *For every function $f : \lambda^{<\omega} \to \lambda$, there is a Namba tree $T \subseteq \lambda^{\uparrow<\omega}$ such that, for every node $s \in T$, it exists*:

- *an ordinal* $\gamma_s > \max(s)$, *and*
- $x_s \in [\max(s)]^\omega$

such that

(1) *$s < t$ implies $\gamma_s < \max t$,*
(2) *$s < t$ implies $x_s = x_t \cap \sup(\{\xi + 1\}_{\xi \in x_s})$,*
(3) *if $b = \langle s_n \rangle_{n \in \omega}$ is a branch of T, then $x_b = \bigcup_{n \in \omega} x_{s_n}$ is closed under f and $x_b \subseteq \bigcup_{n \in \omega} [\max s_n, \gamma_{s_n})$,*
(4) *the set $\{\min(x_{s^\frown\alpha} \backslash \gamma_s) : \alpha \in \lambda \wedge s^\frown\alpha \in T\}$ is unbounded in λ for every $s \in T$.*

So, for f, we take a Namba tree T as in previous Lemma.

In order to build an element x with the desired properties, we are going to build a branch b along the tree T.

We proceed by induction in order to build step by step the branch b. At the same time, we build also a real r by constructing its sections (see Remark 5.1), and such that

$$\mathrm{osc}_x = r.$$

We declare

$$r(0) = 1.$$

Let s_0 be a minimal element of T. In each step n, we will build s_n and r_{n-1} (so, in the case $n = 0$ we have not built yet any section).

Let $\langle N_\xi \rangle_{\xi \in \lambda}$ be an increasing sequence of elementary submodels of cardinality $\aleph_1$ of a certain $\langle H_\theta, \in, T, \ldots \rangle$, with θ sufficiently large and such that $N_\xi \cap \lambda$ is an ordinal for every $\xi \in \lambda$ (see, for example Remark 4.3.2 in [5]).

Observe that for every $\xi \in \lambda$ and for each $s \in T \cap N_\xi$, using the properties of our Namba tree and the elementarity of N_ξ, we can have x_s and γ_s such that

$$\sup(x_s) < \gamma_s \leq N_\xi \cap \lambda. \tag{5}$$

Note also that

$$C = \{\xi < \lambda : \xi = N_\xi \cap \lambda\}$$

is a club in λ. Pick an ordinal $\delta < \lambda$ such that $C_\delta \subseteq C$.

Suppose that r_k is defined for every $k < n$ and that we have already defined s_n for $m \leq n$.

We are now going to build s_{n+1} and r_n.

Let

$$r_n = \varphi^{-1}(\pi_{x_{s_n}}[W \cap x_{s_n}]).$$

Remark that $r(p)$ is already defined for $p \leq n$.

We have three cases:

CASE 1. $r(n) = 1$.

We are going to choose s_{n+1} in such a way that

$$x_{s_{n+1}} \cap [C_\delta(n), C_\delta(n+1)) \neq \emptyset.$$

By property 4 of Lemma 5.3, for $C_\delta(n)$ there is $\alpha \in \lambda$ such that $s_n^\frown\alpha \in T$ and $\min x_{s_n^\frown\alpha} \backslash \gamma_{s_n} \geq C_\delta(n)$. We have the risk that $\min x_{s_n^\frown\alpha} \backslash \gamma_{s_n}$ is not smaller than $C_\delta(n+1)$. In order to avoid this, we use the elementarity of $N_{C_\delta(n+1)}$. Then, by elementarity, there is $s_{n+1} \in T \cap N_{C_\delta(n+1)}$ such that $\min(x_{s_{n+1}} \backslash \gamma_{s_n}) \geq C_\delta(n)$. By (5), $\sup(x_{s_{n+1}}) < N_{C_\delta(n+1)} \cap \lambda = C_\delta(n+1)$, since $C_\delta \subseteq C$.

CASE 2. $r(n) = 0$ and there exists $m > n$ such that $r(m) = 1$, and for every $i \in [n, m)$, $r(i)$ is defined and $r(i) = 0$.

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We are going to choose s_n in a way that, for every $i \in [n, m)$,

$$x_{s_{n+1}} \cap [C_\delta(i), C_\delta(i+1)) = \emptyset$$

and

$$x_{s_{n+1}} \cap [C_\delta(m), C_\delta(m+1)) \neq \emptyset.$$

Again, this choice is possible by the cofinality condition 4 of Lemma 5.3, which gives us the existence of an $\alpha \in \lambda$, such that the node $s_n^\frown \alpha \in T$ verifies

$$x_s \cap [C_\delta(n), C_\delta(m)) = \emptyset \text{ and } \min x_{s_n^\frown \alpha} \backslash \gamma_{s_n} \geq C_\delta(m).$$

The elementarity of the structure $N_{C_\delta(n+1)}$ gives us a similar $s \in T \cap N_{C_\delta(n+1)}$ but also verifying, by (5) $\sup(x_{s_{n+1}}) < N_{C_\delta(n+1)} \cap \lambda = C_\delta(n+1)$, since $C_\delta \subseteq C$.

Case 3. $r(n) = 0$, and there is $m > n$ such that $r(m)$ is not yet defined and $r(i) = 0$ for every $i \in [n, m)$.

Since $r(k)$ is defined for every $k \leq n$, then, there is a natural number $k \leq n$ such that $r(k) = 1$ and for every $p \in (k, n]$, $r(p) = 0$. We can find by the elementarity of $N_{C_\delta(k+1)}$ a node $s_{n+1} \in N_{C_\delta(k+1)}$ such that $x_{s_{n+1}} \subseteq C_\delta(k+1)$.

This construction finished, our real r is defined everywhere, and $\mathrm{osc}_{x_b} = r$. Now we can verify that $\theta_{x_b} = W \cap x_b$:

$$\begin{aligned}
\theta_{x_b} &= \pi_{x_b}^{-1}\left[\mathrm{osc}_{x_b}^*\right] && \text{definition of } \theta_{x_b}, \\
&= \pi_{x_b}^{-1}\left[r^*\right] && \text{by construction of } r, \\
&= \pi_{x_b}^{-1}\left[\bigcup_{n\in\omega} \varphi\left(r_n\right)\right] && \text{definition of }^*, \\
&= \pi_{x_b}^{-1}\left[\bigcup_{n\in\omega} \varphi\left(\varphi^{-1}\left(\pi_{x_{s_n}}\left[W \cap x_{s_n}\right]\right)\right)\right] && \text{construction of } r_n, \\
&= \pi_{x_b}^{-1}\left[\bigcup_{n\in\omega} \pi_{x_{s_n}}\left[W \cap x_{s_n}\right]\right] && \text{since } \varphi \text{ is a bijection}, \\
&= \bigcup_{n\in\omega} W \cap x_{s_n} && \text{by property (2) of Lemma 5.3}, \\
&= W \cap x_b.
\end{aligned}$$

We now prove (b).

Remark that since δ is an ordinal of cofinality ω, the set

$$E_\delta = \{a \in [\delta]^\omega : \mathrm{osc}_a = \chi_\omega\}$$

is a club in $[\delta]^\omega$, where χ_ω denotes the characteristic function of the set ω, i.e., the element r of 2^ω that is constantly equal to 1.

Claim 5.4. *For every $a \in E_\delta$, $\theta_a = \emptyset$.*

Proof. Take $a \in E_\delta$. In particular, the sections of the oscillation are also the characteristic function of ω, i.e., for every $n \in \omega$, $(\mathrm{osc}_a)_n = \chi_\omega$. Then, for every every

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$n \in \omega$, $\varphi((\mathrm{osc}_a)_n) = \varphi(\chi_\omega) = \emptyset$, which implies that $\mathrm{osc}_a^* = \bigcup_{n\in\omega} \varphi((\mathrm{osc}_a)_n) = \emptyset$. Therefore $\theta_a = \pi_a^{-1}[\mathrm{osc}_a^*] = \emptyset$. ⊣

⊣

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