

COMPUTABLE STRUCTURES IN GENERIC EXTENSIONS

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Abstract. In this paper, we investigate connections between structures present in every generic extension of the universe V and computability theory. We introduce the notion of *generic Muchnik reducibility* that can be used to compare the complexity of uncountable structures; we establish basic properties of this reducibility, and study it in the context of *generic presentability*, the existence of a copy of the structure in every extension by a given forcing. We show that every forcing notion making ω_2 countable generically presents some countable structure with no copy in the ground model; and that every structure generically presentable by a forcing notion that does not make ω_2 countable has a copy in the ground model. We also show that any countable structure \mathcal{A} that is generically presentable by a forcing notion not collapsing ω_1 has a countable copy in V , as does any structure \mathcal{B} generically Muchnik reducible to a structure \mathcal{A} of cardinality \aleph_1 . The former positive result yields a new proof of Harrington's result that counterexamples to Vaught's conjecture have models of power \aleph_1 with Scott rank arbitrarily high below ω_2 . Finally, we show that a rigid structure with copies in all generic extensions by a given forcing has a copy already in the ground model.

§1. Introduction. In computable structure theory, one studies the complexity of structures using techniques from computability theory. Almost all of this work concerns countable structures; much less is known about the complexity of uncountable structures. However, the computability theory of uncountable structures has received more attention in the last few years. (See for instance the proceedings volume of the conference *Effective Mathematics of the Uncountable* [8].) One idea for studying the complexity of an uncountable structure that seems new is to consider what happens to the structure when its domain is made countable.

Before making this idea more concrete, we recall the notion of *Muchnik reducibility* between countable structures. This is the standard way in computable structure theory to say that one structure is more complicated than another, in the sense that it is harder to compute.

DEFINITION 1.1. Given countable structures \mathcal{A} and \mathcal{B} , we say that \mathcal{A} is *Muchnik reducible* to \mathcal{B} , and we write $\mathcal{A} \leq_w \mathcal{B}$, if, from any copy of \mathcal{B} , we can compute a copy of \mathcal{A} .

On its face, this notion is limited to countable structures. However, by examining generic extensions of the set-theoretic universe, V , we can extend it further:

DEFINITION 1.2 (Schweber). For a pair of structures \mathcal{A} and \mathcal{B} , not necessarily countable in V , we say that \mathcal{A} is *generically Muchnik reducible* to \mathcal{B} , and we write

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$\mathcal{A} \leq_w^* \mathcal{B}$, if for any generic extension $V[G]$ of the set theoretic universe V in which both structures are countable, we have

$$V[G] \models \mathcal{A} \leq_w \mathcal{B}.$$

In Section 2.1, we will prove the basic properties of this reducibility. We will show that it coincides with Muchnik reducibility on countable structures; i.e., if \mathcal{A} and \mathcal{B} are countable, then $\mathcal{A} \leq_w \mathcal{B}$ if and only if $\mathcal{A} \leq_w^* \mathcal{B}$ (Corollary 2.5). More generally, we do not need to consider all the generic extensions that make \mathcal{A} and \mathcal{B} countable—this is a consequence of Shoenfield absoluteness (Theorem 2.1), a general principle about forcing. We will prove that for any two such generic extensions if $\mathcal{A} \leq_w \mathcal{B}$ holds in one, then it holds in the other (Lemma 2.3). This shows that generic Muchnik reducibility is a very absolute, and hence, natural, notion of computability-theoretic complexity.

We will also show that the equivalence \equiv_w^* , induced from the reducibility \leq_w^* , respects $\mathcal{L}_{\infty\omega}$ -elementary equivalence. In Section 2.3, we will also exhibit some examples of this reducibility. For instance, we show that the countable structures generically Muchnik reducible to the linear order ω_1 are precisely those Muchnik reducible to some countable well-ordering, and we identify two natural structures— \mathcal{W} and \mathcal{R} , the powerset of ω and the field of real numbers—each of which lies above every countable structure in the generic Muchnik reducibility. We show that $\mathcal{W} \leq_w^* \mathcal{R}$; recently Igusa and Knight [10] have shown that $\mathcal{R} \not\leq_w^* \mathcal{W}$, so these two structures are fundamentally different.

Closely related to generic reducibility is *generic presentability*. Intuitively, a countable structure \mathcal{A} is *generically presentable* if there is some forcing notion \mathbb{P} such that any forcing extension by \mathbb{P} contains a copy of \mathcal{A} . We will be interested in when generically presentable structures already have copies in the real universe. To be precise, we define:

DEFINITION 1.3. A *generically presentable structure* is a pair (\mathbb{P}, ν) , where \mathbb{P} is a forcing notion and ν is a \mathbb{P} -name, such that

$$\Vdash_{\mathbb{P}} \text{“}\nu[G] \text{ is a structure with domain } \omega\text{”} \quad \text{and} \quad \Vdash_{\mathbb{P} \times \mathbb{P}} \text{“}\nu[G_0] \cong \nu[G_1]\text{.”}$$

We say (\mathbb{P}, ν) is *generically presented by* \mathbb{P} . When the forcing notion is clear from context, we will abbreviate “ (\mathbb{P}, ν) ” by “ ν ,” or abuse notation and use notation for classical structures (“ \mathcal{A} ,” “ \mathcal{B} ,” etc.) instead. If (\mathbb{P}, ν) is a generically presentable structure and \mathbb{Q} is another forcing notion, we say (\mathbb{P}, ν) is *generically presentable by* \mathbb{Q} if there is a generically presentable structure (\mathbb{Q}, μ) such that

$$\Vdash_{\mathbb{P} \times \mathbb{Q}} \text{“}\nu[G_0] \cong \mu[G_1]\text{,”}$$

and we will elide the distinction between such a pair of generically presentable structures when no confusion will result. Note that every actual structure may be thought of as a generically presented structure.

REMARK 1.4. After submitting, we learned that at around the same time, generic presentability was independently being studied by two other groups. Itay Kaplan and Saharon Shelah [12], addressing a question of Jindrich Zapletal (who has done work [26] on related ideas in the context of Borel reducibility of equivalence relations), defined generic presentability and gave alternate proofs of our Theorems 3.14

and 3.18. Separately, Paul Larson [15] studied the Scott analysis of structures (see Section 3.1); since—roughly—a structure is generically presentable if and only if its Scott sentence exists, his work yields proofs of our Theorems 3.14 and 4.1.

In Section 2.4 below, we give an alternate approach to generic presentability, via countable models of set theory.

REMARK 1.5. Usually, a *copy* of a structure \mathcal{A} is just a structure \mathcal{B} that is isomorphic to \mathcal{A} . However, in this paper we will find ourselves studying structures which may not yet exist, or copies of structures in larger universes, so it is worth making precise what we mean by “copy.” In this paper, we will primarily use the word “copy” in two ways:

- If \mathcal{A} is a structure in V , we will often want to consider copies of \mathcal{A} with domain ω . Although these will not exist in V if \mathcal{A} is uncountable, they will exist in generic extensions; we will use the term “ ω -copy” (of \mathcal{A}) to refer to a copy of \mathcal{A} with domain ω , which may live in a generic extension of the universe.
- Separately, we will also want to ask whether a generically presentable structure is already present, up to isomorphism, in V . Towards that end, if $\mathcal{A} \in V$ is an actual structure and $\mathcal{B} = (\mathbb{P}, \nu)$ is a generically presentable structure, we say that \mathcal{A} is a *copy* of \mathcal{B} if $\Vdash_{\mathbb{P}} \mathcal{A} \cong \nu[G]$.

Although these two uses of the word “copy” are somewhat at odds, we will be careful to make clear at each point what notion of “copy” is meant.

CONVENTION 1.6. *For simplicity, as is standard in set theory, we will frequently abuse notation by referring to generic extensions $V[G]$ of the universe V as if they exist rather than writing everything out in terms of names.*

We will be interested in examining when a generically presentable structure already exists—that is, when it has a copy (or an ω -copy) in the ground model V . It is well-known that if a set S is in $V[G]$ for every \mathbb{P} -generic G , then S must belong to V already (Solovay [25], see Theorem 2.25 below for a precise statement and proof). However, the situation for isomorphic copies of a given structure is more complicated. There are cases in which the analogous fact is true, and there are cases in which it is not. This paper is devoted to analyzing this situation.

In particular, we are interested in the interaction between generic presentability and generic Muchnik reducibility. Generic Muchnik reducibility can be extended to generically presentable structures in a natural way—if \mathcal{A} and \mathcal{B} are generically presentable structures (or one is generically presentable and the other is an actual structure, or etc.), then $\mathcal{A} \leq_w^* \mathcal{B}$ if and only if, whenever \mathbb{P} is a forcing presenting both \mathcal{A} and \mathcal{B} , we have $\Vdash_{\mathbb{P}} \mathcal{A} \leq_w \mathcal{B}$. Now if $\mathcal{A} \leq_w^* \mathcal{B}$, then \mathcal{B} contains all the information necessary to build \mathcal{A} —up to a certain amount of genericity. To what extent is this genericity actually necessary? Ted Slaman formulated this question as follows:

MAIN QUESTION (Slaman). *Suppose \mathcal{A} is a generically presentable structure and $\mathcal{A} \leq_w^* \mathcal{B}$ for some actual structure $\mathcal{B} \in V$. Is there a copy of \mathcal{A} in V ?*

This can be rephrased as a question about inner models, as follows. Suppose $\mathcal{A} \leq_w^* \mathcal{B}$ with \mathcal{B} in Gödel’s constructible universe, L ; must we have some $\mathcal{C} \cong \mathcal{A}$ (in V) with $\mathcal{C} \in L$? Note that if $\mathcal{A} \leq_w^* \mathcal{B}$ with $\mathcal{B} \in L$, then there is a generically presentable structure $(\mathbb{P}, \nu) \in L$ such that—in V —we have $\Vdash_{\mathbb{P}} \nu[G] \cong \mathcal{A}$, so this

really is a special case of the previous question. Of course, L may be replaced with any inner model of ZFC, or even much less than ZFC.

We begin by studying the role of forcing-theoretic properties in generic presentability. We prove:

THEOREM 1.7. *Any structure generically presentable by a forcing notion that does not make ω_2 countable has a copy (not necessarily with domain ω) in V .*

This theorem yields as a corollary a partial positive answer to Slaman’s question.

COROLLARY 1.8. *If \mathcal{A} is a generically presentable structure which is \leq_w^* \mathcal{B} for some actual structure $\mathcal{B} \in V$ with cardinality $\leq \aleph_1$, then \mathcal{A} has a copy in V . Alternately, from an inner model perspective, we have that if \mathcal{B} lives in L and, within L , has size \aleph_1^L , then \mathcal{A} has a copy in L .*

We also give a new proof of the following result of Harrington.

THEOREM 1.9 (Harrington). *If T is a counter-example to Vaught’s conjecture, then it has models of arbitrarily high Scott rank below ω_2 .*

On the other hand, these positive results cannot be extended much further: making ω_2 countable *always* introduces a structure with universe ω that does not have a copy in V , and that moreover has low complexity as measured by the generic Muchnik reducibility. This provides an exact dichotomy among structures generically presentable, and a negative answer to Slaman’s question in general.

THEOREM 1.10. *There is a generically presentable structure \mathcal{M} , which is presented by any notion of forcing that makes ω_2 countable, but which has no copy in V . Moreover, this \mathcal{M} is generically Muchnik reducible to the ordering $(\omega_2, <)$.*

We close with a *structural* approach to the question: what properties ensure that generic presentability implies existence in the ground model? We show that this occurs at least when the structures involved are as “set-like” as possible, in the sense of being rigid—that is, having no nontrivial automorphisms. In Section 4, we show the following:

THEOREM 1.11. *Suppose \mathcal{A} is rigid and is generically presentable. Then there is an isomorphic copy of \mathcal{A} already in V .*

§2. Generic reducibility.

2.1. Basic properties. The key result for analyzing generic presentability and generic reducibility is the Shoenfield Absoluteness Theorem (see [11]). The version we state below is slightly weaker than the actual theorem, but it is all we will need here:

THEOREM 2.1 (Shoenfield). *Suppose φ is a Π_2^1 sentence, with real parameters. Then, for every forcing extension W of V , $V \models \varphi \iff W \models \varphi$.*

An easy fact about (countable) Muchnik reducibility of structures is the following.

OBSERVATION 2.2. *Basic facts about \leq_w are invariant under forcing. Specifically, we have the following.*

- (1) *The relation “ \leq_w ” is Π_2^1 .*
- (2) *For countable \mathcal{A} , the predicate “ $\geq_w \mathcal{A}$ ” is Π_1^1 in a Scott sentence of \mathcal{A} .*

Together with Theorem 2.1, this implies that much of the theory of \leq_w^* is absolute. In particular, we have the next lemma.

LEMMA 2.3. *Fix arbitrary structures \mathcal{M}, \mathcal{N} in V . If there is some generic extension in which \mathcal{M} and \mathcal{N} are countable and $\mathcal{M} \leq_w \mathcal{N}$, then $\mathcal{M} \leq_w^* \mathcal{N}$.*

PROOF. Suppose otherwise. Then there must exist posets \mathbb{P}_0 and \mathbb{P}_1 in V such that forcing with either collapses both \mathcal{M} and \mathcal{N} ,

$$\Vdash_{\mathbb{P}_0} \mathcal{M} \leq_w \mathcal{N} \quad \text{and} \quad \Vdash_{\mathbb{P}_1} \mathcal{M} \not\leq_w \mathcal{N}.$$

Let $G = H_0 \times H_1$ be $\mathbb{P}_0 \times \mathbb{P}_1$ -generic over V . Let \mathcal{M}_0 and \mathcal{N}_0 be reals in $V[H_0]$ coding copies of \mathcal{M} and \mathcal{N} with domain ω , and let \mathcal{M}_1 and \mathcal{N}_1 be reals in $V[H_1]$ coding copies of \mathcal{M} and \mathcal{N} with domain ω . Then, in $V[H_0]$, $\mathcal{M}_0 \leq_w \mathcal{N}_0$, while in $V[H_1]$, $\mathcal{M}_1 \not\leq_w \mathcal{N}_1$. By Shoenfield’s absoluteness, this is still true in $V[H_0][H_1]$. This gives us a contradiction because, in $V[H_0][H_1]$, \mathcal{M}_0 is isomorphic to \mathcal{M}_1 and \mathcal{N}_0 to \mathcal{N}_1 . \dashv

REMARK 2.4. For κ an infinite cardinal, the partial order $Col(\kappa, \omega)$ of finite sequences of ordinals $< \kappa$, ordered in the natural way, collapses κ to ω . This forcing notion is (a special case of) the *Levy collapse*. By 2.3, we may always assume that the forcings we consider are Levy collapses for κ at least as large as each structure under consideration.

As an immediate corollary of Lemma 2.3, we get the following.

COROLLARY 2.5. *For structures \mathcal{A}, \mathcal{B} countable in V , we have $\mathcal{A} \leq_w \mathcal{B}$ if and only if $\mathcal{A} \leq_w^* \mathcal{B}$.*

2.2. Potential isomorphism. Generic Muchnik reducibility also has strong connections with infinitary logic.

DEFINITION 2.6. Let \mathcal{L} be a language; that is, a set of relation and operation symbols.

- $\mathcal{L}_{\infty\omega}$ is the collection of formulas obtained from the atomic \mathcal{L} -formulas by closing under arbitrary set-sized Boolean combinations and single instances of quantification. See [14] for a treatment of the basic properties of $\mathcal{L}_{\infty\omega}$.
- For structures \mathcal{A}, \mathcal{B} of arbitrary cardinality, we say that \mathcal{A} is *$\mathcal{L}_{\infty\omega}$ -elementary equivalent* to \mathcal{B} , and we write $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$, if the structures satisfy the same $\mathcal{L}_{\infty\omega}$ sentences.

There is a structural characterization of $\equiv_{\infty\omega}$, due to Carol Karp:

DEFINITION 2.7. Suppose I is a set of partial maps. We say that I has the *back-and-forth property*—equivalently, I is a *back-and-forth system*—if $\langle \emptyset, \emptyset \rangle \in I$ and for every $\langle \bar{a}, \bar{b} \rangle \in I$,

- (1) \bar{a} and \bar{b} satisfy the same atomic formulas,
- (2) for every $c \in \mathcal{A}$, there is $d \in \mathcal{B}$ such that $\langle \bar{a}c, \bar{b}d \rangle \in I$, and
- (3) for every $d \in \mathcal{B}$, there is $c \in \mathcal{A}$ such that $\langle \bar{a}c, \bar{b}d \rangle \in I$.

An I with the back-and-forth property is called a *back-and-forth system* between \mathcal{A} and \mathcal{B} .

THEOREM 2.8 ([13]). *$\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ iff there is a back-and-forth system between \mathcal{A} and \mathcal{B} .*

It is then not hard to see that for \mathcal{A} and \mathcal{B} countable, we have that $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ if and only if $\mathcal{A} \cong \mathcal{B}$. Additionally, Karp’s characterization shows that $\equiv_{\infty\omega}$ is absolute with respect to forcing. Clearly $\equiv_{\infty\omega}$ is upwards absolute. To show downwards absoluteness, let ν be a name for a back-and-forth system between \mathcal{A} and \mathcal{B} in some forcing extension by \mathbb{P} ; then $I = \{\langle \bar{a}, \bar{b} \rangle : \exists p \in \mathbb{P}(p \Vdash \langle \bar{a}, \bar{b} \rangle \in \nu)\}$ is a back-and-forth system between \mathcal{A} and \mathcal{B} . Thus, for possibly uncountable structures \mathcal{A} and \mathcal{B} , we have $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ iff $\mathcal{A} \cong \mathcal{B}$ once they are made countable:

LEMMA 2.9 (Essentially Barwise [2]). *The following are equivalent:*

- (1) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$,
- (2) in every generic extension where \mathcal{A} and \mathcal{B} are countable, $\mathcal{A} \cong \mathcal{B}$,
- (3) in some generic extension where \mathcal{A} and \mathcal{B} are countable, $\mathcal{A} \cong \mathcal{B}$.

As an immediate corollary, we have the following.

COROLLARY 2.10. $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ implies $\mathcal{A} \equiv_w^* \mathcal{B}$.

This lets us connect $\equiv_{\infty\omega}$ -equivalence and generic Muchnik reducibility in a strong way:

LEMMA 2.11. *Let $\mathcal{A} \in V$ be a structure. The following are equivalent:*

- (1) $\mathcal{A} \leq_w^* \mathcal{B}$ for some countable structure \mathcal{B} .
- (2) $\mathcal{A} \equiv_w^* \mathcal{B}$ for some countable structure \mathcal{B} .
- (3) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ for some countable structure \mathcal{B} .

PROOF. Clearly (3) implies (2) and (2) implies (1). To see that (1) implies (3), suppose $\mathcal{A} \leq_w^* \mathcal{C}$ for \mathcal{C} countable, let C be an ω -copy of \mathcal{C} in V , and let $V[G]$ be a generic extension in which \mathcal{A} is countable. Then in $V[G]$, there is some index e such that, for the e th Turing machine Φ_e , $\Phi_e^C \cong \mathcal{A}$. This means that in V , Φ_e^C must be total, and so Φ_e^C is a copy of \mathcal{A} which lives in V ; that is, the structure $\mathcal{B} = \Phi_e^C$ is $\mathcal{L}_{\infty\omega}$ -equivalent to \mathcal{A} . ⊣

2.3. Examples. We present below some examples of uncountable structures whose complexity in terms of \leq_w^* we have been able to analyze.

EXAMPLE 2.12. Let \mathcal{U} be the structure with domain $\omega \sqcup \mathcal{P}(\omega)$, with signature consisting of only the \in -relation on $\omega \times \mathcal{P}(\omega)$.

PROPOSITION 2.13. $\mathcal{U} \equiv_w^* 0$, where 0 is the empty structure.

PROOF. We will show there is a computable structure \mathcal{S} that is $\equiv_{\infty\omega}$ -equivalent to \mathcal{U} . By the absoluteness of $\equiv_{\infty\omega}$, we will then have that in any generic extension that makes \mathcal{U} countable, \mathcal{S} and \mathcal{U} are still $\equiv_{\infty\omega}$ -equivalent, and, hence, isomorphic.

We note that the orbit in \mathcal{U} of a tuple of sets \vec{X} is determined by the cardinalities of the Boolean combinations of the sets X_i . To guarantee that we have a back-and-forth family of finite partial isomorphisms, we let \mathcal{S} consist of ω together with a family of sets P having the following properties:

- P is an algebra of sets; i.e., it is closed under union, intersection, and complement,
- P includes all finite sets,
- if $X \in P$ is infinite, then there are disjoint $Y, Z \in P$, both infinite, such that $Y \cup Z = X$.

We can easily find such an \mathcal{S} which is computable. We could, for example, take the family of primitive recursive sets. \dashv

Similarly, the field of complex numbers is essentially computable.

EXAMPLE 2.14. Let $\mathcal{C} = (\mathbb{C}; +, \times)$. This is $\equiv_{\infty\omega}$ -equivalent to the algebraically closed field of countably infinite transcendence degree and characteristic zero. By a well-known result of Rabin [21], this has a computable copy. Then \mathcal{C} has minimal complexity; that is, \mathcal{C} has a computable copy in every generic extension in which it is countable.

If we consider a variant of \mathcal{U} in which the elements of ω have names, we reach the opposite end of the complexity spectrum:

EXAMPLE 2.15. Let \mathcal{W} be the expansion of \mathcal{U} to include the successor relation on ω . Then any ω -copy (1.5) of \mathcal{W} computes every real in the ground model V , so given any countable $\mathcal{A} \in V$ we have $\mathcal{A} \leq_w^* \mathcal{W}$.

The situation is the same with respect to the real numbers.

EXAMPLE 2.16. The field of real numbers $\mathcal{R} = (\mathbb{R}; +, \times)$ is, like \mathcal{W} , maximally complicated with respect to countable structures: for every countable structure \mathcal{A} , we have $\mathcal{A} \leq_w^* \mathcal{R}$. To see this, suppose $V[G]$ is a generic extension in which \mathcal{R} has an ω -copy, \mathbb{R} . First, note that the standard ordering $<_{\mathbb{R}}$ is defined both by an existential formula and by a universal formula, and so after collapse the corresponding relation on any ω -copy of \mathbb{R} is computable relative to that copy.

Now fix a real in the ground model $b \in \mathcal{R}$ and let $\hat{b} \in \mathbb{R}$ be the corresponding element of the ω -copy. Since $<_{\mathbb{R}}$ is computable from the atomic diagram of \mathbb{R} and there is a uniform effective procedure for identifying each rational number in \mathbb{R} , the cut corresponding to \hat{b} is also computable from the atomic diagram of \mathbb{R} ; thus, every real in the ground model is computable from the atomic diagram of \mathbb{R} . Since \mathbb{R} was an arbitrary ω -copy of \mathcal{R} in an arbitrary generic extension, it follows that $\mathcal{R} \geq_w^* \mathcal{A}$ for every countable $\mathcal{A} \in V$.

We would now like to compare the structures \mathcal{R} and \mathcal{W} under \leq_w^* . It is easy to show the following.

PROPOSITION 2.17. $\mathcal{R} \geq_w^* \mathcal{W}$.

PROOF. We can use the elements of \mathcal{R} in the interval $[0, 1)$ to enumerate the subsets of ω in V . To each real r in the interval, we associate the set A_r consisting of those n such that the n^{th} term in the binary expansion of r is 1. Minimal care has to be taken for double binary representations: if we assume no binary expansion ends up in an infinite string of 1s, we then need to add those sets. \dashv

Recently, Igusa and Knight [10] have shown that this reduction is strict. However, this relies crucially on the fact that \mathcal{R} is not very saturated (specifically, that \mathcal{R} is Archimedean). For example, for an elementary extension \mathcal{M} of \mathcal{R} that is ω -saturated, we have $\mathcal{W} \geq_w^* \mathcal{M}$. More generally, we have the following.

PROPOSITION 2.18. *Let \mathcal{M} be an ω -saturated model of a complete elementary first order theory T . Then $\mathcal{W} \geq_w^* \mathcal{M}$.*

PROOF. Assume without loss of generality that T is decidable—we may make this assumption since every real can be computed uniformly from the atomic diagram of a single parameter in \mathcal{W} (specifically, itself). Macintyre and Marker [17] showed that for an enumeration \mathcal{E} of a Scott set \mathcal{S} , and an elementary first order theory T in \mathcal{S} , \mathcal{E} computes the complete diagram of a recursively saturated model of T realizing exactly the types in \mathcal{S} that are consistent with T . After we collapse the cardinal so that \mathcal{W} becomes countable, it computes an enumeration \mathcal{E} of the Scott set \mathcal{S} consisting of the subsets of ω in \mathcal{W} . Now, the theory of \mathcal{M} is in \mathcal{S} , and the types realized in \mathcal{M} are exactly those in \mathcal{S} that are consistent with T . Then the result of Macintyre and Marker yields a recursively saturated model realizing exactly these types. This model is isomorphic to the collapse of \mathcal{M} . \dashv

Finally, uncountable well-orderings live strictly between the two extremes.

EXAMPLE 2.19. The linear order $\omega_1 = (\omega_1, <)$ computes—that is, is generically Muchnik above—precisely those countable structures which are Muchnik reducible to some countable well-ordering. One direction is obvious; in the other direction, suppose $\mathcal{A} \leq_w^* \omega_1$ is countable, and let $V[G]$ be a forcing extension in which ω_1 is countable. Then $V[G]$ satisfies “ \mathcal{A} is Muchnik reducible to a countable well-ordering,” which is Σ_2^1 via 2.2, and so already true in V by Shoenfield absoluteness.

PROPOSITION 2.20. $\mathcal{R} >_w^* \omega_1$ and $\mathcal{W} >_w^* \omega_1$, strictly.

PROOF. To see that $\mathcal{R} \not\leq_w^* \omega_1$, fix some noncomputable real $r \in \mathcal{R}$. Then the cut corresponding to r , and hence r itself, is computable in any ω -copy R of \mathcal{R} in any generic extension since the ordering relation is both Σ_1 and Π_1 . On the other hand, by a result of Richter [22], the only sets computable in all copies of a countable linear ordering are the computable sets, so in any generic extension in which ω_1 is countable there will be ω -copies of ω_1 whose atomic diagrams do not compute r .

To see that $\omega_1 \leq_w^* \mathcal{R}$, suppose $V[G]$ is a generic extension in which \mathcal{R} is countable, and let $R \in V[G]$ be a copy of \mathcal{R} with domain ω . Now R computes an enumeration of the sets coded by the cuts in \mathcal{R} —the reals in V . Some of the reals code linear orderings. For an ordering r coded in R , if r is not a well ordering, this is witnessed by a decreasing sequence d , also coded in R . A countable well ordering in V is isomorphic to a countable ordinal, so it stays well ordered in $V[G]$. Using R'' , we get an ω -sequence of well-orderings: For $a \in R$, we take the ordering coded by a , if this is a well ordering, and otherwise, we have a finite ordering. The result is an ordering of type ω_1^V . Now, we apply in $V[G]$ the theorem saying that, for any set X and any linear order \mathcal{L} , if X'' computes a copy of \mathcal{L} then X computes a copy of $\omega \cdot \mathcal{L}$ ([1], Theorem 9.11). Since $\omega_1^V \cong \omega \cdot \omega_1^V$, our R computes a copy of ω_1^V .

The proof that $\mathcal{W} >_w^* \omega_1$ is identical. \dashv

2.4. Generic presentability. In this section we elaborate on the concept of generic presentability.

Recall the definition of generic presentability:

DEFINITION 2.21. A *generically presentable structure* is a pair (\mathbb{P}, ν) , where \mathbb{P} is a forcing notion and ν is a \mathbb{P} -name, such that

$$\Vdash_{\mathbb{P}} \text{“}\nu[G] \text{ is a structure with domain } \omega\text{”} \quad \text{and} \quad \Vdash_{\mathbb{P} \times \mathbb{P}} \text{“}\nu[G_0] \cong \nu[G_1]\text{.”}$$

REMARK 2.22. It may be helpful to note that for any generically presentable structure $\mathcal{A} = (\mathbb{P}, \nu)$, there is some cardinal λ such that, for any $\kappa \geq \lambda$, \mathcal{A} is presented by the Levy collapse $Col(\kappa, \omega)$. To see this, take $\lambda = 2^{|\mathbb{P}|}$. Then forcing with $Col(\kappa, \omega)$ for $\kappa \geq \lambda$ will in turn make the set of dense subsets of \mathbb{P} countable, at which point we can construct a generic filter through \mathbb{P} .

Although this is the definition we will use throughout this paper, it will be useful to note that it can be relativized to arbitrary models of ZFC:

DEFINITION 2.23. For a model M of ZFC, a generically presentable structure over M is a pair $(\mathbb{P}, \nu) \in M$, where \mathbb{P} is a forcing notion in M and ν is a \mathbb{P} -name in M , such that

$$M \models \Vdash_{\mathbb{P}} \text{“}\nu[G] \text{ is a structure with domain } \omega\text{”} \quad \text{and} \quad \Vdash_{\mathbb{P} \times \mathbb{P}} \text{“}\nu[G_0] \cong \nu[G_1]\text{”}.$$

The value of this relativization is the following. Often it is useful to imagine that the set-theoretic universe in which we work is actually countable, and lives inside a larger universe. For instance, this perspective means that the generic filters implicit in forcing arguments have to exist, reducing the need to talk about names directly. The following result shows that generic presentability has an equivalent and perhaps simpler definition if we adopt this viewpoint:

PROPOSITION 2.24. *Suppose M is a countable transitive model of ZFC and \mathcal{A} is a structure in the real universe, V . Then the following are equivalent:*

- (1) *There is a generically presented structure over M , (\mathbb{P}, ν) , such that for every G which is \mathbb{P} -generic over M we have $V \models \nu[G] \cong \mathcal{A}$.*
- (2) *There is a forcing notion \mathbb{P} in M such that, for every G which is \mathbb{P} -generic over M , we have a structure $\mathcal{B} \in M[G]$ such that $V \models \mathcal{A} \cong \mathcal{B}$.*

PROOF. Clearly (1) implies (2). To show (2) implies (1), let \mathbb{P} be a poset such that every generic extension of M by \mathbb{P} contains a copy of \mathcal{A} (as seen in V). Let G and H be mutually \mathbb{P} -generic over M , and let ν and μ be names for copies of \mathcal{A} in $M[G]$ and $M[H]$, respectively. Since G and H are mutually generic, there is some $(p, q) \in G \times H$ such that $(p, q) \Vdash_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \cong \mu[G_1]$. This means that $(p, p) \Vdash_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \cong \nu[G_1]$ by considering the condition (p, q, p) in the triple product $\mathbb{P} \times \mathbb{P} \times \mathbb{P}$. Letting $\mathbb{Q} = \{q \in \mathbb{P} : q \leq p\}$ and $\hat{\nu}$ be the natural restriction of ν to \mathbb{Q} , we have that $(\mathbb{Q}, \hat{\nu})$ is a generically presented structure over M which is as desired. \dashv

An argument similar to the proof of 2.24 shows that the analogue of generic presentability for *sets* is trivial:

THEOREM 2.25 (Solovay). *If a set is present in two mutually generic extensions, then it was already present in the ground model. Formally:*

- (Internal version) *If \mathbb{P} is a forcing notion, $p, q \in \mathbb{P}$, and ν, μ are \mathbb{P} names such that $(p, q) \Vdash_{\mathbb{P}^2} \nu[G_0] = \mu[G_1]$, then there is some set S such that $p \Vdash \nu[G] = S$.*
- (External version) *If M is a countable transitive model of ZFC, \mathbb{P} is a forcing notion in M , G, H are mutually \mathbb{P} -generic filters over M , and $X \in M[G] \cap M[H]$, then $X \in M$.*

PROOF. We will prove (2) only, since the proofs are similar. Suppose M, \mathbb{P}, G, H, X are as hypothesized with X of minimal rank, so $X \subseteq A$ for some $A \in M$. Let

$\mu, \nu \in M$ be \mathbb{P} -names such that $\nu[G] = \mu[H] = X$, and let $(p, q) \in G \times H$ be such that $(p, q) \Vdash_{\mathbb{P}^2} \nu[G_0] = \mu[G_1]$. Suppose towards contradiction there is some $a \in A$ such that $p \not\Vdash a \in \nu$ and $p \not\Vdash a \notin \nu$, and suppose $X(a) = i$; then picking $r \leq p$ with $r \Vdash \nu[G_0](a) = 1 - i$ and $s \leq q$ with $s \Vdash \mu[G_1](a) = i$ (which must exist since $X(a) = i$) yields absurdity. So p already decides membership of each element of A in X , and hence $X = \{a : p \Vdash a \in \nu\} \in M$. \dashv

REMARK 2.26. Note that this argument breaks down completely when we look at structures-up-to-isomorphism instead of sets-up-to-equality, essentially because structures, unlike sets, do not have unique representations. Broadly speaking, in order to adapt this argument to show that a generically presentable structure \mathcal{A} has a copy in the ground model V we need to argue that there is a way to build up \mathcal{A} explicitly from its small substructures. Although this is not always possible, the following model-theoretic perspective will be useful for producing positive results: to any structure we may associate a “Scott sentence,” an infinitary first-order sentence which characterizes the structure and is defined in a suitably absolute fashion. Moreover, if a structure \mathcal{A} is countable, then its Scott sentence provides a reasonably effective recipe for building a copy of \mathcal{A} —specifically, since the satisfiability of $\mathcal{L}_{\omega_1\omega}$ -sentences is absolute, if a model of set theory contains the Scott sentence of \mathcal{A} as an $\mathcal{L}_{\omega_1\omega}$ -sentence then that model contains a copy of \mathcal{A} itself. Intuitively, we are motivated to claim that *a structure is generically presentable if and only if its Scott sentence already exists*. As written of course this is vague, but it is an important intuition for the arguments given in Section 3.

§3. Generic presentability and ω_2 . In this section and the next, we address the question “when do generically presentable structures have copies in V ?” This section focuses on a forcing-theoretic aspect of the question. For which forcing notions \mathbb{P} do we have copies in V for all structures generically presentable by \mathbb{P} with universe ω ? Surprisingly, this is entirely determined by how \mathbb{P} affects cardinals: ω_2 remains uncountable after forcing with \mathbb{P} if and only if every structure generically presentable by \mathbb{P} on ω has a copy in V .

As a consequence of proving the left-to-right direction of this result, we also give a new proof of the result due to Harrington that counterexamples to Vaught’s conjecture must have models of arbitrarily high Scott rank in ω_2 . The right-to-left direction follows from a construction of Laskowski and Shelah [16].

3.1. Scott Analysis. We begin by reviewing the *Scott analysis* of a structure. Scott [24] proved that for every countable structure \mathcal{A} , there is an infinitary sentence σ of $\mathcal{L}_{\omega_1\omega}$ such that the countable models of σ are exactly the isomorphic copies of \mathcal{A} . Such a sentence is called a *Scott sentence*.

There are several definitions of *Scott rank* in the literature (see, in particular, [1, 3, 5, 18, 20]). The definitions give slightly different values. However, all of the definitions assign countable Scott ranks to countable structures. In general, the complexity of the Scott sentence is only a little greater than the Scott rank of the structure. If one definition assigns a computable ordinal Scott rank, then the other definitions do as well, and then there is a Scott sentence that is Σ_α , for some computable ordinal α . The definition that we give below is the one used by Sacks [23]. We begin by defining a family of definable expansions of \mathcal{A} .

DEFINITION 3.1. For each α , we define a fragment \mathfrak{L}_α^A of $\mathcal{L}_{\infty,\omega}$ as follows:

- Let \mathfrak{L}_0^A consist of the elementary first order formulas.
- Given \mathfrak{L}_α^A , for each complete nonprincipal type $\Phi(x) \subseteq \mathfrak{L}_\alpha^A$ realized in \mathcal{A} , add the formula $\bigwedge \Phi(x)$ to $\mathfrak{L}_{\alpha+1}^A$, and close under finite logical connectives and first-order quantifiers.
- At limit levels, take unions.

For each α there is a natural way to expand \mathcal{A} to a \mathfrak{L}_α^A -structure \mathcal{A}_α ; we will abuse notation by omitting the subscript, since no confusion will arise.

At some step α , \mathcal{A} becomes \mathfrak{L}_α^A -atomic, in the sense that all \mathfrak{L}_α^A -types are principal.

DEFINITION 3.2. The *Scott rank* of \mathcal{A} , $sr(\mathcal{A})$, is the least ordinal α such that \mathcal{A} is an \mathfrak{L}_α^A -atomic structure.

LEMMA 3.3. *If \mathcal{A} is generically presentable, then, for every ordinal β , $\mathfrak{L}_\beta^A \in V$.*

PROOF. First, let us remark that we can code the formulas in \mathfrak{L}_β^A by sets: for instance, we code an infinitary conjunction of formulas ψ_i by a pair, the first element being a code that means “conjunction” and the second element being the set of codes for the formulas ψ_i —say, by defining $code(\bigwedge_{i \in I} \psi_i(x)) = \langle 17, \{code(\psi_i(x)) : i \in I\} \rangle$. This is quite standard, so we let the reader fill in the details.

The one important detail is that we are not coding infinitary conjunctions using sequences of formulas, but using sets where the order of the formulas does not matter. The key point is that if we have different presentations of a structure \mathcal{A} , the types realized in each presentation are the same as sets. We can then prove by induction on β , that \mathfrak{L}_β^A is a set that is independent of the presentation of \mathcal{A} . Since \mathcal{A} is generically presentable, say by a forcing notion \mathbb{P} , the language \mathfrak{L}_β^A belongs to all \mathbb{P} -forcing extensions of V , and so by Solovay’s Theorem 2.25, we get that \mathfrak{L}_β^A belongs to V . \dashv

DEFINITION 3.4. Given a structure \mathcal{A} , let $\hat{\mathcal{L}}$ be the language containing a relation symbol for each formula in $\mathfrak{L}_{sr(\mathcal{A})}^A$ (the *Morleyization* of $\mathfrak{L}_{sr(\mathcal{A})}^A$), and let $\hat{\mathcal{A}}$ be the natural expansion of \mathcal{A} to the language $\hat{\mathcal{L}}$. Note that if \mathcal{A} is generically presentable, then $\hat{\mathcal{L}} \in V$ since $\mathfrak{L}_{sr(\mathcal{A})}^A \in V$.

Notice that $\hat{\mathcal{A}}$ is atomic in a very strong way: each $\hat{\mathcal{L}}$ -type is generated by a quantifier-free $\hat{\mathcal{L}}$ -formula.

REMARK 3.5. Throughout this section, we will tacitly assume that \mathcal{L} (and hence $\hat{\mathcal{L}}$ as well) is no larger than \mathcal{A} ; that is, that the statement “ $|\mathcal{L}| \leq |\mathcal{A}|$ ” is true in every forcing extension by \mathbb{P} (where \mathbb{P} is a forcing generically presenting \mathcal{A}). This assumption is used, for example, in 3.7 below, and is necessary for straightforwardly applying the facts about amalgamation we will prove in Section 3.2. Note that this assumption holds for the vast majority of natural structures.

LEMMA 3.6. *If \mathcal{A} is generically presentable, then so is $\hat{\mathcal{A}}$.*

PROOF. We already showed that $\mathfrak{L}_{sr(\mathcal{A})}^A \in V$, so $\hat{\mathcal{L}} \in V$. There is only one way to expand \mathcal{A} to the $\hat{\mathcal{L}}$ -structure $\hat{\mathcal{A}}$. So, $\hat{\mathcal{A}}$ has a presentation with domain ω in every generic extension of V where \mathcal{A} does. \dashv

PROPOSITION 3.7. *Suppose \mathcal{A} is generically presentable by a forcing notion that does not collapse ω_1 . Then \mathcal{A} has a copy in V with domain ω .*

PROOF. Intuitively, the Scott sentence of \mathcal{A} must lie in V , and since ω_1 is not collapsed we can reconstruct \mathcal{A} from its Scott sentence.

In detail, let \mathbb{P} be a forcing notion that does not collapse ω_1 , and for which \mathcal{A} is generically presentable. Since $\hat{\mathcal{A}}$ is generically presentable, and $\hat{\mathcal{L}} \in V$, we have that $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}})$, the $\hat{\mathcal{L}}$ -theory of $\hat{\mathcal{A}}$, is a set of $\hat{\mathcal{L}}$ sentences that belongs to all \mathbb{P} -generic extensions. Thus, $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}}) \in V$.

In all of these extensions, $\hat{\mathcal{L}}$ is countable (because \mathcal{A} is), and, hence, $\hat{\mathcal{L}}$ cannot be uncountable in V . Otherwise, there would be an injection from ω_1 into $\hat{\mathcal{L}}$, and since \mathbb{P} does not collapse ω_1 , $\hat{\mathcal{L}}$ would stay uncountable in $V[G]$.

Now, in each of these generic extensions, $\hat{\mathcal{A}}$ is the unique countable atomic model of $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}})$. The existence of such a model is a Σ_1^1 statement with $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}})$ as parameter. By absoluteness, this must be true in V too, and by the uniqueness of $\hat{\mathcal{A}}$ in $V[G]$, this model must be isomorphic to $\hat{\mathcal{A}}$. ⊢

3.2. Keeping ω_2 uncountable. We now turn to the *Fraïssé limit* construction, first used in [7]:

DEFINITION 3.8. Fix a relational language \mathcal{L} . For an \mathcal{L} -structure \mathcal{B} , we denote by $\mathbf{K}_{\mathcal{B}}$ the set of (structures isomorphic to) finite substructures of \mathcal{B} , and we call $\mathbf{K}_{\mathcal{B}}$ the *age* of \mathcal{B} . For \mathbf{K} a set of finite structures and \mathcal{A} a structure, we say that \mathcal{A} is the *Fraïssé limit* of \mathbf{K} if $\mathbf{K}_{\mathcal{A}} = \mathbf{K}$ and the set of isomorphisms between finite substructures of \mathcal{A} has the back-and-forth property.

CONVENTION 3.9. *When we speak of the cardinality of an age, we will mean the cardinality of the age modulo isomorphism, that is, the number of isomorphism types of finite structures in that age.*

It is clear from the definition that if \mathcal{A} and \mathcal{B} are countable Fraïssé limits for the same age \mathbf{K} , then $\mathcal{A} \cong \mathcal{B}$. A given age may have nonisomorphic uncountable Fraïssé limits. For example, if \mathbf{K} is the set of finite linear orderings, the Fraïssé limits are the dense linear orderings without endpoints, and there are many—in fact, 2^{\aleph_1} many, the most possible—nonisomorphic ones of cardinality \aleph_1 .

LEMMA 3.10. *If \mathcal{A} is generically presentable, then $\mathbf{K}_{\mathcal{A}} \in V$.*

PROOF. This follows from Solovay’s Theorem 2.25: $\mathbf{K}_{\mathcal{A}}$ is a set of finite structures that is independent of the presentation of \mathcal{A} . ⊢

Using the same argument as in Proposition 3.7, we get a bound on the size of $\hat{\mathcal{L}}$ and $\mathbf{K}_{\hat{\mathcal{A}}}$:

COROLLARY 3.11. *If \mathcal{A} is generically presentable by a forcing not making ω_2 countable, then $\hat{\mathcal{L}}$ and $\mathbf{K}_{\hat{\mathcal{A}}}$ have size $\leq \aleph_1$ in V .*

Fraïssé [7] proved that if \mathbf{K} is a countable set of finite structures satisfying the Hereditary Property (*HP*), the Joint Embedding Property (*JEP*) and the Amalgamation Property (*AP*), then it has a Fraïssé limit (see 6.1 of [9] for definitions). The next lemma says that this is still the case when \mathbf{K} has size \aleph_1 . The earliest reference we know is Delhomme, Pouzet, Sagi, and Sauer [6, Corollary 2, p. 1378]. We give the proof because we want to make clear that the result does not automatically

generalize to ages of size $> \aleph_1$; and indeed, we will see in the next subsection that there is an age of size \aleph_2 with no limit (Corollary 3.19).

LEMMA 3.12. *Let \mathbf{K} be a family of \aleph_1 finite structures on a relational language \mathcal{L} of size $\leq \aleph_1$. If \mathbf{K} has HP, JEP, and AP, then there is a Fraïssé limit \mathcal{A} with age \mathbf{K} .*

PROOF. The key is the following:

CLAIM: Suppose we have embeddings $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{A} \rightarrow \mathcal{C}$ where $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ and \mathcal{C} is countable and its age is a subset of \mathbf{K} . Then there is a countable structure \mathcal{D} , whose age is a subset of \mathbf{K} , and which amalgamates these embeddings.

To prove the claim, write \mathcal{C} as the union of an increasing sequence $\{\mathcal{C}_n : n \in \omega\}$ where each $\mathcal{C}_n \in \mathbf{K}$, and with $\mathcal{C}_0 = \mathcal{A}$. Let $\mathcal{D}_0 = \mathcal{B}$, and note that we have an embedding from \mathcal{C}_0 to \mathcal{D}_0 . Given \mathcal{D}_n , by induction we will have an embedding from \mathcal{C}_n into \mathcal{D}_n , and \mathcal{D}_n will be an element of \mathbf{K} ; and by definition we have an embedding from \mathcal{C}_n into \mathcal{C}_{n+1} . We then form \mathcal{D}_{n+1} by amalgamating the embeddings $\mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$ and $\mathcal{C}_n \rightarrow \mathcal{D}_n$ within \mathbf{K} . The direct limit \mathcal{D} of the \mathcal{D}_i is the desired amalgamation.

Now we prove the lemma. Suppose \mathbf{K} is such a family of finite structures. There is a sequence $(\mathcal{A}_\xi)_{\xi \in \omega_1}$ of structures such that:

- $\xi_0 < \xi_1 \Rightarrow \mathcal{A}_{\xi_0} \subseteq \mathcal{A}_{\xi_1}$;
- each \mathcal{A}_ξ is countable and its age is a subset of \mathbf{K} ; and
- for every $\xi \in \omega_1$ and $\mathcal{B}, \mathcal{C} \in \mathbf{K}$ and every pair of embeddings $\mathcal{B} \rightarrow \mathcal{C}$ and $\mathcal{B} \rightarrow \mathcal{A}_\xi$, there is $\gamma > \xi$ and an embedding $\mathcal{C} \rightarrow \mathcal{A}_\gamma$ compatible with the inclusion $\mathcal{A}_\xi \rightarrow \mathcal{A}_\gamma$.

The union \mathcal{A} of the \mathcal{A}_ξ clearly has age \mathbf{K} . It is clear from the construction that the set of finite partial isomorphisms has the back-and-forth property. ⊢

Note that the limit \mathcal{A} constructed above need not be \aleph_1 -homogeneous or unique.

COROLLARY 3.13. *Let \mathcal{B} be an \mathcal{L} -structure that lives in an extension of the universe and is ω -homogeneous in the sense that the family of isomorphisms between finite substructures has the back-and-forth property. Suppose \mathcal{B} is generically presentable, and $|\mathbf{K}_\mathcal{B}|, |\mathcal{L}| \leq \aleph_1$ in V . Then in V there is a structure $\mathcal{L}_{\infty\omega}$ -equivalent to \mathcal{B} .*

PROOF. Since \mathcal{B} is generically presentable, we have that $\mathbf{K}_\mathcal{B} \in V$ by Lemma 3.10. Since \mathcal{B} is ω -homogeneous, $\mathbf{K}_\mathcal{B}$ has HP, JEP, and AP in any model where \mathcal{B} lives; since these properties are absolute, we conclude that $\mathbf{K}_\mathcal{B}$ has these properties in V . Since $|\mathbf{K}_\mathcal{B}| \leq \aleph_1$ and $|\mathcal{L}| \leq \aleph_1$ in V , by Lemma 3.12 we have that $\mathbf{K}_\mathcal{B}$ has a Fraïssé limit \mathcal{F} in V . In a generic extension presenting \mathcal{B} , the age $\mathbf{K}_\mathcal{B}$ —and, hence, the Fraïssé limit \mathcal{F} —will be countable. Then $\mathcal{F} \cong \mathcal{B}$, by the uniqueness of countable Fraïssé limits, so \mathcal{F} is the required structure $\mathcal{L}_{\infty\omega}$ -equivalent to \mathcal{B} which lives in V . ⊢

We are now ready to prove the main positive result of this section.

THEOREM 3.14. *Suppose \mathcal{A} is generically presentable by a forcing notion \mathbb{P} that does not make ω_2 countable. Then there is a copy of \mathcal{A} in V , with cardinality at most \aleph_1 in V .*

More precisely, if (\mathbb{P}, ν) is a generically presentable structure and \mathbb{P} does not make ω_2 countable, then there is a copy $\mathcal{B} \in V$ of (\mathbb{P}, ν) , with $|\mathcal{B}| \leq \aleph_1$.

PROOF. Let \mathcal{L} be the language of \mathcal{A} . Since \mathcal{A} is generically presentable, by Lemmas 3.3 and 3.6 we know that $\hat{\mathcal{L}}$ is in V and $\hat{\mathcal{A}}$ is generically presentable. Consider some generic extension $V[G]$ by a forcing which generically presents \mathcal{A} and which does not make ω_2 countable. Using in $V[G]$ the fact that Scott ranks of countable structures are countable, since ω_2^V is still uncountable in $V[G]$ the language $\hat{\mathcal{L}}$ has size $\leq \aleph_1$ in V . This implies that $\mathbf{K}_{\hat{\mathcal{A}}}$ is in V by Lemma 3.10 and has size $\leq \aleph_1$ in V by Corollary 3.11. Now, we can apply Corollary 3.13 to get a copy of $\hat{\mathcal{A}}$ which lives in V (of course, $\hat{\mathcal{A}}$ need not be countable in V). Intuitively, we now want to take the reduct of this copy to \mathcal{L} , but $\hat{\mathcal{L}}$ need not include \mathcal{L} (for instance, if two \mathcal{L} -symbols have the same interpretation); instead, from $\hat{\mathcal{A}}$ we can now “decode” the correct interpretations of each of the symbols in \mathcal{L} , and thus produce a copy of \mathcal{A} itself. \dashv

Note that Theorem 3.14 does not directly imply Proposition 3.7, since the latter concludes that the generically presentable structure in question has a *countable* copy in V .

We may apply Theorem 3.14 to prove the following.

THEOREM 3.15 (Harrington, unpublished). *If T is a counterexample to Vaught’s Conjecture, then for each $\beta < \omega_2$, T has a model of size \aleph_1 with Scott rank $\geq \beta$.*

PROOF. Recall that if T is a counterexample to Vaught’s conjecture it has countable models of arbitrary Scott rank below ω_1 . Being a counterexample to Vaught’s conjecture is a Π_2^1 property ([19]; see also [23], Proposition 5.1) and hence absolute. Let $\mathbb{P} = \omega_1^{<\omega}$ be the usual Levy collapse of ω_1 and let G be \mathbb{P} -generic. Note that \mathbb{P} is *homogeneous* in the following sense: the partial orders \mathbb{P} and $\{q \in \mathbb{P} : q \leq p\}$ are isomorphic for any $p \in \mathbb{P}$. Since T is a counterexample to Vaught’s conjecture, in $V[G]$ we have a countable model \mathcal{B} of Scott rank $\alpha \geq \beta$. We claim that \mathcal{B} is generically presentable over V by \mathbb{P} , that is, that there is a generically presentable structure (\mathbb{P}, μ) such that $V[G] \models \mu[G] \cong \mathcal{B}$. This would give us the claimed result: since \mathbb{P} does not collapse ω_2 , by Theorem 3.14, we would have a copy of \mathcal{B} of size \aleph_1 in V , and since Scott rank is absolute, this copy is as wanted.

So fix a \mathbb{P} -generic G and a name $v \in V$ for a structure \mathcal{B} in $V[G]$ which is (in $V[G]$) a countable model of T with Scott rank $\geq \alpha$, and suppose towards contradiction that \mathcal{B} is *not* generically presentable in the sense of the previous paragraph. This will let us produce a size-continuum set of countable models of T of bounded Scott rank, thus contradicting the assumption that T is a counterexample to Vaught’s conjecture.

We proceed as follows. First, suppose without loss of generality that

$$\Vdash “v[G] \models T \text{ and } sr(v[G]) = \alpha;”$$

we can make this assumption since *some* condition in G must force this, and \mathbb{P} is homogeneous so we may take that condition to be the empty condition. We now claim that whenever H_0, H_1 are mutually \mathbb{P} -generic, we have $v[H_0] \not\cong v[H_1]$. This immediately follows from the assumption that \mathcal{B} is not generically presentable—otherwise, taking an H_0, H_1 mutually generic with $v[H_0] \cong v[H_1]$, we must have some $p_i \in H_i$ such that $(p_0, p_1) \Vdash_{\mathbb{P} \times \mathbb{P}} v[G_0] \cong v[G_1]$. Since \mathbb{P} is homogeneous we may assume $p_0 = p_1 = \emptyset$; but then this contradicts our assumption that \mathcal{B} is not generically presentable.

So we have that mutually generic filters through \mathbb{P} yield nonisomorphic models of T of Scott rank α . Now, consider a forcing notion \mathbb{Q} that adds perfectly many mutually \mathbb{P} -generics. (This is quite standard: for instance let \mathbb{Q} be the set of finite partial maps from $2^{<\omega}$ to $\omega_1^{<\omega}$ and then obtain the \mathbb{P} generics by concatenating the $\omega_1^{<\omega}$ -strings along each path in 2^ω .) After forcing with \mathbb{Q} , by the arguments above we obtain continuum many pairwise-nonisomorphic countable models of T , each of Scott rank $\alpha < \omega_1$. Since being a counterexample to Vaught's conjecture is absolute, this is a contradiction. \dashv

REMARK 3.16. Recently, Baldwin, S.-D. Friedman, Koerwien, and Laskowski [4] have given a new proof of Harrington's result using similar genericity arguments; their proof uses a generic version of the Morley tree, which they show is invariant across forcing extensions.

Finally, we can use Theorem 3.14 to give a partial positive answer to Slaman's question:

COROLLARY 3.17. *Suppose \mathcal{A} is a generically presentable structure with $\mathcal{A} \leq_w^* \mathcal{B}$ for some $\mathcal{B} \in V$ with cardinality $\leq \aleph_1$. Then \mathcal{A} has a copy in V .*

PROOF. Let \mathbb{P} be a forcing notion that collapses ω_1 while keeping ω_2 uncountable, such as $\mathbb{P} = \omega_1^{<\omega}$. Let $V[G]$ be a generic extension by \mathbb{P} . Then \mathcal{B} is countable in $V[G]$, and, a fortiori, there is a copy of \mathcal{A} in $V[G]$. It follows that \mathcal{A} is \mathbb{P} -generically presentable. Then by Theorem 3.14, there is a copy of \mathcal{A} in V . \dashv

3.3. Collapsing ω_2 to ω . We close this section by presenting a strong negative result, coming from a construction due to Shelah and Laskowski [16]. Throughout the rest of this section, we abbreviate the linear order $(\omega_2, <)$ by " ω_2 ."

THEOREM 3.18. *There is a structure \mathcal{A} , generically presentable by any forcing making ω_2 countable, but with no copy in V .*

PROOF. Laskowski and Shelah [16] gave an example of an elementary first order theory T , in a countable language, such that:

- (1) The language has a sort V such that, for every model \mathcal{M} of T and every subset $A \subseteq V^{\mathcal{M}}$, $T(A)$ has an atomic model if and only if $|A| \leq \aleph_1$.
- (2) T has a countable model \mathcal{M}_0 such that $V^{\mathcal{M}_0}$ is totally indiscernible in the sense that any permutation of $V^{\mathcal{M}_0}$ extends to an automorphism of \mathcal{M}_0 . Furthermore, \mathcal{M}_0 is atomic over $V^{\mathcal{M}_0}$.

For \mathcal{C} a countable structure, let $\mathcal{M}_{\mathcal{C}}$ be the two-sorted structure with one sort corresponding to a copy of \mathcal{C} , one sort corresponding to a copy of \mathcal{M}_0 , and with a function symbol f providing a bijection between \mathcal{C} and $V_0^{\mathcal{M}}$. Since the elements of $V^{\mathcal{M}_0}$ are totally indiscernible, any two choices of f yield isomorphic structures, so $\mathcal{M}_{\mathcal{C}}$ is well-defined.

Now consider the "structure" \mathcal{M}_{ω_2} which lives in any extension of the universe where ω_2 is countable. Thus, \mathcal{M}_{ω_2} is generically presentable by $Col(\omega_2, \omega)$. However, there is no copy of \mathcal{M}_{ω_2} in V : Since if the first sort is really ω_2 , of size \aleph_2 , then in the second sort, the predicate V has size \aleph_2 . But, by the assumption on \mathcal{M}_0 , $\mathcal{M}_{\mathcal{C}}$ is always atomic over \mathcal{C} (a fact that is absolute), and by the assumption on T , $T(V^{\mathcal{M}_{\omega_2}})$ has no atomic models. \dashv

The structure of Laskowski and Shelah also provides a counterexample to a natural extension of Lemma 3.12.

COROLLARY 3.19. *There is an age S of size \aleph_2 with the Hereditary, Joint Embedding, and Amalgamation properties but for which there is no Fraïssé limit.*

PROOF. Consider the theory $T(A) = Th(\mathcal{M}_0, a_{a \in A})$, where $A = A^M$ has size \aleph_2 . The principal types are dense, but $T(A)$ has no atomic model. We add predicate symbols for the principal types. For $B \subseteq A$ of size up to \aleph_1 , there is an atomic model of the corresponding theory $T(B) = Th(\mathcal{M}_0, a_{a \in B})$. Let K consist of the finite substructures of the atomic models of the theories $T(B)$. In total, what we have is appropriate to be the age for an atomic model of $T(A)$. That is, we have the Hereditary, Joint Embedding, and Amalgamation properties (essentially [16], pg. 3). However, any Fraïssé limit of S would yield an atomic model of $T(A)$, so the Fraïssé limit cannot exist. \dashv

§4. Generically presentable rigid structures. In the previous section, we gave a complete characterization of those posets \mathbb{P} with the property that every structure generically presentable by \mathbb{P} has a copy already in the ground model. In this section, we examine the dual question: what properties of structures ensure that generic presentability implies the existence of a copy in the ground model? Specifically, we extend Solovay’s Theorem 2.25 to structures that are sufficiently “set-like:”

THEOREM 4.1. *If a generically presentable structure is rigid, then it has a copy in the ground model.*

More precisely, suppose (\mathbb{P}, ν) is a generically presentable structure such that $\Vdash_{\mathbb{P}} \nu[G]$ has no nontrivial automorphisms.” Then (\mathbb{P}, ν) has a copy in V .

PROOF. We assume the language \mathcal{L} of the rigid generically presentable structure $\mathcal{N} = (\mathbb{P}, \nu)$ is relational. On $\omega \times \mathbb{P}$, we define the relation \equiv as follows:

$$(a, p) \equiv (b, q) \iff (p, q) \Vdash_{\mathbb{P}^2} \text{“}\{(a, b)\} \text{ extends to an isomorphism } \nu[g_0] \cong \nu[g_1].\text{”}$$

If $(a, p) \equiv (a, p)$, we say a is *stable* in p , and we write \mathbb{M} for the set $\{(a, p) : a \text{ is stable in } p\}$.

LEMMA 4.2. *The relation \equiv is an equivalence relation on \mathbb{M} .*

PROOF. Symmetry is clear, and reflexivity is immediate from the definition of \mathbb{M} . For transitivity, suppose $(a, p) \equiv (b, q) \equiv (c, r)$, and let $G_0 \times G_1$ be \mathbb{P}^2 -generic over V with $p \in G_0, r \in G_1$; and let H be \mathbb{P} -generic over $V[G_0 \times G_1]$ -generic, with $q \in H$. Then clearly in $V[G_0 \times G_1][H]$, there is an isomorphism between $\nu[G_0]$ and $\nu[G_1]$ taking a to c ; but this is a Σ_1^1 property, and so already true in $V[G_0 \times G_1]$. Thus, $(a, p) \equiv (c, r)$. \dashv

Now let M be the set of \equiv -classes of elements of \mathbb{M} . The basic properties of M , which parallel the properties of ages needed for Fraïssé constructions, are:

LEMMA 4.3. *For $p \in \mathbb{P}, a \in \omega$,*

- (1) (Extension) *if a is stable in p and $q \leq p$, then a is stable in q and $(a, p) \equiv (a, q)$;*
and
- (2) (Genericity) *there is some $q \leq p$ with a stable in q .*

PROOF.

- (1) That a is stable in q is immediate from the definition of stability. To see that $(a, p) \equiv (a, q)$, note that any pair of generics H_0, H_1 witnessing the failure of $(a, p) \equiv (a, q)$ would also witness the instability of (a, p) .
- (2) Consider the condition $(p, p) \in \mathbb{P}^2$. By our assumption on v , there must be some condition $(q, q') \leq (p, p)$ and $a' \in \omega$ such that

$$(q, q') \Vdash_{\mathbb{P}^2} \{(a, a')\} \text{ extends to an isomorphism } v[g'_0] \cong v[g'_1].$$

It now follows that a is stable in q : given $G_0 \times G_1$ \mathbb{P}^2 -generic over V extending (q, q) , fix some H which is \mathbb{P} -generic over $V[G_0 \times G_1]$ with $q' \in H$. Then in $V[G_0 \times G_1][H]$ there is an isomorphism between $v[G_0]$ and $v[G_1]$ extending $\{(a, a)\}$; but this is a Σ_1^1 fact, so already true in $V[G_0 \times G_1]$. \dashv

Finally, the following result is where rigidity is used. Intuitively, rigidity plays the role in our proof that ω -homogeneity plays in standard Fraïssé limit constructions.

LEMMA 4.4 (Simultaneity). *Suppose $p, q \in \mathbb{P}$ and $i_1, \dots, i_n: \subseteq \omega \rightarrow \omega$ are partial maps in V with disjoint domains which are each forced by (p, q) in \mathbb{P}^2 to extend to isomorphisms $j_1, \dots, j_n: v[G_0] \cong v[G_1]$. Then*

$$(p, q) \Vdash_{\mathbb{P}^2} \bigcup_{1 \leq j \leq n} i_j \text{ extends to an isomorphism } v[G_0] \cong v[G_1].$$

Note that this result immediately implies the seemingly stronger result in which disjointness of domains is not assumed.

PROOF. We will prove the lemma in the case where $n = 2$, $p = q$, $i_1 = \{(a, a)\}$, and $i_2 = \{(b, b)\}$ for some distinct $a, b \in \omega$; the general result is no different. Note that the assumption on i_j in this case means just that a and b are stable in p .

Let $G_0 \times G_1$ be \mathbb{P}^2 -generic extending (p, p) . Then, forced by (p, p) , there are isomorphisms $j_1, j_2: v[G_0] \cong v[G_1]$ with $j_1(a) = a$ and $j_2(b) = b$. Consider the map $j = j_1 \circ j_2^{-1}$. This is an automorphism of $v[G_1]$, and hence by rigidity must be the identity; so $j_1(b) = b$, since $j_2^{-1}(b) = b$ by assumption on j_2 . But then j_1 is an isomorphism extending $\{(a, a), (b, b)\}$, so (p, p) forces that there is an isomorphism between $v[G_0]$ and $v[G_1]$ extending $\{(a, a), (b, b)\}$. \dashv

Now we come to the body of the proof of Theorem 4.1. We can turn \mathbb{M} into an \mathcal{L} -structure, \mathcal{M} , as follows: writing (\mathbf{a}, \mathbf{p}) for the equivalence class of $(a, p) \in \mathbb{M}$, for each n -ary relation symbol $R \in \mathcal{L}$ we let $R^{\mathcal{M}}$ be the set of tuples $((\mathbf{a}_1, \mathbf{p}_1), \dots, (\mathbf{a}_n, \mathbf{p}_n))$ such that

$$\exists q \in \mathbb{P}, c_1, \dots, c_n \text{ stable in } q \ (\forall i \leq n [(\mathbf{a}_i, \mathbf{p}_i) = (\mathbf{c}_i, \mathbf{q})] \wedge q \Vdash "v \models R(c_1, \dots, c_n)").$$

Informally, this definition ensures that each relation R holds whenever it ought to hold; we will also need the converse result, that each R fails whenever it ought to fail, and this is where Simultaneity will come in.

LEMMA 4.5. *Let G be \mathbb{P} -generic over V . Then $V[G] \models v[G] \cong \mathcal{M}$.*

PROOF. For $a \in v[G]$, let $Stab_a^G = \{p \in G : (a, p) \in \mathbb{M}\}$. Then for every $p, q \in Stab_a^G$, we must have $(a, p) \equiv (a, q)$: since $p, q \in G$, there must be a common strengthening $r \leq p, q$; by 4.3(1), we have $(a, p) \equiv (a, r)$ and $(a, q) \equiv (a, r)$, and

hence $(a, p) \equiv (a, q)$ by transitivity. So the set $\{(a, p) : p \in \text{Stab}_a^G\}$ is contained in a single \equiv -class, and hence corresponds to a single element of \mathcal{M} .

Consider the map $i : v[G] \rightarrow \mathcal{M} : a \mapsto \{a\} \times \text{Stab}_a^G$. We claim that i is an isomorphism. Surjectivity is an immediate consequence of genericity (Lemma 4.3(2)), and injectivity follows from the rigidity of $v[G]$.

Finally, we must show that i is a homomorphism. Let R be a relation symbol in \mathcal{L} and $\bar{a} \in v[G]$. First, suppose $v[G] \models R(\bar{a})$. Let $p \in G$ be such that $p \in \bigcap_{a \in \bar{a}} \text{Stab}_a^G$ and $p \Vdash v[G] \models R(\bar{a})$. Then p witnesses that $\mathcal{M} \models R(i(\bar{a}))$. Conversely, suppose $\mathcal{M} \models R(i(\bar{a}))$ and fix $p \in \bigcap_{a \in \bar{a}} \text{Stab}_a^G$. Then we must have some $q \in \mathbb{P}$ and \bar{c} stable in q such that $(c_i, q) \equiv (a_i, p)$ for each i and $q \Vdash R(\bar{c})$. But then by simultaneity (Lemma 4.4) we must have $p \Vdash R(\bar{a})$. ⊢

This finishes the proof of Theorem 4.1. ⊢

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